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ON THE K-THEORY AND HOMOTOPY THEORY OF THE KLEIN BOTTLE GROUP

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Abstract

We construct infinitely many chain homotopically distinct algebraic 2-complexes for the Klein bottle group and give various topological applications. We compare our examples to other examples in the literature and address the question of geometric realizability.

1. Introduction

Let G be a group. A (G,n)-complex X is a finite n-dimensional CW-complex with $\pi_1(X)=G$ and $\pi_i(X)=0$ for $2\leqslant i<n$. Its directed Euler characteristic is the alternating sum $\chi(X)=c_n-c_{n-1}+\cdots\pm c_0$, where c_i is the number of i-cells. An algebraic (G,n)-complex $\mathscr{X}$ is an exact sequence

$$\mathbb{Z}G^{m_n}\to\mathbb{Z}G^{m_{n-1}}\to\cdots\to\mathbb{Z}G^{m_0}\to\mathbb{Z}\to 0.$$

Thus, an algebraic (G,n)-complex is a partial free resolution of the trivial module $\mathbb{Z}$ of length n. Its directed Euler characteristic is defined as the alternating sum $\chi(\mathscr{X})=m_n-m_{n-1}+\cdots\pm m_0$. Note that the augmented cellular chain complex $C_*(\widetilde{X})\to\mathbb{Z}\to 0$ of the universal covering $\widetilde{X}$ of a (G,n)-complex X is an algebraic (G,n)-complex. The *geometric realization problem* poses the question: Given an algebraic $(G,2)$-complex $\mathscr{X}$, does there exist a $(G,2)$-complex X so that $\mathscr{X}$ and $C_*(\widetilde{X})\to\mathbb{Z}\to 0$ are chain homotopy equivalent? The realization question came out of work of Wall [**21**]. Closely related to the geometric realization problem is Wall's $D(2)$ *problem*: Suppose that X is a finite 3-complex such that $H_3(\widetilde{X},\mathbb{Z})=H^3(X,\mathscr{B})=0$ for all local coefficient systems $\mathscr{B}$ on X. Is X homotopy equivalent to a finite 2-complex? In [**10**, appendix B], Johnson proved the following realization theorem: Let G be a finitely presented group of type $FL(3)$; then the $D(2)$-property holds for G if and only if every algebraic $(G,2)$-complex admits a geometric realization. More information on the history of the geometric realization problem and the $D(2)$-problem can be found in the introduction of Johnson's book [**10**]. See also [**6**].

The theorem below is the main contribution of this article.

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Theorem 1.1. *Let*

$$G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$$

be the Klein bottle group. There exists infinitely many chain homotopically distinct algebraic $(G, 2)$*-complexes of Euler characteristic* 1*. Thus, if there are only finitely many homotopically distinct* $(G, 2)$*-complexes of Euler characteristic* 1*, then both the geometric realization and the* $D(2)$*-problem can be answered negatively.*

The last statement in the theorem follows from Johnson's realization theorem mentioned above. Other potential counterexamples to the geometric realization and the $D(2)$ problem are in the literature. See, for instance, Beyl-Waller [**3**]. A finitely generated G-module M is *stably free* if $M \oplus \mathbb{Z}G^a \cong \mathbb{Z}G^b$ for some $a, b \in \mathbb{N}$. The difference $b - a$ is called the *stably free rank* of M. It follows from work of Artamonov [**1**] and Stafford [**20**] that there are infinitely many stably free G-modules of rank 1, where G is the Klein bottle group (the fact that the Klein bottle group admits any such module has independently been observed by Lewin [**13**]). We use these modules to construct infinitely many chain homotopically distinct algebraic $(G, 2)$-complexes with Euler characteristic 1 mentioned in the above theorem. We do not know at the time of writing whether any of them can be geometrically realized. The result has various other topological consequences. It implies that there are infinitely many homotopically distinct $(G, 3)$-complexes with fixed Euler characteristic, and that there is a finite $(G, 3)$-complex that dominates infinitely many homotopically distinct $(G, 3)$-complexes.

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2. Stably free modules for the Klein bottle group

In this section we review work of Artamonov [**1**] and Stafford [**20**] on K-theoretic aspects of polycyclic groups. Let R be a ring (with unity). Let $\sigma\colon R \to R$ be a ring automorphism and let $R[x, x^{-1}, \sigma]$ be the skewed Laurent polynomial ring over R. Elements in $R[x, x^{-1}, \sigma]$ are polynomials in x and x^{-1} with coefficients in R, but coefficients and variables do not commute. We have $xr = \sigma(r)x$ and $x^{-1}r = \sigma^{-1}(r)x^{-1}$.

Theorem 2.1 ([**20**, Thm. 1.2])**.** *Let R be a commutative Noetherian domain and let $S = R[x, x^{-1}, \sigma]$ be a skew Laurent polynomial ring over R. Suppose $r, s \in R$ and the following conditions hold:*

1. *r is not a unit in R.*
2. *$Sr + S(x + s) = S$, that is, r and $x + s$ generate S as a left S-module.*
3. *$\sigma(r)s \notin Rr$.*

Let $\phi\colon S \oplus S \to S$ be the S-module epimorphism $[r, x + s]$ (that is, $\phi(1, 0) = r$ and $\phi(0, 1) = x + s$). Then $\ker \phi$ *is a stably free, non-free, S-module of stably free rank* 1*.*

Let $G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ be the Klein bottle group. Let $R = \mathbb{Z}[y, y^{-1}]$ and let $\sigma\colon R \to R$ be the ring automorphism that sends y to y^{-1} and fixes $\mathbb{Z}$. Then the group ring $\mathbb{Z}G$ is the skewed Laurent polynomial ring $R[x, x^{-1}, \sigma]$. Let $r_n = 1 + ny + ny^3$ and $s_n = \sigma(r_n) = 1 + ny^{-1} + ny^{-3}$, $n \in \mathbb{N}$. Note that the elements $r_n, x + s_n$ of $R[x, x^{-1}, \sigma]$ satisfy the three conditions stated in the above theorem. Condition 1 holds because units in skewed Laurent polynomial rings are monomials. For Condition 2 notice that

$$(x - r_n)(x + s_n) + \sigma(r_n)r_n = x^2 + x\sigma(r_n) - r_n x - r_n\sigma(r_n) + \sigma(r_n)r_n.$$

Since $x\sigma(r_n) = \sigma^2(r_n)x$ and $\sigma^2 = id$ and $R = \mathbb{Z}[y, y^{-1}]$ is commutative, we see that the sum equals the unit x^2. Thus r_n and $x + s_n$ generates S. Let us address the final Condition 3. Assume that $\sigma(r_n)s_n \in Rr_n$. Then $(1 + ny^{-1} + ny^{-3})^2 \in Rr_n$, and thus $(y^3 + ny^2 + n)^2 \in Rr_n$. If we multiply with an appropriate y^k, we obtain an equation $y^k(y^3 + ny^2 + n)^2 = f(y)(ny^3 + ny + 1)$ in $\mathbb{Z}[y]$, where $f(y)$ is some polynomial in $\mathbb{Z}[y]$. So $ny^3 + ny + 1$ divides $y^k(y^3 + ny^2 + n)^2$. Since $ny^3 + ny + 1$ is irreducible in $\mathbb{Q}[y]$ (it does not have a rational root) and does not divide a monomial, it divides $y^3 + ny^2 + n$, which is a contradiction (long division). Stafford's theorem now implies that $\ker\phi_n$ is stably free, non-free of rank 1, where $\phi\colon S \oplus S \to S$, defined by the matrix $[r_n, x + s_n]$.

Artamonov [**1**] studied K-theoretic properties of solvable groups. Slight adaptations to the Klein bottle group yield the following result.

Theorem 2.2 (Artamonov [**1**])**.** *Let $G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ be the Klein bottle group and let $\phi_n\colon \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G$, $n \in \mathbb{N}$, be the epimorphism given by the matrix $[r_n, x + s_n]$ (that is, $\phi(1,0) = r_n$ and $\phi(0,1) = x + s_n$), where $r_n = 1 + ny + ny^3$ and $s_n = 1 + ny^{-1} + ny^{-3}$. Then the set $\{\ker\phi_n\}_{n\in\mathbb{N}}$ contains infinitely many isomorphically distinct stably free non-free modules of rank* 1*.*

Proof. Recall Dirichlet's theorem on primes in arithmetic progressions [**19**], which states that for any relatively prime $a, b \in \mathbb{Z}$, there are infinitely many primes p equal to a mod b. This guarantees, in particular, that for every m there is a prime p that is 1 mod m. We recursively define a set Q of prime numbers in the following way: Let $q_1 = 2$ and assume q_i has been defined for $i \leqslant n$. Let q_{n+1} be a prime that is 1 modulo the product $q_1 \cdots q_n$.

Let $K_q = \ker\phi_q$ and $K_{q,p} = K_q/pK_q$. Note that since $\phi_p\colon \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G$ splits, $K_{q,p}$ is the kernel of the map $\phi_{q,p}\colon \mathbb{Z}_pG \oplus \mathbb{Z}_pG \to \mathbb{Z}_pG$ (ϕ_q mod p). Note that $\phi_{q,q} = [1, x + 1]$; thus the kernel $K_{q,q}$ is free of rank 1. Now assume that $p, q \in Q$ and $p < q$. By construction of Q we have $q = 1$ mod p. Thus $\phi_{q,p} = [1 + y + y^3, x + 1 + y^{-1} + y^{-3}]$. We can apply Stafford's Theorem 2.1 with $R = \mathbb{Z}_p[y, y^{-1}]$, $\sigma(y) = y^{-1}$ (as before), $r = 1 + y + y^3$ and $s = 1 + y^{-1} + y^{-3}$ to see that $K_{q,p}$ is not free. This shows that K_p and K_q are not isomorphic because $K_{p,p}$ is free but $K_{q,p}$ is not. $\square$

3. Algebraic 2-complexes for the Klein bottle group

Let X be the standard 2-complex built from the presentation $\langle x, y \mid xyx^{-1} = y^{-1} \rangle$ of the Klein bottle group. Note that X is the usual description of the Klein bottle as a cell complex: X has a single 0-cell, two 1-cells denoted by x and y and one 2-cell d

with boundary $xyx^{-1}y$. Let $\widetilde{X}$ be the universal covering of X. Fix a vertex 1 in $\widetilde{X}$. Let $\tilde{x}$ and $\tilde{y}$ be lifts of x and y, respectively, starting at 1. Let $\tilde{d}$ be the lift of d to 1. Consider the augmented cellular chain complex $C_*(\widetilde{X}) \to \mathbb{Z} \to 0$:

$$0 \to \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} 0$$

with bases $\{\tilde{d}\}$ and $\{\tilde{x}, \tilde{y}\}$ for $C_2(\widetilde{X})$ and $C_1(\widetilde{X})$, respectively. Note that since X is aspherical ($\widetilde{X}$ is $\mathbb{R}^2$), ∂_2 is injective, so the image $\mathbb{Z}G\partial_2(\tilde{d})$ is free of rank 1. Let $r_n, s_n \in \mathbb{Z}G$ be the elements defined in the previous section. Define an epimorphism $\psi_n \colon \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G\partial_2(\tilde{d})$ by $\psi_n = [r_n\partial_2(\tilde{d}), (x + s_n)\partial_2(\tilde{d})]$. Let $j \colon \mathbb{Z}G\partial_2(\tilde{d}) \hookrightarrow \mathbb{Z}G \oplus \mathbb{Z}G$ be the inclusion and define $\partial_2^n = j \circ \psi_n$. Let $\mathscr{X}_n$ be the algebraic 2-complex

$$\mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_2^n} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Notice that $H_2(\mathscr{X}_n)$ is isomorphic to $K_n = \ker \phi_n$ and hence is stably free non-free of rank 1 by Theorem 2.2. We have the following result.

Theorem 3.1. *The set $\{\mathscr{X}_n\}_{n\in\mathbb{N}}$ contains infinitely many chain homotopically distinct algebraic $(G, 2)$-complexes of directed Euler characteristic 1, where G is the Klein bottle group.*

4. Topological applications

Let X be a (G, n)-complex. Then the homotopy type of X is captured by the triple $(\pi_1(X), \pi_n(X), \kappa_X)$, where $\kappa_X \in H^{n+1}(X, \pi_n(X))$ is the k-invariant of X. See MacLane-Whitehead [**16**] and also [**9**, Chap. 2, §4]. In particular, if $H^{n+1}(G, M) = 0$ for every G-module M, then two (G, n)-complexes are homotopic if and only if their nth homotopy modules are isomorphic.

As before, let X be the standard 2-complex built from the presentation

$$\langle x, y \mid xyx^{-1} = y^{-1}\rangle$$

of the Klein bottle group and let $Y = X \vee S^2$, where S^2 is a 2-sphere. Note that $\pi_2(Y) = \mathbb{Z}G$. Let $f_n, g_n \colon S^2 \to Y$ be maps that represent the generators r_n, $x + s_n$ as defined in Section 2, respectively. Attach two 3-balls to X, using the attaching maps f_n and g_n to obtain a $(G, 3)$-complex Z_n. Note that $\pi_3(Z_n)$ is isomorphic to $K_n = \ker \phi_n$, the stably free non-free module of Section 2.

Theorem 4.1. *The set $\{Z_n\}_{n\in\mathbb{N}}$ contains infinitely many homotopically distinct $(G, 3)$-complexes of directed Euler characteristic 1, where G is the Klein bottle group.*

Note that $\pi_3(Z_n \vee S^3) = \mathbb{Z}G \oplus \mathbb{Z}G$. Thus, all $(G, 3)$-complexes $Z_n \vee S^3$ are homotopically equivalent to the $(G, 3)$-complex $W = Z_1 \vee S^3$ by what was said in the first paragraph of this section ($H^4(G, M) = 0$ for all G-modules M because the Klein bottle group G is 2-dimensional). Since every Z_n is a retract of $Z_n \vee S^3$, it follows that Z_n is a homotopy retract of W. So the finite 3-complex W dominates infinitely many homotopically distinct 3-complexes. See Kołodziejczyk [**11**] for a similar result.

Theorem 4.2. *The finite 3-complex $W = ((X \vee S^2) \cup_{f_1} B^3 \cup_{g_1} B^3) \vee S^3$ dominates infinitely many homotopically distinct finite 3-complexes.*

5. Relation modules

Given a generating tuple $\mathbf{g} = (g_1, \ldots, g_n)$ for a group G, we denote by

$$p_{\mathbf{g}} \colon F(a_1, \ldots, a_n) \to G$$

the projection from the free group to G defined by $p_{\mathbf{g}}(a_i) = g_i$, $1 \leqslant i \leqslant n$. The kernel R of the projection $p_{\mathbf{g}}$ is the relator group associated with $\mathbf{g}$, and the $\mathbb{Z}G$-module $R/[R, R]$ is the relation module associated with $\mathbf{g}$.

Stably free non-free G-modules often arise as relation modules (see Dunwoody [**5**], Harlander and Jensen [**8**], Lustig [**15**]). It was shown in [**8**] that for groups that are fundamental groups of 2-dimensional aspherical complexes, relation modules can always be realized as second homotopy modules $\pi_2(X)$ for some $(G, 2)$-complex X. In particular, if the stably free G-modules K_n constructed earlier for the Klein bottle group G could be shown to be isomorphic to relation modules, then all the algebraic 2-complexes $\mathscr{X}_n$ of Theorem 3.1 could be geometrically realized. In this section we show that relation modules for the Klein bottle group are free. Hence none of the K_n is a relation module unless it is free.

Theorem 5.1. *Relation modules for the Klein bottle group G are free.*

Proof. The above theorem follows from the fact that there is only one Nielsen class of generating n-tuples for G, and that Nielsen equivalent generating n-tuples give rise to isomorphic relation modules. This is indeed true for all closed surface groups; see Louder [**14**]. We will present the proof for the Klein bottle group which is considerably simpler than the general case. Recall that two generating n-tuples $\mathbf{g} = (g_1, \ldots, g_n)$ and $\mathbf{g'} = (g'_1, \ldots, g'_n)$ are called *Nielsen equivalent* if the following holds: There exists an automorphism $\phi \in \mathrm{Aut}(F_n)$, where $F_n = F(a_1, \ldots, a_n)$ is a free group of rank n, such that $(p_{\mathbf{g'}} \circ \phi(a_1), \ldots, p_{\mathbf{g'}} \circ \phi(a_n)) = (p_{\mathbf{g}}(a_1), \ldots, p_{\mathbf{g}}(a_n))$. Note that any generating n-tuple $(x^{i_1}, \ldots, x^{i_n})$ of the infinite cyclic group $\langle x \rangle$ is Nielsen equivalent to the tuple $(x, 1, \ldots, 1)$. This follows from the Euclidean algorithm.

Let $G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ be the Klein bottle group and consider a generating n-tuples $\mathbf{g} = (x^{i_1}y^{j_1}, \ldots, x^{i_n}y^{j_n})$. We will show that it is Nielsen equivalent to $(x, y, 1, \ldots, 1)$. Hence the relation module associated with $\mathbf{g}$ is isomorphic to $\mathbb{Z}G^{n-2}$. First note that $\mathbf{h} = (x^{i_1}, \ldots, x^{i_n})$ is a generating tuple for the infinite cyclic group $\langle x \rangle$, and hence the Nielsen equivalent to $\mathbf{h'} = (x, 1, \ldots, 1)$ under some automorphism ϕ. Let $\alpha \colon G \to \langle x \rangle$, sending x to x and y to 1. Since $\alpha \circ p_{\mathbf{g}} = p_{\mathbf{h}}$ and $\alpha \circ p_{\mathbf{g'}} = p_{\mathbf{h'}}$, it follows that $\mathbf{g}$ is Nielsen equivalent to $\mathbf{g'} = (xy^{k_1}, y^{k_2}, \ldots, y^{k_n})$. Next notice that $\mathbf{i} = (y^{k_2}, \ldots, y^{k_n})$ is a generating tuple for the (normal) subgroup $\langle y \rangle$ of G. Indeed, let H be the subgroup of G generated by $y^{k_2}, \ldots, y^{k_n}$. Note that H is normal, and if d is the greatest common divisor of $k_2, \ldots, k_n$, then $H = \langle y^d \rangle$. Note further that G/H is cyclic, generated by xy^{k_1}. Now $G/H = \langle x, y \mid xyx^{-1} = y^{-1}, y^d \rangle$, which is cyclic only if $d = 1$.

Now the generating tuple $\mathbf{i}$ is Nielsen equivalent to $\mathbf{i'} = (y, 1, \ldots, 1)$ under some $\psi' \colon F(a_2, \ldots, a_n) \to F(a_2, \ldots, a_n)$. We extend the automorphism ψ' to an automorphism $\psi \in \mathrm{Aut}(F_n)$ by defining $\psi(a_1) = a_1$. Now note that $\mathbf{g'}$ is Nielsen equivalent to $(xy^{k_1}, y, 1, \ldots, 1)$ under ψ which is clearly Nielsen equivalent to $(x, y, 1, \ldots, 1)$. $\square$

6. Geometric realization

In this section we introduce the relation lifting problem which is a stronger version of the geometric realization problem. We prove a theorem that can be used to construct potential counterexamples to both problems. We call $[x_1, \ldots, x_n \mid s_1, \ldots, s_m]$ an *almost presentation of* G if there is an epimorphism $q\colon \langle x_1, \ldots, x_n \mid s_1, \ldots, s_m\rangle \to G$ with perfect kernel P, that is $P_{ab} = P/[P,P] = 1$ (see also Mannan [**17**]). Let $S = \langle\langle s_1, \ldots, s_m\rangle\rangle$. Then $P = R/S$ for some normal subgroup R of $F(x_1, \ldots, x_n)$ and $G = F(x_1, \ldots, x_n)/R$. Note that $P_{ab} = R/S[R,R] = 1$ implies that $R = S[R,R]$. Hence the s_i, $i = 1, \ldots, m$, generate the relation $\mathbb{Z}G$-module $R/[R,R]$. The other direction is also true: If the elements $s_1, \ldots, s_m$ generate the relation $\mathbb{Z}G$-module $R/[R,R]$, then the kernel of the epimorphism $q\colon \langle x_1, \ldots, x_n \mid s_1, \ldots, s_m\rangle \to G = F(x_1, \ldots, x_n)/R$ is perfect, and hence $[x_1, \ldots, x_n \mid s_1, \ldots, s_m]$ is an almost presentation of G.

Note that if $[x_1, \ldots, x_n \mid s_1, \ldots, s_k]$ is an almost presentation of G and X is the standard 2-complex $|\langle x_1, \ldots, x_n \mid s_1, \ldots, s_k\rangle|$, then $\mathscr{X} = \{C_*(X_P) \to \mathbb{Z} \to 0\}$ is an algebraic $(G,2)$-complex, where X_P is the covering of X associated with P. We also write $\mathscr{X} = |[x_1, \ldots, x_n \mid s_1, \ldots, s_k]|$.

In this section we address a version of the geometric realization problem, called the *relation lifting problem*: Given a presentation $F(x_1, \ldots, x_n)/R$ of a group G and relation module generators $s_1, \ldots, s_m$. Do there exist elements $r_1, \ldots, r_m$ so that $s_i[R,R] = r_i[R,R]$, $i = 1, \ldots, m$, and $\langle\langle r_1, \ldots, r_m\rangle\rangle = R$? We remark that, in general, the relation lifting problem has a negative answer (see Dunwoody [**4**]). Note that an affirmative answer implies that the algebraic 2-complex $\mathscr{X} = |[x_1, \ldots, x_n \mid s_1, \ldots, s_m]|$ can be geometrically realized. Indeed, the chain complex of the universal covering of $Y = |\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle|$ is identical to $\mathscr{X}$.

It turns out that in case G is the fundamental group of an aspherical 2-complex, almost presentations for G can be easily constructed. In fact, any choice of generator of the cyclic module $\mathbb{Z}G$ yields an almost presentation for G. In the following we give the details of this construction and discuss the trefoil group and the Klein bottle group in that context.

Let G be a group and $\mathbb{F}G$ be the free group on basis the elements of G. Note that $\mathbb{F}G$ is a G-group which yields the group ring $\mathbb{Z}G$ when abelianized. If $\alpha = (g_1)\cdots(g_n) \in \mathbb{F}G$ and $h \in G$, then we write ${}^{\alpha}h = (g_1 h g_1^{-1})\cdots(g_n h g_n^{-1})$. If $f\colon F \to G$ is a group epimorphism and $\alpha = (w_1)\cdots(w_n) \in \mathbb{F}F$, then we write $\bar{\alpha} = g_1 + \cdots + g_n \in \mathbb{Z}G$, where $g_i = f(w_i)$, $i = 1, \ldots, n$.

Theorem 6.1. *Let $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$ be a presentation for a group G whose associated standard 2-complex is aspherical. Let $\alpha_1, \ldots, \alpha_k \in \mathbb{F}F(x_1, \ldots, x_n)$. Let $\hat{\alpha}_i$ be the image of α_i under the map $\mathbb{F}F(x_1, \ldots, x_n) \to \mathbb{F}G$ and let $\bar{\alpha}_i$ be the image of $\hat{\alpha}_i$ under the map $\mathbb{F}G \to \mathbb{Z}G$. Let $P = \mathbb{F}G/\langle\langle g\hat{\alpha}_i, g \in G, i = 1, \ldots, k\rangle\rangle$.*

1. *P is isomorphic to the kernel of the map*
$$\hat{G} = \langle x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, {}^{\alpha_1}y, \ldots, {}^{\alpha_k}y\rangle \to G = \langle x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, y\rangle.$$
2. *$\mathscr{X} = |[x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, {}^{\alpha_1}y, \ldots, {}^{\alpha_k}y]|$ is an algebraic $(G,2)$-complex if and only if the elements $\bar{\alpha}_1, \ldots, \bar{\alpha}_k \in \mathbb{Z}G$ generate $\mathbb{Z}G$ as a left module (or,*

equivalently, P is perfect). In that case $H_2(\mathscr{X}) = \ker\phi$, where $\phi\colon \mathbb{Z}G^k \to \mathbb{Z}G$ is defined by the matrix $[\bar{\alpha}_1, \ldots, \bar{\alpha}_k]$.

3. *$X = |\langle x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, {}^{\alpha_1}y, \ldots, {}^{\alpha_k}y\rangle|$ is a $(G,2)$-complex if and only if P is trivial. In that case $\pi_2(X) = \ker\phi$, where $\phi\colon \mathbb{Z}G^k \to \mathbb{Z}G$ is defined by the matrix $[\bar{\alpha}_1, \ldots, \bar{\alpha}_k]$.*

Proof. Let

$$Y = |\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle|$$

and

$$X = |\langle x_1, \ldots, x_n, y \mid r_1, \ldots, r_m, {}^{\alpha_1}y, \ldots, {}^{\alpha_k}y\rangle|.$$

Let $\hat{G}$ be the fundamental group of X and let $P = \langle\langle y\rangle\rangle$ be the normal closure of y in $\hat{G}$. Note that P is the kernel of an epimorphism $\hat{G} \to G$. Let X_P be the covering of X associated with P. We analyze the cell structure of X_P. Since Y is a retract of X, the complex X_P contains $\widetilde{Y}$, the universal covering of Y, as a subcomplex. Indeed, $X_P^{(0)} = \widetilde{Y}^{(0)} = G$, and $X_P^{(1)}$ is the 1-skeleton of $\widetilde{Y}$ with a loop y_g attached at the vertex g, $g \in G$. Furthermore, the 2-cells in X_P are the 2-cells in $\widetilde{Y}$ together with the lifts of the 2-cells for the relations ${}^{\alpha_i}y$, $i = 1, \ldots, k$. If we smash the contractible subcomplex $\widetilde{Y}$ of X_P to a point, then we obtain a 2-complex Z with a single vertex that is homotopically equivalent to X_P. In particular, its fundamental group is P. The 1-skeleton is a wedge of circles y_g in one-to-one correspondence with the elements of G. Consider a 2-cell d in X with boundary ${}^{\alpha_i}y = w_1 y w_1^{-1} \cdots w_l y w_l^{-1}$, where $\alpha_i = (w_1)\cdots(w_l)$. If we lift d to a vertex g and then smash $\widetilde{Y}$, then we obtain a 2-cell d_g with boundary $y_{g\hat{w}_1} \cdots y_{g\hat{w}_l}$. Thus $P = \pi_1(Z) \to \mathbb{F}G/\langle\langle g\hat{\alpha}_1, \ldots, g\hat{\alpha}_k, g \in G\rangle\rangle$, sending y_g to (g), is an isomorphism. We have established (1).

We turn to statement (2). Note that by definition $\mathscr{X}$ is the augmented cellular chain complex $C_*(X_P) \to \mathbb{Z} \to 0$. It is an algebraic 2-complex, that is, it is exact, if and only if $P = H_1(\mathscr{X}) = 0$. The above analysis of the cell structure reveals that $C_*(X_P) \to \mathbb{Z} \to 0$ is of the form

$$\mathbb{Z}G^k \oplus \mathbb{Z}G^m \xrightarrow{\phi\oplus\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G^n \xrightarrow{0\oplus\partial_1} \mathbb{Z}G \to \mathbb{Z} \to 0,$$

where

$$0 \to \mathbb{Z}G^m \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G^n \xrightarrow{\partial_1} \mathbb{Z}G \to \mathbb{Z} \to 0$$

is the augmented cellular chain complex $C_*(\widetilde{Y}) \to \mathbb{Z} \to 0$, which is exact because Y is assumed to be aspherical. It follows that $\ker\phi \oplus \partial_2 = \ker\phi$, and that the former chain complex is exact if and only if ϕ is onto, which is the case if and only if the elements $\bar{\alpha}_1, \ldots, \bar{\alpha}_k$ generate $\mathbb{Z}G$. We have shown (2).

Finally, we prove (3). The first statement is clear. Note that X_P is now the universal covering of X, so by the Hurewicz's Theorem that $\pi_2(X) = H_2(X_P)$. The remaining statements in (3) follow from (2). $\square$

The following example illustrates the use of the Theorem 6.1. Let $G = \langle x\rangle$ be the infinite cyclic group. Then the group ring $\mathbb{Z}G = \mathbb{Z}[x^{\pm 1}]$ is a Laurent polynomial ring.

Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ be two polynomials that generate the group ring as a left module. Then

$$[x, y \mid y^{a_0}xy^{a_1}x^{-1} \cdots x^n y^{a_n}x^{-n}, y^{b_0}xy^{b_1}x^{-1} \cdots x^m y^{b_m}x^{-m}]$$

is an almost presentation of G, by Theorem 6.1. Is it a presentation of G? It is conceivable that one can choose the polynomials $f(x)$ and $g(x)$ so that the resulting almost presentation does not lift to an actual presentation. This would imply that the infinite cyclic group does not have the relation lifting property despite the fact that every algebraic $(G, 2)$-complex is geometrically realizable (see Johnson [**10**, Proposition A.10 in appendix A]. Dunwoody showed in [**4**] that the group $\mathbb{Z} * \mathbb{Z}_5$ does not have the relation lifting property, that is, there exists an almost presentation for $\mathbb{Z} * \mathbb{Z}_5$ that does not lift to an actual presentation. His construction relies on non-trivial units in the group ring. As far as we know, no torsion free groups that do not have the relation lifting property have been constructed.

We next look at Dunwoody's well-known trefoil group examples. Consider the elements $\alpha = (1)(a)(a^2), \beta = (1)(b)(b^2)(b^3) \in \mathbb{F}F(a, b)$. Dunwoody showed that $\bar{\alpha} = 1 + a + a^2, \bar{\beta} = 1 + b + b^2 + b^3 \in \mathbb{Z}G$ generate the group ring $\mathbb{Z}G$ of the trefoil group $G = \langle a, b \mid a^2 = b^3 \rangle$. Hence the relation module $R/[R, R]$, $R = \langle\langle r \rangle\rangle$, $r = a^2b^{-3}$ is generated by ${}^{\alpha}r$ and ${}^{\beta}r$. He further showed that the kernel of the homomorphism $\phi\colon \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G$, defined by the matrix $[\bar{\alpha}, \bar{\beta}]$, is non-free stably free of rank 1. This results in an algebraic $(G, 2)$-complex $\mathscr{X} = [a, b \mid {}^{\alpha}r, {}^{\beta}r]$ with $H_2(\mathscr{X}) = \ker \phi$ and Euler characteristic 1. Dunwoody proceeded to show that this algebraic 2-complex is geometric, that is $|\langle a, b \mid^{\alpha} r, {}^{\beta}r\rangle|$ is a $(G, 2)$-complex. Since the second homotopy module of this complex is $\ker \phi$, it is not free. So this complex is not homotopically equivalent to $|\langle a, b \mid r, 1\rangle|$. We will recall Dunwoody's realization results from our viewpoint. Note that since the elements $\bar{\alpha}, \bar{\beta}$ generate $\mathbb{Z}G$, we know by Theorem 6.1 that $\mathscr{X} = |[a, b, c \mid a^2 = b^3, {}^{\alpha}c, {}^{\beta}c]|$ is an algebraic $(G, 2)$-complex.

Theorem 6.2. *$X = |\langle a, b, c \mid a^2 = b^3, {}^{\alpha}c, {}^{\beta}c\rangle|$ is a $(G, 2)$-complex for the trefoil group G. Since $\pi_2(X) = \ker \phi$ is not free, X is not homotopically equivalent to $|\langle a, b, c \mid a^2 = b^3, c, 1\rangle|$.*

Proof. We will only show that X is a $(G, 2)$-complex. In the light of Theorem 6.1, the only thing we have to show is that $P = \mathbb{F}G/\langle\langle g\alpha, g\beta, g \in G\rangle\rangle = 1$. Note that $(1)(a)(a^2) = 1$ in P and $(a)(a^2)(a^3) = 1$ in P imply that $(a^3) = (1)$ in P. Also, $(1)(b)(b^2)(b^3) = 1$ and $(b)(b^2)(b^3)(b^4) = 1$ imply that $(b^4) = (1)$. Since a^3 and b^4 generate G, it follows that $(g) = (1)$ in P for all $g \in G$. So P is cyclic, generated by (1). Now $1 = (1)(a)(a^2) = (1)^3$ and $1 = (1)(b)(b^2)(b^3) = (1)^4$; hence $(1) = 1$ and P is trivial. □

We next look at the Klein bottle group $G = \langle x, y \mid xyx^{-1} = y^{-1}\rangle$ again. Let $\alpha_n = (1)(y)^n(y^3)^n$, $\beta_n = (1)(y^{-1})^n(y^{-3})^n(x) \in \mathbb{F}F(x, y)$. It was shown in Section 2 that the elements $\bar{\alpha}_n = 1 + ny + ny^3$, $\bar{\beta}_n = 1 + ny^{-1} + ny^{-3} + x$ generate $\mathbb{Z}G$, where G is the Klein bottle group. Hence $\mathscr{X}_n = [x, y, z \mid xyx^{-1} = y^{-1}, {}^{\alpha_n}z, {}^{\beta_n}z]$ is an algebraic $(G, 2)$-complex by Theorem 6.1.

Question. Is there an $n \in \mathbb{N}$ so that $|\langle x, y, z \mid xyx^{-1} = y^{-1}, {}^{\alpha_n}z, {}^{\beta_n}z\rangle|$ is a $(G, 2)$-complex?

By Theorem 6.1 one would have to show that $P = \mathbb{F}G/\langle\langle g\alpha, g\beta, g \in G\rangle\rangle = 1$. We have not been able to adapt the arguments given for the trefoil group to the Klein bottle group.

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