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REMARK ON RIGIDITY OVER SEVERAL FIELDS

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(*communicated by J.F. Jardine*)

Abstract

It is shown that T-spectrum representable cohomology theories on smooth algebraic varieties satisfy normalization condition over nonreal fields. As a consequence, one can see that the rigidity property holds for all representable theories over considered fields.

1. Introduction

Consider some category of schemes (spaces) $\mathcal{S}$ over a base scheme (space) B together with a cohomology theory $E^*\colon \mathcal{S}^{\mathrm{op}} \to \mathbf{Ab}$. We say that E^* satisfies *rigidity* if for every irreducible scheme (arc-connected space) $X \xrightarrow{\chi} B$, any two sections $\sigma_0, \sigma_1\colon B \to X$ of the structure morphism χ induce the same homomorphism $\sigma_0^* = \sigma_1^*\colon E^*(X) \to E^*(B)$. (Here and below we will omit the bigrading of scheme cohomology groups.) In classical topology, the rigidity property is an obvious consequence of homotopy invariance of cohomology theories. However, in algebraic geometry $\mathbb{A}^1$-invariance does not always imply rigidity. (See [**16**] for details.) It only holds under certain restrictions on the category $\mathcal{S}$ and the cohomology theory E^*.

Rigidity property played an important role in the calculation of algebraic K-groups of fields [**12**] and Henselian rings. One should mention corresponding results obtained for algebraic K-functor with finite coefficients by Suslin, Gabber, and others (see [**1, 2, 13, 14**]). Similar results for hermitian K-theory are given in [**4, 6**], and for Witt groups in [**7**], [**10**, p. 208].

In [**3**], the author, together with Jens Hornbostel, established the rigidity property and some of its corollaries for every cohomology theory represented by a T-spectrum and satisfying the so-called normalization condition (see loc. cit., Definition 1.3). This gives a complete investigation of the orientable case; however, unorientable cohomology theories (such as Witt groups or cohomotopy) are more subtle. Generally, for a given cohomology theory it is not obvious whether the normalization property is satisfied over a concrete field. Below we will show that for the category of smooth separated schemes of finite type (algebraic varieties) over a nonreal field the normalization condition automatically holds for every T-representable theory that makes the main result of [**3**] a powerful tool to study unorientable theories.

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2. The main result

Consider a category Sm/k of smooth algebraic varieties over a field k. Also consider a category of pointed varieties Sm_+/k whose objects are pairs $(X, * \to X)$ consisting of a variety and a morphism of the basepoint to it.[1] Morphisms in this category are ones of Sm/k, preserving the basepoint choice. By default, the basepoints of $\mathbb{G}_m := \operatorname{Spec} k[t, t^{-1}]$ and $\mathbb{A}^1 := \operatorname{Spec} k[t]$ are $\{1\}$ and of the projective line $\mathbb{P}^1$ is $[1:1]$. The forgetful functor $Sm_+/k \to Sm/k$ admits a left-adjoint functor $()_+ \colon Sm/k \to Sm_+/k$ that adds an external basepoint to a scheme.

The category Sm/k can be embedded in the category of sheaves in the Nisnevich topology or *spaces* $Spc/k := Shv_{Nis}(Sm/k)$. The embedding functor sends a scheme V to a presheaf $\operatorname{Hom}_{Sm/k}(-, V)$ represented by V that automatically happens to be a sheaf in the Nisnevich topology (see details in [**15**]). The same procedure works for pointed varieties and one finishes in the category Spc_+/k of pointed spaces. Below, we will identify varieties with corresponding representable sheaves and call them spaces.

Since the category Spc_+/k admits all small limits and colimits, one can define the wedge sum $X \vee Y$ of two pointed spaces as the colimit of the diagram:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \\ Y. & \end{tikzcd}$$

For a given morphism $Y \xrightarrow{f} X$ one sets X/Y as the colimit of the diagram:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & X \\ *. & \end{tikzcd}$$

One also defines the smash-product of two pointed spaces as

$$X \wedge Y := X \times Y / \left((* \times Y) \vee (X \times *) \right).$$

Finally, following Voevodsky, we define the 0-sphere S^0 as $\operatorname{Spec} k_+$ and the simplicial sphere $S^1 := \mathbb{A}^1/S^0$, where the embedding map $S^0 \hookrightarrow \mathbb{A}^1$ sends $\operatorname{Spec} k$ to the point $\{0\}$.

From now on we pass to the stable homotopy category of symmetric T-spectra $\mathbf{Sp}_T^{\Sigma}/k$ (see [**5**]). The functor Σ_T^{∞} sends a pointed space to a suspension T-spectrum.

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[1] Here and below $*$ denotes the terminal object of a category.

Abusing the notation we will often use the same letters for spaces like $\mathbb{G}_m, \mathbb{P}^1$, etc. and the corresponding T-suspension symmetric spectra, unless this leads to possible misunderstanding.

Following Morel's notation, we denote by $S^p(q)$ the sphere $(S^1)^{\wedge p} \wedge \mathbb{G}_m^{\wedge q}$. Recall the definition of the Hopf morphism [**8**]. Consider a standard morphism $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ induced by the map: $(a, b) \mapsto [a : b]$. One can see that the spaces above are homotopy equivalent to corresponding motivic spheres. Hence, we get the morphism: $S^1(2) \to S^1(1)$.

Desuspending in the category $\mathbf{Sp}_T^\Sigma/k$, one also obtains the morphism $\mathcal{H}\colon S^0(1) := \mathbb{G}_m \to *$ that will be called *Hopf morphism* below. Let E^* denote a cohomology theory represented by a symmetric T-spectrum.

Lemma 2.1. *If the map $\mathcal{H}^*$ induced in the theory E^* by Hopf morphism vanishes, then the map $E^*(\mathbb{P}^2) \to E^*(\mathbb{P}^1)$ given by the standard embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is an epimorphism.*

Proof. Following Morel, consider the distinguished square:

$$\begin{tikzcd} \mathbb{A}^2 - \{0\} \arrow[r, hook] \arrow[d] & \mathbb{A}^2 \arrow[d, hook] \\ \mathbb{P}^1 \arrow[r, hook] & \mathbb{P}^2. \end{tikzcd}$$

Here the left vertical arrow is the suspension of the Hopf map. Its cone, as it follows from the square above, is isomorphic to $\mathbb{P}^2/\mathbb{A}^2 \simeq \mathbb{P}^2$. If the induced Hopf map in cohomology vanishes, then the map $E^*(\mathbb{P}^2) \to E^*(\mathbb{P}^1)$ is an epimorphism. $\square$

Let us define several important homotopy classes of morphisms and clarify their interrelation. Let $\varepsilon\colon \mathbb{G}_m \to \mathbb{G}_m \in [S^0, S^0]$ be the homotopy class induced by the morphism $\alpha \mapsto \alpha^{-1}$. For every $u \in k^\times$ denote by $\langle u \rangle \in [S^0, S^0]$ the homotopy class of the morphism $\mathbb{P}^1 \to \mathbb{P}^1$ given as: $[x : y] \mapsto [x : uy]$.

Further, we will need the following fact that is well-known in the classical algebraic topology (see [**8**, Lemma 6.2.2]). Since the sphere S^1 is a cogroup object in $\mathbf{Sp}_T^\Sigma/k$, the cofiber sequence of spectra induced by the sequence $\mathbb{G}_m \vee \mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$ splits. Therefore, in the category $\mathbf{Sp}_T^\Sigma/k$ the spectrum $\Sigma_T^\infty(\mathbb{G}_m \times \mathbb{G}_m)$ can be decomposed as a wedge of spectra $\mathbb{G}_m \vee \mathbb{G}_m \vee \Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m)$. Consider the morphism of spectra $\tilde{\eta}\colon \Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m) \to \mathbb{G}_m$ determined on the component $\Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m)$ by the product map $\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\times} \mathbb{G}_m$ and denote by $\eta \in [\mathbb{G}_m, S^0]$ its homotopy class.

The homotopy class of $\langle -1 \rangle$ can be represented by the morphism $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ sending $[x : y]$ to $[y : x]$. One can easily see that the corresponding matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{SL_2}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are SL_2-equivalent and, therefore, induce the same homotopy class.

The morphism φ acts as ε being restricted to $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ and swaps two standard copies of $\mathbb{A}^1$ lying into $\mathbb{P}^1$. Since $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$, one can see that $\varepsilon = -\langle -1 \rangle$.

Using similar matrix argument as above, one sees that the morphism $\tau\colon \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1$, interchanging two copies of $\mathbb{P}^1$ in the smash-product, corresponds to the element $\varepsilon \in [\mathbb{P}^1, \mathbb{P}^1] = [S^0, S^0]$. Since $\mathbb{P}^1$ is a suspension of $\mathbb{G}_m$, this implies that the homotopy class of the induced morphism $\tilde{\tau}\colon \Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m) \to \Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m)$ equals to $-\varepsilon \in [S^0, S^0]$.

Composing $\tilde{\tau}$ and $\tilde{\eta}$ and taking into account commutativity of the product in $\mathbb{G}_m$, one sees that $\eta\varepsilon = \eta$.

Finally, embedding $\mathbb{G}_m \times \mathbb{G}_m$ into $\mathbb{A}^2 - \{0\}$, we obtain that the Hopf morphism is homotopy equivalent to S^1-suspension of the morphism $\Sigma_T^\infty(\mathbb{G}_m \wedge \mathbb{G}_m) \to \mathbb{G}_m$ induced by $(\alpha, \beta) \mapsto \alpha^{-1}\beta$. This identifies homotopy classes $\mathcal{H} \simeq \eta\varepsilon \simeq \eta$. The discussion altogether yields the following simple lemma, which is also definitely a formal consequence of relation [**9**, 6.1.2.4].

Lemma 2.2. *If the endomorphism* $\langle -1\rangle^* - \mathrm{id}$ *has* 2*-primary (additive) order in the group* $\mathrm{End}\,(E^*(\mathbb{G}_m))$*, then so does the map* $\mathcal{H}^*$*.*

Proof. Let us mention, first, that every cohomology group $E^*(\mathbb{G}_m)$ is a $\pi_0(S^0) = [S^0, S^0]$-module, so that the class $\langle -1\rangle^* - \mathrm{id}$ determines a well-defined group endomorphism.

The statement $\mathcal{H} \simeq \eta\varepsilon \simeq \eta$ above together with the explicit calculation of the morphism ε implies that $(\langle -1\rangle^* + \mathrm{id})\mathcal{H}^* = 0$. Subtracting $(\langle -1\rangle^* - \mathrm{id})\mathcal{H}^*$ from the former and multiplying by some large enough power of 2, we get the desired result. □

Let us recall that a field k is called *formally real* if -1 is **not** a sum of squares in k. Otherwise, k is said to be *nonreal*.

Lemma 2.3. *Let* k *be a field and* $\hat{I}(k)$ *denote the fundamental ideal in its Grothendieck–Witt ring* $GW(k)$*. Then, the following conditions are equivalent:*

1. *The field* k *is nonreal;*
2. *The ideal* $\hat{I}(k)$ *is a* 2*-group;*
3. *The element* $\langle -1\rangle - 1 \in GW(k)$ *has* 2*-primary order;*[2]
4. *The element* $\langle -1\rangle - 1 \in GW(k)$ *has finite order.*

Proof. *(1)* $\Rightarrow$ *(2)*. See [**10**, Theorem 6.4(i,ii)].

(2) $\Rightarrow$ *(3)*. Obvious, since $\dim(\langle -1\rangle - 1) = 0$ and, therefore, $\langle -1\rangle - 1 \in \hat{I}(k)$.

(3) $\Rightarrow$ *(4)*. Trivial.

(4) $\Rightarrow$ *(1)*. Assume that the element $\langle -1\rangle - 1$ has finite order in the additive group $GW(k)$. It means, by definition, that for some large enough integer n and some $m > 0$ the quadratic forms $\sum_{i=1\cdots n} -x_i^2 + \sum_{j=1\cdots m} y_j^2$ and $\sum_{k=1\cdots m+n} z_k^2$ are equivalent. In particular, they represent the same sets of numbers. But, the first form represents -1 and therefore so does the second. □

Combining the three lemmata above with Morel's computation of the $\mathbb{A}^1$-homotopy ring $[\mathbb{G}_m, \mathbb{G}_m]$, one obtains the following proposition.

Proposition 2.4. *Let* k *be a perfect nonreal field of* $\mathrm{char} \neq 2$*. Let* E^* *be a cohomology theory represented by a symmetric* T*-spectrum and assume that* 2 *is invertible in* $E^{0,0}(S^0)$*. Then, the map* $E^*(\mathbb{P}^2) \to E^*(\mathbb{P}^1)$ *induced by the standard embedding* $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ *is an epimorphism.*

[2]Here $\langle -1\rangle$ denotes the class of the quadratic form $-x^2$ in the Grothendieck–Witt ring $GW(k)$.

Proof. By Morel's theorem[3] [**9**, 6.2.2], the ring of $\mathbb{A}^1$-homotopy classes $[\mathbb{G}_m, \mathbb{G}_m]$ is naturally isomorphic to $GW(k)$. Under this isomorphism, the class of the self-map $u\colon \mathbb{P}^1 \to \mathbb{P}^1$ $([x\colon y] \to [x\colon uy])$ passes to the class of the corresponding quadratic form $\langle u\rangle \in GW(k)$. By Lemma 2.3, the element of the Grothendieck–Witt ring corresponding to the homotopy class of $\langle -1\rangle - 1$ has 2-primary order. Taking into account that $E^*(\mathbb{G}_m)$ is a $\pi_0(S^0)$-module and applying Lemma 2.2, one sees that $\mathcal{H}^* = 0$. Finally, by Lemma 2.1, the natural map $E^*(\mathbb{P}^2) \to E^*(\mathbb{P}^1)$ is an epimorphism. □

This proposition shows that cohomology theories over nonreal fields satisfy the normalization criterion (see [**3**, Definition 1.3]). Hence, the rigidity result of *loc. cit.* holds for these fields without any additional conditions. For completeness we reproduce modified forms of the rigidity theorem [**3**, 0.3] and its corollary [**3**, 0.4] here.

Theorem 2.5. *Let L be an infinite field of* char $\neq 2$. *Let also R be a Henselian local ring essentially smooth over L with a perfect nonreal field of fractions* $\operatorname{Frac}(R) = k$. *Consider a T-representable cohomology theory E on the category of smooth schemes of finite type over L that satisfies the condition $\ell E = 0$ for some odd integer ℓ that is invertible in R. Let $f\colon M \to \operatorname{Spec} R$ be a smooth affine morphism of (pure) relative dimension d, and $s_0, s_1\colon \operatorname{Spec} R \to M$ two sections of f such that $s_0(P) = s_1(P)$ at the closed point P of* $\operatorname{Spec} R$. *Then, the maps s_0^*, s_1^* induced in cohomology groups E^* are equal.*

Corollary 2.6. *Let E and L be as in the previous theorem, V a smooth variety over L, $P \in V(L)$ a L-rational point of V, and R be a henselisation of V at P with a perfect nonreal fraction field. Then, $E^*(\operatorname{Spec} R) \xrightarrow{\cong} E^*(\operatorname{Spec} L)$ is an isomorphism.*

Finally, let us give some explicit examples of fields satisfying the conditions of Proposition 2.4.

Example 2.7. As it was well-known before, every cohomology theory has rigidity property over quadratically closed fields. Certainly, this case is covered (at least for perfect fields) by our result.

Example 2.8. Every perfect field of finite characteristic ($\neq 2$) also satisfies the conditions of our proposition, as -1 is obviously a sum of squares.

Example 2.9. Let k be a field, containing for some prime p a field $\mathbb{Q}_p$ of rational p-adic numbers. Let us show that this field is nonreal. Certainly, it is sufficient to check that every p-adic integer is a sum of squares. For example, one can write any $x \in \mathbb{Z}_p$ as $x = 1 + 1 + \cdots + 1 + y$ such that $y \equiv 1 \mod n(p)$, where $n(p) = p$ for $p > 2$ and $n(2) = 8$. By a well-known property of p-adic numbers (see, for example, [**11**, II.3.3]) y is a square in $\mathbb{Z}_p$ that proves the desired property.

[3] Here we use the requirement that the field k is perfect of char $\neq 2$.

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