

FREE 2-RANK OF SYMMETRY OF PRODUCTS OF MILNOR MANIFOLDS

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Abstract

A real Milnor manifold is the non-singular hypersurface of degree $(1, 1)$ in the product of two real projective spaces. These manifolds were introduced by Milnor to give generators for the unoriented cobordism algebra, and they admit free actions by elementary abelian 2-groups. In this paper, we obtain some results on the free 2-rank of symmetry of products of finitely many real Milnor manifolds under the assumption that the induced action on mod 2 cohomology is trivial. Similar results are obtained for complex Milnor manifolds that are defined analogously. Here the free 2-rank of symmetry of a topological space is the maximal rank of an elementary abelian 2-group that acts freely on that space.

1. Introduction

One of the basic problems in the theory of transformation groups is to determine the structure of a group that acts in a specific way on a given topological space. The problem of determining finite groups that can act freely on spheres has been of special interest. A classical result of Smith [29] says that if a group acts freely on a sphere, then all its abelian subgroups are cyclic. Conversely, Swan [30] proved that any group satisfying this condition acts freely on a finite complex with the homotopy type of a sphere. Notice that a finite abelian group is cyclic if and only if it does not contain a subgroup of the form $\mathbb{Z}/p \oplus \mathbb{Z}/p$ for any prime p . Thus, $\mathbb{Z}/p \oplus \mathbb{Z}/p$ cannot act freely on a sphere. However, \mathbb{Z}/p does act freely on a sphere, where the sphere must be odd dimensional for odd p . These results motivated the concept of free p -rank of symmetry of a topological space X for a prime p , introduced in [1], and defined as

$$\text{frk}_p(X) = \max\{r \mid (\mathbb{Z}/p)^r \text{ acts freely on } X\}.$$

Determining the free p -rank of symmetry of a topological space is an interesting problem and has been considered for many spaces. In view of the Smith theory, we

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have

$$\mathrm{frk}_p(\mathbb{S}^n) = \begin{cases} 1 & \text{if } n \text{ is odd and } p \text{ is arbitrary} \\ 1 & \text{if } n \text{ is even and } p = 2 \\ 0 & \text{if } n \text{ is even and } p > 2. \end{cases}$$

The problem of extending this result to products of finitely many spheres has been of great interest to many topologists. Oliver [28] proved that every finite group acts freely on a product of spheres. For a finite group G , define

$$h(G) = \min\{s \mid G \text{ acts freely on } \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_s}\} \text{ and} \\ r(G) = \max\{t \mid (\mathbb{Z}/p)^t \leq G \text{ for some prime } p\}.$$

Attempts to extend Smith's result to products of finitely many spheres led to the following conjecture by Benson and Carlson [6].

Conjecture I. $h(G) = r(G)$ for each finite group G .

Conjecture I implies the following well-known conjecture [1, 2, 9] regarding free actions of elementary abelian p -groups on products of spheres.

Conjecture II. $\mathrm{frk}_p(\mathbb{S}^{2n_1+1} \times \cdots \times \mathbb{S}^{2n_k+1}) = k$ for each prime p and integer $k \geq 0$.

For a single sphere it is simply the result of Smith [29]. For products of two spheres it was already proved by Conner [11], and for products of three spheres it was proved to be true by Heller [20]. Carlsson [9, 10] proved the conjecture for products of equidimensional spheres under the assumption of trivial induced action on cohomology. Adem and Browder [1] proved that $\mathrm{frk}_p((\mathbb{S}^n)^k) = k$ with the only remaining cases as $p = 2$ and $n = 1, 3, 7$. Some time later, Yalçın [32] proved that $\mathrm{frk}_2((\mathbb{S}^1)^k) = k$. For an integer n , let

$$\eta(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Then the most comprehensive result is due to Hanke [19], who proved that if $p > 3(n_1 + \cdots + n_k)$, then

$$\mathrm{frk}_p(\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}) = \eta(n_1) + \cdots + \eta(n_k).$$

In a very recent work [27], Okutan and Yalçın proved Conjecture II in the case where the dimensions $\{n_i\}$ are high compared to all the differences $|n_i - n_j|$ between the dimensions.

Recall that $\mathbb{Z}/2$ is the only finite group that can act freely on an even-dimensional sphere. This result was extended to products of finitely many even-dimensional spheres by Cusick [14, 15]. He proved that if G is a finite group acting freely on $\mathbb{S}^{2n_1} \times \cdots \times \mathbb{S}^{2n_k}$ with trivial induced action on mod 2 cohomology, then $G \cong (\mathbb{Z}/2)^r$ for some $r \leq k$. In particular, when the induced action on mod 2 cohomology is trivial, then $\mathrm{frk}_2(\mathbb{S}^{2n_1} \times \cdots \times \mathbb{S}^{2n_k}) = k$.

Although a lot of work has been done for products of spheres, free p -rank of symmetry of many other interesting spaces is still not known. An immediate extension of the problem from spheres and their products is to consider spherical space forms and their products. Let p be an odd prime. A lens space L_p^{2n-1} is an odd-dimensional

spherical space form obtained as the quotient of the standard \mathbb{Z}/p action on \mathbb{S}^{2n-1} . Allday [5] conjectured that

$$\text{frk}_p(L_p^{2n_1-1} \times \cdots \times L_p^{2n_k-1}) = k.$$

The equidimensional case of the conjecture was proved by Yalçın [31]. The general case of the conjecture seems still open.

The problem of computing the free 2-rank of symmetry of products of projective spaces was considered by Cusick. It is known that $\mathbb{C}P^n$ admits a free action by a finite group if and only if n is odd, in which case the only possible group is $\mathbb{Z}/2$. Cusick [16] proved that

$$\text{frk}_2(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}) = \eta(n_1) + \cdots + \eta(n_k).$$

For an integer n , let

$$\theta(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \equiv 1 \pmod{4} \\ 2 & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Cusick [13] investigated the real case and conjectured that if the induced action on mod 2 cohomology is trivial, then

$$\text{frk}_2(\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}) = \theta(n_1) + \cdots + \theta(n_k).$$

He proved the conjecture when $n_i \not\equiv 3 \pmod{4}$ for each $1 \leq i \leq k$. Adem and Yalçın [3] proved the conjecture for products of equidimensional real projective spaces without the assumption of trivial induced action on mod 2 cohomology. Later, Yalçın [32] proved the conjecture when n_i is odd for each $1 \leq i \leq k$. The general case of the conjecture is still open.

A product of two projective spaces can be considered as the total space of a trivial projective space bundle over a projective space. It is an interesting question to determine the free rank of symmetry of the total space of a twisted projective space bundle over a projective space. Milnor manifolds are fundamental examples of such spaces. These manifolds were introduced by Milnor [25] in search for generators for the unoriented cobordism algebra, and are non-singular hypersurfaces of degree $(1, 1)$ in the product of two projective spaces (see Section 2 for detailed definitions). The purpose of this paper is to obtain some results regarding the free 2-rank of symmetry of products of finitely many Milnor manifolds. We will consider both real and complex Milnor manifolds. We will prove our results by adopting Cusick's method, which depends on some results of Carlsson.

Let $\mathbb{R}H_{r,s}$ and $\mathbb{C}H_{r,s}$ denote the real and the complex Milnor manifold, respectively (see Section 2 for notation). Let $X \simeq_2 Y$ mean that X and Y are topological spaces having isomorphic mod 2 cohomology algebra. Then the main results are as follows.

Theorem 1.1. *Let $(\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. Suppose that the induced action on mod 2 cohomology of X is trivial. Then*

1. $r \leq 2(\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$,
2. $r \leq (\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$ if $s_i, r_i \not\equiv 3 \pmod{4}$ for each $1 \leq i \leq n$.

Theorem 1.2. *Let $(\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. Suppose that the induced action on mod 2 cohomology of X is trivial. Then*

1. $r \leq 3(\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$,
2. $r \leq (\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$ if $s_i, r_i \not\equiv 3 \pmod{4}$ for each $1 \leq i \leq n$.

2. Definition and cohomology of Milnor manifolds

Milnor manifolds were introduced by Milnor [25] in search for generators for the unoriented cobordism algebra. Let r and s be integers such that $0 \leq s \leq r$. A real Milnor manifold, denoted by $\mathbb{R}H_{r,s}$, is the non-singular hypersurface of degree $(1, 1)$ in the product $\mathbb{R}P^r \times \mathbb{R}P^s$. It is a $(s+r-1)$ -dimensional closed smooth manifold and can also be described in terms of homogeneous coordinates of real projective spaces as

$$\mathbb{R}H_{r,s} = \left\{ ([x_0, \dots, x_r], [y_0, \dots, y_s]) \in \mathbb{R}P^r \times \mathbb{R}P^s \mid x_0 y_0 + \cdots + x_s y_s = 0 \right\}.$$

Equivalently, a real Milnor manifold can be defined as the total space of the fiber bundle

$$\mathbb{R}P^{r-1} \xrightarrow{i} \mathbb{R}H_{r,s} \xrightarrow{\pi} \mathbb{R}P^s.$$

This is actually the projectivization of the vector bundle

$$\mathbb{R}^r \hookrightarrow E^\perp \longrightarrow \mathbb{R}P^s,$$

which is the orthogonal complement in $\mathbb{R}P^s \times \mathbb{R}^{r+1}$ of the canonical line bundle

$$\mathbb{R} \hookrightarrow E \longrightarrow \mathbb{R}P^s,$$

where $E = \left\{ ([x], y) \in \mathbb{R}P^s \times \mathbb{R}^{r+1} \mid y \in [x] \right\}$.

These manifolds are important as the unoriented cobordism algebra of smooth manifolds is generated by the cobordism classes of $\mathbb{R}P^k$ and $\mathbb{R}H_{r,s}$ [25, Lemma 1]. Therefore, determining their various invariants is an important problem.

In a similar way, a complex Milnor manifold, denoted by $\mathbb{C}H_{r,s}$, is the non-singular hypersurface of degree $(1, 1)$ in the product $\mathbb{C}P^r \times \mathbb{C}P^s$. It is a $2(s+r-1)$ -dimensional closed smooth manifold and can also be described in terms of homogeneous coordinates as

$$\mathbb{C}H_{r,s} = \left\{ ([z_0, \dots, z_r], [w_0, \dots, w_s]) \in \mathbb{C}P^r \times \mathbb{C}P^s \mid z_0 \bar{w}_0 + \cdots + z_s \bar{w}_s = 0 \right\}.$$

Equivalently, as in the real case, a complex Milnor manifold can be defined as the total space of the fiber bundle

$$\mathbb{C}P^{r-1} \xrightarrow{i} \mathbb{C}H_{r,s} \xrightarrow{\pi} \mathbb{C}P^s.$$

It is known due to Conner and Floyd [12, p. 63] that $\mathbb{C}H_{r,s}$ is unoriented cobordant to $\mathbb{R}H_{r,s} \times \mathbb{R}H_{r,s}$.

These manifolds have been well studied in the past. See, for example, [17, 21, 26] for recent results. Their cohomology algebra is also well known, and we will use it in the proofs of our main theorems.

Lemma 2.1 ([8, 26]). *Let $0 \leq s \leq r$. Then the mod 2 cohomology algebra of a Milnor manifold is given as follows:*

1. $H^*(\mathbb{R}H_{r,s}; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]/\langle a^{s+1}, b^r + ab^{r-1} + \cdots + a^s b^{r-s} \rangle$,
where a and b are homogeneous elements of degree 1 each.
2. $H^*(\mathbb{C}H_{r,s}; \mathbb{Z}/2) \cong \mathbb{Z}/2[g, h]/\langle g^{s+1}, h^r + gh^{r-1} + \cdots + g^s h^{r-s} \rangle$,
where g and h are homogeneous elements of degree 2 each.

Note that $\mathbb{R}H_{r,0} = \mathbb{R}P^{r-1}$ and $\mathbb{C}H_{r,0} = \mathbb{C}P^{r-1}$. Since the free 2-rank of symmetry of products of projective spaces has already been considered by Cusick [13], we henceforth assume that $1 \leq s \leq r$.

3. Examples of free actions of elementary abelian 2-groups

Just like projective spaces, Milnor manifolds also admit free actions by elementary abelian 2-groups.

3.1. The real case

First, we construct free actions on $\mathbb{R}H_{r,s}$ for various values of s and r .

3.1.1. When $s = r$

The involution on $\mathbb{R}P^s \times \mathbb{R}P^s$ given by

$$([x_0, \dots, x_s], [y_0, \dots, y_s]) \mapsto ([y_0, \dots, y_s], [x_0, \dots, x_s])$$

restricts to a free involution $A : \mathbb{R}H_{s,s} \rightarrow \mathbb{R}H_{s,s}$.

3.1.2. When $s, r \equiv 1 \pmod{4}$

Let $n = s, r$ and $S_1 : \mathbb{R}P^n \rightarrow \mathbb{R}P^n$ be the free involution given by

$$[x_0, x_1, \dots, x_{n-1}, x_n] \mapsto [-x_1, x_0, \dots, -x_n, x_{n-1}].$$

Then the restriction of $S_1 \times S_1 : \mathbb{R}P^r \times \mathbb{R}P^s \rightarrow \mathbb{R}P^r \times \mathbb{R}P^s$ on $\mathbb{R}H_{r,s}$ gives a free involution

$$A_1 : \mathbb{R}H_{r,s} \rightarrow \mathbb{R}H_{r,s}.$$

3.1.3. When $s, r \equiv 3 \pmod{4}$

First, notice that the free involution A_1 is also defined in this case. Let $n = s, r$ and $S_2 : \mathbb{R}P^n \rightarrow \mathbb{R}P^n$ be the free involution given by

$$[x_0, x_1, x_2, x_3, \dots, x_{r-3}, x_{r-2}, x_{r-1}, x_r] \mapsto [-x_2, x_3, x_0, -x_1, \dots, -x_{r-1}, x_r, x_{r-3}, -x_{r-2}].$$

Then the restriction of $S_2 \times S_2 : \mathbb{R}P^r \times \mathbb{R}P^s \rightarrow \mathbb{R}P^r \times \mathbb{R}P^s$ on $\mathbb{R}H_{r,s}$ gives a free involution

$$A_2 : \mathbb{R}H_{r,s} \rightarrow \mathbb{R}H_{r,s}.$$

Notice that $A_1 \neq A_2$, $A_1 A_2 = A_2 A_1$, and $A_1 A_2$ acts freely on $\mathbb{R}H_{r,s}$. Therefore, $(\mathbb{Z}/2)^2$ acts freely on $\mathbb{R}H_{r,s}$ when $s, r \equiv 3 \pmod{4}$.

3.2. The complex case

Now we construct free actions on $\mathbb{C}H_{r,s}$ for various values of s and r .

3.2.1. When $s = r$

The involution on $\mathbb{C}P^s \times \mathbb{C}P^s$ given by

$$([z_0, \dots, z_s], [w_0, \dots, w_s]) \mapsto ([w_0, \dots, w_s], [z_0, \dots, z_s])$$

restricts to a free involution $B : \mathbb{C}H_{s,s} \rightarrow \mathbb{C}H_{s,s}$.

3.2.2. When both s and r are odd

It is known that $\mathbb{C}P^n$ admits a free action by a finite group if and only if n is odd and in that case the only possible group is $\mathbb{Z}/2$. Let $n = s, r$ and $T_1 : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ be the free involution given by

$$[z_0, z_1, \dots, z_{n-1}, z_n] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_n, \bar{z}_{n-1}].$$

Then the restriction of $T_1 \times T_1 : \mathbb{C}P^r \times \mathbb{C}P^s \rightarrow \mathbb{C}P^r \times \mathbb{C}P^s$ on $\mathbb{C}H_{r,s}$ gives a free involution

$$B_1 : \mathbb{C}H_{r,s} \rightarrow \mathbb{C}H_{r,s}.$$

3.2.3. When s is odd and r is even

Let $T_2 : \mathbb{C}P^r \rightarrow \mathbb{C}P^r$ be the involution given by

$$[z_0, z_1, \dots, z_{r-2}, z_{r-1}, z_r] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_{r-1}, \bar{z}_{r-2}, \iota z_r],$$

where $\iota^2 = -1$. Notice that T_2 is not a free involution. But the restriction of $T_2 \times T_1 : \mathbb{C}P^r \times \mathbb{C}P^s \rightarrow \mathbb{C}P^r \times \mathbb{C}P^s$ on $\mathbb{C}H_{r,s}$ gives a free involution

$$B_2 : \mathbb{C}H_{r,s} \rightarrow \mathbb{C}H_{r,s}.$$

4. Preliminary results

Here we recall some facts that we will use in this paper. We refer the reader to [4, 7, 24] for details on the cohomology theory of transformation groups and spectral sequences. We refer to [23] for basic properties of Steenrod algebra. Throughout, we will use cohomology with $\mathbb{Z}/2$ coefficients and will suppress it from cohomology notation. All spaces under consideration will be finite-dimensional CW-complexes. More generally, we can also consider paracompact spaces of finite cohomological dimension or finitistic spaces (which include paracompact spaces of finite covering dimension and compact Hausdorff spaces).

All group actions under consideration are assumed to be continuous. Let G be a finite group acting on a space X , and let

$$G \hookrightarrow E_G \rightarrow B_G$$

be the universal principal G -bundle. Let

$$X_G = (X \times E_G)/G$$

be the orbit space of the diagonal action on $X \times E_G$. Then the projection

$$X \times E_G \rightarrow E_G$$

is G -equivariant and gives a fibration

$$X \hookrightarrow X_G \longrightarrow B_G$$

called the Borel fibration [7, Chapter IV]. Recall that for $G = (\mathbb{Z}/2)^r$, we have

$$B_G = \underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{r \text{ times}},$$

and hence

$$H^*(B_G; \mathbb{Z}/2) = \mathbb{Z}/2[\alpha_1, \dots, \alpha_r],$$

where α_i is a homogeneous element of degree 1 for each $1 \leq i \leq r$.

We will exploit the Leray–Serre spectral sequence associated to a fibration as given by the following theorem.

Theorem 4.1 ([24], Theorem 5.2). *Let $X \hookrightarrow E \longrightarrow B$ be a fibration. Then there is a first quadrant spectral sequence of algebras $\{E_r^{*,*}, d_r\}$, converging to $H^*(E)$ as an algebra, with*

$$E_2^{k,l} = H^k(B; \mathcal{H}^l(X)),$$

the cohomology of the base B with local coefficients in the cohomology of the fiber X .

The product in $E_{r+1}^{*,*}$ is induced by the product in $E_r^{*,*}$ and the differentials are derivations. Further, there is an isomorphism of graded commutative algebra

$$H^*(E) \cong \text{Tot}E_\infty^{*,*},$$

where $\text{Tot}E_\infty^{*,*}$ is the total complex of $E_\infty^{*,*}$.

Theorem 4.2 ([24], Theorem 5.9). *Let $X \xrightarrow{i} E \xrightarrow{\pi} B$ be a fibration. Suppose that the system of local coefficients on B is simple; then the edge homomorphisms*

$$H^k(B) = E_2^{k,0} \longrightarrow E_3^{k,0} \longrightarrow \cdots \longrightarrow E_k^{k,0} \longrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(E) \quad \text{and}$$

$$H^l(E) \longrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \cdots \subset E_2^{0,l} = H^l(X)$$

are the homomorphisms

$$\pi^* : H^k(B) \longrightarrow H^k(E) \quad \text{and} \quad i^* : H^l(E) \longrightarrow H^l(X).$$

Next we recall some results regarding elementary abelian 2-group actions on finite-dimensional CW-complexes.

Theorem 4.3 ([4], Theorem 3.10.4). *Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex X . Suppose that $\sum_{i \geq 0} \text{rk}(H^i(X)) < \infty$ and the induced action on $H^*(X)$ is trivial; then the Leray–Serre spectral sequence associated to $X \hookrightarrow X_G \longrightarrow B_G$ does not degenerate at the E_2 term.*

Proposition 4.4 ([4], Proposition 3.10.9 and Lemma 3.10.16). *Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex X . Then $H^*(X/G) \cong H^*(X_G)$. Further, if $H^i(X) = 0$ for all $i > n$, then $H^i(X_G) = 0$ for all $i > n$.*

We will also use the following results regarding non-trivial common zeros of a system of homogeneous polynomials.

Proposition 4.5 ([9], Proposition 1). *Let $G = (\mathbb{Z}/2)^r$ and f_1, \dots, f_n be elements of $H^m(B_G)$ regarded as homogeneous polynomials of degree m in r variables. Then they have a non-trivial common zero in $(\mathbb{Z}/2)^r$ if and only if there is a subgroup inclusion $j : \mathbb{Z}/2 \hookrightarrow (\mathbb{Z}/2)^r$ such that $j^*(f_i) = 0$ for each $1 \leq i \leq n$, where $j^* : H^*(B_G) \rightarrow H^*(B_{\mathbb{Z}/2})$ is the induced map on cohomology.*

Proposition 4.6 ([9], Proposition 4). *Let $G = (\mathbb{Z}/2)^r$ and f_1, \dots, f_n be elements of $H^*(B_G)$ regarded as homogeneous polynomials in r variables. Suppose that the ideal $\langle f_1, \dots, f_n \rangle$ is invariant under the action of the Steenrod algebra. Then they have a non-trivial common zero in $(\mathbb{Z}/2)^r$ if $r > n$.*

Proposition 4.7 ([18]). *Let f_1, \dots, f_n be homogeneous polynomials of degree m in r variables with coefficients in $\mathbb{Z}/2$. Then they have a non-trivial common zero in $(\mathbb{Z}/2)^r$ if $r > mn$.*

5. Induced action on cohomology

Given a continuous map of topological spaces, determining the induced map on cohomology is a difficult problem in general, even for nice spaces such as spheres. In this section, we show that there are involutions on Milnor manifolds for which the induced action on mod 2 cohomology is non-trivial. First we consider the real case.

Proposition 5.1. *Let $A : \mathbb{R}H_{s,s} \rightarrow \mathbb{R}H_{s,s}$ be the free involution given by*

$$A([x_0, \dots, x_s], [y_0, \dots, y_s]) \mapsto ([y_0, \dots, y_s], [x_0, \dots, x_s]).$$

Then $A^ : H^*(\mathbb{R}H_{s,s}) \rightarrow H^*(\mathbb{R}H_{s,s})$ is non-trivial.*

Proof. By Lemma 2.1,

$$H^*(\mathbb{R}H_{s,s}; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b] / \langle a^{s+1}, b^s + ab^{s-1} + \dots + a^s \rangle,$$

where a and b are homogeneous elements of degree 1 each. By the Künneth formula,

$$H_1(\mathbb{R}P^s \times \mathbb{R}P^s) \cong H_1(\mathbb{R}P^s) \oplus H_1(\mathbb{R}P^s).$$

Let $\sigma = (\sigma_1, \sigma_2) : \Delta^1 \rightarrow \mathbb{R}P^s \times \mathbb{R}P^s$ be a singular 1-simplex, and let

$$A : \mathbb{R}P^s \times \mathbb{R}P^s \rightarrow \mathbb{R}P^s \times \mathbb{R}P^s$$

denote the same involution. Then $A_*([\sigma]) = [A \circ \sigma] = [(\sigma_2, \sigma_1)]$. This shows that if a_1 and b_1 are generators of $H_1(\mathbb{R}P^s) \oplus H_1(\mathbb{R}P^s)$, then $A_*(a_1) = b_1$ and hence the induced action on $H_1(\mathbb{R}P^s \times \mathbb{R}P^s)$ is non-trivial.

Further, we have $H^1(\mathbb{R}P^s \times \mathbb{R}P^s) \cong H^1(\mathbb{R}P^s) \oplus H^1(\mathbb{R}P^s)$. If $f \in H^1(\mathbb{R}P^s \times \mathbb{R}P^s)$, then $A^*(f) = f \circ A_*$. In particular, $A^*(f)(a_1) = f \circ A_*(a_1) = f(b_1)$. Choosing f such that $f(a_1) \neq f(b_1)$, we see that A^* acts non-trivially on cohomology. Thus, if a_2 and b_2 are generators of $H^1(\mathbb{R}P^s) \oplus H^1(\mathbb{R}P^s)$, then $A^*(a_2) = b_2$.

Recall that the real Milnor manifold is also given by the fiber bundle

$$\mathbb{R}P^{s-1} \xrightarrow{i} \mathbb{R}H_{s,s} \xrightarrow{\pi} \mathbb{R}P^s.$$

Let $\mathbb{R}H_{s,s} \xrightarrow{j_1} \mathbb{R}P^s \times \mathbb{R}P^s$ be the canonical inclusion, and let

$$\mathbb{R}P^s \xrightarrow{j_2} \mathbb{R}P^s \times \mathbb{R}P^s \xrightarrow{pr_1} \mathbb{R}P^s$$

be the trivial fiber bundle, where j_2 is inclusion on to the second factor and pr_1 is projection on to the first factor. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{R}P^s \times \mathbb{R}P^s) & \xleftarrow{pr_1^*} & H^1(\mathbb{R}P^s) \\ \downarrow j_1^* & \swarrow \pi^* & \\ H^1(\mathbb{R}H_{s,s}) & & \end{array}$$

Applying Theorem 4.1 to the fiber bundles given by π and pr_1 and using Theorem 4.2, we get $\pi^*(a_2) = a$ and $pr_1^*(a_2) = a_2$. Further, by commutativity of the diagram, we get $j_1^*(a_2) = j_1^*(pr_1^*(a_2)) = \pi^*(a_2) = a$.

Let $\mathbb{R}P^{s-1} \xrightarrow{j} \mathbb{R}P^s$ be the canonical inclusion. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^1(\mathbb{R}P^s \times \mathbb{R}P^s) & \xrightarrow{A^*} & H^1(\mathbb{R}P^s \times \mathbb{R}P^s) & \xrightarrow{j_2^*} & H^1(\mathbb{R}P^s) \\ \downarrow j_1^* & & \downarrow j_1^* & & \downarrow j^* \\ H^1(\mathbb{R}H_{s,s}) & \xrightarrow{A^*} & H^1(\mathbb{R}H_{s,s}) & \xrightarrow{i^*} & H^1(\mathbb{R}P^{s-1}). \end{array}$$

If $b_3 \in H^1(\mathbb{R}P^{s-1})$ is the generator, then $j^*(b_2) = b_3$. Again by Theorems 4.1 and 4.2, we get $i^*(b) = b_3$ and $j_2^*(b_2) = b_2$. The commutativity of the right square shows that $j_1^*(b_2) = b$. The commutativity of the left square gives $A^*(j_1^*(a_2)) = j_1^*(A^*(a_2))$. This implies $A^*(a) = j_1^*(b_2) = b$. Thus the induced map A^* is non-trivial. \square

We have a similar result for the complex case.

Proposition 5.2. *Let $B : \mathbb{C}H_{s,s} \rightarrow \mathbb{C}H_{s,s}$ be the free involution given by*

$$B([z_0, \dots, z_s], [w_0, \dots, w_s]) \mapsto ([w_0, \dots, w_s], [z_0, \dots, z_s]).$$

Then $B^ : H^*(\mathbb{C}H_{s,s}) \rightarrow H^*(\mathbb{C}H_{s,s})$ is non-trivial.*

Proof. The proof is similar to that of the real case and left to the reader. \square

Note that using propositions 5.1 and 5.2, we can construct free actions of $(\mathbb{Z}/2)^n$ on $\prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$ and $\prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ for $1 \leq s_i = r_i$ and $n \geq 2$, such that the induced action on mod 2 cohomology is non-trivial.

6. Proofs of theorems

6.1. The real case

Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. Further, assume that the induced action on mod

2 cohomology of X is trivial. Using the Künneth formula and Lemma 2.1, we get

$$H^*(X) \cong \mathbb{Z}/2[a_1, b_1, \dots, a_n, b_n]/I,$$

where

$$I = \langle a_1^{s_1+1}, b_1^{r_1} + a_1 b_1^{r_1-1} + \dots + a_1^{s_1} b_1^{r_1-s_1}, \dots, a_n^{s_n+1}, b_n^{r_n} + a_n b_n^{r_n-1} + \dots + a_n^{s_n} b_n^{r_n-s_n} \rangle$$

and $a_1, b_1, \dots, a_n, b_n$ are homogeneous elements of degree 1 each. Consider the Leray–Serre spectral sequence associated to the Borel fibration

$$X \hookrightarrow X_G \longrightarrow B_G.$$

Then we have

$$E_2^{k,l} \cong E_2^{k,0} \otimes E_2^{0,l},$$

where $E_2^{k,0} = H^k(B_G)$ and $E_2^{0,l} = H^0(B_G; \mathcal{H}^l(X)) = H^l(X)^G = H^l(X)$ because the induced action on cohomology is trivial. Thus we have

$$\begin{aligned} E_2^{*,*} &\cong H^*(B_G) \otimes H^*(X) \\ &\cong \mathbb{Z}/2[\alpha_1, \dots, \alpha_r] \otimes \mathbb{Z}/2[a_1, b_1, \dots, a_n, b_n]/I. \end{aligned}$$

By Theorem 4.3, the spectral sequence does not degenerate at the E_2 term, and hence $d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$ is non-zero. Let $d_2(1 \otimes a_i) = u_i \otimes 1$ and $d_2(1 \otimes b_i) = v_i \otimes 1$ for $1 \leq i \leq n$ with at least one of them being non-zero. Notice that d_2 is completely determined by $d_2(1 \otimes a_i)$ and $d_2(1 \otimes b_i)$ as it is a derivation. Consider the ideal

$$J = \langle u_1, v_1, \dots, u_n, v_n \rangle$$

in $H^*(B_G)$. Then, using arguments as in [13], we have the following lemma.

Lemma 6.1. *Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. Further, assume that the induced action on the mod 2 cohomology of X is trivial and that $s_i, r_i \not\equiv 3 \pmod{4}$ for each $1 \leq i \leq n$. Then the ideal J in $H^*(B_G)$ is invariant under the action of the Steenrod algebra.*

Proof. Fix some $1 \leq i \leq n$. Since $s_i \geq 1$, we have $a_i \neq 0$, and hence $b_i^{r_i} \neq 0$. Notice that

$$\begin{aligned} b_i^{r_i+1} &= (a_i b_i^{r_i-1} + \dots + a_i^{s_i} b_i^{r_i-s_i}) b_i \\ &= a_i b_i^{r_i} + \dots + a_i^{s_i} b_i^{r_i-s_i+1} \\ &= a_i (a_i b_i^{r_i-1} + \dots + a_i^{s_i} b_i^{r_i-s_i}) + a_i^2 b_i^{r_i-1} + \dots + a_i^{s_i} b_i^{r_i-s_i+1} \\ &= a_i^2 b_i^{r_i-1} + \dots + a_i^{s_i} b_i^{r_i-s_i+1} + a_i^{s_i+1} b_i^{r_i-s_i} + a_i^2 b_i^{r_i-1} + \dots + a_i^{s_i} b_i^{r_i-s_i+1} \\ &= 0. \end{aligned}$$

If r_i is even, then

$$0 = d_2(1 \otimes b_i^{r_i+1}) = (1 \otimes b_i^{r_i}) d_2(1 \otimes b_i) = v_i \otimes b_i^{r_i}.$$

Since the map $- \otimes b_i^{r_i} : E_2^{*,0} \rightarrow E_2^{*,r_i}$ is injective, we get $v_i \otimes 1 = 0$. Recall that $a_i^{s_i+1} = 0$. Just as above, if s_i is even, then $u_i \otimes 1 = 0$. Such u_i and v_i are obviously invariant under the action of the Steenrod algebra.

Next let $r_i = 4m + 1$ and $v_i \otimes 1 \neq 0$. Notice that $Sq^1(1 \otimes b_i) = 1 \otimes b_i^2$ and $d_2(1 \otimes b_i^{2m}) = 0$ by the derivation property of d_2 . Thus $1 \otimes b_i^{2m}$ represents an element in $E_3^{0,2m}$ and we have

$$0 = d_3(1 \otimes b_i^{4m+2}) = (1 \otimes b_i^{4m})d_3(1 \otimes b_i^2) \quad \text{in } E_3^{3,4m}. \quad (1)$$

Recall that the transgression operator $d_r : E_r^{0,r-1} \rightarrow E_r^{r,0}$ commutes with the Steenrod operations. In other words, the following diagram commutes:

$$\begin{array}{ccc} E_2^{0,1} & \xrightarrow{d_2} & E_2^{2,0} \\ \downarrow Sq^1 & & \downarrow Sq^1 \\ E_3^{0,2} & \xrightarrow{d_3} & E_3^{3,0} \end{array}.$$

This shows that $d_3(1 \otimes b_i^2)$ is represented by $Sq^1(v_i \otimes 1)$. By equation (1) we have that $(1 \otimes b_i^{4m})Sq^1(v_i \otimes 1)$ lies in the image of $d_2 : E_2^{1,4m+1} \rightarrow E_2^{3,4m}$, and hence

$$(1 \otimes b_i^{4m})Sq^1(v_i \otimes 1) = (1 \otimes b_i^{4m})d_2(w)$$

for some $w \in E_2^{1,1}$. Since $1 \otimes b_i^{4m} \neq 0$, the map $- \otimes b_i^{4m} : E_2^{*,0} \rightarrow E_2^{*,4m}$ is injective, we get $Sq^1(v_i \otimes 1) = d_2(w)$. Let

$$w = \sum_{j,k} \lambda_{j,k}(\alpha_j \otimes a_k) + \sum_{j,k} \mu_{j,k}(\alpha_j \otimes b_k),$$

where $\lambda_{j,k}, \mu_{j,k} \in \mathbb{Z}/2$. Then

$$d_2(w) = \sum_{j,k} \lambda_{j,k}(\alpha_j u_k \otimes 1) + \sum_{j,k} \mu_{j,k}(\alpha_j v_k \otimes 1) \in J.$$

This shows that $Sq^1(v_i \otimes 1) \in J$. Similarly, if $s_i = 4m + 1$ and $u_i \otimes 1 \neq 0$, then $Sq^1(u_i \otimes 1) \in J$. Hence J is invariant under the action of the Steenrod algebra. \square

For an integer n , define

$$\eta(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Theorem 1.1

We first prove (1). As noticed in the proof of Lemma 6.1, if some s_i, r_i is even, then the corresponding u_i, v_i is zero. Regard $u_1, v_1, \dots, u_n, v_n \in H^2(B_G)$ as homogeneous polynomials of degree two in r variables. Suppose that they have a non-trivial common zero in $(\mathbb{Z}/2)^r$. Then by Proposition 4.5, there is a subgroup inclusion

$$j : \mathbb{Z}/2 \hookrightarrow (\mathbb{Z}/2)^r$$

such that $j^*(u_i) = 0 = j^*(v_i)$ for each $1 \leq i \leq n$. Restrict the G action on X to $\mathbb{Z}/2$ action on X , and consider the Leray-Serre spectral sequence $\{\overline{E}_r^{*,*}, \overline{d}_r\}$ associated to

the Borel fibration

$$X \hookrightarrow X_{\mathbb{Z}/2} \longrightarrow B_{\mathbb{Z}/2}.$$

The naturality of the Leray–Serre spectral sequence gives the following commutative diagram:

$$\begin{array}{ccc} E_r^{k,l} & \xrightarrow{j^*} & \overline{E}_r^{k,l} \\ \downarrow d_r & & \downarrow \overline{d}_r \\ E_r^{k+r,l-r+1} & \xrightarrow{j^*} & \overline{E}_r^{k+r,l-r+1} \end{array} \quad (2)$$

Observe that $j^* : E_2^{0,l} \rightarrow \overline{E}_2^{0,l}$ is the identity map. This, together with the commutative diagram, gives $\overline{d}_2(1 \otimes a_i) = 0 = \overline{d}_2(1 \otimes b_i)$ for each $1 \leq i \leq n$. Hence $\overline{d}_r = 0$ for each $r \geq 2$ and

$$\overline{E}_2^{*,*} = \overline{E}_\infty^{*,*}.$$

But we have $H^*(X_G) \cong \text{Tot} E_\infty^{*,*}$. This implies that $H^*(X_G)$ is infinite dimensional, a contradiction by Proposition 4.4. Hence the system of homogeneous polynomials does not have any non-trivial common zero in $(\mathbb{Z}/2)^r$. Thus, by Proposition 4.7, $r \leq 2(\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$.

Next, we prove (2). By Lemma 6.1, the ideal J is invariant under the action of the Steenrod algebra. By the discussion above, the system of homogeneous polynomials $u_1, v_1, \dots, u_n, v_n$ do not have any non-trivial common zero in $(\mathbb{Z}/2)^r$. Thus, by Proposition 4.6, we have $r \leq (\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n))$. This completes the proof of Theorem 1.1. \square

6.2. The complex case

Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq \prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. And suppose that the induced action on mod 2 cohomology of X is trivial. Using Lemma 2.1, we get

$$H^*(X) \cong \mathbb{Z}/2[g_1, h_1, \dots, g_n, h_n]/K,$$

where K is the ideal

$$\langle g_1^{s_1+1}, h_1^{r_1} + g_1 h_1^{r_1-1} + \cdots + g_1^{s_1} h_1^{r_1-s_1}, \dots, g_n^{s_n+1}, h_n^{r_n} + g_n h_n^{r_n-1} + \cdots + g_n^{s_n} h_n^{r_n-s_n} \rangle$$

and $g_1, h_1, \dots, g_n, h_n$ are all homogeneous elements of degree 2 each. As in the real case, we have

$$E_2^{*,*} \cong \mathbb{Z}/2[\alpha_1, \dots, \alpha_r] \otimes \mathbb{Z}/2[g_1, h_1, \dots, g_n, h_n]/K.$$

Again by Theorem 4.3, the spectral sequence does not degenerate at the E_2 term. Notice that $d_r = 0$ for all even r . In particular, $d_2 = 0$, and hence $d_3 : E_3^{0,2} \rightarrow E_3^{3,0}$ must be non-zero. Let $d_3(1 \otimes g_i) = x_i \otimes 1$ and $d_3(1 \otimes h_i) = y_i \otimes 1$ for $1 \leq i \leq n$ with at least one of them being non-zero. Consider the ideal

$$L = \langle x_1, y_1, \dots, x_n, y_n \rangle$$

in $H^*(B_G)$. Then we have the following lemma.

Lemma 6.2. *Let $G = (\mathbb{Z}/2)^r$ act freely on a finite-dimensional CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ with $1 \leq s_i \leq r_i$ for each $1 \leq i \leq n$. Suppose that the induced action on mod 2 cohomology of X is trivial and $s_i, r_i \not\equiv 3 \pmod{4}$ for each $1 \leq i \leq n$. Then the ideal L in $H^*(B_G)$ is invariant under the action of the Steenrod algebra.*

Proof. We describe the proof briefly as it is similar to the proof of Lemma 6.1. Fix some $1 \leq i \leq n$. Since $s_i \geq 1$, we have $h_i^{r_i} \neq 0$. Notice that $h_i^{r_i+1} = 0$. If r_i is even, then

$$0 = d_3(1 \otimes h_i^{r_i+1}) = y_i \otimes h_i^{r_i}.$$

Since the map $-\otimes h_i^{r_i} : E_3^{*,0} \rightarrow E_3^{*,r_i}$ is injective, we get $y_i \otimes 1 = 0$. Since $g_i^{s_i+1} = 0$, it follows that if s_i is even, then $x_i \otimes 1 = 0$. Such x_i and y_i are obviously invariant under the action of the Steenrod algebra.

Let $r_i = 4m + 1$ and $y_i \otimes 1 \neq 0$. Notice that $Sq^1(1 \otimes h_i) = 0$ and the following diagram commutes:

$$\begin{array}{ccc} E_3^{0,2} & \xrightarrow{d_3} & E_3^{3,0} \\ \downarrow Sq^1 & & \downarrow Sq^1 \\ E_4^{0,3} & \xrightarrow{d_4} & E_4^{4,0}. \end{array}$$

Since $d_4 = 0$, the commutativity of the diagram shows that $0 = Sq^1(y_i \otimes 1) \in L$.

Next we have $Sq^2(1 \otimes h_i) = 1 \otimes h_i^2$. Further, $d_3(1 \otimes h_i^{2m}) = 0$, and hence $1 \otimes h_i^{2m}$ represents an element in $E_5^{0,4m}$. Since $h_i^{4m+2} = 0$, we have

$$0 = d_5(1 \otimes h_i^{4m+2}) = (1 \otimes h_i^{4m})d_5(1 \otimes h_i^2) \quad \text{in } E_5^{5,4m}. \quad (3)$$

Consider the following commutative diagram:

$$\begin{array}{ccc} E_3^{0,2} & \xrightarrow{d_3} & E_3^{3,0} \\ \downarrow Sq^2 & & \downarrow Sq^2 \\ E_5^{0,4} & \xrightarrow{d_5} & E_5^{5,0}. \end{array}$$

This shows that $d_5(1 \otimes h_i^2)$ is represented by $Sq^2(y_i \otimes 1)$. By equation (3) we have that $(1 \otimes h_i^{4m})Sq^2(y_i \otimes 1)$ lies in the image of $d_3 : E_3^{2,4m+2} \rightarrow E_3^{5,4m}$ and hence

$$(1 \otimes h_i^{4m})Sq^2(y_i \otimes 1) = (1 \otimes h_i^{4m})d_3(z)$$

for some $z \in E_3^{2,2}$. Since $1 \otimes h_i^{4m} \neq 0$, the map $-\otimes h_i^{4m} : E_3^{*,0} \rightarrow E_3^{*,4m}$ is injective, and we get $Sq^2(y_i \otimes 1) = d_3(z)$. Let

$$z = \sum_{j,k,l} \lambda_{j,k,l} (\alpha_j \alpha_k \otimes g_l) + \sum_{j,k,l} \mu_{j,k,l} (\alpha_j \alpha_k \otimes h_l),$$

where $\lambda_{j,k,l}, \mu_{j,k,l} \in \mathbb{Z}/2$. Then

$$d_3(z) = \sum_{j,k,l} \lambda_{j,k,l} (\alpha_j \alpha_k x_l \otimes 1) + \sum_{j,k,l} \mu_{j,k,l} (\alpha_j \alpha_k y_l \otimes 1) \in L.$$

This shows that $Sq^2(y_i \otimes 1) \in L$. Similarly, if $s_i = 4m + 1$ and $x_i \otimes 1 \neq 0$, then

$Sq^1(x_i \otimes 1), Sq^2(x_i \otimes 1) \in L$. Hence L is invariant under the action of the Steenrod algebra. \square

Proof of Theorem 1.2

If some s_i, r_i is even, then the corresponding x_i, y_i is zero. Suppose $x_1, y_1, \dots, x_n, y_n$ have a non-trivial common zero in $(\mathbb{Z}/2)^r$. Then by Proposition 4.5 there is a subgroup inclusion

$$j : \mathbb{Z}/2 \hookrightarrow (\mathbb{Z}/2)^r$$

such that $j^*(x_i) = 0 = j^*(y_i)$ for each $1 \leq i \leq n$. Restrict the G action on X to $\mathbb{Z}/2$ action on X , and consider the Leray–Serre spectral sequence $\{\overline{E}_r^{*,*}, \overline{d}_r\}$ associated to the Borel fibration

$$X \hookrightarrow X_{\mathbb{Z}/2} \rightarrow B_{\mathbb{Z}/2}.$$

Observe that $j^* : E_3^{0,l} \rightarrow \overline{E}_3^{0,l}$ is the identity map. This, together with the commutative diagram (2), gives $\overline{d}_3(1 \otimes g_i) = 0 = \overline{d}_3(1 \otimes h_i)$ for each $1 \leq i \leq n$. Hence $\overline{d}_r = 0$ for each $r \geq 2$ and $\overline{E}_2^{*,*} = \overline{E}_\infty^{*,*}$. This gives a contradiction by Proposition 4.4. Hence the system of homogeneous polynomials does not have any non-trivial common zero in $(\mathbb{Z}/2)^r$. Thus, by Proposition 4.7, $r \leq 3(\eta(s_1) + \eta(r_1) + \dots + \eta(s_n) + \eta(r_n))$. This proves (1).

By Lemma 6.2, the ideal L is invariant under the action of the Steenrod algebra. By the above discussion, the system of homogeneous polynomials $x_1, y_1, \dots, x_n, y_n$ does not have any non-trivial common zero in $(\mathbb{Z}/2)^r$. Thus, by Proposition 4.6, we have $r \leq (\eta(s_1) + \eta(r_1) + \dots + \eta(s_n) + \eta(r_n))$. This proves Theorem 1.2(2). \square

Restricting to actions of elementary abelian 2-groups on $\prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ for which the induced action on mod 2 cohomology is trivial, we obtain the following corollary.

Corollary 6.3. *Let $1 \leq s_i < r_i$ for each $1 \leq i \leq n$. Then*

$$\text{frk}_2 \left(\prod_{i=1}^n \mathbb{C}H_{r_i, s_i} \right) = \eta(s_1) + \eta(r_1) + \dots + \eta(s_n) + \eta(r_n)$$

whenever $s_i \equiv 1 \pmod{4}$ and $r_i \equiv 0, 2 \pmod{4}$.

Proof. Theorem 1.2(2) gives the upper bound. In Section 3, we constructed a $\mathbb{Z}/2$ action on $\mathbb{C}H_{r_i, s_i}$ when s_i is odd and r_i is even, for which the induced action on $H^*(\mathbb{C}H_{r_i, s_i})$ is trivial. The products of these actions on $\prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$ achieve the desired bound when $s_i \equiv 1 \pmod{4}$ and $r_i \equiv 0, 2 \pmod{4}$. \square

7. Some concluding remarks

We conclude with the following remarks on our results. Adem and Yalçın asked the following question [3, p. 70].

Question. *If $(\mathbb{Z}/2)^r$ acts freely on a finite CW-complex X with mod 2 cohomology generated by one-dimensional classes, does it follow that $r \leq 2 \dim H_1(X; \mathbb{Z}/2)$?*

Let $(\mathbb{Z}/2)^r$ act freely on a finite CW-complex $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$ such that the induced action on mod 2 cohomology is trivial. If $1 \leq s_i < r_i$ for each $1 \leq i \leq n$, then

$$H_1(\mathbb{R}H_{r_i, s_i}; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and hence $\dim H_1(X; \mathbb{Z}/2) = 2n$. Therefore, by Theorem 1.1(1),

$$r \leq 2(\eta(s_1) + \eta(r_1) + \cdots + \eta(s_n) + \eta(r_n)) \leq 2(2n) = 2 \dim H_1(X; \mathbb{Z}/2).$$

Thus the above question has a positive answer for $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$.

Let X be as in Theorem 1.1 and Theorem 1.2. Let s_i be even and r_i be odd for each $1 \leq i \leq n$. If $X \simeq_2 \prod_{i=1}^n \mathbb{R}H_{r_i, s_i}$, then the Euler characteristic $\chi(X) = 1$. Similarly, if $X \simeq_2 \prod_{i=1}^n \mathbb{C}H_{r_i, s_i}$, then $\chi(X)$ is odd. Hence no elementary abelian 2-group can act freely on X and our theorems are weak in this case.

It is well known that if a closed smooth manifold does not bound mod 2, then it does not admit any free involution. It was shown in [22] that, $\mathbb{R}H_{r, s}$ does not bound for $s = 2k + 1$ and $r = 2^\beta(2l + 1)$ if and only if one of the following holds:

- $\beta \geq 2$ and $k \geq 1$,
- $\beta = 1$, $l + 1 = 2^\delta(2t + 1)$ and $k \geq 2^{\delta+1} - 1$.

Thus $\mathbb{R}H_{r, s}$ does not admit any free involution in these cases.

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