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# ON GROEBNER BASES AND IMMERSIONS OF GRASSMANN MANIFOLDS $G_{2,n}$

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(*communicated by Donald M. Davis*)

*Abstract*

Mod 2 cohomology of the Grassmann manifold $G_{2,n}$ is a polynomial algebra modulo a certain well-known ideal. A Groebner basis for this ideal is obtained. Using this basis, some new immersion results for Grassmannians $G_{2,n}$ are established.

## 1. Introduction

Mod 2 cohomology of Grassmann manifolds $G_{k,n} = O(n+k)/O(n) \times O(k)$ has a rather simple description. It is the polynomial algebra on the Stiefel-Whitney classes $w_1, w_2, \ldots, w_k$ of the canonical vector bundle $\gamma_k$ over $G_{k,n}$ modulo the ideal $I_{k,n}$ generated by the dual classes $\overline{w}_{n+1}, \overline{w}_{n+2}, \ldots, \overline{w}_{n+k}$. Alas, from this description it is not at all easy to establish whether a certain cohomology class is zero or not. In [**6**], Monks found Groebner bases for the ideal $I_{2,n}$ in the cases $n = 2^s - 3$ and $n = 2^s - 4$. Using these bases, some new results concerning the mod 2 cohomology of $G_{2,2^s-3}$ and $G_{2,2^s-4}$ were established in that paper. Also, the author used the method of modified Postnikov towers and gave an immersion result for the spaces $G_{2,2^s-3}$ into $\mathbb{R}^d$. In [**9**], Shimkus improved this immersion result by the same method.

Motivated by these results, we have found a reduced Groebner basis for the ideal $I_{2,n}$ for all $n$. This result is stated in Theorem 2.7. In Corollary 2.8 we present a convenient vector space bases for $H^*(G_{2,n};\mathbb{Z}_2)$.

Using these bases and modified Postnikov towers, in Theorem 3.11 we generalize the immersion result established in [**9**] and prove that $G_{2,n}$ immerses into $\mathbb{R}^{4n-5}$ where $n$ is any odd integer $\geqslant 7$. Our result improves upon the previously known best result (obtained by Cohen in [**2**]) whenever $\alpha(n) = \alpha(2n) < 5$ (where $\alpha(n)$ denotes the number of ones in the binary expansion of $n$).

The lower bounds for the immersion dimension of $G_{2,n}$ (which is defined by $\mathrm{imm}(G_{2,n}) := \min\{d \mid G_{2,n} \text{ immerses into } \mathbb{R}^d\}$) were established by Oproiu ([**8**]) using the method of the Stiefel-Whitney classes. For example, he has shown that $\mathrm{imm}(G_{2,2^{s-1}+1}) \geqslant 2^{s+1} - 2$. Our result states that $\mathrm{imm}(G_{2,2^{s-1}+1}) \leqslant 2^{s+1} - 1$ for $s \geqslant 4$, so it only remains to check whether $G_{2,2^{s-1}+1}$ can be immersed into $\mathbb{R}^{2^{s+1}-2}$. One more example where the lower bound from [**8**] almost reaches the upper bound obtained in Theorem 3.11 is $G_{2,7}$, $22 \leqslant \mathrm{imm}(G_{2,7}) \leqslant 23$.

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In addition to these main results, in Theorem 3.1 we use Groebner bases to give a simple proof of the previous result of Oproiu concerning lower bounds for $\mathrm{imm}(G_{2,n})$.


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## 2. Groebner bases

For positive integer $b$ and arbitrary integer $a$, the binomial coefficient is defined by $\binom{a}{b} := \frac{a(a-1)\cdots(a-b+1)}{b!}$. Also, $\binom{a}{0} := 1$. If $b$ is a negative integer, we define $\binom{a}{b}$ to be equal to zero. Then it is easy to see that the well-known formula

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1} \tag{1}$$

is valid for all $a, b \in \mathbb{Z}$.

From formula (1) we deduce directly that $\binom{a}{b} + \binom{a-1}{b-1} \equiv \binom{a-1}{b} \pmod 2$, $a, b \in \mathbb{Z}$, or equivalently $\binom{a-1}{b-1} \equiv \binom{a}{b} + \binom{a-1}{b} \pmod 2$, $a, b \in \mathbb{Z}$.

Henceforth, all binomial coefficients are considered mod 2.

Let $G_{k,n}$ be the Grassmann manifold of unoriented $k$-dimensional vector subspaces in $\mathbb{R}^{n+k}$. It is known that the cohomology algebra $H^*(G_{k,n}; \mathbb{Z}_2)$ is isomorphic to the quotient $\mathbb{Z}_2[w_1, w_2, \dots, w_k]/I_{k,n}$ of the polynomial algebra $\mathbb{Z}_2[w_1, w_2, \dots, w_k]$ by the ideal $I_{k,n}$ generated by polynomials $\overline{w}_{n+1}, \overline{w}_{n+2}, \dots, \overline{w}_{n+k}$, which are obtained from the equation

$$(1 + w_1 + w_2 + \cdots + w_k)(1 + \overline{w}_1 + \overline{w}_2 + \cdots) = 1.$$

For $k = 2$ (which is the case from now on), we have

$$\begin{aligned} 1 + \overline{w}_1 + \overline{w}_2 + \cdots &= \frac{1}{1 + w_1 + w_2} \\ &= \sum_{t \geqslant 0} (w_1 + w_2)^t = \sum_{t \geqslant 0} \sum_{a+b=t} \binom{a+b}{a} w_1^a w_2^b \\ &= \sum_{a,b \geqslant 0} \binom{a+b}{a} w_1^a w_2^b. \end{aligned}$$

By identifying the homogenous parts of (cohomological) degree $r \geqslant 0$, we obtain

$$\overline{w}_r = \sum_{a+2b=r} \binom{a+b}{a} w_1^a w_2^b.$$

It is understood that $a$ and $b$ are nonnegative integers.

We use the grlex ordering on the monomials in $\mathbb{Z}_2[w_1, w_2]$ with $w_1 > w_2$. That is, $w_1^a w_2^b \preceq w_1^c w_2^d$ if either $a + b < c + d$ or else $a + b = c + d$ and $a \leqslant c$. Of course, we will write $w_1^a w_2^b \prec w_1^c w_2^d$ when $w_1^a w_2^b \preceq w_1^c w_2^d$ and $w_1^a w_2^b \neq w_1^c w_2^d$.

We shall prove that, with respect to this ordering, the reduced Groebner basis for the ideal $I_{2,n} = (\overline{w}_{n+1}, \overline{w}_{n+2})$ is of the form $G = \{g_0, g_1, \ldots, g_{n+1}\}$ where $\mathrm{LT}(g_m) = w_1^{n+1-m} w_2^m$, $0 \leqslant m \leqslant n+1$. From this it follows immediately that a vector space basis for $H^*(G_{2,n}; \mathbb{Z}_2)$ is the set of all monomials $w_1^a w_2^b$ such that $a + b \leqslant n$.

Let us now define the polynomials $g_m$ $(0 \leqslant m \leqslant n+1)$.

**Definition 2.1.** For $0 \leqslant m \leqslant n+1$, let

$$g_m := \sum_{a+2b=n+1+m} \binom{a+b-m}{a} w_1^a w_2^b.$$

As before, it is understood that $a, b \geqslant 0$. Note that the (cohomological) degree of the polynomial $g_m$ is $n + 1 + m$.

By comparing with the above formula for $\overline{w}_r$, it is obvious that $g_0 = \overline{w}_{n+1}$. Also,

$$w_2 \overline{w}_n = \sum_{a+2b=n} \binom{a+b}{a} w_1^a w_2^{b+1} = \sum_{a+2b=n+2} \binom{a+b-1}{a} w_1^a w_2^b = g_1.$$

The change of variable $b \mapsto b - 1$ does not affect the requirement that $b \geqslant 0$ since for $b = 0$ the binomial coefficient $\binom{a+b-1}{a} = \binom{n+1}{n+2}$ is equal to 0.

From the defining formula, one can see that $b$ must be such that $m \leqslant b \leqslant \frac{n+1+m}{2}$. Namely, $a + b - m$ cannot be negative since $a + b - m < 0$ implies $a + 2b \leqslant 2(a + b) < 2m \leqslant n + 1 + m$, contradicting the requirement that $a + 2b = n + 1 + m$. Now, $a + b - m$ must be $\geqslant a$ in order for $\binom{a+b-m}{a}$ to be nonzero, and we conclude that $b \geqslant m$. The second inequality comes from the condition $a + 2b = n + 1 + m$. Therefore, we have

$$g_m = \sum_{b=m}^{[\frac{n+1+m}{2}]} \binom{n+1-b}{b-m} w_1^{n+1+m-2b} w_2^b. \tag{2}$$

It is obvious that the summand obtained for $b = m$ provides the leading term $\mathrm{LT}(g_m) = w_1^{n+1-m} w_2^m$.

In order to show that $G = \{g_0, g_1, \ldots, g_{n+1}\}$ is a Groebner basis for $I_{2,n}$, we define the ideal $I_G := (G) = (g_0, g_1, \ldots, g_{n+1})$ in $\mathbb{Z}_2[w_1, w_2]$. As we have already noticed, $\overline{w}_{n+1} = g_0 \in I_G$, $\overline{w}_{n+2} = w_1 \overline{w}_{n+1} + w_2 \overline{w}_n = w_1 g_0 + g_1 \in I_G$, so $I_{2,n} \subseteq I_G$.

It remains to prove that $I_G \subseteq I_{2,n}$ and that $G$ is a Groebner basis. It turns out that the following proposition plays the crucial role in proving these facts.

**Proposition 2.2.** *For each* $m \in \{0, 1, \ldots, n-1\}$, $w_2 g_m + w_1 g_{m+1} = g_{m+2}$. *Also, we have that* $w_2 g_n + w_1 g_{n+1} = 0$.

*Proof.* We calculate

$$\begin{aligned}
&w_2g_m + w_1g_{m+1}\\
&\quad= \sum_{a+2b=n+1+m} \binom{a+b-m}{a} w_1^a w_2^{b+1} + \sum_{a+2b=n+m+2} \binom{a+b-m-1}{a} w_1^{a+1} w_2^b\\
&\quad= \sum_{a+2b=n+m+3} \binom{a+b-m-1}{a} w_1^a w_2^b + \sum_{a+2b=n+m+3} \binom{a+b-m-2}{a-1} w_1^a w_2^b\\
&\quad= \sum_{a+2b=n+m+3} \binom{a+b-m-2}{a} w_1^a w_2^b = g_{m+2}.
\end{aligned}$$

We note that, for the similar reasons as above, the change of variable $b \mapsto b-1$ ($a \mapsto a-1$) does not affect the requirement that $b \geqslant 0$ ($a \geqslant 0$).

The second statement is a consequence of the equalities $g_n = \mathrm{LT}(g_n) = w_1w_2^n$ and $g_{n+1} = \mathrm{LT}(g_{n+1}) = w_2^{n+1}$ which are easily seen from (2). $\square$

**Corollary 2.3.** $I_G \subseteq I_{2,n}$.

*Proof.* We already know that

$$g_0 = \overline{w}_{n+1} \in I_{2,n} \quad \text{and} \quad g_1 = w_1\overline{w}_{n+1} + \overline{w}_{n+2} \in I_{2,n}.$$

Proposition 2.2 applies, and by induction on $m$ we have that $g_m \in I_{2,n}$ ($0 \leqslant m \leqslant n+1$). The corollary follows. $\square$

Therefore, $G$ is a basis for $I_{2,n}$, and we wish to prove that it is a Groebner basis. We recall that (for a fixed monomial ordering) the $S$-polynomial of polynomials $f, g \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k]$ is given by (we work with mod 2 coefficients)

$$S(f,g) = \frac{L}{\mathrm{LT}(f)} \cdot f + \frac{L}{\mathrm{LT}(g)} \cdot g,$$

where $L = \mathrm{lcm}(\mathrm{LT}(f), \mathrm{LT}(g))$ denotes the least common multiple of $\mathrm{LT}(f)$ and $\mathrm{LT}(g)$. If $0 \leqslant m < m+s \leqslant n+1$, we see that

$$\mathrm{lcm}(\mathrm{LT}(g_m), \mathrm{LT}(g_{m+s})) = \mathrm{lcm}(w_1^{n+1-m}w_2^m, w_1^{n+1-m-s}w_2^{m+s}) = w_1^{n+1-m}w_2^{m+s},$$

and so we have

$$S(g_m, g_{m+s}) = w_2^s g_m + w_1^s g_{m+s}. \tag{3}$$

We are going to prove that $G$ satisfies a sufficient condition (see [**1**]) for being a Groebner basis. In order to do that, we recall the following definition and theorem ([**1**, p. 219]). We formulate them for the field $R = \mathbb{Z}_2$. It is assumed that we have an ordering $\preceq$ on the monomials in $\mathbb{Z}_2[x_1, x_2, \ldots, x_k]$.

**Definition 2.4.** Let $F$ be a finite subset of $\mathbb{Z}_2[x_1, x_2, \ldots, x_k]$, $f \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k]$ a nonzero polynomial and $t$ a fixed monomial. If $f$ can be written as a finite sum of the form $\sum\limits_i m_if_i$, where $f_i \in F$ and $m_i \in \mathbb{Z}_2[x_1, x_2, \ldots, x_k]$ are nonzero monomials such that $\mathrm{LT}(m_if_i) \preceq t$ for all $i$, we say that $\sum\limits_i m_if_i$ is a *$t$-representation of $f$ with respect to $F$.*

**Theorem 2.5.** *Let $F$ be a finite subset of $\mathbb{Z}_2[x_1, x_2, \ldots, x_k]$, $0 \notin F$. If for all $f_1, f_2 \in F$, $S(f_1, f_2)$ either equals zero or has a $t$-representation with respect to $F$ for some monomial $t \prec \operatorname{lcm}(\operatorname{LT}(f_1), \operatorname{LT}(f_2))$, then $F$ is a Groebner basis.*

We need the following lemma.

**Lemma 2.6.** *For $0 \leqslant m < m+s \leqslant n+1$, $S(g_m, g_{m+s}) = \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i}$.*

It is understood that for $m+s = n+1$, the last summand in this sum (for $i = s-1$) is zero.

*Proof.* We proceed by induction on $s$. For $s = 1$, we obtain

$$S(g_m, g_{m+1}) = w_2 g_m + w_1 g_{m+1} = g_{m+2} = \sum_{i=0}^{0} w_1^i w_2^{-i} g_{m+2+i},$$

using (3) and Proposition 2.2. For the inductive step, we have

$$\begin{aligned} S(g_m, g_{m+s}) &= w_2^s g_m + w_1^s g_{m+s} \\ &= w_2^s g_m + w_2 w_1^{s-1} g_{m+s-1} + w_2 w_1^{s-1} g_{m+s-1} + w_1^s g_{m+s} \\ &= w_2 S(g_m, g_{m+s-1}) + w_1^{s-1} g_{m+s+1} \\ &= w_1^{s-1} g_{m+s+1} + \sum_{i=0}^{s-2} w_1^i w_2^{s-1-i} g_{m+2+i} \\ &= \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i}, \end{aligned}$$

again by (3), Proposition 2.2 and the induction hypothesis. It is clear that if $m+s = n+1$ then the summand $w_1^{s-1} g_{m+s+1}$ does not appear in the sum (Proposition 2.2) and so $0 \leqslant i \leqslant s-2$ in this case. $\square$

**Theorem 2.7.** *Let $n \geqslant 2$. Then $G = \{g_0, g_1, \ldots, g_{n+1}\}$ defined above is the reduced Groebner basis for the ideal $I_{2,n}$ in $\mathbb{Z}_2[w_1, w_2]$ with respect to the grlex ordering $\preceq$.*

*Proof.* We have already shown that $G$ is a basis for $I_{2,n}$. We wish to apply Theorem 2.5. Let $g_m$ and $g_{m+s}$ $(0 \leqslant m < m+s \leqslant n+1)$ be two arbitrary elements of $G$. If $m = n$, then $m+s$ must be $n+1$ and, using (3) and Proposition 2.2, one obtains $S(g_m, g_{m+s}) = S(g_n, g_{n+1}) = w_2 g_n + w_1 g_{n+1} = 0$. If $m \leqslant n-1$, then according to Lemma 2.6,

$$S(g_m, g_{m+s}) = \sum_{i=0}^{s-1} w_1^i w_2^{s-1-i} g_{m+2+i}.$$

Define $t = t(m,s) := w_1^{n-1-m} w_2^{m+s+1}$. First of all, observe that

$$t \prec w_1^{n+1-m} w_2^{m+s} = \operatorname{lcm}(\operatorname{LT}(g_m), \operatorname{LT}(g_{m+s})).$$

Now, for all $i \in \{0, 1, \dots, s-1\}$,

$$\begin{aligned}\mathrm{LT}(w_1^i w_2^{s-1-i} g_{m+2+i}) &= w_1^i w_2^{s-1-i} \mathrm{LT}(g_{m+2+i}) \\ &= w_1^i w_2^{s-1-i} w_1^{n+1-m-2-i} w_2^{m+2+i} \\ &= w_1^{n-1-m} w_2^{m+s+1} = t.\end{aligned}$$

Theorem 2.5 applies, and we conclude that $G$ is a Groebner basis for $I_{2,n}$.

To see that it is the reduced one, we observe that $\{\mathrm{LT}(g) \mid g \in G\}$ is the set of all monomials $w_1^a w_2^b$ such that $a + b = n + 1$. Also, by looking at formula (2), we see that all other terms appearing in $g_m$ have the sum of the exponents $< n + 1$, and so they cannot be divisible by any of the leading terms in $G$. □

Since $G$ is a Groebner basis for $I_{2,n}$, a vector space basis for $\mathbb{Z}_2[w_1, w_2]/I_{2,n}$ could be formed by taking all the monomials in $\mathbb{Z}_2[w_1, w_2]$ (more precisely, their classes) which are not divisible by any of $\mathrm{LT}(g_0), \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_{n+1})$. As we have noticed in the proof of Theorem 2.7, the set $\{\mathrm{LT}(g) \mid g \in G\}$ is the set of all monomials $w_1^a w_2^b$ such that $a + b = n + 1$. From this it is obvious that $w_1^a w_2^b$ is not divisible by any of the leading terms $\mathrm{LT}(g_m)$ if and only if $a + b \leqslant n$. By collecting all these facts, we have proved the following corollary.

**Corollary 2.8.** *Let $n \geqslant 2$. If $w_i$ is the $i$-th Stiefel-Whitney class of the canonical vector bundle $\gamma_2$ over $G_{2,n}$, then the set $\{w_1^a w_2^b \mid a + b \leqslant n\}$ is a vector space basis for $H^*(G_{2,n}; \mathbb{Z}_2)$.*

Let us now determine a few elements of the Groebner basis $G$ which will be used in our later calculations. As we have already noticed, by formula (2), $g_{n+1} = w_2^{n+1}$ and $g_n = w_1 w_2^n$. Using this and Proposition 2.2, we obtain $w_2 g_{n-1} = w_1 g_n + g_{n+1} = w_1^2 w_2^n + w_2^{n+1} = w_2(w_1^2 w_2^{n-1} + w_2^n)$, and so we deduce that $g_{n-1} = w_1^2 w_2^{n-1} + w_2^n$. Continuing in the same manner, one gets

$$\begin{aligned}g_{n-2} &= w_1^3 w_2^{n-2}; \\ g_{n-3} &= w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} + w_2^{n-1}; \\ g_{n-4} &= w_1^5 w_2^{n-4} + w_1 w_2^{n-2}; \\ g_{n-5} &= w_1^6 w_2^{n-5} + w_1^4 w_2^{n-4} + w_2^{n-2}.\end{aligned}$$

## 3. Immersions

In order to construct the immersions of Grassmannians $G_{2,n}$ into Euclidean spaces, we recall the theorem of Hirsch ([**4**]) which states that a smooth compact $m$-manifold $M^m$ immerses in $\mathbb{R}^{m+l}$ if and only if the classifying map $f_\nu \colon M^m \to BO$ of the stable normal bundle $\nu$ of $M^m$ lifts up to $BO(l)$.

$$\begin{tikzcd} & BO(l) \arrow[d, "p"] \\ M^m \arrow[ur, dashed] \arrow[r, "f_\nu"] & BO \end{tikzcd}$$

Let $\operatorname{imm}(M^m)$ denote the least integer $d$ such that $M^m$ immerses into $\mathbb{R}^d$. By Hirsch's theorem, if $w_k(\nu) \neq 0$ then $\operatorname{imm}(M^m) \geqslant m+k$.

As in Corollary 2.8, let $w_i$ be the $i$-th Stiefel-Whitney class of the canonical vector bundle $\gamma_2$ over $G_{2,n}$ ($n \geqslant 2$). It is well known (see [**8**, p. 179]) that, if $2^s$ is the least power of 2 exceeding $n$, i.e., $2^{s-1} \leqslant n < 2^s$, then for the total Stiefel-Whitney class $w(\nu)$ of the stable normal bundle $\nu$ of $G_{2,n}$, one has

$$w(\nu) = (1+w_1^2)(1+w_1+w_2)^{2^{s+1}-2-n}. \tag{4}$$

For $n = 2^s - 2$, from formula (2) we have that

$$g_0 = \sum_{b=0}^{2^{s-1}-1} \binom{2^s-1-b}{b} w_1^{2^s-1-2b} w_2^b = w_1^{2^s-1}$$

since the binomial coefficient $\binom{2^s-1-b}{b}$ is odd only for $b = 0$ (by Lucas formula). This means that $w_1^{2^s-1} = 0$ in $H^*(G_{2,2^s-2}; \mathbb{Z}_2)$. But then $w_1^{2^s-1} = 0$ in $H^*(G_{2,n}; \mathbb{Z}_2)$ for all $n \leqslant 2^s - 2$ since the inclusion $i\colon G_{2,n} \to G_{2,2^s-2}$ is obviously covered by a map of canonical bundles $\gamma_2$.

If $2^{s-1} \leqslant n \leqslant 2^s - 2$, then by formula (4) we have

$$\begin{aligned} w(\nu) &= (1+w_1^2)(1+w_1+w_2)^{2^s}(1+w_1+w_2)^{2^s-2-n} \\ &= (1+w_1^2)(1+w_1^{2^s}+w_2^{2^s})(1+w_1+w_2)^{2^s-2-n}. \end{aligned}$$

Now, $w_2^{2^s} = 0$ because it is a class of degree $2^{s+1} > 2^{s+1} - 4 \geqslant 2n = \dim(G_{2,n})$. Also, by the previous discussion $w_1^{2^s} = 0$ and (4) simplifies to

$$w(\nu) = (1+w_1^2)(1+w_1+w_2)^{2^s-2-n}. \tag{5}$$

If $n = 2^s - 1$, then from (4) we obtain

$$\begin{aligned} w(\nu) &= (1+w_1^2)(1+w_1+w_2)^{2^s-1} \\ &= (1+w_1)^2 \sum_{i=0}^{2^s-1} \binom{2^s-1}{i} (1+w_1)^i w_2^{2^s-1-i} \\ &= \sum_{i=0}^{2^s-1} (1+w_1)^{i+2} w_2^{2^s-1-i} \\ &= \sum_{i=0}^{2^s-1} \sum_{j=0}^{i+2} \binom{i+2}{j} w_1^j w_2^{2^s-1-i}. \end{aligned} \tag{6}$$

We now recall a theorem of Oproiu ([**8**]), and we prove it using the Groebner basis from Theorem 2.7.

**Theorem 3.1** (Oproiu [**8**])**.** *For $2 \leqslant 2^{s-1} \leqslant n < 2^s$, we have:*

*(a) If $n \leqslant 2^s - 2$, then $\operatorname{imm}(G_{2,n}) \geqslant 2^{s+1} - 2$.*

*(b) $\operatorname{imm}(G_{2,2^s-1}) \geqslant 3 \cdot 2^s - 2$.*

*Proof.* (a) The top class in the expression (5) is $w_{2^{s+1}-2-2n}(\nu) = w_1^2 w_2^{2^s-2-n}$, and since the sum of the exponents $2 + 2^s - 2 - n = 2^s - n \leqslant 2^{s-1} \leqslant n$, by Corollary 2.8

we have that $w_{2^{s+1}-2-2n}(\nu) \neq 0$ and we conclude that

$$\operatorname{imm}(G_{2,n}) \geqslant \dim(G_{2,n}) + 2^{s+1} - 2 - 2n = 2^{s+1} - 2.$$

(b) From the equality (6) we calculate

$$\begin{aligned}
w_{2^s}(\nu) &= \sum_{i=2^{s-1}-1}^{2^s-1} \binom{i+2}{2i+2-2^s} w_1^{2i+2-2^s} w_2^{2^s-1-i} \\
&= \sum_{l=0}^{2^{s-1}} \binom{2^s+1-l}{2^s-2l} w_1^{2^s-2l} w_2^l = \sum_{l=0}^{2^{s-1}} \binom{2^s+1-l}{l+1} w_1^{2^s-2l} w_2^l \\
&= \binom{2^s+1}{1} w_1^{2^s} + \binom{2^s}{2} w_1^{2^s-2} w_2 + \binom{2^s-1}{3} w_1^{2^s-4} w_2^2 + \cdots \\
&= w_1^{2^s} + w_1^{2^s-4} w_2^2 + \cdots,
\end{aligned}$$

where the unwritten monomials (if there are any) have the sum of the exponents $\leqslant 2^s - 3 = n - 2$. Note that, since $2^{s-1} \geqslant 2$, three written summands must appear in the sum.

On the other hand, from the equality (2) we see that the first element of the Groebner basis in this case is

$$\begin{aligned}
g_0 &= \sum_{b=0}^{2^{s-1}} \binom{2^s-b}{b} w_1^{2^s-2b} w_2^b = w_1^{2^s} + (2^s-1) w_1^{2^s-2} w_2 + \binom{2^s-2}{2} w_1^{2^s-4} w_2^2 + \cdots \\
&= w_1^{2^s} + w_1^{2^s-2} w_2 + w_1^{2^s-4} w_2^2 + \cdots .
\end{aligned}$$

Again, the unwritten monomials have the sum of the exponents $\leqslant n-2$, and three written ones must be here.

By adding these two equalities together, using the fact that $g_0 = 0$ in $H^*(G_{2,n};\mathbb{Z}_2)$ we obtain

$$w_{2^s}(\nu) = w_1^{2^s-2} w_2 + \cdots .$$

The sum of the exponents in the monomial $w_1^{2^s-2} w_2$ is $2^s - 1 = n$, and in the remaining monomials (if there are any) this sum is $\leqslant n-2$, so none of these monomials is divisible by any of the leading terms $\operatorname{LT}(g_m)$. This means that we have obtained the remainder of dividing $w_{2^s}(\nu)$ by $G$. Since $w_1^{2^s-2} w_2$ must appear in this remainder, we conclude that $w_{2^s}(\nu) \neq 0$. Finally, this implies that

$$\operatorname{imm}(G_{2,2^s-1}) \geqslant \dim(G_{2,2^s-1}) + 2^s = 3 \cdot 2^s - 2$$

and we are done. □

*Example 3.2.* If $n = 2^{s-1} > 2$, then $\operatorname{imm}(G_{2,2^{s-1}}) \geqslant 2^{s+1} - 2 = 2 \cdot \dim(G_{2,2^{s-1}}) - 2$. By the result of Massey [**5**, Theorem V], if $M^m$ is orientable, $m > 4$ and $w_2(\nu) \cdot w_{m-2}(\nu) = 0$, then $M^m$ immerses into $\mathbb{R}^{2m-2}$. Now, $G_{2,2^{s-1}}$ is orientable (Grassmannian $G_{k,n}$ is orientable if and only if $n+k$ is even; see [**8**, p. 179]), and from formula (5) we have

$$w(\nu) = (1 + w_1^2)(1 + w_1 + w_2)^{2^{s-1}-2},$$

so

$$w_2(\nu) = \left(1 + \binom{2^{s-1}-2}{2}\right) w_1^2 + (2^{s-1}-2)w_2 = 0$$

since $2^{s-1} > 2$. This implies that $G_{2,2^{s-1}}$ immerses into $\mathbb{R}^{2^{s+1}-2}$, i.e., $\operatorname{imm}(G_{2,2^{s-1}}) \leqslant 2^{s+1}-2$, so for $2^{s-1} > 2$, we actually have the equality

$$\operatorname{imm}(G_{2,2^{s-1}}) = 2^{s+1} - 2.$$

Also, we note that for $G_{2,3}$, Oproiu's Theorem 3.1(b) gives $\operatorname{imm}(G_{2,3}) \geqslant 10$, and by Cohen's theorem ([**2**]), $\operatorname{imm}(G_{2,3}) \leqslant 10$, so $\operatorname{imm}(G_{2,3}) = 10$. For $G_{2,5}$, the results of Oproiu ([**8**]) and Monks ([**6**]) provide inequalities $14 \leqslant \operatorname{imm}(G_{2,5}) \leqslant 17$.

We now turn to the proof of the immersion result.

**Lemma 3.3.** *Let $n$ be an odd integer $\geqslant 5$. For the stable normal bundle $\nu$ of $G_{2,n}$ we have:*

*(a)* $w_i(\nu) = 0$ *for* $i \geqslant 2n-5$;

*(b)* $w_1(\nu) = w_1$;

*(c)* $w_2(\nu) = w_2$ *if* $n \equiv 3 \pmod 4$; $w_2(\nu) = w_1^2 + w_2$ *if* $n \equiv 1 \pmod 4$.

*Proof.* As above, let $s$ be the integer such that $2^{s-1} \leqslant n < 2^s$. Since $n$ is odd, we have that $n \geqslant 2^{s-1}+1$. This implies $4n \geqslant 2^{s+1}+4$, i.e., $2^{s+1}-2-2n < 2n-5$.

If $n \neq 2^s - 1$, then from formula (5) we see that the top class in the expression for $w(\nu)$, namely $w_1^2 w_2^{2^s-2-n}$, is of degree $2^{s+1}-2-2n$, and by the previous inequality, we deduce that $w_i(\nu) = 0$ for $i \geqslant 2n-5$.

For $n = 2^s - 1$, Oproiu shows [**8**, p. 182] that the top nonzero class in the expression (4) is of degree $2^s$ and, since $n \geqslant 7$ in this case, we conclude that $2^s = n+1 < 2n-5$ obtaining (a).

From formula (4) we read off

$$w_1(\nu) = (2^{s+1}-2-n)w_1;$$

$$w_2(\nu) = \left(1 + \binom{2^{s+1}-2-n}{2}\right) w_1^2 + (2^{s+1}-2-n)w_2$$

and obtain (b) and (c). □

A few more lemmas will be useful.

**Lemma 3.4.** *In $H^*(G_{2,n};\mathbb{Z}_2)$, for all nonnegative integers $a$ and $b$, the following relations hold:*

*(a)* $Sq^1(w_1^a w_2^b) = (a+b)w_1^{a+1}w_2^b$;

*(b)* $Sq^2(w_1^a w_2^b) = b w_1^a w_2^{b+1} + \binom{a+b}{2} w_1^{a+2} w_2^b$.

*Proof.* Since $Sq^j(w_1^a) = \binom{a}{j} w_1^{a+j}$, the formulas are true for $b = 0$. We proceed by induction on $b$.

(a) By the Wu formula, $Sq^1 w_2 = w_1 w_2$. Using the Cartan formula and the induction hypothesis, we have

$$\begin{aligned}Sq^1(w_1^a w_2^b) &= Sq^1(w_2 w_1^a w_2^{b-1})\\ &= w_1 w_2 w_1^a w_2^{b-1} + (a+b-1) w_2 w_1^{a+1} w_2^{b-1}\\ &= (a+b) w_1^{a+1} w_2^b.\end{aligned}$$

(b) For the induction step we use again formulas of Cartan and Wu, the statement (a) and the fact that $Sq^2 w_2 = w_2^2$. We calculate

$$\begin{aligned}Sq^2(w_1^a w_2^b) &= Sq^2(w_2 w_1^a w_2^{b-1})\\ &= w_1^a w_2^{b+1} + (a+b-1) w_1^{a+2} w_2^b + (b-1) w_1^a w_2^{b+1}\\ &\quad + \binom{a+b-1}{2} w_1^{a+2} w_2^b\\ &= b w_1^a w_2^{b+1} + \binom{a+b}{2} w_1^{a+2} w_2^b,\end{aligned}$$

and the proof is complete. ∎

**Lemma 3.5.** *The map* $(Sq^2 + w_2(\nu))\colon H^{2n-5}(G_{2,n};\mathbb{Z}_2) \to H^{2n-3}(G_{2,n};\mathbb{Z}_2)$*, where* $n$ *is an odd integer* $\geqslant 5$*, is determined by the equalities*

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1^5 w_2^{n-5}) &= (Sq^2 + w_2(\nu))(w_1^3 w_2^{n-4}) = (Sq^2 + w_2(\nu))(w_1 w_2^{n-3})\\ &= w_1 w_2^{n-2}.\end{aligned}$$

*Proof.* By Corollary 2.8, the set $\{w_1^5 w_2^{n-5}, w_1^3 w_2^{n-4}, w_1 w_2^{n-3}\}$ is a vector space basis for $H^{2n-5}(G_{2,n};\mathbb{Z}_2)$.

Now, if $n \equiv 3 \pmod 4$, using Lemma 3.3, Lemma 3.4 and Groebner basis from Theorem 2.7, we calculate

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1^5 w_2^{n-5}) &= Sq^2(w_1^5 w_2^{n-5}) + w_2 w_1^5 w_2^{n-5}\\ &= (n-5) w_1^5 w_2^{n-4} + \binom{n}{2} w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4}\\ &= w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4}\\ &= w_1(g_{n-5} + w_1^4 w_2^{n-4} + w_2^{n-2}) + w_1^5 w_2^{n-4} = w_1 w_2^{n-2};\end{aligned}$$

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1^3 w_2^{n-4}) &= Sq^2(w_1^3 w_2^{n-4}) + w_2 w_1^3 w_2^{n-4}\\ &= (n-4) w_1^3 w_2^{n-3} + \binom{n-1}{2} w_1^5 w_2^{n-4} + w_1^3 w_2^{n-3}\\ &= w_1^5 w_2^{n-4} = g_{n-4} + w_1 w_2^{n-2} = w_1 w_2^{n-2};\end{aligned}$$

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1 w_2^{n-3}) &= Sq^2(w_1 w_2^{n-3}) + w_2 w_1 w_2^{n-3}\\ &= (n-3) w_1 w_2^{n-2} + \binom{n-2}{2} w_1^3 w_2^{n-3} + w_1 w_2^{n-2}\\ &= w_1 w_2^{n-2}.\end{aligned}$$

Similarly, if $n \equiv 1 \pmod 4$, we have

$$\begin{aligned}
(Sq^2 + w_2(\nu))(w_1^5 w_2^{n-5}) &= Sq^2(w_1^5 w_2^{n-5}) + w_1^2 w_1^5 w_2^{n-5} + w_2 w_1^5 w_2^{n-5} \\
&= (n-5) w_1^5 w_2^{n-4} + \binom{n}{2} w_1^7 w_2^{n-5} + w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4} \\
&= w_1^7 w_2^{n-5} + w_1^5 w_2^{n-4} \\
&= w_1(g_{n-5} + w_1^4 w_2^{n-4} + w_2^{n-2}) + w_1^5 w_2^{n-4} = w_1 w_2^{n-2}; \\
(Sq^2 + w_2(\nu))(w_1^3 w_2^{n-4}) &= Sq^2(w_1^3 w_2^{n-4}) + w_1^2 w_1^3 w_2^{n-4} + w_2 w_1^3 w_2^{n-4} \\
&= (n-4) w_1^3 w_2^{n-3} + \binom{n-1}{2} w_1^5 w_2^{n-4} + w_1^5 w_2^{n-4} + w_1^3 w_2^{n-3} \\
&= w_1^5 w_2^{n-4} = g_{n-4} + w_1 w_2^{n-2} = w_1 w_2^{n-2}; \\
(Sq^2 + w_2(\nu))(w_1 w_2^{n-3}) &= Sq^2(w_1 w_2^{n-3}) + w_1^2 w_1 w_2^{n-3} + w_2 w_1 w_2^{n-3} \\
&= (n-3) w_1 w_2^{n-2} + \binom{n-2}{2} w_1^3 w_2^{n-3} + w_1^3 w_2^{n-3} + w_1 w_2^{n-2} \\
&= w_1 w_2^{n-2},
\end{aligned}$$

and the proof of the lemma is complete. □

**Lemma 3.6.** *The map $Sq^1\colon H^{2n-2}(G_{2,n};\mathbb{Z}_2) \to H^{2n-1}(G_{2,n};\mathbb{Z}_2)$, where $n$ is an odd integer $\geqslant 5$, is trivial.*

*Proof.* The set $\{w_1^2 w_2^{n-2}, w_2^{n-1}\}$ is a vector space basis for $H^{2n-2}(G_{2,n};\mathbb{Z}_2)$ (Corollary 2.8). Using Lemma 3.4, we obtain

$$Sq^1(w_1^2 w_2^{n-2}) = n w_1^3 w_2^{n-2} = w_1^3 w_2^{n-2} = g_{n-2} = 0;$$

$$Sq^1(w_2^{n-1}) = (n-1) w_1 w_2^{n-1} = 0,$$

which proves the lemma. □

**Lemma 3.7.** *The map $(Sq^2 + w_2(\nu))\colon H^{2n-3}(G_{2,n};\mathbb{Z}_2) \to H^{2n-1}(G_{2,n};\mathbb{Z}_2)$, where $n$ is an odd integer $\geqslant 5$, is determined by the equalities:*

$$\begin{aligned}
&(Sq^2 + w_2(\nu))(w_1^3 w_2^{n-3}) = w_1 w_2^{n-1} \neq 0; \\
&(Sq^2 + w_2(\nu))(w_1 w_2^{n-2}) = 0.
\end{aligned}$$

*Proof.* Again by Corollary 2.8, the classes $w_1^3 w_2^{n-3}$ and $w_1 w_2^{n-2}$ form a vector space basis for $H^{2n-3}(G_{2,n};\mathbb{Z}_2)$, and the class $w_1 w_2^{n-1}$ is nontrivial in $H^{2n-1}(G_{2,n};\mathbb{Z}_2) \cong \mathbb{Z}_2$.

By Lemma 3.3 and Lemma 3.4, for $n \equiv 3 \pmod 4$ we have

$$\begin{aligned}
(Sq^2 + w_2(\nu))(w_1^3 w_2^{n-3}) &= Sq^2(w_1^3 w_2^{n-3}) + w_2 w_1^3 w_2^{n-3} \\
&= (n-3) w_1^3 w_2^{n-2} + \binom{n}{2} w_1^5 w_2^{n-3} + w_1^3 w_2^{n-2} \\
&= w_1^5 w_2^{n-3} + w_1^3 w_2^{n-2} \\
&= w_1(g_{n-3} + w_1^2 w_2^{n-2} + w_2^{n-1}) + w_1^3 w_2^{n-2} \\
&= w_1 w_2^{n-1};
\end{aligned}$$

$$\begin{aligned}(Sq^2+w_2(\nu))(w_1w_2^{n-2}) &= Sq^2(w_1w_2^{n-2})+w_2w_1w_2^{n-2}\\ &= (n-2)w_1w_2^{n-1}+\binom{n-1}{2}w_1^3w_2^{n-2}+w_1w_2^{n-1}\\ &= w_1^3w_2^{n-2}=g_{n-2}=0.\end{aligned}$$

Likewise, for $n\equiv 1 \pmod 4$, we obtain

$$\begin{aligned}(Sq^2+w_2(\nu))(w_1^3w_2^{n-3}) &= Sq^2(w_1^3w_2^{n-3})+w_1^2w_1^3w_2^{n-3}+w_2w_1^3w_2^{n-3}\\ &= (n-3)w_1^3w_2^{n-2}+\binom{n}{2}w_1^5w_2^{n-3}+w_1^5w_2^{n-3}+w_1^3w_2^{n-2}\\ &= w_1^5w_2^{n-3}+w_1^3w_2^{n-2}\\ &= w_1(g_{n-3}+w_1^2w_2^{n-2}+w_2^{n-1})+w_1^3w_2^{n-2}\\ &= w_1w_2^{n-1};\\ (Sq^2+w_2(\nu))(w_1w_2^{n-2}) &= Sq^2(w_1w_2^{n-2})+w_1^2w_1w_2^{n-2}+w_2w_1w_2^{n-2}\\ &= (n-2)w_1w_2^{n-1}+\binom{n-1}{2}w_1^3w_2^{n-2}+w_1^3w_2^{n-2}+w_1w_2^{n-1}\\ &= w_1^3w_2^{n-2}=g_{n-2}=0,\end{aligned}$$

which was to be proved. □

**Lemma 3.8.** *The map $Sq^1\colon H^{2n-3}(G_{2,n};\mathbb{Z}_2)\to H^{2n-2}(G_{2,n};\mathbb{Z}_2)$, where $n$ is an odd integer $\geqslant 5$, is given by the equalities:*

$$\begin{aligned}&Sq^1(w_1^3w_2^{n-3})=w_1^2w_2^{n-2}+w_2^{n-1};\\ &Sq^1(w_1w_2^{n-2})=0.\end{aligned}$$

*Proof.* As we have noticed in the proof of the previous lemma, the set

$$\{w_1^3w_2^{n-3},w_1w_2^{n-2}\}$$

is a vector space basis for $H^{2n-3}(G_{2,n};\mathbb{Z}_2)$. So, we calculate

$$\begin{aligned}Sq^1(w_1^3w_2^{n-3})=nw_1^4w_2^{n-3}=w_1^4w_2^{n-3}=g_{n-3}+w_1^2w_2^{n-2}+w_2^{n-1}=w_1^2w_2^{n-2}+w_2^{n-1},\\ Sq^1(w_1w_2^{n-2})=(n-1)w_1^2w_2^{n-2}=0\end{aligned}$$

by Lemma 3.4. □

**Lemma 3.9.** *If $n$ is an odd integer $\geqslant 5$, then in $H^*(G_{2,n};\mathbb{Z}_2)$ we have*

$$(Sq^2+w_1(\nu)^2+w_2(\nu))Sq^1(w_1^3w_2^{n-4}+w_1w_2^{n-3})=w_1^2w_2^{n-2}.$$

*Proof.* By Lemma 3.4(a),

$$Sq^1(w_1^3w_2^{n-4}+w_1w_2^{n-3})=(n-1)w_1^4w_2^{n-4}+(n-2)w_1^2w_2^{n-3}=w_1^2w_2^{n-3}.$$

If $n\equiv 3 \pmod 4$, by Lemma 3.3 and Lemma 3.4(b), one obtains

$$\begin{aligned}(Sq^2+w_1(\nu)^2+w_2(\nu))(w_1^2w_2^{n-3}) &= Sq^2(w_1^2w_2^{n-3})+w_1^2w_1^2w_2^{n-3}+w_2w_1^2w_2^{n-3}\\ &= (n-3)w_1^2w_2^{n-2}+\binom{n-1}{2}w_1^4w_2^{n-3}+w_1^4w_2^{n-3}\\ &\quad +w_1^2w_2^{n-2}=w_1^2w_2^{n-2}.\end{aligned}$$

If $n \equiv 1 \pmod 4$, again by Lemma 3.3 and Lemma 3.4(b), we have

$$\begin{aligned}(Sq^2 + w_1(\nu)^2 + w_2(\nu))(w_1^2 w_2^{n-3}) &= Sq^2(w_1^2 w_2^{n-3}) + w_2 w_1^2 w_2^{n-3} \\ &= (n-3)w_1^2 w_2^{n-2} + \binom{n-1}{2} w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} \\ &= w_1^2 w_2^{n-2},\end{aligned}$$

and we are done. □

**Lemma 3.10.** *If $n$ is an odd integer $\geqslant 5$, then in $H^*(G_{2,n};\mathbb{Z}_2)$ the following equality holds:*

$$(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) = w_2^{n-1}.$$

*Proof.* As before, we use Lemma 3.3, Lemma 3.4 and Groebner basis from Theorem 2.7.

If $n \equiv 3 \pmod 4$,

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) &= Sq^2(w_1^2 w_2^{n-3}) + w_2 w_1^2 w_2^{n-3} \\ &= (n-3)w_1^2 w_2^{n-2} + \binom{n-1}{2} w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} \\ &= w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = g_{n-3} + w_2^{n-1} = w_2^{n-1}.\end{aligned}$$

If $n \equiv 1 \pmod 4$,

$$\begin{aligned}(Sq^2 + w_2(\nu))(w_1^2 w_2^{n-3}) &= Sq^2(w_1^2 w_2^{n-3}) + w_1^2 w_1^2 w_2^{n-3} + w_2 w_1^2 w_2^{n-3} \\ &= (n-3)w_1^2 w_2^{n-2} + \binom{n-1}{2} w_1^4 w_2^{n-3} + w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} \\ &= w_1^4 w_2^{n-3} + w_1^2 w_2^{n-2} = g_{n-3} + w_2^{n-1} = w_2^{n-1},\end{aligned}$$

completing the proof. □

We are now ready to prove our immersion result.

**Theorem 3.11.** *If $n$ is an odd integer $\geqslant 7$, then $G_{2,n}$ immerses into $\mathbb{R}^{4n-5}$.*

*Proof.* Let $f_\nu\colon G_{2,n} \to BO$ be the classifying map for the stable normal bundle $\nu$ of $G_{2,n}$. We want to show that $f_\nu$ can be lifted up to $BO(2n-5)$. We will use the $2n$-MPT for the fibration $p\colon BO(2n-5) \to BO$ which can be constructed by the method of Gitler and Mahowald ([**3**]) using the result of Nussbaum ([**7**]) who proved that their method is applicable to the fibrations $p\colon BO(l) \to BO$ when $l$ is odd. The tower is presented in Figure 1 ($K_m$ stands for the Eilenberg-MacLane space $K(\mathbb{Z}_2, m)$).

The relations that produce the $k$-invariants are

$$k_1^2 : (Sq^2 + w_2)w_{2n-4} = 0,$$

$$k_2^2 : (Sq^2 + w_1^2 + w_2)Sq^1 w_{2n-4} + Sq^1 w_{2n-2} = 0,$$

$$k_3^2 : \begin{cases}(Sq^4 + w_4)w_{2n-4} + w_2 w_{2n-2} = 0, & n \equiv 3 \pmod 4\\ (Sq^4 + w_4)w_{2n-4} + Sq^2 w_{2n-2} = 0, & n \equiv 1 \pmod 4,\end{cases}$$

$$k_1^3 : (Sq^2 + w_2)k_1^2 + Sq^1 k_2^2 = 0.$$

$$
\begin{tikzcd}
 & E_3 \arrow[d, "q_3"] & \\
 & E_2 \arrow[r, "k_1^3"] \arrow[d, "q_2"] & K_{2n-2} \\
 & E_1 \arrow[r, "k_1^2\times k_2^2\times k_3^2"] \arrow[d, "q_1"] & K_{2n-3}\times K_{2n-2}\times K_{2n-1} \\
G_{2,n} \arrow[r, "f_\nu"] \arrow[uur, dashed, "h"] \arrow[ur, dashed, "g"] & BO \arrow[r, "w_{2n-4}\times w_{2n-2}"] & K_{2n-4}\times K_{2n-2}.
\end{tikzcd}
$$

Figure 1: $2n$-MPT for $p : BO(2n-5) \to BO$

Since $\dim(G_{2,n}) = 2n$, $f_\nu$ lifts up to $BO(2n-5)$ if and only if it lifts up to $E_3$.

By Lemma 3.3(a), $f_\nu^*(w_{2n-4}) = w_{2n-4}(\nu) = 0$, $f_\nu^*(w_{2n-2}) = w_{2n-2}(\nu) = 0$, so $f_\nu$ can be lifted up to $E_1$, i.e., there is a map $g_1\colon G_{2,n} \to E_1$ such that $q_1 \circ g_1 = f_\nu$.

In order to make the next step (to lift $f_\nu$ up to $E_2$), we need to modify (if necessary) the lifting $g_1$ to a lifting $g$ such that $g^*(k_1^2) = g^*(k_2^2) = g^*(k_3^2) = 0$. By choosing a map $\alpha\times\beta\colon G_{2,n} \to K_{2n-5}\times K_{2n-3} = \Omega(K_{2n-4}\times K_{2n-2})$ (i.e., classes $\alpha \in H^{2n-5}(G_{2,n};\mathbb{Z}_2)$ and $\beta \in H^{2n-3}(G_{2,n};\mathbb{Z}_2)$), we get another lifting $g\colon G_{2,n} \to E_1$ as the composition

$$G_{2,n} \xrightarrow{\triangle} G_{2,n}\times G_{2,n} \xrightarrow{(\alpha\times\beta)\times g_1} K_{2n-5}\times K_{2n-3}\times E_1 \xrightarrow{\mu} E_1,$$

where $\triangle$ is the diagonal mapping and $\mu\colon \Omega(K_{2n-4}\times K_{2n-2})\times E_1 \to E_1$ is the action of the fibre in the principal fibration $q_1\colon E_1 \to BO$. So, we are looking for classes $\alpha$ and $\beta$ such that $g^*(k_1^2) = g^*(k_2^2) = g^*(k_3^2) = 0$. By looking at the relations that produce the $k$-invariants $k_1^2, k_2^2$ and $k_3^2$, we conclude that the following equalities hold (see [**3**, p. 95]):

$$
\begin{aligned}
g^*(k_1^2) &= g_1^*(k_1^2) + (Sq^2 + w_2(\nu))(\alpha);\\
g^*(k_2^2) &= g_1^*(k_2^2) + (Sq^2 + w_1(\nu)^2 + w_2(\nu))Sq^1\alpha + Sq^1\beta;\\
g^*(k_3^2) &= \begin{cases} g_1^*(k_3^2) + (Sq^4 + w_4(\nu))(\alpha) + w_2\cdot\beta, & n \equiv 3 \pmod 4\\ g_1^*(k_3^2) + (Sq^4 + w_4(\nu))(\alpha) + Sq^2\beta, & n \equiv 1 \pmod 4). \end{cases}
\end{aligned}
$$

First, we need to prove that the class $g_1^*(k_1^2)$ is in the image of the map $(Sq^2 + w_2(\nu))\colon H^{2n-5}(G_{2,n};\mathbb{Z}_2) \to H^{2n-3}(G_{2,n};\mathbb{Z}_2)$. The $k$-invariant $k_1^3$ is produced by the relation $(Sq^2 + w_2)k_1^2 + Sq^1k_2^2 = 0$ which holds in $H^*(E_1;\mathbb{Z}_2)$. Applying $g_1^*$, we get

$$(Sq^2 + w_2(\nu))g_1^*(k_1^2) = Sq^1g_1^*(k_2^2).$$

But, by Lemma 3.6, $Sq^1g_1^*(k_2^2) = 0$. Hence, $g_1^*(k_1^2)$ is in the kernel of the map $(Sq^2 +$

$w_2(\nu))\colon H^{2n-3}(G_{2,n};\mathbb{Z}_2)\to H^{2n-1}(G_{2,n};\mathbb{Z}_2)$, and according to Lemmas 3.5 and 3.7, this kernel coincides with the image of the map $(Sq^2+w_2(\nu))\colon H^{2n-5}(G_{2,n};\mathbb{Z}_2)\to H^{2n-3}(G_{2,n};\mathbb{Z}_2)$. Thus, we can find a class $\alpha\in H^{2n-5}(G_{2,n};\mathbb{Z}_2)$ such that $g^*(k_1^2)=0$.

Since $H^{2n-2}(G_{2,n};\mathbb{Z}_2)$ is generated by the classes $w_1^2w_2^{n-2}$ and $w_2^{n-1}$ (Corollary 2.8), by Lemma 3.8 and Lemma 3.9 we see that we can choose a class $\beta\in H^{2n-3}(G_{2,n};\mathbb{Z}_2)$ and modify $\alpha$ (by adding, if necessary, the class $w_1^3w_2^{n-4}+w_1w_2^{n-3}$) to obtain $g$ such that $g^*(k_2^2)=0$. Since $w_1^3w_2^{n-4}+w_1w_2^{n-3}$ is in the kernel of the map $(Sq^2+w_2(\nu))\colon H^{2n-5}(G_{2,n};\mathbb{Z}_2)\to H^{2n-3}(G_{2,n};\mathbb{Z}_2)$ (Lemma 3.5), adding this class to the previous $\alpha$ will not spoil the equality $g^*(k_1^2)=0$.

Finally, observe the class $\beta':=w_1w_2^{n-2}\in H^{2n-3}(G_{2,n};\mathbb{Z}_2)$. According to Corollary 2.8, $w_2\cdot\beta'=w_1w_2^{n-1}\neq 0$ and if $n\equiv 1 \pmod 4$, by Lemma 3.4,

$$Sq^2\beta'=(n-2)w_1w_2^{n-1}+\binom{n-1}{2}w_1^3w_2^{n-2}=w_1w_2^{n-1}\neq 0.$$

Since $\beta'$ is in the kernel of the map $Sq^1\colon H^{2n-3}(G_{2,n};\mathbb{Z}_2)\to H^{2n-2}(G_{2,n};\mathbb{Z}_2)$ (Lemma 3.8), we can add this class to the previous $\beta$ (if necessary) and obtain a lifting $g$ such that $g^*(k_1^2)=g^*(k_2^2)=g^*(k_3^2)=0$.

Therefore, we can lift $f_\nu$ up to $E_2$, i.e., there is a map $h_1\colon G_{2,n}\to E_2$ such that $q_1\circ q_2\circ h_1=q_1\circ g=f_\nu$.

We need to make one more step: to prove that the lifting $h_1$ can be modified to a lifting $h$ which lifts up to $E_3$, i.e., such that $h^*(k_1^3)=0$. Arguing as before, we see that it suffices to find classes $a\in H^{2n-4}(G_{2,n};\mathbb{Z}_2)$ and $b\in H^{2n-3}(G_{2,n};\mathbb{Z}_2)$ such that $(Sq^2+w_2(\nu))(a)+Sq^1b=h_1^*(k_1^3)\in H^{2n-2}(G_{2,n};\mathbb{Z}_2)$. But, since $w_1^2w_2^{n-2}$ and $w_2^{n-1}$ generate $H^{2n-2}(G_{2,n};\mathbb{Z}_2)$, according to Lemma 3.8 and Lemma 3.10, such classes $a$ and $b$ exist (that is, the indeterminacy of $k_1^3$ is all of $H^{2n-2}(G_{2,n};\mathbb{Z}_2)$). This completes the proof of the theorem. $\square$

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