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MORPHIC COHOMOLOGY OF TORIC VARIETIES

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(*communicated by Charles A. Weibel*)

Abstract

In this paper we construct a spectral sequence computing a modified version of morphic cohomology of a toric variety (even in the singular case) in terms of combinatorial data coming from the fan of the toric variety.

1. Introduction

Morphic cohomology is a cohomological theory on complex algebraic varieties introduced by Friedlander and Lawson in [**7**]. On one hand this theory has a very geometric definition, and on the other it is strongly related to the abstractly defined motivic cohomology theory.

In this paper, we describe, following [**5**], a modification of morphic cohomology on quasi-projective varieties (not only smooth) in order to have Mayer-Vietoris and homotopy invariance properties. Then, we use these properties to construct a spectral sequence computing the (modified) morphic cohomology of a toric variety in terms of its combinatorial data.

In Section 2 we review some results we need about morphic cohomology and define the modification of the theory we will use.

In Section 3 we set up the notation we use for toric varieties and write down an explicit computation of the morphic cohomology of algebraic tori.

Finally, in Section 4 we build a resolution of the constant sheaf $\mathbb{Z}_X$ on a toric variety in terms of the combinatorial data. This resolution allows us to construct a spectral sequence (Theorem 4.16) computing the hypercohomology of a complex of sheaves $\mathcal{F}^*$ on the toric variety $X(\Delta)$ in terms of the combinatorial data and the value of this hypercohomology on algebraic tori. Then we specialize this to the case of morphic cohomology, giving a very explicit spectral sequence converging to the morphic cohomology of $X(\Delta)$ and whose second page involves only combinatorial data. Moreover, we prove its rational degeneration (Theorem 4.18). To conclude, we present a sample computation with this spectral sequence and two applications: one comparing the modified morphic cohomology to the morphic cohomology with cdh descent and the other to the Suslin conjecture for toric varieties.

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2. Morphic cohomology

2.1. Definitions and fundamental results

In this section we recall some facts about morphic cohomology. Let Y be a projective variety over $\mathbb{C}$, with a fixed projective embedding. It is a classical fact (see Chapter 1 in [**13**]) that for any natural number k, the set of effective k-cycles on Y has the structure of an algebraic variety. This variety is called the *Chow variety*, and we will denote it by $\mathrm{C}(Y,k)$. It may have infinitely many connected components, corresponding to the homology classes of the cycles. The Chow variety has a natural operation given by the sum of cycles. This operation is algebraic, so $\mathrm{C}(Y,k)$ is a commutative monoid in the category $\mathbf{Proj}_{\mathbb{C}}$ of projective varieties. Then, the free group $\mathrm{Z}(Y,k)$ of algebraic k-cycles on Y is the group completion of $\mathrm{C}(Y,k)$, inheriting the topology and making it an abelian topological group. Its homotopy groups are interesting invariants of Y: they are called the *Lawson homology* groups (see [**3**] for the details) and are denoted by

$$\mathrm{L}_k\mathrm{H}_n(Y) = \pi_{n-2k}\mathrm{Z}(Y,k).$$

The topological group structure on $\mathrm{Z}(Y,k)$ can be defined for any Y, not necessarily projective (see [**14**] and [**15**]).

In [**4**] and [**7**], Friedlander and Lawson define a cohomological version of Lawson homology for quasi-projective varieties, the *morphic cohomology*. They define it as

$$\mathrm{L}^q\mathrm{H}^n(X) = \pi_{2q-n}\mathrm{M}(X,\mathbb{A}^q,0),$$

where $\mathrm{M}(X,Y,k)$ is a topological abelian group. If X is projective, normal and Y is projective, then this topological group is defined as

$$\mathrm{M}(X,Y,k) = \mathrm{Hom}(X,\mathrm{C}(Y,k))^+,$$

with the compact open topology on the space of morphisms. See the references above for more details on the general definition. We now recall some of its properties.

Theorem 2.1.

1. *(Functoriality). The spaces* $\mathrm{M}(X,Y,k)$ *are contravariantly functorial in* X *for arbitrary morphisms and covariantly functorial in* Y *for proper morphisms.*
2. *(Homotopy invariance). For any quasi-projective variety* X*, the projection to the first factor of* $X \times \mathbb{A}^1$ *induces an isomorphism* $\mathrm{L}^q\mathrm{H}^n(X) \cong \mathrm{L}^q\mathrm{H}^n(X \times \mathbb{A}^1)$.
3. *(Duality). There is a natural map* $\Gamma\colon \mathrm{L}^q\mathrm{H}^n(X) \to \mathrm{L}_{d-q}\mathrm{H}_{2d-n}(X)$*, which is an isomorphism when* X *is smooth.*

4. *(Gysin morphism). Let $i\colon X \to Y$ be a closed embedding of smooth varieties of relative dimension c. Then there is a functorial Gysin map $i_!\colon \mathrm{L}^q\mathrm{H}^n(X) \to \mathrm{L}^{q+c}\mathrm{H}^{n+2c}(Y)$, given by $i_! = \Gamma^{-1} i_* \Gamma$.*
5. *(Mayer-Vietoris). Let X be smooth and $U, V \subset X$ an open cover. Then there is a Mayer-Vietoris long exact sequence*

$$\cdots \longrightarrow \mathrm{L}^q\mathrm{H}^n(X) \longrightarrow \mathrm{L}^q\mathrm{H}^n(U) \oplus \mathrm{L}^q\mathrm{H}^n(V) \longrightarrow \mathrm{L}^q\mathrm{H}^n(U \cap V) \longrightarrow \cdots.$$

Proof. (See [**4**]). Point 1 follows from Propositions 3.1 and 3.3, point 2 from Proposition 3.5, point 3 from Theorem 5.2, point 4 from Proposition 6.1, and point 5 comes from point 3 and the localization long exact sequence for Lawson homology ([**14**, Proposition 4.8]). □

2.2. Descent and homotopy invariance

We are interested in using homotopy invariance and Mayer-Vietoris properties for morphic cohomology. Homotopy invariance holds for singular varieties (point 2 in Theorem 2.1 above), but the Mayer-Vietoris property is not known to hold in general. It does hold for smooth varieties thanks to duality (point 3 in Theorem 2.1 above). In [**5**] Friedlander defines a modified morphic cohomology, called topological cycle cohomology, which satisfies a Mayer-Vietoris property for Zariski open covers and coincides with the previous version on smooth schemes. However, there is a catch. The sheafification involved in the construction of Friedlander's topological cycle cohomology seems to interfere with the argument for homotopy invariance. As a result, we do not have a proof of homotopy invariance for Friedlander's theory in full generality (for smooth varieties we can use duality). As we need to use homotopy invariance, we use a standard construction to force this property back into the theory.

First of all, taking singular chains on $\mathrm{M}(X, Y, k)$ we get a simplicial abelian group, which via the Dold-Kan correspondence, produces a cochain complex. Now, taking the special case $\mathrm{M}(X, \mathbb{A}^q, 0)$, the space defining morphic cohomology, and taking into account the functoriality in X, we have in fact a cochain complex of presheaves of abelian groups on the category of quasi-projective varieties $\mathbf{qProj}_{\mathbb{C}}$. We denote this complex of presheaves by $\mathcal{M}^*(q)$. More precisely, we have

$$\mathcal{M}^n(q)(X) = \mathrm{Sing}_{2q-n}\, \mathrm{M}(X, \mathbb{A}^q, 0). \tag{1}$$

Recall that the Dold-Kan functor sends a simplicial abelian group $A_\bullet$ to a chain complex whose homology groups are isomorphic to the homotopy groups $\pi_n A_\bullet$, so the cohomology of the global sections of our cochain complex of presheaves gives exactly the morphic cohomology

$$\mathrm{L}^q\mathrm{H}^n(X, \mathbb{Z}) \cong \mathrm{H}^n\Gamma(X, \mathcal{M}^*(q)).$$

Let $\Delta^\bullet$ be the standard cosimplicial scheme with

$$\Delta^n = \big\{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \mid x_0 + \cdots + x_n = 1\big\}$$

and let $\mathrm{Sh}_{\mathrm{zar}}$ be the Zariski sheafification functor. Then we define the following complexes of sheaves:

$$\begin{aligned} \mathcal{M}^n_{\mathrm{zar}}(q) &= \mathrm{Sh}_{\mathrm{zar}}\mathcal{M}^n(q), \\ \mathcal{M}^n_{\mathrm{zar,hi}}(q) &= \mathrm{Tot}^n\, \mathcal{M}^*_{\mathrm{zar}}(\text{-} \times \Delta^\bullet, q). \end{aligned}$$

Remark 2.2. The complexes $\mathcal{M}^*(q)$ are unbounded below and bounded above by $2q$. This poses some homological algebra troubles as they are bounded on the wrong side. However, due to a result of Spaltenstein [**17**], one can still have resolutions $\mathcal{M}^*(q) \to \mathcal{I}^*$ playing the role of injective resolutions. Those are called K-injective in [**17**]. Later, we will use cohomological finiteness arguments to prove convergence of the spectral sequences we encounter.

Remark 2.3. We think about the complexes $\mathcal{M}^*_{\mathrm{zar}}(q)$ and $\mathcal{M}^*_{\mathrm{zar,hi}}(q)$ as objects in the derived category $\mathbf{D}^-\mathbf{Sh}(\mathbf{qProj}_{\mathbb{C}})$ of abelian sheaves on the Zariski site $\mathbf{qProj}_{\mathbb{C}}$. For any variety X, and a complex of sheaves $\mathcal{F}^* \in \mathbf{D}^-\mathbf{Sh}(\mathbf{qProj}_{\mathbb{C}})$, we will denote by $\mathcal{F}^*|_X$ the object in $\mathbf{D}^-\mathbf{Sh}(X)$ obtained by restricting $\mathcal{F}^*$ to the small Zariski site of X.

This sheaf-theoretic interpretation leads to natural reformulations of the morphic cohomology groups.

Definition 2.4. The *topological cycle cohomology groups* are the hypercohomology groups

$$\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar}}(X) = \mathbb{H}^n(X, \mathcal{M}^*_{\mathrm{zar}}(q)|_X).$$

The *homotopy invariant topological cycle cohomology* is defined by

$$\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X) = \mathbb{H}^n(X, \mathcal{M}^*_{\mathrm{zar,hi}}(q)|_X).$$

Remark 2.5. The hypercohomologies in Definition 2.4 can be rewritten as an ext-group either on the derived category $\mathbf{D}^-\mathbf{Sh}(X)$ of sheaves of abelian groups on X, or on the derived categories of sheaves on the big Zariski site $\mathbf{qProj}_{\mathbb{C}}$:

$$\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X) \cong \mathrm{Ext}^n_{\mathbf{Sh}(X)}(\mathbb{Z}_X, \mathcal{M}^*_{\mathrm{zar,hi}}(q)|_X) \cong \mathrm{Ext}^n_{\mathbf{Sh}(\mathbf{qProj}_{\mathbb{C}})}(\mathbb{Z}h_X, \mathcal{M}^*_{\mathrm{zar,hi}}(q)),$$

where $\mathbb{Z}_X$ denotes the constant sheaf on X, and $\mathbb{Z}h_X$ is the free abelian group generated by the sheaf on $\mathbf{qProj}_{\mathbb{C}}$ represented by X, i.e., $\mathbb{Z}h_X(U) = \mathbb{Z}\,\mathrm{Hom}(U, X)$.

There are comparison morphisms

$$\mathcal{M}^*(q) \xrightarrow{a} \mathcal{M}^*_{\mathrm{zar}}(q) \xrightarrow{b} \mathcal{M}^*_{\mathrm{zar,hi}}(q)\ .$$

The first is the inclusion of a presheaf in its associated sheaf, and the second is the inclusion into the summand of the total complex corresponding to the algebraic simplex Δ^0. These maps induce comparison maps

$$\mathrm{L}^q\mathrm{H}^n(X) \xrightarrow{a} \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar}}(X) \xrightarrow{b} \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X)\ . \tag{2}$$

Proposition 2.6. *Assume X is a smooth quasi-projective variety. Then the comparison morphisms* (2) *are isomorphisms.*

Proof. From Theorem 2.1 we know that the presheaf $\mathcal{M}^*(q)$ restricted to smooth varieties satisfies the Mayer-Vietoris property, that is, the square

$$\begin{tikzcd} \mathcal{M}^*(q)(X) \arrow[r] \arrow[d] & \mathcal{M}^*(q)(U) \arrow[d] \\ \mathcal{M}^*(q)(V) \arrow[r] & \mathcal{M}^*(q)(U \cap V) \end{tikzcd}$$

is homotopy cartesian. Take a K-injective resolution $\mathcal{M}^*_{\mathrm{zar}}(q) \to \mathcal{I}^*_{\mathrm{zar}}$. Using the Brown-Gersten Theorem ([**19**, Lemma 3.5]) we conclude that the presheaf $\mathcal{M}^*(q)$ is globally weakly equivalent to the K-injective resolution $\mathcal{I}^*_{\mathrm{zar}}$, so

$$\mathrm{H}^n\Gamma(X, \mathcal{M}^*(q)) \cong \mathrm{H}^n\Gamma(X, \mathcal{I}^*_{\mathrm{zar}}) = \mathbb{H}^n(X, \mathcal{M}^*(q)),$$

and this settles the first isomorphism a.

As for b, by Theorem 2.1 we know that $\mathrm{L}^q\mathrm{H}^n(\text{-})$ is a homotopy invariant functor, and by the previous isomorphism coincides with $\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar}}(\text{-})$ on smooth varieties, so the last one is also homotopy invariant on smooth varieties. Then the spectral sequence associated to the double complex defining $\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(\text{-})$ degenerates on the second page and this gives the isomorphism b. □

Now, this leads to

Theorem 2.7.

1. *(Mayer-Vietoris). Let U, V be an open cover of a quasi-projective variety X. Then there is a Mayer-Vietoris long exact sequence for the groups* $\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(\text{-})$

$$\cdots \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X) \longrightarrow \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(U) \oplus \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(V) \longrightarrow \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(U \cap V) \cdots.$$

2. *(Homotopy invariance). For any quasi-projective variety X the projection to the first factor induces an isomorphism* $\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X \times \mathbb{A}^1) \cong \mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X)$.

Proof. In point 1 the open cover $\{U, V\}$ gives an exact triangle in $\mathbf{D}^-\mathbf{Sh}(X)$

$$\mathbb{Z}_{U\cap V} \longrightarrow \mathbb{Z}_U \oplus \mathbb{Z}_V \longrightarrow \mathbb{Z}_X \longrightarrow \mathbb{Z}_{U\cap V}[1]$$

and applying the functor $\mathrm{Ext}^n(\text{-}, \mathcal{M}^*_{\mathrm{zar,hi}}(q))$ produces the desired long exact sequence.

Point 2 is a standard argument. See, for example, Corollary 2.19 in [**16**]. □

Remark 2.8. In the remainder of this paper we will deal only with the modified version $\mathrm{L}^q\mathrm{H}^n_{\mathrm{zar,hi}}(X)$ of morphic cohomology. We will drop the subindices "zar" and "hi" from the notation for readability and refer to it as "morphic cohomology".

It will be useful to deal with the morphic complexes all at once, so we define the complex of sheaves $\mathcal{M}^* = \bigoplus_{q\geqslant 0} \mathcal{M}^*(q)$. This is a bigraded object, having the geometric degree q and the cohomological degree n with a differential in the n direction. We use the notation $\mathrm{L}^*\mathrm{H}^n(X)$ for the graded group

$$\mathrm{L}^*\mathrm{H}^n(X) = \bigoplus_{q\geqslant 0} \mathrm{L}^q\mathrm{H}^n(X) \cong \mathbb{H}^n(X, \mathcal{M}^*).$$

2.3. Cup product and Künneth homomorphism

In this section we assume all varieties are smooth and quasi-projective. In this case, Proposition 2.6 tells us that all three versions of morphic cohomology coincide.

Let X, X' be varieties over $\mathbb{C}$. Following [**8**, Proposition 3.2], the projections from $X \times X'$ to the factors induce an exterior product

$$\mathrm{L}^q\mathrm{H}^n(X) \otimes \mathrm{L}^{q'}\mathrm{H}^{n'}(X') \longrightarrow \mathrm{L}^{q+q'}\mathrm{H}^{n+n'}(X \times X'). \tag{3}$$

Theorem 2.9. *Composing the exterior product* (3) *with the diagonal embedding defines a cup product in morphic cohomology*

$$\mathrm{L}^q\mathrm{H}^n(X) \otimes \mathrm{L}^{q'}\mathrm{H}^{n'}(X) \longrightarrow \mathrm{L}^{q+q'}\mathrm{H}^{n+n'}(X),$$

which is graded commutative: $a \cdot b = (-1)^{nn'} b \cdot a$ *for* $a \in \mathrm{L}^q\mathrm{H}^n(X)$*,* $b \in \mathrm{L}^{q'}\mathrm{H}^{n'}(X)$*.*

Proof. See [**7**, Corollary 6.2]. □

Remark 2.10. Let us denote by LH the morphic cohomology ring of a point. As stated in Proposition 2.11 below, LH is a graded ring concentrated in cohomological degree 0, where the grading comes from the q-index. Then, the structure map $X \to \operatorname{Spec}\mathbb{C}$ provides $\mathrm{L}^*\mathrm{H}^n(X)$ with the structure of a graded LH-module.

We can now recall some basic computations of morphic cohomology rings.

Proposition 2.11.

1. *For* $k \geqslant 0$, $\mathrm{L}^*\mathrm{H}^*(\mathbb{A}^k) \cong \mathbb{Z}[s]$*, where* s *is a free generator of bidegree* $(1,0)$ *(degree* 1 *with respect to the* q*-grading).*
2. *For* $k \geqslant 0$*,* $\mathrm{L}^*\mathrm{H}^*(\mathbb{P}^k) \cong \mathbb{Z}[s,h]/(h^{k+1})$*, where* s *has bidegree* $(1,0)$ *and* h *has bidegree* $(1,2)$*.*
3. $\mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \cong \mathbb{Z}[s,e]/(e^2)$*, where* s *is a generator of bidegree* $(1,0)$ *and* e *is a generator of bidegree* $(1,1)$*.*

Proof. Points 1 and 2 follow from duality and the computation of Lawson homology of $\mathbb{P}^k$ (see [**3**, Corollary 4.4]).

For point 3, we take the open cover of $\mathbb{P}^1$ by two affine spaces. Then we have the following piece of Mayer-Vietoris sequence:

$$\mathrm{L}^*\mathrm{H}^n(\mathbb{P}^1) \longrightarrow \mathrm{L}^*\mathrm{H}^n(\mathbb{A}^1)^{\oplus 2} \longrightarrow \mathrm{L}^*\mathrm{H}^n(\mathbb{G}_m) \longrightarrow \mathrm{L}^*\mathrm{H}^{n+1}(\mathbb{P}^1) \longrightarrow \mathrm{L}^*\mathrm{H}^{n+1}(\mathbb{A}^1)^{\oplus 2},$$

which, using points 1 and 2 for the computations of $\mathbb{P}^1$ and $\mathbb{A}^1$, gives the result. □

Remark 2.12. Note that, in particular, $\mathrm{LH} \cong \mathbb{Z}[s]$. Then, by Remark 2.10 the morphic cohomology groups $\mathrm{L}^*\mathrm{H}^n(X)$ are $\mathbb{Z}[s]$-modules. The action by s on $\mathrm{L}^*\mathrm{H}^n(X)$ corresponds to the s-maps in morphic cohomology.

As this structure of LH-module in morphic cohomology is functorial, the exterior product (3) factors through

$$\mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(Y) \longrightarrow \mathrm{L}^*\mathrm{H}^*(X \times Y). \tag{4}$$

In rather special circumstances, this Künneth homomorphism (4) is an isomorphism. We need a very special case of this Künneth isomorphism, which we now prove.

Proposition 2.13. *Let X be a smooth quasi-projective variety. The Künneth homomorphism*

$$\mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \longrightarrow \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{G}_m)$$

is an isomorphism.

Proof. Let $i\colon \mathrm{pt} \to \mathbb{A}^1$ be the inclusion of a point and $j\colon \mathbb{G}_m \to \mathbb{A}^1$ its open complement. We have the following commutative diagram of long exact sequences:

$$\begin{tikzcd}
\cdots \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{LH} \arrow[r, "id\otimes i_!"] \arrow[d, "\cong"] & \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{LH} \arrow[r, "id\otimes j^*"] \arrow[d, "\cong"] & \mathrm{L}^*\mathrm{H}^*(X) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \cdots \arrow[d] \\
\cdots \mathrm{L}^*\mathrm{H}^*(X \times \mathrm{pt}) \arrow[r, "(id\times i)_!"] & \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{A}^1) \arrow[r, "(id\times j)^*"] & \mathrm{L}^*\mathrm{H}^*(X \times \mathbb{G}_m) \cdots .
\end{tikzcd}$$

The vertical maps are the Künneth morphisms, and $i_!$ is the Gysin map defined by duality (Theorem 2.1, 4) as $i_! = \Gamma^{-1} i_* \Gamma$. The exactness of the rows comes, by duality, from the localization theorem (Proposition 4.8 in [**14**]). For the first row, we also need the computation in 2.11 to ensure that all LH-modules in the exact sequence

$$\cdots \longrightarrow \mathrm{L}^*\mathrm{H}^*(\mathrm{pt}) \xrightarrow{i_!} \mathrm{L}^*\mathrm{H}^*(\mathbb{A}^1) \xrightarrow{j^*} \mathrm{L}^*\mathrm{H}^*(\mathbb{G}_m) \longrightarrow \cdots$$

are flat, so that when tensoring with $\mathrm{L}^*\mathrm{H}^*(X)$ the exactness is preserved.

Now the first two vertical maps are isomorphisms. The first by definition, while the second as a consequence of homotopy invariance (point 2 in Theorem 2.1). Then a standard application of the five lemma proves the desired isomorphism. $\square$

Remark 2.14. I became aware of a construction of a Künneth spectral sequence in the case in which X or Y is a linear variety in a private communication with Mircea Voineagu [**20**]. A similar construction is done in [**12**] for higher Chow groups and K-theory.

3. Toric varieties

3.1. Definitions and notation

First we set the notation following [**10**]. Let $N \cong \mathbb{Z}^n$ be a free $\mathbb{Z}$-module of rank n and M its dual $\mathbb{Z}$-module. We will denote by $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. In this way, N and M are to be thought of as lattices on $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$. Moreover, there is the duality pairing $\langle u, v\rangle$ for $u \in N_{\mathbb{R}}$ and $v \in M_{\mathbb{R}}$.

Let Δ be a fan with associated toric variety $X(\Delta)$ defined over $\mathbb{C}$. For any cone $\sigma \in \Delta$ there is a Zariski open subset X_σ such that

$$X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M],$$

where $\sigma^\vee = \{v \in M_{\mathbb{R}} \mid \langle u, v\rangle \geqslant 0,\ \forall u \in \sigma\}$ is the dual cone. We denote by $i_\sigma\colon X_\sigma \hookrightarrow X(\Delta)$ these open embeddings. Moreover, inside X_σ there is a distinguished closed subvariety T_σ such that

$$T_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\perp \cap M],$$

where now $\sigma^\perp = \{v \in M_{\mathbb{R}} \mid \langle u, v\rangle = 0,\ \forall u \in \sigma\}$ is the orthogonal cone. Observe that $\sigma^\perp$ is a vector space of dimension $\operatorname{codim}\sigma$. This means that $T_\sigma \cong \mathbb{G}_m^{\operatorname{codim}\sigma}$ is an algebraic torus. In fact, T_0 is a torus of dimension n acting on $X(\Delta)$, and the torus T_σ is the lowest dimensional orbit for this action contained in X_σ.

If $\tau \leqslant \sigma$, then $X_\tau \subset X_\sigma$. We denote by $i_{\tau,\sigma}\colon X_\tau \hookrightarrow X_\sigma$ this inclusion, which is induced by the morphism of rings $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\tau^\vee \cap M]$.

For any cone σ, the closed embedding $j_\sigma\colon T_\sigma \hookrightarrow X_\sigma$ is induced by the morphism of rings $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\perp \cap M]$ which sends the lattice vectors in $\sigma^\perp$ to themselves and the others to 0.

Finally, there is a retraction $r_\sigma\colon X_\sigma \to T_\sigma$ induced by the monomorphism of rings $\mathbb{C}[\sigma^\perp \cap M] \to \mathbb{C}[\sigma^\vee \cap M]$.

Proposition 3.1. *Let $X(\Delta)$ be a toric variety and σ a cone in Δ. There is a morphism*

$$h\colon X_\sigma \times \mathbb{A}^1_{\mathbb{C}} \longrightarrow X_\sigma$$

such that $h(-,1) = \mathrm{id}$, $h(-,0) = j_\sigma r_\sigma$ and $h(-,t)$ restricts to the identity on T_σ for every t. That is, the morphism h gives an algebraic homotopy equivalence between X_σ and T_σ.

Proof. Recall that $X_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$, $T_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\perp \cap M]$ and the inclusion $T_\sigma \to X_\sigma$ is given by the quotient $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\perp \cap M]$, which is the identity on $\sigma^\perp$ and sends any element $v \in \sigma^\vee$ not in $\sigma^\perp$ to $0 \in \mathbb{C}[\sigma^\perp \cap M]$.

Pick $u_0 \in \sigma$ such that $\sigma^\perp = \sigma^\vee \cap u_0^\perp$. Then define

$$h^*\colon \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M] \otimes \mathbb{C}[t]$$

by $h^*(v) = v \otimes t^{\langle u_0, v\rangle}$ for every $v \in \sigma^\vee$. This gives a morphism of schemes $h\colon X_\sigma \times \mathbb{A}^1_{\mathbb{C}} \to X_\sigma$ with the desired properties. □

We will use the notation $\Delta^{(k)}$ for the set of all cones of codimension k in Δ.

Definition 3.2. An *orientation of a cone* σ is an orientation of the vector spaces $\mathbb{R}\sigma$. An *orientation of a fan* Δ will be a choice of an orientation for every cone in Δ.

We will always use fans with a fixed orientation.

Remark 3.3. Recall that, by definition, any face $\tau \leqslant \sigma$ is given as $\tau = \sigma \cap u^\perp$ for some $u \in M_{\mathbb{R}}$. Let $\tau \leqslant \sigma$ be a face of codimension 1, given as $\tau = \sigma \cap u^\perp$. Then there exists $v \in \sigma$ such that $\langle v, u\rangle > 0$ and

$$\mathbb{R}\sigma = \mathbb{R}v + \mathbb{R}\tau \tag{5}$$

as subspaces of $N_{\mathbb{R}}$. This last identity allows us to transfer the orientation of σ to τ as follows: the orientation induced on τ by σ is the one compatible with the identity (5) and taking the orientation on $\mathbb{R}v$ given by the vector v.

3.2. Morphic cohomology of an algebraic torus

Now we compute the morphic cohomology ring of an algebraic torus. As we will need this computation for subtori of a toric variety, it will be useful to have a canonical description of this ring in terms of the lattice defining the toric variety.

Let N be a lattice of rank n, and $L_{\mathbb{R}} \subset M_{\mathbb{R}} = N_{\mathbb{R}}^\vee$ a subspace of dimension r generated by vectors in the lattice M. Consider the rank r sublattice $L = L_{\mathbb{R}} \cap M$ and its associated torus $T_L = \operatorname{Spec}\mathbb{C}[L]$.

We introduce a bit of notation. Let K be a graded LH-module. We denote by $K[l]_q$ the graded LH-module obtained from K by shifting it l steps into the increasing direction for the q-degree, that is, $(K[l]_q)_i = K_{i-l}$.

It follows from Proposition 2.11 that the piece of homological degree 1, $\mathrm{L}^*\mathrm{H}^1(\mathbb{G}_m)$, is isomorphic, as an LH-module, to $\mathrm{LH}[1]_q$. This is a free graded LH-module with one generator in q-degree 1; we called this generator e in Proposition 2.11. It corresponds, by duality, to a radial Borel-Moore chain joining 0 and ∞ in $\mathbb{G}_m$. Now, any $v \in L$ defines a character $\chi_v \colon T_L \to \operatorname{Spec} \mathbb{C}[v, v^{-1}] = \mathbb{G}_m$. Then, we define a graded morphism of rings

$$\varphi\colon \bigoplus_{n\geqslant 0}(\textstyle\bigwedge^n L \otimes \mathrm{LH})[n]_q \longrightarrow \mathrm{L}^*\mathrm{H}^*(T_L)$$

by

$$\varphi(v \otimes 1) = \chi_v^*(e),$$

for $v \in L$, and extended in the obvious way to the exterior algebra because $\mathrm{L}^*\mathrm{H}^*(T_L)$ is a graded commutative algebra (Theorem 2.9).

Theorem 3.4. *The morphism φ is an isomorphism.*

Proof. We argue by induction on the rank of L. The isomorphism is clear when $\operatorname{rank} L = 1$ by the computation in 2.11. Let $L = L_0 \oplus \mathbb{Z}v$. This gives a product decomposition $T_L = T_{L_0} \times \mathbb{G}_m$. Now, because the Künneth isomorphism in 2.13 preserves the cup product, we get a commutative diagram

$$\begin{tikzcd}
(\bigwedge^n L \otimes \mathrm{LH})[n]_q \arrow[r] \arrow[d, "\varphi_n"] & \bigoplus_{r+s=n}(\bigwedge^r L_0 \otimes \mathrm{LH})[r]_q \otimes_{\mathrm{LH}} (\bigwedge^s \mathbb{Z}v \otimes \mathrm{LH})[s]_q \arrow[d] \\
\mathrm{L}^*\mathrm{H}^n(T_L) \arrow[r] & \bigoplus_{r+s=n} \mathrm{L}^*\mathrm{H}^r(T_{L_0}) \otimes_{\mathrm{LH}} \mathrm{L}^*\mathrm{H}^s(\mathbb{G}_m).
\end{tikzcd}$$

The upper row is an isomorphism by multilinear algebra results, while the lower row is an isomorphism by the Künneth isomorphism 2.13. The right vertical map is a sum of tensor products of φ's corresponding to lower dimensional tori, so it is an isomorphism by induction hypothesis. We conclude then that the left vertical map is an isomorphism. $\square$

4. Spectral sequence associated to a toric variety

Let $X(\Delta)$ be a toric variety of dimension n, R a ring and $\mathcal{F}^*$ a cochain complex of sheaves of R-modules on X. As usual, the hypercohomology of $\mathcal{F}^*$ is

$$\mathbb{H}^n(X(\Delta), \mathcal{F}^*) = \mathrm{H}^n\Gamma(X(\Delta), \mathcal{I}^*),$$

where $\mathcal{I}^*$ is a K-injective resolution $\mathcal{F}^* \to \mathcal{I}^*$.

In this section we will write down a spectral sequence converging to the hypercohomology $\mathbb{H}^n(X(\Delta), \mathcal{F}^*)$ whose E_2 page is computable in terms of the combinatorics of the toric variety, and the hypercohomology of $\mathcal{F}^*$ on algebraic tori. The spectral sequence comes from the identification

$$\mathbb{H}^n(X(\Delta), \mathcal{F}^*) = \operatorname{Ext}^n(R_X, \mathcal{F}^*)$$

and the fact that the hyper-ext can be computed resolving either variable. We will choose to resolve the constant sheaf R_X producing a Čech-like resolution $\check{\mathcal{C}}_*(\Delta, R) \to R_X$ from the combinatorics of the toric variety.

A similar idea, applied to singular homology and cohomology, was exploited in the thesis [**11**].

4.1. Resolution associated to a fan

Let $X(\Delta)$ be a toric variety defined by a fan Δ, and let R be a commutative ring.

Definition 4.1. Let $\check{\mathcal{C}}_k(\Delta, R)$ for $k \geqslant 0$ be the sequence of sheaves of R-modules on $X(\Delta)$ given by

$$\check{\mathcal{C}}_k(\Delta, R) = \bigoplus_{\sigma \in \Delta^{(k)}} i_{\sigma !} i_\sigma^* R_X,$$

where R_X is the constant sheaf on $X(\Delta)$ and $i_\sigma \colon X_\sigma \to X(\Delta)$ is the inclusion of X_σ.

Moreover, we define a sequence of morphisms $d_k \colon \check{\mathcal{C}}_k(\Delta, R) \to \check{\mathcal{C}}_{k-1}(\Delta, R)$ given by

$$d_k = \bigoplus_{\substack{\sigma \in \Delta^{(k-1)} \\ \tau \in \Delta^{(k)} \\ \tau \leqslant \sigma}} \epsilon(\tau, \sigma) \mu_{\tau,\sigma},$$

where $\mu_{\tau,\sigma} \colon i_{\tau !} i_\tau^* R_X \to i_{\sigma !} i_\sigma^* R_X$ is the natural inclusion of sheaves inducing the identity on the nonzero stalks, and $\epsilon(\tau, \sigma) = \pm 1$ according to whether the orientation induced by σ on τ coincides or not with the fixed orientation in τ.

Definition 4.2. Given a fan Δ and a cone $\sigma \in \Delta$ of codimension k, there is a fan Δ_σ defined on the lattice $N_\sigma = N/(\mathbb{R}\sigma \cap N)$ of dimension k, whose cones are the projection of cones in Δ having σ as a face.

This way, the cones in Δ_σ correspond bijectively with the cones $\tau \in \Delta$ having σ as a face.

Remark 4.3. Let $x \in X(\Delta)$ be a point. We denote by $\sigma(x)$ the unique cone in Δ such that $x \in T_{\sigma(x)}$. Observe that

$$(i_{\tau !} i_\tau^* R_X)_x = \begin{cases} R & \text{if } \sigma(x) \leqslant \tau, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the stalk $(i_{\tau !} i_\tau^* R_X)_x$ is nonzero exactly for the cones $\tau \in \Delta$ which represent cones in $\Delta_{\sigma(x)}$.

Proposition 4.4. *Let $X(\Delta)$ be a toric variety associated to a fan Δ.*

1. *There is a canonical isomorphism*

$$\check{\mathcal{C}}_k(\Delta, R)_x \cong \bigoplus_{\substack{\sigma \in \Delta^{(k)} \\ \sigma(x) \leqslant \sigma}} R,$$

and the morphism induced on this stalk by d_k is given by

$$d_{k,x}([\tau]) = \sum_{\substack{\sigma \in \Delta^{(k-1)} \\ \tau \leqslant \sigma}} \epsilon(\tau, \sigma)[\sigma].$$

2. *The sequence of sheaves $\check{\mathcal{C}}_k(\Delta, R)$ together with the morphisms d_k form a chain complex of sheaves of R-modules.*
3. *Let $\Delta \subset \overline{\Delta}$ be an inclusion of fans on the same lattice, giving an open embedding $u\colon X(\Delta) \hookrightarrow X(\overline{\Delta})$. Then, there is a canonical isomorphism of complexes of sheaves $\check{\mathcal{C}}_*(\Delta, R) \cong u^*\check{\mathcal{C}}_*(\overline{\Delta}, R)$.*

Proof. Point 1 follows from 4.1 and Remark 4.3.

For point 2 we need only to check that $(d_{k-1}d_k) = 0$ on stalks. Using the identification of these stalks in 1), we see that for $[\tau]$ an element of the basis of $\check{\mathcal{C}}_k(\Delta, R)_x$, we have

$$d_{k-1,x}d_{k,x}([\tau]) = \sum_{\tau<\sigma<\eta} \epsilon(\tau, \sigma)\epsilon(\sigma, \eta)[\eta].$$

Then, for fixed τ and η there are exactly two faces in between, giving opposite signs. One can see this by looking at the image of η in $N_{\mathbb{R}}/\mathbb{R}\tau$. This image is a two-dimensional cone that obviously has exactly two faces with opposite orientation. This shows that $d_{k-1,x}d_{k,x} = (d_{k-1}d_k)_x = 0$.

Point 3 follows from the following computation:

$$\begin{aligned} u^*\check{\mathcal{C}}_k(\overline{\Delta}, R) &= u^*\Big(\bigoplus_{\overline{\sigma}\in\overline{\Delta}^{(k)}} i_{\overline{\sigma}!} i^*_{\overline{\sigma}} R_{X(\overline{\Delta})} \Big) \\ &= u^*\Big(\bigoplus_{\sigma\in\Delta^{(k)}} (ui_\sigma)_!(ui_\sigma)^* R_{X(\overline{\Delta})} \Big) \\ &= \bigoplus_{\sigma\in\Delta^{(k)}} u^*u_! i_{\sigma!} i^*_\sigma u^* R_{X(\overline{\Delta})} \\ &\cong \bigoplus_{\sigma\in\Delta^{(k)}} i_{\sigma!} i^*_\sigma R_{X(\Delta)} = \check{\mathcal{C}}_k(\Delta, R). \end{aligned}$$ □

Definition 4.5. Let $a\colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ be the augmentation morphism induced by the morphisms $i_{\sigma!}i^*_\sigma R_X \to R_{X(\Delta)}$.

We will prove that $a\colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ is a quasi-isomorphism. To do so, we will relate the stalk complexes $\check{\mathcal{C}}_*(\Delta, R)_x$ with the cellular homology complex of a cellular decomposition on a ball of dimension $\operatorname{codim} \sigma(x)$.

Let Δ be a fan on a lattice N. We define

$$\begin{aligned} B(\Delta) &= \{p \in N_{\mathbb{R}} \mid \|p\| \leqslant 1\}, \\ S(\Delta) &= \{p \in N_{\mathbb{R}} \mid \|p\| = 1\}. \end{aligned}$$

Pick $x \in X(\Delta)$. Then, the space $S(\Delta_{\sigma(x)})$ is a sphere of dimension $\operatorname{codim} \sigma(x) - 1$, and every non-zero cone $\sigma \in \Delta_{\sigma(x)}$ gives a cell of dimension $\dim \sigma - 1$ on $S(\Delta_{\sigma(x)})$, defined by $e_\sigma = \sigma \cap S(\Delta_{\sigma(x)})$. The set $\{e_\sigma\}_{\sigma\in\Delta_{\sigma(x)}}$, together with the entire ball, gives a cellular decomposition of $B(\Delta_{\sigma(x)})$. However, we are interested in a dual cellular decomposition $e^\vee_\sigma$ which we now proceed to describe.

Definition 4.6. To any complete fan Δ we associate an abstract simplicial complex $K(\Delta)$ as follows:

1. The vertices in $K(\Delta)$ correspond to the cones in Δ.
2. The k-simplexes in $K(\Delta)$ are the sets of vertices belonging to flags in Δ of length k, that is, sequences of strictly included cones

$$\tau_0 < \tau_1 < \cdots < \tau_k.$$

Remark 4.7. If we had omitted the cone 0 in the definition of $K(\Delta)$, we would have obtained a combinatorial model of the barycentric subdivision of the fan Δ.

For every 1-dimensional cone $\tau \in \Delta^{(n-1)}$ let $u_\tau \in N_{\mathbb{R}}$ be the unique unit vector generating it. Then, for any non-zero cone $\sigma \in \Delta$, let v_σ be the vector

$$v_\sigma = \sum_{\substack{\tau \in \Delta^{(n-1)} \\ \tau \leqslant \sigma}} u_\tau .$$

Definition 4.8. For every k-simplex $(\tau_0, \ldots, \tau_k) \in K(\Delta)$ given by a flag of cones $\tau_0 < \cdots < \tau_k$, we define a subset $d_{(\tau_0,\ldots,\tau_k)} \subset B(\Delta)$ as follows:

$$d_{(\tau_0,\ldots,\tau_k)} = \begin{cases} \{0\} & \text{if } \tau_0 = 0 \text{ and } k = 0, \\ \mathbb{R}_{\geqslant 0} \langle v_{\tau_1}, \ldots, v_{\tau_k} \rangle \cap B(\Delta) & \text{if } \tau_0 = 0 \text{ and } k > 0, \\ \mathbb{R}_{\geqslant 0} \langle v_{\tau_0}, \ldots, v_{\tau_k} \rangle \cap S(\Delta) & \text{if } \tau_0 \neq 0. \end{cases}$$

Proposition 4.9. *Let Δ be a complete fan. The subsets $d_{(\tau_0,\ldots,\tau_k)} \subset B(\Delta)$ are homeomorphic to closed balls of dimension k. Together they form a cellular decomposition of the ball $B(\Delta)$, giving a geometric realization of the abstract simplicial complex $K(\Delta)$.*

Proof. Let $(\tau_0, \ldots, \tau_k) \in K(\Delta)$. Because the vectors v_{τ_i} all belong to the cone τ_k, the subsets $\mathbb{R}_{\geqslant 0} \langle v_{\tau_0}, \ldots, v_{\tau_k} \rangle$ are strictly convex cones, so in either case of Definition 4.8, the resulting set $d_{(\tau_0,\ldots,\tau_k)}$ is a cell: it is either a connected convex subset of $B(\Delta)$ or a connected and geodesically convex subset of $S(\Delta)$. The statement about the dimension of $d_{(\tau_0,\ldots,\tau_k)}$ follows from the linear independence of the vectors v_{τ_i} associated to the flag $0 \neq \tau_0 < \cdots < \tau_k$.

Finally, observe that the boundary of a cell $d_{(\tau_0,\ldots,\tau_k)}$ is formed by the cells resulting from removing one cone in the flag, all of lower dimension. This proves that the cells $d_{(\tau_0,\ldots,\tau_k)}$ give a cellular decomposition of the ball $B(\Delta)$. □

Definition 4.10. For every cone $\sigma \in \Delta$, let $e_\sigma^\vee \subset B(\Delta)$ be the subset defined by

$$e_\sigma^\vee = \bigcup_{\substack{k \geqslant 0 \\ (\tau_0,\ldots,\tau_k) \in K(\Delta) \\ \sigma \leqslant \tau_0}} d_{(\tau_0,\ldots,\tau_k)} .$$

Proposition 4.11. *Let Δ be a complete fan. The subsets $e_\sigma^\vee \subset B(\Delta)$ are homeomorphic to closed balls of dimension* $\operatorname{codim} \sigma$ *and form a cellular decomposition of the ball $B(\Delta)$. The ball together with this decomposition will be denoted by $B(\Delta)^\vee$.*

Proof. Observe that $e_\sigma^\vee$ is a geometric realization of a subcomplex of $K(\Delta)$ which is isomorphic to $K(\Delta_\sigma)$ (follows directly from the definitions). Now, Proposition 4.9

applied to the simplicial complex $K(\Delta_\sigma)$ realizes $K(\Delta_\sigma)$ as a $(\operatorname{codim}\sigma)$-dimensional ball $B(\Delta_\sigma)$. So, $e_\sigma^\vee$ is homeomorphic to this ball.

The cells $e_\sigma^\vee$ cover all the ball $B(\Delta)$ by completeness of the fan, and they are attached properly because the $d_{(\tau_0,\ldots,\tau_k)}$ are. □

Proposition 4.12. *Let Δ be a complete fan. There is a canonical isomorphism of chain complexes*

$$\check{\mathcal{C}}_*(\Delta, R)_x \cong \mathrm{C}_*^{\mathrm{cell}}(B(\Delta_{\sigma(x)})^\vee, R).$$

Proof. There is a canonical isomorphism of R-modules

$$\check{\mathcal{C}}_k(\Delta, R)_x \cong \mathrm{C}_k^{\mathrm{cell}}(B(\Delta_{\sigma(x)})^\vee, R),$$

as both are generated by the cones in $\Delta_{\sigma(x)}$ of codimension k (see Proposition 4.4).

It only remains to check that the differentials in $\check{\mathcal{C}}_*(\Delta, R)_x$ coincide with the cellular ones. Note that the attaching maps $f_\tau\colon \partial e_\tau^\vee \to \mathrm{Sk}_{\operatorname{codim}\tau-1} B(\Delta_{\sigma(x)})^\vee$ are homeomorphisms with the image. So, for any lower dimensional cell $e_\sigma^\vee$ on the boundary of $e_\tau^\vee$, the corresponding matrix element in the cellular differential is a sign, according to the relative orientation of the cells $e_\sigma^\vee$ and $e_\tau^\vee$. This is exactly the differential in $\check{\mathcal{C}}_*(\Delta, R)_x$. □

Corollary 4.13. *Let Δ be an arbitrary fan. Then, the augmentation $a\colon \check{\mathcal{C}}_*(\Delta, R) \to R_X$ is a quasi-isomorphism.*

Proof. First take $\Delta \subset \overline{\Delta}$ a completion of the fan Δ. Because of point 3 in Proposition 4.4, it is enough to check that $\check{\mathcal{C}}_*(\overline{\Delta}, R) \to R_{X(\overline{\Delta})}$ is a quasi-isomorphism for the complete fan $\overline{\Delta}$.

Now, Proposition 4.12 tells us that the stalk complex $\check{\mathcal{C}}_*(\overline{\Delta}, R)_x$ is isomorphic to the cellular complex associated to the cellular decomposition of the ball $B(\overline{\Delta}_{\sigma(x)})^\vee$, so its homology is

$$\mathrm{H}_k\check{\mathcal{C}}_*(\overline{\Delta}, R) \cong \mathrm{H}_k\mathrm{C}_*^{\mathrm{cell}}(B(\overline{\Delta}_{\sigma(x)})^\vee, R) = \begin{cases} R & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

We conclude that the augmentation $a_x\colon \check{\mathcal{C}}_*(\overline{\Delta}, R)_x \to R$ on the stalks is a quasi-isomorphism. As quasi-isomorphisms of complexes of sheaves are detected on stalks, we are done. □

4.2. The spectral sequence

Let $X(\Delta)$ be a toric variety and $\mathcal{F}^*$ be a complex of sheaves on X. We describe a spectral sequence converging to the hypercohomology $\mathbb{H}^n(X(\Delta), \mathcal{F}^*)$.

Definition 4.14. A complex of sheaves $\mathcal{F}^*$ is said to have *homotopy invariant cohomology* if for every variety X the projection $p\colon X \times \mathbb{A}^1 \to X$ induces an isomorphisms in hypercohomology $\mathbb{H}^n(X, \mathcal{F}^*) \cong \mathbb{H}^n(X \times \mathbb{A}^1, \mathcal{F}^*)$.

Remark 4.15. The complex of sheaves $\mathcal{M}^*$ defining morphic cohomology has homotopy invariant cohomology by point 2 in Theorem 2.1.

Theorem 4.16. *Let $X(\Delta)$ be a toric variety associated to a fan Δ and $\mathcal{F}^*$ a bounded above cochain complex of sheaves. There is a convergent spectral sequence*

$$E_1^{r,s} = \operatorname{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*) \Longrightarrow \mathbb{H}^{r+s}(X(\Delta), \mathcal{F}^*).$$

Moreover, if $\mathcal{F}^*$ has homotopy invariant cohomology, then

$$E_1^{r,s} \cong \bigoplus_{\sigma \in \Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{F}^*), \tag{6}$$

and the differentials on the first page $d_1 \colon E_1^{r,s} \to E_1^{r+1,s}$ are given by

$$d_1 = \sum_{\substack{\sigma \in \Delta^{(r)} \\ \tau \in \Delta^{(r+1)} \\ \tau \leqslant \sigma}} \epsilon(\tau, \sigma) r^*_{\tau,\sigma}, \tag{7}$$

where

$$r_{\tau,\sigma} \colon T_\tau = \operatorname{Spec} \mathbb{C}[\tau^\perp] \longrightarrow T_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\perp]$$

are the morphisms induced by the natural inclusion $\sigma^\perp \to \tau^\perp$.

Proof. Let $\mathcal{F}^* \to \mathcal{I}^*$ be a K-injective resolution of $\mathcal{F}^*$ (see Remark 2.2). Let $\check{\mathcal{C}}_*(\Delta, \mathbb{Z}) \to \mathbb{Z}_X$ be the resolution of the constant sheaf $\mathbb{Z}_X$ from Corollary 4.13. We build a double complex

$$C^{r,s} = \operatorname{Hom}(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{I}^s)$$

with the induced differentials (going in the increasing direction of r and s). The homology of this double complex in the s direction is $\operatorname{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*)$, giving the spectral sequence

$$E_1^{r,s} = \operatorname{Ext}^s(\check{\mathcal{C}}_r(\Delta, \mathbb{Z}), \mathcal{F}^*) \Longrightarrow \mathbb{H}^{r+s}(X(\Delta), \mathcal{F}^*).$$

As for the convergence, the complex of sheaves $\mathcal{I}^*$ is bounded above, and the schemes X_σ have finite cohomological dimension. Using the hypercohomology spectral sequence, we conclude that $\mathbb{H}^k(X_\sigma, \mathcal{F}^*)$ vanishes for large k. In other words, the first page is bounded above in the s direction. By construction, it is bounded (from both sides) in the r direction, and this is enough to establish the convergence.

If $\mathcal{F}^*$ is homotopy invariant, as the immersion $T_\sigma \to X_\sigma$ are algebraic homotopy equivalences, then we get the isomorphism (6).

Finally, the differentials on the first page are induced by the r-differentials in the double complex $C^{r,s}$, which are given by the formula

$$d_1 = \sum_{\substack{\sigma \in \Delta^{(r)} \\ \tau \in \Delta^{(r+1)} \\ \tau \leqslant \sigma}} \epsilon(\tau, \sigma) i^*_{\tau,\sigma},$$

where $i_{\tau,\sigma} \colon X_\tau \to X_\sigma$ is the inclusion. Formula (7) follows from the equation $r^*_{\tau,\sigma} = j^*_\tau i^*_{\tau,\sigma} r^*_\sigma$ and the fact that j^*_τ and r^*_σ are mutually inverse isomorphisms giving the identification $\mathbb{H}^s(T_\sigma, \mathcal{F}^*) \cong \mathbb{H}^s(X_\sigma, \mathcal{F}^*)$. $\square$

We have a rather explicit description of the first page and differentials of the spectral sequence in 4.16. Together with the computation in Theorem 3.4 of the morphic cohomology of a torus we can make it more explicit.

Corollary 4.17. *Let $\mathcal{M}^*$ be the complex defining morphic cohomology. Then, the first page of the spectral sequence in 4.16 is*

$$E_1^{r,s} \cong \bigoplus_{\sigma\in\Delta^{(r)}} \Big(\bigwedge^s(\sigma^\perp \cap M)\otimes \mathrm{LH}\Big)[s]_q,$$

and the differentials $d_1^r\colon E_1^{r,s}\to E_1^{r+1,s}$ are given by

$$d_1^r\Big(\sum_{\sigma\in\Delta^{(r)}} x_\sigma v_{1,\sigma}\wedge\cdots\wedge v_{s,\sigma}\Big) = \sum_{\sigma\in\Delta^{(r)}} x_\sigma \sum_{\substack{\tau\in\Delta^{(r+1)}\\ \tau\leqslant\sigma}} \epsilon(\tau,\sigma) v_{1,\sigma}\wedge\cdots\wedge v_{s,\sigma}.$$

Proof. The proof follows from Theorem 4.16 and the computation 3.4. □

Finally, using an idea from [**11**] which can be traced back to [**18**], we show that this spectral sequence degenerates rationally.

Theorem 4.18. *The spectral sequence from Corollary 4.17 degenerates on the second page when tensored with $\mathbb{Q}$.*

Proof. Let $\mathrm{LH}_\mathbb{Q} = \mathrm{LH}\otimes_\mathbb{Z}\mathbb{Q}$. The toric variety $X(\Delta)$ admits an $\mathbb{N}$-action. Let $m\in\mathbb{N}$, then $[m]\colon X(\Delta)\to X(\Delta)$ is the morphism which on the open sets $X_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee\cap M]$ is defined through the ring homomorphism $[m]^*\colon \mathbb{C}[\sigma^\vee\cap M]\to\mathbb{C}[\sigma^\vee\cap M]$ given by $[v]\mapsto[mv]$ (see [**18**] for details).

The $\mathbb{N}$-action on $X(\Delta)$ induces an $\mathbb{N}$-action on the spectral sequence from Corollary 4.17. As the rational morphic cohomology of a torus $T_L = \operatorname{Spec}\mathbb{C}[L]$ is

$$\mathrm{L}^*\mathrm{H}^s(T_L)_\mathbb{Q} = \Big(\bigwedge^s L\otimes \mathrm{LH}_\mathbb{Q}\Big)[s]_q,$$

The $\mathbb{N}$-action on the page $E_1^{r,s}$ is just multiplication by m^s. As the next pages $E_k^{r,s}$ of the spectral sequence are subquotients of $E_1^{*,*}$, the action on those pages is also given by m^s. On the other hand the differentials go $d_k\colon E_k^{r,s}\to E_k^{r+k,s+1-k}$ and the $\mathbb{N}$-action commutes with them, so

$$m^s d_k(x) = d_k(m^s x) = d_k([m]x) = [m]d_k(x) = m^{s+1-k}d_k(x),$$

where $x\in E_k^{r,s}$. Rationally, this forces $d_k(x)=0$ when $k\geqslant 2$. □

4.3. cdh descent

We now describe a cdh-descent version of morphic cohomology following Section 3 in [**1**]. This theory will have the Mayer-Vietoris property for cdh covers built-in by definition. Let $\mathcal{M}^*$ be the presheaf of complexes defined in equation (1) and let

$$\mathcal{M}^* \longrightarrow \mathcal{M}^*_{\mathrm{cdh}}$$

be a cdh-fibrant replacement on the big cdh-site on $\mathbf{Var}_\mathbb{C}$. Then we can define a cdh-fibrant version of morphic cohomology as

$$\mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(X) = \mathbb{H}^n(X, \mathcal{M}^*_{\mathrm{cdh}}(q)|_X). \tag{8}$$

Proposition 4.19. *The cdh version of morphic cohomology in* (8) *satisfies the following properties:*

1. *For X smooth,* $\mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(X) \cong \mathrm{L}^q\mathrm{H}^n(X)$.
2. $\mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(\text{-})$ *has the Mayer-Vietoris property for Zariski covers of any quasi-projective X, as in Theorem 2.7.*
3. *For any quasi-projective X,* $\mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(X \times \mathbb{A}^1) \cong \mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(X)$ *(homotopy invariance).*

Proof. Point 1 is a consequence of the fact that the version of morphic cohomology $\mathrm{L}^q\mathrm{H}^n(\text{-})$ already has descent for cdh covers on the smooth site. This goes as follows. From Corollary 3.9 in [**1**] and resolution of singularities, we know that in order to have descent for the cdh topology restricted to the smooth site it is enough to have a Mayer-Vietoris property for Nisnevich covers and blow-up squares with smooth centers. Using this, and the comparison with motivic cohomology from Corollary 3.5 in [**9**] it follows that $\mathrm{L}^q\mathrm{H}^n(X) \cong \mathbb{H}^n(X, r\mathcal{M}^*_{\mathrm{cdh}}(q))$. Here $r\mathcal{M}^*_{\mathrm{cdh}}(q)$ is the restriction to the smooth site of the complex of sheaves $\mathcal{M}^*_{\mathrm{cdh}}(q)$. Finally, arguing as in the first paragraph of the proof of Theorem 3.12 in [**1**], we conclude that $\mathbb{H}^n(X, r\mathcal{M}^*_{\mathrm{cdh}}(q)) \cong \mathbb{H}^n(X, \mathcal{M}^*_{\mathrm{cdh}}(q)) = \mathrm{L}^q\mathrm{H}^n_{\mathrm{cdh}}(X)$.

Point 2 can be argued as in Theorem 2.7. It is essentially built in by definition, as Zariski covers are cdh covers.

Point 3 follows from Hironaka's theorem on resolution of singularities, as an arbitrary variety is locally smooth in the cdh topology. This means that for any X we can take a proper smooth hypercovering $U_\bullet \to X$ (6.2.8 in [**2**]). Then we have a descent spectral sequence (Section 5.3 in [**2**])

$$E_1^{r,s} = \mathbb{H}^s(U_r, \mathcal{M}^*_{\mathrm{cdh}}(q)|_{U_r}) \Longrightarrow \mathbb{H}^{r+s}(X, \mathcal{M}^*_{\mathrm{cdh}}(q)|_X).$$

As the hypercovering $U_\bullet$ is smooth, using point 1 above and homotopy invariance for $\mathrm{L}^q\mathrm{H}^n(\text{-})$ (Point 1 in Theorem 2.1), we get a comparison morphism of spectral sequences with isomorphisms on the first page

$$\begin{tikzcd}
E_1^{r,s} = \mathbb{H}^s(U_r, \mathcal{M}^*_{\mathrm{cdh}}(q)) \arrow[r, Rightarrow] \arrow[d, "\cong"] & \mathbb{H}^{r+s}(X, \mathcal{M}^*_{\mathrm{cdh}}(q)) \arrow[d] \\
E_1'^{r,s} = \mathbb{H}^s(U_r \times \mathbb{A}^1, \mathcal{M}^*_{\mathrm{cdh}}(q)) \arrow[r, Rightarrow] & \mathbb{H}^{r+s}(X \times \mathbb{A}^1, \mathcal{M}^*_{\mathrm{cdh}}(q))
\end{tikzcd}$$

from which we deduce that the map on the right is also an isomorphism. □

Proceeding as in Section 2.2 we obtain a chain of comparison morphism

$$\mathcal{M}^*_{\mathrm{zar,hi}} \longrightarrow (\mathcal{M}^*_{\mathrm{cdh}})_{\mathrm{zar,hi}} \xleftarrow{\ \simeq\ } \mathcal{M}^*_{\mathrm{cdh}}. \tag{9}$$

The right map is a quasi-isomorphism as a consequence of Proposition 4.19. This leads to a comparison map on morphic cohomology

$$\mathrm{L}^*\mathrm{H}^*_{\mathrm{zar,hi}}(X) \longrightarrow \mathrm{L}^*\mathrm{H}^*_{\mathrm{cdh}}(X)\ . \tag{10}$$

Now, the spectral sequence from Theorem 4.16 has the following corollary:

Corollary 4.20. *Let $X(\Delta)$ be a toric variety associated to a fan Δ. Then, the comparison morphism* (10) *evaluated at $X(\Delta)$ is an isomorphism.*

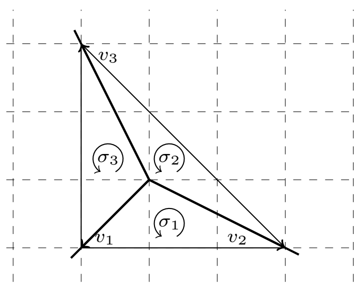
Proof. We apply Theorem 4.16 to both complexes of sheaves $\mathcal{M}^*_{\mathrm{zar,hi}}$ and $\mathcal{M}^*_{\mathrm{cdh}}$. Then, the comparison morphism (9) gives a morphism of spectral sequences

$$\begin{array}{ccc} E_1^{r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^*_{\mathrm{zar,hi}}(q)) & \Longrightarrow & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^*_{\mathrm{zar,hi}}(q)) \\ \downarrow & & \downarrow \\ E_1^{\prime r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^*_{\mathrm{cdh}}(q)) & \Longrightarrow & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^*_{\mathrm{cdh}}(q)). \end{array}$$

Because of point 1 in Proposition 4.19 above, this gives an isomorphism on the first page so the vertical map on the right is also an isomorphism. □

4.4. An example and an application

As an example of how the spectral sequence works, we give a sample computation. Consider the following fan Δ in $\mathbb{Z}^2$ as pictured in Figure 1:



$$\begin{aligned} v_1 &= (-1,-1), & v_1^\perp &= \langle(1,-1)\rangle\,, \\ v_2 &= (2,-1), & v_2^\perp &= \langle(1,2)\rangle\,, \\ v_3 &= (-1,2), & v_3^\perp &= \langle(-2,-1)\rangle\,. \end{aligned}$$

Figure 1: Fan Δ.

The associated toric variety $X(\Delta)$ is the quotient $\mathbb{P}^2/\mu_3$, where the action of a cubic root of unity $\xi\in\mu_3$ is given by $\xi[x:y:z] = [x:\xi y:\xi^2 z]$.

Let $R = \mathrm{LH} \cong \mathbb{Z}[s]$. Then, the spectral sequence is represented in Figures 2 and 3. The differentials on the first page are given by the matrices

$$d_1^{00} : \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad d_1^{10} : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad d_1^{11} : \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & -1 \end{pmatrix}.$$

So, the convergence of the spectral sequence tells us that

$$\mathrm{L}^*\mathrm{H}^n(X) = \begin{cases} R[k]_q & \text{for } n = 2k \text{ and } k\in\{0,1,2\}, \\ R/3[1]_q & \text{for } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now we describe an application to the Suslin conjecture. Let $\varepsilon\colon \mathbf{Top}\to\mathbf{qProj}_{\mathbb{C}}$ be the morphism of sites, with the usual topology in **Top** and the Zariski topology on $\mathbf{qProj}_{\mathbb{C}}$. Let $\mathbf{R}\varepsilon_*\mathbb{Z}$ be the derived push-forward of the constant sheaf $\mathbb{Z}$ on **Top** to the Zariski site $\mathbf{qProj}_{\mathbb{C}}$. There is a natural map $\mathcal{M}^*(q)\to\mathbf{R}\varepsilon_*\mathbb{Z}$ which, on smooth varieties, factors as

$$\mathcal{M}^*(q) \longrightarrow \tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z}. \tag{11}$$

See [**6**] for details.

$$\begin{tikzcd}
s & & & \\
2 & 0 \arrow[r] & 0 \arrow[r] & R \\
1 & 0 \arrow[r] & R^3 \arrow[r, "d_1^{11}"] & R^2 \\
0 & R^3 \arrow[r, "d_1^{00}"] & R^3 \arrow[r, "d_1^{10}"] & R \\
 & 0 & 1 & 2 \quad r
\end{tikzcd}$$

Figure 2: Page $E_1^{r,s}$.

$$\begin{tikzcd}
s & & & \\
2 & 0 \arrow[drr] & 0 & R \\
1 & 0 \arrow[drr] & R & R/3 \\
0 & R & 0 & 0 \\
 & 0 & 1 & 2 \quad r
\end{tikzcd}$$

Figure 3: Page $E_2^{r,s}$.

There is the following conjecture, a morphic analogue of the Beilinson-Lichtenbaum conjecture in the motivic world.

Conjecture 4.21 (Suslin)**.** *The comparison morphism* (11) *above is a quasi-isomorphism on smooth varieties.*

This conjecture is proved for the class of smooth linear varieties (which include smooth toric varieties) in [**6**, Theorem 7.14].

The spectral sequence 4.16 has the following corollary:

Corollary 4.22. *The Suslin conjecture holds for all quasi-projective toric varieties (not necessarily smooth).*

Proof. First of all, we have to check that $\mathcal{M}^*(q)|_{X(\Delta)}$ is exact above degree q, in order to have a factorization $\mathcal{M}^*(q)|_{X(\Delta)} \to \tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z}|_{X(\Delta)}$ as in (11). This is a local statement on $X(\Delta)$, so we can restrict to an open X_σ. Now the inclusion $j_\sigma\colon T_\sigma \to X_\sigma$ is an algebraic homotopy equivalence, and they induce isomorphisms on hypercohomology

$$\mathbb{H}^n(X_\sigma, \mathcal{M}^*(q)|_{X_\sigma}) \xrightarrow{\cong} \mathbb{H}^n(T_\sigma, \mathcal{M}^*(q)|_{T_\sigma}) \ ,$$

so the natural map $\mathcal{M}^*(q)|_{X_\sigma} \to \mathbf{R}j_{\sigma *}\mathcal{M}^*(q)|_{T_\sigma}$ is a quasi-isomorphism. As T_σ is smooth, its cohomology vanishes above q, and we have the desired factorization.

Now, $\tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z}$ has homotopy invariant cohomology, because $\mathbf{R}\varepsilon_*\mathbb{Z}$ does, and the truncation preserves the homotopy invariance of the cohomology sheaves. We can apply Theorem 4.16 and get a spectral sequence converging to $\mathbb{H}^n(X(\Delta), \tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z})$. Moreover, the comparison map (11) gives a morphism of spectral sequences

$$\begin{tikzcd}
E_1^{r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \mathcal{M}^*(q)) \arrow[r, Rightarrow] \arrow[d] & \mathbb{H}^{r+s}(X(\Delta), \mathcal{M}^*(q)) \arrow[d] \\
E_1^{'r,s} = \bigoplus_{\sigma\in\Delta^{(r)}} \mathbb{H}^s(T_\sigma, \tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z}) \arrow[r, Rightarrow] & \mathbb{H}^{r+s}(X(\Delta), \tau_{\leqslant q}\mathbf{R}\varepsilon_*\mathbb{Z}),
\end{tikzcd}$$

which is an isomorphism on the first page by Theorem 7.14 in [**6**], so it gives an isomorphism on the right, as claimed. ∎

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