

STABLE SPLITTINGS OF CLASSIFYING SPACES OF FINITE
GROUPS: BRIDGING THE WORK OF BENSON–FESHBACH
AND MARTINO–PRIDDY

JILL DIETZ

(communicated by N.J. Kuhn)

Abstract

We provide a direct connection between the work of David Benson and Mark Feshbach [2], and the work of John Martino and Stewart Priddy [9] on stable splittings of classifying spaces of finite groups.

1. Introduction

The journal *Topology* published two papers in 1992 on stable splittings of classifying spaces of finite groups. One was written by the pair of David Benson and Mark Feshbach [2], and the other by John Martino and Stewart Priddy [9]. The two papers address exactly the same question and essentially reach the same conclusion: that the p -complete stable splitting of a classifying space of a finite group, BG , is obtained from information about the contribution of summands from classifying spaces of subgroups of G , which in turn is obtained by studying certain simple modules. However, the approaches taken by the two pairs of authors are quite different.

Stable decompositions of BG_+ can be obtained via idempotent decompositions of the identity in the ring of stable self maps, $\{BG_+, BG_+\}$. In general, if R is a ring and $1 = e_1 + e_2 + \cdots + e_n$ is a primitive orthogonal idempotent decomposition in R , then each $e_i R$ is an indecomposable R -module. If $J(R)$ is the Jacobson radical of R , then each $e_i R / e_i J(R)$ is a simple R -module. The number of copies of a particular indecomposable module $e_i R$ in a decomposition of R is equal to the dimension of the corresponding simple module over its endomorphism ring. Neither Benson–Feshbach nor Martino–Priddy directly searches for simple $\{BG_+, BG_+\}$ -modules. Rather, each pair analyzes an associated ring and its simple modules.

Studying the ring $\{BG_+, BG_+\}$ was made more feasible with Carlsson’s [3] solution of the Segal Conjecture. Using this solution, Lewis, May, and McClure [8] showed that $\{BG_+, BG_+\}$ is ring isomorphic to the completion of a kind of double Burnside ring. That is, $\{BG_+, BG_+\} \cong A(G, G)_I^\wedge$, where completion is with respect to filtration by a certain ideal I .

In [2], Benson and Feshbach show that when $G = P$ is a p -group, indecomposable summands of BP_+ correspond to simple $\bar{A}(P, P)$ -modules, where $\bar{A}(P, P) =$

Received January 26, 2006, revised June 23, 2006; published on July 17, 2006.

2000 Mathematics Subject Classification: 55R35.

Key words and phrases: Classifying spaces of groups, stable homotopy theory.

Copyright © 2006, International Press. Permission to copy for private use granted.

$\mathbb{F}_p \otimes_{\mathbb{Z}} A(P, P)$. They define a “coadjoint” module for $\bar{A}(P, P)$ whose subquotients are either zero or simple. The simple modules are parametrized by “types” of subgroups $Q \leq P$ and by simple $\mathbb{F}_p \text{Out } Q$ -modules, and form a set of representatives for the homotopy types of indecomposable summands of BP_+ . Again, the multiplicity of a summand in BP_+ is equal to the dimension of the corresponding simple $\bar{A}(P, P)$ -module over its endomorphism ring. Further details on the Benson–Feshbach method are given in Section 3.

In [9], Martino and Priddy use Nishida’s [10] application of the Segal Conjecture showing that for all finite p -groups P , $\mathbb{Z}_p \text{Out } P \cong \{BP, BP\}/J(P)$, where $J(P)$ is the ideal of $\{BP, BP\}$ generated by all maps of the form $BP \rightarrow BK \rightarrow BP$, with $K \lneq P$. They obtain splittings of BP from information on simple $\mathbb{F}_p \text{Out } Q$ -modules, $Q \leq P$. The multiplicity of a summand X in BP , denoted $m(X, BP)$, is determined by measuring the degree of linear independence among the contributions to $m(X, BP)$ from classifying spaces of subgroups of P . Further details on the Martino–Priddy method are given in Section 4.

In his reviews of the two papers, John Harris [7] wrote: “It would be interesting to see a direct proof that these two [computations of $m(X, BG)$] are the same.” Similarly, in Benson’s [1] survey of developments in the study of stable splittings of classifying spaces of finite groups, he asked “What is the precise relationship between the matrices of Martino and Priddy and the modules of Benson and Feshbach?” This paper answers Harris and Benson’s questions.

The organization of this paper is as follows: In Section 2, we remind the reader of the basic theory of stable splittings. This basic theory is the launching point for the two papers [2] and [9]. Sections 3 and 4 give further details on the Benson–Feshbach and Martino–Priddy methods respectively. Section 5 provides a direct connection between the two theories. We give an example in Section 6 which shows how to compute the multiplicity of a summand in the splitting of a certain BP using each method, and illustrate the connection between the two methods.

2. Stable Splittings

Stably, we have

$$BG_+ \simeq \bigvee_{p \mid |G|} (BG_+)_p^{\wedge},$$

where $(BG_+)_p^{\wedge}$ denotes the p -completion of BG_+ . It makes sense, then, to fix a prime p and study the p -local stable decomposition of BG_+ . Moreover, $(BG_+)_p^{\wedge}$ appears as a summand of BP_+ , where P is a p -Sylow subgroup of G . Thus, we will concentrate on the case when $G = P$ is a p -group. Throughout this paper p will be a prime, P a p -group, BP_+ the p -completion of the suspension spectrum of the classifying space with disjoint basepoint, and $\{BP_+, BP_+\}$ the ring of p -complete self maps. We will always work in the category of p -complete spectra.

A decomposition $BP_+ \simeq X_1 \vee X_2 \vee \cdots \vee X_n$ corresponds to an idempotent decomposition in $\{BP_+, BP_+\}$, $1 = e_1 + e_2 + \cdots + e_n$. A summand $X_i = e_i BP_+$ is the mapping telescope $\text{Tel}(BP_+ \xrightarrow{e_i} BP_+ \xrightarrow{e_i} \cdots)$.

The Segal Conjecture relates the ring $\{BP_+, BP_+\}$ to a double Burnside ring $A(P, P)$ as follows. Let G and G' be two finite groups. $A(G, G')$ is the Grothendieck group of isomorphism classes of finite $G \times G'$ -sets with free right G' action. Given $H \leq G$ and a homomorphism $\phi: H \rightarrow G'$, let $X_{H,\phi} = (G \times G') / \triangle_{H,\phi}$, where $\triangle_{H,\phi} = \{(h, \phi(h)) \mid h \in H\}$. Every transitive G' -free $G \times G'$ -set is of this form, so $A(G, G')$ is free abelian with basis elements $\zeta_{H,\phi}$ corresponding to conjugacy classes of pairs (H, ϕ) .

Multiplication, $A(G, G') \times A(G'', G) \rightarrow A(G'', G')$, is given by a double coset formula (see [2], Formula 2.1). When $G = G' = G''$, we get a ring structure on $A(G, G)$.

There is a homomorphism $\alpha: A(G, G') \rightarrow \{BG_+, BG'_+\}$ sending $\zeta_{H,\phi}$ to the composite

$$BG_+ \xrightarrow{tr_+} BH_+ \xrightarrow{B\phi_+} BG'_+,$$

where tr_+ is the unreduced transfer map. The map α is an isomorphism only after completion with respect to a filtration given by powers of a certain ideal I :

$$A(G, G')_I^\wedge \xrightarrow{\cong} \{BG_+, BG'_+\}.$$

It is convenient to eliminate the disjoint basepoint. Since $BG_+ \simeq S^0 \vee BG$, splitting BG_+ and splitting BG are equivalent problems.

There is an augmentation homomorphism $\varepsilon: A(G, G') \rightarrow A(G, 1)$. Let $\tilde{A}(G, G') = \text{Ker } \varepsilon$. Then we have an isomorphism

$$\tilde{A}(G, G')_I^\wedge \xrightarrow{\cong} \{BG, BG'\}.$$

When P is a p -group, I -adic completion on $\tilde{A}(P, P)$ is the same as p -adic completion so we have

$$\tilde{A}(P, P)_p^\wedge := \mathbb{Z}_p^\wedge \otimes \tilde{A}(P, P) \cong \tilde{A}(P, P)_I^\wedge \cong \{BP, BP\}.$$

Further reducing $\mathbb{Z}_p^\wedge$ to its residue field $\mathbb{F}_p$ and letting $\tilde{A}_p(P, P) = \mathbb{F}_p \otimes_{\mathbb{Z}} \tilde{A}(P, P)_p^\wedge$, we get maps

$$\{BP, BP\} \xrightarrow{\cong} \tilde{A}(P, P)_p^\wedge \rightarrow \tilde{A}_p(P, P).$$

From the idempotent refinement theorem, we see that a primitive orthogonal idempotent decomposition of the identity in $\tilde{A}_p(P, P)$ lifts to one in $\tilde{A}(P, P)_p^\wedge$. Thus, there is a one-to-one correspondence between stable homotopy types of indecomposable summands of BP and isomorphism types of simple $\tilde{A}_p(P, P)$ -modules.

As mentioned in Section 1, Nishida [10] used the Segal Conjecture to show that for all finite p -groups P , the composite

$$\mathbb{Z}_p^\wedge \text{Out } P \rightarrow \{BP, BP\} \rightarrow \{BP, BP\} / J(P) \quad (1)$$

is an isomorphism of rings.

Definition 2.1. If a primitive idempotent $e \in \{BP, BP\}$ is not in $J(P)$, then the summand eBP is said to **originate** in BP . Every indecomposable summand of BP originates in some BQ , $Q \leq P$ (see [10], where the term “dominant” is used instead of “originate”).

If $X = eBQ$ originates in BQ , then there is a corresponding idempotent $\hat{e} \in \mathbb{Z}_p \text{Out } Q$. Reducing mod p , we get an idempotent $\bar{e} \in \mathbb{F}_p \text{Out } Q$. Though it is not necessarily primitive, we do obtain a simple right $\mathbb{F}_p \text{Out } Q$ -module $M = \bar{e}R/\bar{e}J(R)$, where $R = \mathbb{F}_p \text{Out } Q$. The relationship $X \leftrightarrow M$ is a one-to-one correspondence between homotopy classes of original indecomposable summands of BQ and isomorphism classes of simple right $\mathbb{F}_p \text{Out } Q$ -modules (see [11], Proposition 1.2).

A truly seminal theorem in the field of stable splittings came from Priddy in 1988. Before we restate his result, we need some notation. If $R = \mathbb{F}_p G$ is a group ring and M is a right R -module, let M^* denote the corresponding left R -module with action given by $r \cdot m^* = m \cdot r^*$, where r^* denotes the image of $r \in R$ under the anti-automorphism $\sum a_g g \mapsto \sum a_g g^{-1}$. Other notation: the conjugation homomorphism is $c_g(a) = gag^{-1}$, ${}^g K = gKg^{-1}$, and for $H, K \leq P$ we define $N(H, K) = \{x \in P \mid H \leq {}^x K\}$.

Theorem 2.2. ([11], Theorem 0.1) *Let X be an original summand of BQ , $Q \leq P$. Then X is a summand of BP if and only if there exist subgroups $Q' \leq P' \leq P$, $Q' \cong Q$, and a retraction $\pi: P' \rightarrow Q'$, such that $\gamma \cdot M^* \neq 0$, where M is the simple right $\mathbb{F}_p \text{Out } Q'$ -module corresponding to X , and*

$$\gamma = \sum_x Q' \xrightarrow{c_{x^{-1}}} x^{-1}Q'x \leq P' \xrightarrow{\pi} Q'$$

with the sum running over coset representatives of $N(Q', P')/P'$ such that the composition $\pi \circ c_{x^{-1}}$ is in $\text{Out } Q'$.

This same theorem appears in [2] (as Theorem 5.2) with slightly different notation and a different proof. There the sum above runs over double coset representatives x of Q' and P' in P , satisfying the additional condition that $Q' \leq {}^x P'$.

Catalano explains that the Benson–Feshbach conditions can be replaced by $x \in P/P'$ and $Q' \leq {}^x P'$, that, in turn, are equivalent to $x \in N(Q', P')/P'$ (see remarks in [4] subsequent to Lemma 2.1 and Theorem 2.3).

Definition 2.3. Any subgroup Q' of P satisfying $\gamma \cdot M^* \neq 0$ as above is called a **contributor**, and Q' **contributes** to the multiplicity of X in BP . If a summand X of BK appears in the splitting of BP , then K is isomorphic to a contributor (see [4], p. 35).

3. The Benson–Feshbach Method

Benson–Feshbach define a *coadjoint* module $M(P, P)$ over $A(P, P)$ as follows. Let $Q \leq P$, $\psi: Q \rightarrow P'$, and $X = X_{Q, \psi}$, and define

$$\begin{aligned} f_{Q, \psi}: A(P, P') &\rightarrow \mathbb{Z} && \text{by} \\ X &\mapsto |X^{\Delta_{Q, \psi}}|. \end{aligned}$$

Define $M(P, P')$ to be the free abelian group with one basis element $f_{Q, \psi}$ for each conjugacy class of pairs (Q, ψ) . There is an action

$$A(P_1, P_2) \times M(P_1, P_3) \rightarrow M(P_2, P_3)$$

given by

$$\zeta_{H,\phi} \cdot f_{Q,\psi} = \sum_{\substack{x \in Q \setminus P_1/H \\ Q \leq^x H \\ Q \cap^x (\text{Ker } \phi) \leq \text{Ker } \psi}} f_{\phi(x^{-1}Q), \psi \circ c_x \circ \phi^{-1}} \quad ([2], \text{Prop. 3.1}).$$

If $P_1 = P_2 = P_3 = P$, then $M(P, P)$ is an $A(P, P)$ -module.

Let $\bar{M}(P, P) = \mathbb{F}_p \otimes_{\mathbb{Z}} M(P, P)$ and $\bar{A}(P, P) = \mathbf{F}_p \otimes_{\mathbb{Z}} A(P, P)$. Every simple $\bar{A}(P, P)$ -module is a composition factor of $\bar{M}(P, P)$ ([2], Lemma 3.4). There is an $\bar{A}(P, P)$ -invariant filtration of $\bar{M}(P, P)$ given by what Benson–Feshbach call “types.” The subquotients arising from the filtration will be the simple modules we want.

Definition 3.1. ([2], Definition 4.3) Let $Q_1, Q_2 \leq P$ and let $\psi_i: Q_i \rightarrow P$ be homomorphisms. Write $(Q_1, \psi_1) \succeq (Q_2, \psi_2)$ if there is a surjective homomorphism $\alpha: Q_1 \rightarrow Q_2$ which extends to a homomorphism $Q_1 C_P(Q_1) \rightarrow P$, and there is an element $g \in P$ such that $\psi_1 = c_g \circ \psi_2 \circ \alpha$. If $(Q_1, \psi_1) \succeq (Q_2, \psi_2)$ and $(Q_2, \psi_2) \succeq (Q_1, \psi_1)$, then we write $(Q_1, \psi_1) \sim (Q_2, \psi_2)$ and say that (Q_1, ψ_1) has the same **type** as (Q_2, ψ_2) . If ψ_1 and ψ_2 are isomorphisms, then by abuse of notation we write $Q_1 \sim Q_2$ and talk of types of subgroups.

The partial order $\succeq$ defines an $\bar{A}(P, P)$ -invariant filtration of $\bar{M}(P, P')$ ([2], Proposition 4.4). Filtered quotients are denoted $\bar{L}(P, P')_{Q,\psi}$ and $\bar{L}(P, Q) = \oplus \bar{L}(P, Q)_{Q,\psi}$, where the sum is taken over all types with ψ an isomorphism.

Proposition 3.2. ([2], Proposition 4.8) *The module $\bar{L}(P, Q)$ has a basis consisting of the $\bar{f}_{Q',\psi'}$, where Q' has the same type as Q and ψ' is an isomorphism from Q' to Q . A left action of $\mathbb{F}_p \text{Out } Q$ on $\bar{L}(P, Q)$ is defined by $\eta \cdot \bar{f}_{Q',\psi'} = \bar{f}_{Q',\eta \circ \psi'}$, for $\eta \in \text{Out } Q$. Moreover, if M is a simple right $\mathbb{F}_p \text{Out } Q$ -module then $M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{L}(P, Q)$ is a left $\bar{A}(P, P)$ -module with action given by*

$$\zeta_{H,\phi} \cdot (s \otimes \bar{f}_{Q',\psi'}) = \sum_{\substack{x \in Q' \setminus P/H \\ Q' \leq^x \text{Stab}_P M \leq^x H \\ Q' \cap^x (\text{Ker } \phi) = 1}} s \otimes f_{\phi(x^{-1}Q'), \psi \circ c_x \circ \phi^{-1}}.$$

We write $\text{Stab}_P M$ for the kernel of the composite

$$N_P(Q') \rightarrow \text{Out } Q' \xrightarrow{\cong} \text{Out } Q \rightarrow \text{Aut } M.$$

From Michael Catalano we have a bit more information as follows. Let Q_i be a conjugacy class representative of Q in P that has the same type as Q , and let $\overline{N_P(Q_i)}$ be the image of the map

$$\begin{array}{ccccc} N_P(Q_i) & \rightarrow & \text{Out } Q_i & \rightarrow & \text{Out } Q \\ g & \mapsto & c_g & \mapsto & \psi_i \circ c_g \circ \psi_i^{-1} \end{array}$$

(see [2], Lemma 5.3). In [4], Lemma 2.1, we see that

$$M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{L}(P, Q) \cong \oplus_{i=1}^k M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{f}_{Q_i, \psi_i}.$$

Furthermore, $M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{f}_{Q_i, \psi_i} \cong \overline{M/N_P(Q_i)}$ (see [5], p. 16). We end up with the following proposition.

Proposition 3.3. (Catalano) *As $\mathbb{F}_p$ -spaces, we have*

$$M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{L}(P, Q) \cong \oplus_{i=1}^k M / \overline{N_P(Q_i)}$$

where k is the number of conjugacy classes of subgroups of P of the same type as Q , Q_i is a conjugacy class representative, and $\psi_i: Q_i \rightarrow Q$ is an isomorphism.

Benson–Feshbach define a certain maximal submodule $\mathcal{M}$ of $M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{L}(P, Q)$ inductively ([2], Definition 4.9), but Catalano proves $\mathcal{M} = \mathcal{M}_0$ ([4], Lemma 3.1). Thus

$$\mathcal{M} = \mathcal{M}_0 = \cap_{\text{Im } \phi \cong Q} \text{Ker } \zeta_{L, \phi}.$$

Let $\bar{L}(P, Q, M) = (M \otimes_{\mathbb{F}_p \text{Out } Q} \bar{L}(P, Q)) / \mathcal{M}$.

Theorem 3.4. ([2], Theorem 5.7) *The modules $\bar{L}(P, Q, M)$ are either simple or zero. As Q runs through the types of subgroups of P and M runs through simple $\mathbb{F}_p \text{Out } Q$ -modules, the non-zero modules of the form $\bar{L}(P, Q, M)$ run through a complete set of representatives for the isomorphism types of simple $\bar{A}(P, P)$ -modules.*

The module $\bar{L}(P, Q, M)$ is zero if and only if the original summand of BQ_+ corresponding to M does not appear as a summand of BP_+ . Alternatively, the module $\bar{L}(P, Q, M)$ is non-zero if and only if the conditions of Theorem 2.2 hold (specifically, there must exist $Q \cong Q' \leq P'$ such that $\gamma \cdot M^* \neq 0$). Finally, if X originates in BQ then the multiplicity of X in BP equals the dimension of $\bar{L}(P, Q, M)$ as a module over its endomorphism ring, $\text{End}_{\bar{A}(P, P)} \bar{L}(P, Q, M)$.

There is a ring homomorphism $\rho: \bar{A}(P, P) \rightarrow \mathbb{F}_p \text{Out } P$ defined by $\rho(\zeta_{H, \phi}) = 0$ unless $H = P$ and ϕ is an automorphism, in which case $\rho(\zeta_{H, \phi}) = \bar{\phi}$ where $\bar{\phi}$ is the outer automorphism corresponding to ϕ ([2], p.166). Clearly ρ is a split surjection. A primitive idempotent is in $\text{Ker } \rho$ if and only if the corresponding indecomposable summand of BP does not originate in BP .

If $X = eBP$ is an original indecomposable summand of BP , then there is a primitive idempotent $\bar{e} \in \bar{A}(P, P)$ that is not in $\text{Ker } \rho$. Thus, there is a non-trivial idempotent $\bar{e} \in \mathbb{F}_p \text{Out } P$ such that $\rho(\bar{e}) = \bar{e}$. We see that $m(X, BP)$ equals the dimension of $M = \bar{e}R/\bar{e}J(R)$ over its endomorphism ring $\text{End}_{\mathbb{F}_p \text{Out } P} M$.

4. The Martino–Priddy Method

Let X be an original indecomposable summand of BQ , $Q \leq P$, with corresponding simple right $\mathbb{F}_p \text{Out } P$ -module M . As in [9], define $\text{Split } Q$ to be the set of conjugacy classes of retractions $\pi_j: P_j \rightarrow Q_j$, where $Q_j \cong Q$. For all j , fix isomorphisms $\psi_j: Q_j \rightarrow Q$.

Let

$$\omega_{ij} = \sum_{N(Q_i, P_j)/P_j} \psi_j \circ \pi_j \circ c_{x^{-1}} \circ \psi_i^{-1}$$

where the sum runs over coset representatives of $N(Q_i, P_j)/P_j$ such that the composition

$$Q \xrightarrow{\psi_i^{-1}} Q_i \xrightarrow{c_{x^{-1}}} x^{-1}Q_i x \leq P_j \xrightarrow{\pi_j} Q_j \xrightarrow{\psi_j} Q$$

is in $\text{Out}Q$.

Let $n = |\text{Split } Q|$, then $\mathcal{A}(Q) = (\omega_{ij})$ is in $\text{Mat}_n(R)$, where $R = \mathbb{F}_p \text{Out } Q$. Let $\mathcal{K} = \text{End}_R M$. If $\dim_{\mathcal{K}} M^* = m$, then each ω_{ij} can be viewed as an $m \times m$ matrix over $\mathcal{K}$, where ω_{ij} acts on the left of the module M^* . Define W_{ij} to be the matrix in $\text{Mat}_m(\mathcal{K})$ corresponding to ω_{ij} , and let $\mathcal{A}(Q, M^*) = (W_{ij}) \in \text{Mat}_{nm}(\mathcal{K})$.

Theorem 4.1. ([9], Theorem 0.1) *Let X be an original summand of BQ , $Q \leq P$, with corresponding simple right $\mathbb{F}_p \text{Out } Q$ -module M . Then, the multiplicity of X in BP is $m(X, BP) = \text{rank}_{\mathcal{K}} \mathcal{A}(Q, M^*)$.*

Note that if X is an original summand of BP , then $\text{Split } P = \{id: P \rightarrow P\}$ and $m(X, BP) = \text{rank}_{\mathcal{K}} \mathcal{A}(Q, M^*) = \dim_{\text{End}_{\mathbb{F}_p \text{Out } P} M^*}$.

5. Connecting Benson–Feshbach and Martino–Priddy

We have seen that Benson–Feshbach and Martino–Priddy compute the multiplicity of an original summand of BP in exactly the same way as $\dim_{\text{End}_{\mathbb{F}_p \text{Out } P} M}$, but their computations of $m(X, BP)$ for non-original summands differ vastly. This section will show a direct connection between the two computations for $m(X, BP)$ when X originates in BQ for some $Q \leq P$.

5.1. Reducing Martino–Priddy’s Matrix

If X is a non-original summand of BP , Benson–Feshbach show there is only one type of contributor Q as follows.

Lemma 5.1. (see Proposition 5.9 of [2] and Proposition 2.4 of [4]) *If Q' is a contributor to $m(X, BP)$, then Q' is a direct summand of $Q'C_P(Q')$.*

The proof of this result in [2] relies almost entirely on that paper’s Theorem 5.2, which has a Martino–Priddy analogue (as noted after Theorem 2.2 in this paper). Thus, we may use this Lemma in the Martino–Priddy context.

Theorem 5.2. ([2], p.158) *Let X be a summand of BP that originates in BQ , $Q \leq P$. If both $Q_1 \cong Q$ and $Q_2 \cong Q$ contribute to $m(X, BP)$, then Q_1 and Q_2 have the same type.*

Proof. To show $(Q_1, \psi_1) \succeq (Q_2, \psi_2)$, we let α be the composite $Q_1 \xrightarrow{\psi_1} Q \xrightarrow{\psi_2^{-1}} Q_2$. Since Q_1 is a contributor, Lemma 5.1 implies $Q_1 C_P(Q_1) = Q_1 \times A_1$ for some $A_1 \leq P$. We see that α extends to $Q_1 C_P(Q_1)$ via

$$Q_1 C_P(Q_1) = Q_1 \times A_1 \xrightarrow{\text{proj}} Q_1 \xrightarrow{\alpha} Q_2,$$

and $\psi_1 = c_e \circ \psi_2 \circ \alpha$. Similarly, $(Q_2, \psi_2) \succeq (Q_1, \psi_1)$. $\square$

Since only contributors play a role in determining $m(X, BP)$, we can refine the definition of $\text{Split } Q$ as follows: Define

$$\text{Split}_T Q = \{\pi_j: P_j \rightarrow Q_j\}$$

to be conjugacy classes of retractions where $Q_j \leq P_j \leq P$ and Q_j has the same type as Q . As before, we will fix isomorphisms $\psi_j: Q_j \rightarrow Q$.

As in Section 4, let

$$\omega_{ij} = \sum_{N(Q_i, P_j)/P_j} \psi_j \circ \pi_j \circ c_{x-1} \circ \psi_i^{-1} \in \mathbb{F}_p \text{Out } Q.$$

For the rest of this section, let $n = |\text{Split}_T Q|$. Define $\mathcal{A}_T(Q) = (\omega_{ij}) \in \text{Mat}_n(R)$ and $\mathcal{A}_T(Q, M^*) = (W_{ij}) \in \text{Mat}_{nm}(\mathcal{K})$ as analogues to $\mathcal{A}(Q)$ and $\mathcal{A}(Q, M^*)$ respectively.

Theorem 5.3. $m(X, BP) = \text{rank}_{\mathcal{K}} \mathcal{A}_T(Q, M^*)$.

Proof. The proof of Theorem 0.1 in [9] works in this refined setting. $\square$

Among the n retractions in $\text{Split}_T Q$ are the k identity maps $Q_j \rightarrow Q_j$. Order the retractions in $\text{Split}_T Q$ in such a way that the first k retractions are the k identity maps.

Let $\mathcal{B}(Q)$ be the $k \times n$ submatrix of $\mathcal{A}_T(Q)$ obtained by considering only the first k rows of $\mathcal{A}_T(Q)$. Define $\mathcal{B}(Q, M^*)$ to be the corresponding $km \times nm$ submatrix of $\mathcal{A}_T(Q, M^*)$. More specifically, we think of $\mathcal{B}(Q)$ as a map $(M^*)^n \rightarrow (M^*)^k$, and $\mathcal{B}(Q, M^*)$ as a map $\mathcal{K}^{nm} \rightarrow \mathcal{K}^{km}$.

Proposition 5.4. $\text{rank}_{\mathcal{K}} \mathcal{B}(Q, M^*) = \text{rank}_{\mathcal{K}} \mathcal{A}_T(Q, M^*)$.

Proof. Consider row t of the matrix $\mathcal{A}_T(Q)$, where $t > k$. We know that $Q_t = Q_i$ for some $i = 1, \dots, k$. Thus, $\omega_{tj} = \omega_{ij}$ for all $j = 1, \dots, n$. The matrix $\mathcal{A}_T(Q, M^*)$ row reduces to a matrix that has $\mathcal{B}(Q, M^*)$ in the top $km \times nm$ block, followed by a $(n - k)m \times nm$ block of zeroes. Clearly $\text{rank}_{\mathcal{K}} \mathcal{B}(Q, M^*) = \text{rank}_{\mathcal{K}} \mathcal{A}_T(Q, M^*)$. $\square$

5.2. The map from Benson–Feshbach to Martino–Priddy

The basic idea is to define $\Phi: M \otimes_R \bar{L}(P, Q) \rightarrow (M^*)^n$ by sending $s \otimes \bar{f}_{Q_i, \psi_i}$ to the image of s^* under the action of row i of $\mathcal{B}(Q)$. We will essentially show that $\text{Ker } \Phi = \mathcal{M}$ and $\text{Im } \Phi = \text{Im } \mathcal{B}(Q, M^*)$. Thus, $\bar{L}(P, Q, M) \cong \text{Im } \mathcal{B}(Q, M^*)$ and the dimension of $\bar{L}(P, Q, M)$ equals the rank of $\mathcal{B}(Q, M^*)$.

Consider elements of M as rows, while elements of M^* will be written as columns. Let $\bar{f}_{Q', \psi'} \in \bar{L}(P, Q)$ where Q' has the same type as Q and $\psi': Q' \rightarrow Q$ is an isomorphism. Define

$$\Phi: M \otimes_R \bar{L}(P, Q) \rightarrow (M^*)^n$$

by $\Phi(s \otimes \bar{f}_{Q', \psi'}) = [\omega_{\psi'1} \cdot s^*, \omega_{\psi'2} \cdot s^*, \dots, \omega_{\psi'n} \cdot s^*]^T$ where

$$\omega_{\psi'j} = \sum_{x \in N(Q', P_j)/P_j} (Q \xrightarrow{(\psi')^{-1}} Q' \xrightarrow{c_{x-1}} P_j \xrightarrow{\pi_j} Q_j \xrightarrow{\psi_j} Q)$$

is an element of $\mathbb{F}_p \text{Out } Q$. (Note: If $\psi' = \psi_i$ for some i , then $\omega_{\psi'j} = \omega_{ij}$.)

If $\nu = \sum_{i=1}^k s_i \otimes \bar{f}_{Q_i, \psi_i}$ is a typical element of $M \otimes_R \bar{L}(P, Q)$, then

$$\begin{aligned}
 \Phi(\nu) &= \sum_{i=1}^k \Phi(s_i \otimes \bar{f}_{Q_i, \psi_i}) \\
 &= \begin{bmatrix} \omega_{11} \cdot s_1^* \\ \omega_{12} \cdot s_1^* \\ \vdots \\ \omega_{1n} \cdot s_1^* \end{bmatrix} + \cdots + \begin{bmatrix} \omega_{k1} \cdot s_k^* \\ \omega_{k2} \cdot s_k^* \\ \vdots \\ \omega_{kn} \cdot s_k^* \end{bmatrix} \\
 &= \begin{bmatrix} \sum_{i=1}^k \omega_{i1} \cdot s_i^* \\ \vdots \\ \sum_{i=1}^k \omega_{in} \cdot s_i^* \end{bmatrix} \\
 &= \mathcal{B}(Q)^T \cdot \begin{bmatrix} s_1^* \\ s_2^* \\ \vdots \\ s_k^* \end{bmatrix} \in (M^*)^n
 \end{aligned}$$

Proposition 5.5. *The map Φ is well-defined.*

Proof. Let $\alpha \in \overline{N_P(Q_i)}$, where $\alpha = \psi_i \circ c_g \circ \psi_i^{-1}$ for some $g \in N_P(Q_i)$. We need $\Phi(s \otimes \bar{f}_{Q_i, \psi_i}) = \Phi(s\alpha \otimes \bar{f}_{Q_i, \psi_i})$.

Now $\Phi(s\alpha \otimes \bar{f}_{Q_i, \psi_i}) = \Phi(s \otimes \bar{f}_{Q_i, \alpha \circ \psi_i}) = [\oplus_{j=1}^n \omega_{(\alpha \circ \psi_i)j} \cdot s^*]^T$, where

$$\omega_{(\alpha \circ \psi_i)j} = \sum_{x \in N(Q_i, P_j)/P_j} (Q \xrightarrow{(\alpha \circ \psi_i)^{-1}} Q_i \xrightarrow{c_{x-1}} P_j \xrightarrow{\pi_j} Q_j \xrightarrow{\psi_j} Q).$$

Now

$$\begin{aligned}
 \sum_{x \in N(Q_i, P_j)/P_j} \psi_j \circ \pi_j \circ c_{x-1} \circ (\alpha \circ \psi_i)^{-1} &= \sum \psi_j \circ \pi_j \circ c_{x-1} \circ \psi_i^{-1} \circ \alpha^{-1} \\
 &= \sum \psi_j \circ \pi_j \circ c_{x-1} \circ \psi_i^{-1} \circ \psi_i \circ c_{g^{-1}} \circ \psi_i^{-1} \\
 &= \sum \psi_j \circ \pi_j \circ c_{x-1} \circ c_{g^{-1}} \circ \psi_i^{-1} \\
 &= \sum \psi_j \circ \pi_j \circ c_{x-1} g^{-1} \circ \psi_i^{-1}.
 \end{aligned}$$

Since $g \in N_P(Q_i)$ and $x \in N(Q_i, P_j)$, we know $gx \in N(Q_i, P_j)$. Furthermore, the map $x \mapsto gx$ is a bijection of $N_P(Q_i)$. Thus

$$\sum_{x \in N(Q_i, P_j)} \psi_j \circ \pi_j \circ c_{x-1} g^{-1} \circ \psi_i^{-1} = \sum_{x \in N(Q_i, P_j)} \psi_j \circ \pi_j \circ c_{x-1} \circ \psi_i^{-1}.$$

We have $\omega_{(\alpha \circ \psi_i)j} = \omega_{ij}$ for all $j = 1, \dots, n$, and so $\Phi(s\alpha \otimes \bar{f}_{Q_i, \psi_i}) = \Phi(s \otimes \bar{f}_{Q_i, \psi_i})$. $\square$

We need to consider the action of elements of $\bar{A}(P, P)$ on $\bar{L}(P, Q, M)$, especially those corresponding to the retracts.

Lemma 5.6. $\zeta_{P_j, \pi_j}(s_i \otimes \bar{f}_{Q_i, \psi_i}) = s_i \cdot \omega_{ij}^* \otimes \bar{f}_{Q_j, \psi_j}$.

Proof. We have

$$\zeta_{P_j, \pi_j}(s_i \otimes \bar{f}_{Q_i, \psi_i}) = \sum_{\substack{x \in Q_i \setminus P/P_j \\ Q_i \leq \text{Stab}_P M \leq {}^x P_j \\ Q_i \cap {}^x (\text{Ker } \pi_j) = 1}} s_i \otimes f_{\pi_j(x^{-1}Q_i), \psi_i \circ c_x \circ \pi_j^{-1}}.$$

The condition $x \in Q_i \setminus P/P_j$ can be replaced by $x \in P/P_j$ ([4], p.34), and the condition $Q_i \leq \text{Stab}_P M \leq {}^x P_j$ can be replaced by $Q_i \leq {}^x P_j$ ([2], Lemma 4.2). The two replacement conditions are equivalent to the condition $x \in N(Q_i, P_j)/P_j$.

Now $x^{-1}Q_i x \leq P_j$ and $x^{-1}Q_i x \cap \text{Ker } \pi_j = 1$, so π_j is one-to-one, hence onto, when restricted to $x^{-1}Q_i x$. Thus we have $\pi_j(x^{-1}Q_i) = Q_j$.

Consider a summand of ω_{ij} ,

$$\omega_{ij}(x): Q \xrightarrow{\psi_i^{-1}} Q_i \xrightarrow{c_x^{-1}} x^{-1}Q_i x \leq P_j \xrightarrow{\pi_j} Q_j \xrightarrow{\psi_j} Q,$$

so $\omega_{ij} = \sum_{x \in N(Q_i, P_j)/P_j} \omega_{ij}(x)$. Each $\omega_{ij}(x)$ in the sum is an element of $\text{Out } Q$. As in the proof of Lemma 5.1, we can write

$$\omega_{ij} = \sum_{x \in N(Q_i, P_j)/P_j} \omega_{ij}(x) = \sum_{\substack{x \in N(Q_i, P_j)/P_j \\ Q_i \cap {}^x (\text{Ker } \pi_j) = 1}} \omega_{ij}(x).$$

Each $\omega_{ij}(x)$ is invertible. Let

$$\omega_{ij}^{-1}(x) = Q \xrightarrow{\psi_j^{-1}} Q_j \xrightarrow{\pi_j^{-1}} x^{-1}Q_i x \xrightarrow{c_x} Q_i \xrightarrow{\psi_i} Q.$$

$$\text{Then } \omega_{ij}^* = \sum_{\substack{x \in N(Q_i, P_j)/P_j \\ Q_i \cap {}^x (\text{Ker } \pi_j) = 1}} \omega_{ij}^{-1}(x).$$

Thus, we have

$$\begin{aligned} \zeta_{P_j, \pi_j}(s_i \otimes \bar{f}_{Q_i, \psi_i}) &= \sum_{\substack{x \in N(Q_i, P_j)/P_j \\ Q_i \cap {}^x (\text{Ker } \pi_j) = 1}} s_i \otimes \bar{f}_{Q_j, \psi_i \circ c_x \circ \pi_j^{-1}} \\ &= \sum s_i \otimes \bar{f}_{Q_j, \psi_i \circ c_x \circ \pi_j^{-1} \circ \psi_j^{-1} \circ \psi_j} \\ &= \sum s_i \otimes \bar{f}_{Q_j, \omega_{ij}^{-1}(x) \circ \psi_j} \\ &= \sum s_i \cdot \omega_{ij}^{-1}(x) \otimes \bar{f}_{Q_j, \psi_j} \\ &= s_i \cdot \omega_{ij}^* \otimes \bar{f}_{Q_j, \psi_j}. \end{aligned}$$

□

Proposition 5.7. $\text{Ker } \Phi = \mathcal{M}$.

Proof. Let $\nu = \sum_{i=1}^k s_i \otimes \bar{f}_{Q_i, \psi_i} \in \mathcal{M} = \cap_{\text{Im } \phi \cong Q} \text{Ker } \zeta_{L, \phi}$, where $L \leq P$ and $\phi: L \rightarrow P$.

In particular, for all $j = 1, \dots, n$ we have

$$\begin{aligned} 0 &= \zeta_{P_j, \pi_j}(\nu) \\ &= \sum_{i=1}^k \zeta_{P_j, \pi_j}(s_i \otimes \bar{f}_{Q_i, \psi_i}) \\ &= \sum_{i=1}^k (s_i \cdot \omega_{ij}^* \otimes \bar{f}_{Q_j, \psi_j}) \\ &= (\sum_{i=1}^k s_i \cdot \omega_{ij}^*) \otimes \bar{f}_{Q_j, \psi_j}. \end{aligned}$$

Now by ([5], Lemma 5.21), this implies that for all $j = 1, \dots, n$

$$\begin{aligned} 0 &= \sum_{i=1}^k s_i \cdot \omega_{ij}^* \\ &= \sum_{i=1}^k \omega_{ij} \cdot s_i^*. \end{aligned}$$

Thus we have $\Phi(\nu) = 0$ and $\mathcal{M} \subseteq \text{Ker } \Phi$ is proved.

Now let $\nu = \sum_{i=1}^k s_i \otimes \bar{f}_{Q_i, \psi_i} \in \text{Ker } \Phi$. If $\nu \notin \mathcal{M}$, then there is a pair (L, ϕ) , where $\phi: L \rightarrow P$ and $\text{Im } \phi \cong Q$, for which $\nu \notin \text{Ker } \zeta_{L, \phi}$. Consider

$$\begin{aligned} \zeta_{L, \phi}(\nu) &= \sum_{i=1}^k \zeta_{L, \phi}(s_i \otimes \bar{f}_{Q_i, \psi_i}) \\ &= \sum_{i=1}^k \left[\sum_{\substack{x \in N(Q_i, L)/L \\ Q_i \cap x(\text{Ker } \phi) = 1}} s_i \otimes \bar{f}_{\phi(x^{-1}Q_i), \psi_i \circ c_x \circ \phi^{-1}} \right]. \end{aligned}$$

There must exist at least one pair (Q_i, x) satisfying the conditions in the sum. Let (Q_i, x) be such a pair. Note that $x \in N(Q_i, L)$ implies $x^{-1}Q_i x \leq L$, and $x^{-1}Q_i x \cap \text{Ker } \phi = 1$ implies $\phi|_{x^{-1}Q_i x}$ is one-to-one, hence onto $\phi(L) = \bar{Q}$. Let $\bar{\phi} = \phi|_{x^{-1}Q_i x}$ so $\bar{\phi}: x^{-1}Q_i x \rightarrow \bar{Q}$ is an isomorphism. Consider the composition

$$r_x: L \xrightarrow{\phi} \bar{Q} \xrightarrow{\bar{\phi}^{-1}} x^{-1}Q_i x.$$

Now r_x is a retraction onto a conjugate of Q_i , so the pair (L, r_x) is conjugate to a pair $(P_t, \pi_t) \in \text{Split}_T Q$ for some $t = 1, \dots, n$. Since $\nu \in \text{Ker } \Phi$, we know that $0 = \sum_{i=1}^k \omega_{ij} \cdot s_i^* = \sum_{i=1}^k s_i \cdot \omega_{ij}^*$ for all $j = 1, \dots, n$. Because (P_t, π_t) is conjugate to (L, r_x) , we have $\zeta_{L, r_x}(\nu) = \zeta_{P_t, \pi_t}(\nu) = (\sum_{i=1}^k s_i \cdot \omega_{ij}^*) \otimes \bar{f}_{Q_j, \psi_j} = 0 \otimes \bar{f}_{Q_j, \psi_j}$. Since $\text{Im } \phi \cong Q$ and $\text{Ker } \phi = \text{Ker } r_x$, we again see from Lemma 5.21 of [5] that $\zeta_{L, \phi}(\nu) = 0$. This is a contradiction, so we must have $\nu \in \mathcal{M}$.

This completes the proof that $\text{Ker } \Phi = \mathcal{M}$. $\square$

5.3. Alternative Transformations

Define a $\mathcal{K}$ -linear version of the homomorphism Φ as follows. Let $\nu = \sum_{i=1}^k \vec{s}_i \otimes \bar{f}_{Q_i, \psi_i} \in M \otimes_R \bar{L}(P, Q)$, where we now let $\vec{s}_i$ represent an m -dimensional vector in the $\mathcal{K}$ -module M . Define $\Psi: M \otimes_R \bar{L}(P, Q) \rightarrow \mathcal{K}^{nm}$ by

$$\Psi(\nu) = \mathcal{B}(Q, M^*)^T \cdot \begin{bmatrix} \vec{s}_1^* \\ \vec{s}_2^* \\ \vdots \\ \vec{s}_k^* \end{bmatrix} \in \mathcal{K}^{nm}.$$

Following the proofs of Propositions 5.5 and 5.7, we see that Ψ is well-defined and $\text{Ker } \Psi = \mathcal{M}$.

Taking this one step further, let $d = \dim_{\mathbb{F}_p} \mathcal{K}$ and let $\bar{\mathcal{B}}(Q, M^*)$ denote the matrix $\mathcal{B}(Q, M^*)$ considered as a $dkm \times dnm$ matrix over $\mathbb{F}_p$. Define $\chi: M \otimes_R \bar{L}(P, Q) \rightarrow \mathbb{F}_p^{dnm}$ as an $\mathbb{F}_p$ -linear analog to Φ and Ψ . Again, we have that χ is well-defined and $\text{Ker } \chi = \mathcal{M}$. Since $\mathbb{F}_p$ is a field, $\text{rank}_{\mathbb{F}_p} \bar{\mathcal{B}}(Q, M^*)^T = \text{rank}_{\mathbb{F}_p} \bar{\mathcal{B}}(Q, M^*)$.

5.4. Computing Multiplicity

Again, according to Benson–Feshbach, the multiplicity of X in BP , where X originates in BQ , is

$$m_{\text{BF}}(X, BP) = \dim_{\text{End}_{\bar{A}(P,P)} \bar{L}(P,Q,M)} \bar{L}(P, Q, M).$$

(The “BF” subscript simply refers to Benson and Feshbach.)

On the other hand, according to Martino–Priddy, the multiplicity of X in BP is

$$m_{\text{MP}}(X, BP) = \text{rank}_{\mathcal{K}} \mathcal{B}(Q, M^*).$$

Catalano proves that when $\bar{L}(P, Q, M) \neq 0$, $\text{End}_{\bar{A}(P,P)} \bar{L}(P, Q, M) \cong \mathcal{K}$ (see [4], Lemma 3.2).

Finally,

$$\begin{aligned} m_{\text{BF}}(X, BP) &= \dim_{\text{End}_{\bar{A}(P,P)} \bar{L}(P,Q,M)} \bar{L}(P, Q, M) \\ &= \dim_{\mathcal{K}} \bar{L}(P, Q, M) \\ &= \left(\frac{1}{d}\right) \dim_{\mathbb{F}_p} \bar{L}(P, Q, M) \\ &= \left(\frac{1}{d}\right) \dim_{\mathbb{F}_p} \text{Im } \chi \\ &= \left(\frac{1}{d}\right) \dim_{\mathbb{F}_p} \text{Im } \bar{\mathcal{B}}(Q, M^*)^T \\ &= \left(\frac{1}{d}\right) \text{rank}_{\mathbb{F}_p} \bar{\mathcal{B}}(Q, M^*)^T \\ &= \left(\frac{1}{d}\right) \text{rank}_{\mathbb{F}_p} \bar{\mathcal{B}}(Q, M^*) \\ &= \text{rank}_{\mathcal{K}} \mathcal{B}(Q, M^*) \\ &= m_{\text{MP}}(X, BP). \end{aligned}$$

We see, then, a bridge from the work of Benson–Feshbach to the work of Martino–Priddy.

6. An Example

Using the notation of Hall and Senior [6], let $P = 16\Gamma_2 c_1 = (A \times B) \rtimes C$, where $A = \mathbb{Z}_4 \langle a \rangle$, $B = \mathbb{Z}_2 \langle b \rangle$, and $C = \mathbb{Z}_2 \langle c \rangle$ with $cac^{-1} = ab$. Martino and Priddy determined the complete stable splitting of BP in [9].

Let $Q = \mathbb{Z}_2 \langle c \rangle$. The elements of order 2 in P are $a^2, b, c, a^2b, a^2c, bc, a^2bc$. These elements fall into 5 conjugacy classes: $\text{Cl}(a^2) = \{a^2\}$, $\text{Cl}(b) = \{b\}$, $\text{Cl}(c) = \{c, bc\}$, $\text{Cl}(a^2b) = \{a^2b\}$, and $\text{Cl}(a^2c) = \{a^2c, a^2bc\}$. Take as conjugacy class representatives of groups of the same type as Q the elements c and a^2c . Let $Q_1 = \langle c \rangle$ and $Q_2 = \langle a^2c \rangle$.

6.1. The Martino–Priddy Matrix

If $P_j \rightarrow Q_j$ is a retract, then ω_{ij} will be zero for all i unless $\text{Stab}_P M$ is a subgroup of a conjugate of P_j . Since $\text{Out } Q = 1$, there is only one simple $\mathbb{F}_2 \text{Out } Q$ -module, namely $M = \mathbb{F}_2$ with trivial action. In this case, $\text{Stab}_P M = N_P(Q_i)$ which equals $\langle a^2, b, c \rangle$ for $i = 1, 2$. The only subgroups L of P for which $\langle a^2, b, c \rangle$ is a subgroup of a conjugate of L are $L = \langle a^2, b, c \rangle$ and $L = P$.

There are 4 retracts from $\langle a^2, b, c \rangle$ to each of Q_1 and Q_2 . For the retracts to Q_1 , the generators a^2, b, c map to e, e, c respectively under the first retract π_1 , to e, c, c under the second π_2 , to c, e, c under π_3 , and to c, c, c under π_4 . For the retracts to Q_2 , the generators a^2, b, c map to e, e, a^2c under the first retract π_5 , to e, a^2c, a^2c under the second π_6 , to a^2c, e, e under π_7 , and to a^2c, a^2c, e under π_8 .

There are two retracts from P to Q_1 (and two to Q_2). The first retract sends the generators a, b, c to e, e, c respectively (and to e, e, a^2c in Q_2), while the second retract maps the generators to c, e, c respectively (and to a^2c, e, a^2c in Q_2). The retracts to Q_1 will be denoted π_9 and π_{10} , while the retracts to Q_2 are π_{11} and π_{12} .

There are a total of 12 elements in $\text{Split}_T Q$.

Since $\text{Out } Q = 1$, the entries of the 2×12 matrix $\mathcal{B}(Q)$ will be sums of 0's and 1's. When $P_j = \langle a^2, b, c \rangle$, $N(Q_i, P_j)/P_j = P/P_j = \langle \bar{a} \rangle \cong \mathbb{Z}_2$. When $P_j = P$, $N(Q_i, P_j)/P_j = P/P = \bar{e}$. We get

$$\begin{aligned} B(Q) &= \begin{bmatrix} 1+1 & 1+0 & 1+1 & 1+0 & 1+1 & 1+0 & 0+0 & 0+1 & 1 & 1 & 1 & 1 \\ 1+1 & 1+0 & 0+0 & 1+0 & 1+1 & 1+0 & 1+1 & 1+0 & 1 & 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Since $M = \mathbb{F}_2$ has dimension 1 and trivial action, $\mathcal{B}(Q) = \mathcal{B}(Q, M^*)$. We see that the matrix has rank 1, so there is one copy of $B\mathbb{Z}_2$ in BP .

6.2. The Benson–Feshbach Modules

Let $\psi_1: Q_1 \rightarrow Q$ and $\psi_2: Q_2 \rightarrow Q$ be isomorphisms. As above $M = \mathbb{F}_2$ is the trivial $\mathbb{F}_2 \text{Out } Q$ -module. We know that $\mathbb{F}_2 \otimes \bar{L}(P, Q) = (\mathbb{F}_2 \otimes \bar{f}_{Q_1, \psi_1}) \oplus (\mathbb{F}_2 \otimes \bar{f}_{Q_2, \psi_2})$.

When computing $\zeta_{L, \phi}(1 \otimes \bar{f}_{Q_i, \psi_i})$, we need only consider those subgroups L of P for which $\text{Stab}_P \mathbb{F}_2$ is a subgroup of a conjugate of L . As we saw in the previous section, this implies $L = \langle a^2, b, c \rangle$ or $L = P$.

Suppose $\phi: L \rightarrow Q_i$ is surjective. If $L = P$, then $\text{Ker } \phi$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$.

There is only one copy of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in P , namely $\langle a^2, b, c \rangle$. In this case, when computing $\zeta_{P, \phi}(1 \otimes \bar{f}_{Q_i, \psi_i})$, the sum runs over all $x \in N(Q_i, P)/P = P/P$ such that $Q_i \cap^x \text{Ker } \phi = 1$. But when $x = e$, $Q_i \cap^e \text{Ker } \phi = Q_i \cap \langle a^2, b, c \rangle \neq 1$. So $\text{Ker } \zeta_{P, \phi} = M \otimes \bar{L}(P, Q)$.

There are two conjugacy classes of subgroups of P isomorphic to $\mathbb{Z}_4$. Choose conjugacy class representatives $L_1 = \langle a \rangle$ and $L_2 = \langle ac \rangle$. No matter what i and j are, $L_i \times Q_j = \langle a, c \rangle$. In this case, when computing $\zeta_{P, \phi}(1 \otimes \bar{f}_{Q_i, \psi_i})$, the sum runs over all $x \in N(Q_i, P)/P = P/P$ such that $Q_i \cap^x \text{Ker } \phi = 1$. But when $x = e$, $Q_i \cap^e \text{Ker } \phi = Q_i \cap \langle a, c \rangle \neq 1$. So $\text{Ker } \zeta_{P, \phi} = M \otimes \bar{L}(P, Q)$.

Now suppose $L = \langle a^2, b, c \rangle$, then $\text{Ker } \phi$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are 7 subgroups of P isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$: $K_1 = \langle a^2, b \rangle$, $K_2 = \langle a^2, c \rangle$, $K_3 = \langle a^2, bc \rangle$, $K_4 = \langle b, c \rangle$, $K_5 = \langle b, a^2c \rangle$, $K_6 = \langle a^2b, c \rangle$, and $K_7 = \langle a^2b, a^2c \rangle$.

No matter what the kernel is, when computing $\zeta_{L, \phi}(1 \otimes \bar{f}_{Q_i, \psi_i})$, the sum runs over all $x \in N(Q_i, L)/L = P/L = \{\bar{e}, \bar{a}\}$ such that $Q_i \cap^x \text{Ker } \phi = 1$. Conjugation by a on the K_i yields: ${}^a K_1 = K_1$, ${}^a K_2 = K_3$, ${}^a K_3 = K_2$, ${}^a K_4 = K_4$, ${}^a K_5 = K_5$, ${}^a K_6 = K_7$, and ${}^a K_7 = K_6$.

Suppose the image of ϕ is Q_j , then we get the following intersections and actions:

$$\begin{aligned}
 Q_1 \cap K_1 = 1, \quad Q_1 \cap {}^a K_1 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 2(1 \otimes \bar{f}_{Q_j,\psi_j}) = 0 \\
 Q_1 \cap K_2 \neq 1, \quad Q_1 \cap {}^a K_2 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_1 \cap K_3 = 1, \quad Q_1 \cap {}^a K_3 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_1 \cap K_4 \neq 1, \quad Q_1 \cap {}^a K_4 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 0 \\
 Q_1 \cap K_5 = 1, \quad Q_1 \cap {}^a K_5 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 2(1 \otimes \bar{f}_{Q_j,\psi_j}) = 0 \\
 Q_1 \cap K_6 \neq 1, \quad Q_1 \cap {}^a K_6 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_1 \cap K_7 = 1, \quad Q_1 \cap {}^a K_7 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_1,\psi_1}) = 1 \otimes \bar{f}_{Q_j,\psi_j}
 \end{aligned}$$

$$\begin{aligned}
 Q_2 \cap K_1 = 1, \quad Q_2 \cap {}^a K_1 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 2(1 \otimes \bar{f}_{Q_j,\psi_j}) = 0 \\
 Q_2 \cap K_2 \neq 1, \quad Q_2 \cap {}^a K_2 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_2 \cap K_3 = 1, \quad Q_2 \cap {}^a K_3 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_2 \cap K_4 = 1, \quad Q_2 \cap {}^a K_4 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 2(1 \otimes \bar{f}_{Q_j,\psi_j}) = 0 \\
 Q_2 \cap K_5 \neq 1, \quad Q_2 \cap {}^a K_5 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 0 \\
 Q_2 \cap K_6 = 1, \quad Q_2 \cap {}^a K_6 \neq 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 1 \otimes \bar{f}_{Q_j,\psi_j} \\
 Q_2 \cap K_7 \neq 1, \quad Q_2 \cap {}^a K_7 = 1 &\Rightarrow \zeta_{L,\phi}(1 \otimes \bar{f}_{Q_2,\psi_2}) = 1 \otimes \bar{f}_{Q_j,\psi_j}
 \end{aligned}$$

We see that $\text{Ker } \zeta_{L,\phi}$ is either all of $M \otimes \bar{L}(P, Q)$, or just the submodule generated by $1 \otimes \bar{f}_{Q_1,\psi_1} + 1 \otimes \bar{f}_{Q_2,\psi_2}$. Thus $\mathcal{M} = \langle 1 \otimes \bar{f}_{Q_1,\psi_1} + 1 \otimes \bar{f}_{Q_2,\psi_2} \rangle$, and $\bar{L}(P, Q, M)$ is one-dimensional.

6.3. The map Φ

We will make a couple of computations to show how the map Φ works, and to confirm that $\mathcal{M} = \text{Ker } \Phi$.

Consider $\nu = 1 \otimes \bar{f}_{Q_1,\psi_1} + 1 \otimes \bar{f}_{Q_2,\psi_2} \in \mathcal{M}$.

$$\Phi(\nu) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1^* \\ 1^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So we see that $\mathcal{M} \subseteq \text{Ker } \Phi$.

Now let $\nu = \sum_{i=1}^2 s_i \otimes \bar{f}_{Q_i, \psi_i} \in \text{Ker } \Phi$. Then

$$\Phi(\nu) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_1^* \\ s_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ s_1^* + s_2^* \\ 0 \\ s_1^* + s_2^* \\ 0 \\ s_1^* + s_2^* \\ 0 \\ s_1^* + s_2^* \\ s_1^* + s_2^* \\ s_1^* + s_2^* \\ s_1^* + s_2^* \\ s_1^* + s_2^* \end{bmatrix} = \vec{0}.$$

We see that either $s_1 = s_2 = 0$ or $s_1 = s_2 = 1$.

Assume $s_1 = s_2 = 1$, and consider $\zeta_{L, \phi}(\nu)$ where $L = \langle a^2, b, c \rangle$ and $\phi: L \rightarrow Q_2$ sends each of the generators a^2 , b , and c to a^2c in Q_2 . In particular, note that ϕ is not a retract.

Now

$$\begin{aligned} \zeta_{L, \phi}(\nu) &= \zeta_{L, \phi}(1 \otimes \bar{f}_{Q_1, \psi_1}) + \zeta_{L, \phi}(1 \otimes \bar{f}_{Q_2, \psi_2}) \\ &= \sum 1 \otimes \bar{f}_{\phi(x^{-1}Q_1), \psi_1 \circ c_x \circ \phi^{-1}} + \sum 1 \otimes \bar{f}_{\phi(x^{-1}Q_2), \psi_2 \circ c_x \circ \phi^{-1}}, \end{aligned}$$

where the first sum runs over $x \in N(Q_1, L)/L$ with $Q_1 \cap^x (\text{Ker } \phi) = 1$ and the second sum runs over $x \in N(Q_2, L)/L$ with $Q_2 \cap^x (\text{Ker } \phi) = 1$

As we saw in Section 6.2, for $i = 1, 2$ we have $N(Q_i, L)/L = P/L = \mathbb{Z}_2 \langle \bar{a} \rangle$. Since $\text{Ker } \phi = K_7$, we know that $Q_1 \cap^e (\text{Ker } \phi) = 1$ and $Q_1 \cap^a (\text{Ker } \phi) \neq 1$. Also, $\phi(e^{-1}Q_1) = Q_2$ and $\psi_1 \circ c_e \circ \phi^{-1} = \psi_2$. Thus, $\zeta_{L, \phi}(1 \otimes \bar{f}_{Q_1, \psi_1}) = 1 \otimes \bar{f}_{Q_2, \psi_2}$. On the other hand, $Q_2 \cap^e (\text{Ker } \phi) \neq 1$, while $Q_2 \cap^a (\text{Ker } \phi) = 1$. Also, $\phi(a^{-1}Q_2) = Q_2$ and $\psi_2 \circ c_a \circ \phi^{-1} = \psi_2$. Thus, $\zeta_{L, \phi}(1 \otimes \bar{f}_{Q_2, \psi_2}) = 1 \otimes \bar{f}_{Q_2, \psi_2}$. We conclude that $1 \otimes \bar{f}_{Q_1, \psi_1} + 1 \otimes \bar{f}_{Q_2, \psi_2} \in \text{Ker } \zeta_{L, \phi}$.

Now $\phi: L \rightarrow Q_2$ is not a retract, but $r_a = \bar{\phi}^{-1} \circ \phi$ (as defined in the proof of Proposition 5.7) is a retract to $a^{-1}Q_2a$. In that proof we argued that $\zeta_{L, \phi}$ acts like ζ_{P_j, π_j} for some j . It is easy to confirm that the pair (L, r_a) is conjugate to the pair (P_8, π_8) . Moreover, one can see from the work above that $\zeta_{P_8, \pi_8}(\nu) = 2(1 \otimes \bar{f}_{Q_2, \psi_2}) = 0$.

Acknowledgements

A special “thank you” goes out to each of Dave, Mark, John and Stewart. Their support encouraged the undertaking of this project and helped secure local funding for it. Also, it is clear that without their fundamental contributions to the subject of stable splittings this paper would be pointless. Finally, thanks to the referee for a careful reading of the paper and several helpful clarifications.

References

- [1] Benson, D. 1996, 'Stably splitting BG ', *Bull. Amer. Math. Soc.*, vol. 33, pp. 189–198.
- [2] Benson, D. & Feshbach, M. 1992, 'Stable splittings of classifying spaces of finite groups', *Topology*, vol. 31, pp. 157–176.
- [3] Carlsson, G. 1984, 'Equivariant Stable Homotopy and Segal's Burnside Ring Conjecture,' *Ann. of Math.*, vol. 120, pp. 189–224.
- [4] Catalano, M. 1996, 'Stable splittings of BP for some P of order 32', *Jour. Pure and App. Alg.* vol. 111, pp. 31–50.
- [5] Catalano, M. 1991, 'The stable splitting of BP for some groups P of order 32', Ph.D. Dissertation, University of Minnesota.
- [6] Hall, M. & Senior, J. 1964, *The groups of order 2^n ($n < 6$)*, MacMillan Co., New York.
- [7] Harris, J. 1993, '93d:55013,' in *Mathematical Reviews: MR1153243*, American Mathematical Society.
- [8] Lewis, G., May, P., & McClure, J. 1982, 'Classifying G -spaces and the Segal Conjecture,' *CMS Conference Proceedings 2: Current Trends in Algebraic Topology, Part 2*, American Mathematical Society, Providence, pp. 165–179.
- [9] Martino, J. & Priddy, S. 1992, 'The complete stable splitting for the classifying space of a finite group,' *Topology*, vol. 31, pp. 143–156.
- [10] Nishida, G. 1985, 'Stable Homotopy Type of Classifying Spaces of Finite Groups,' *Algebraic and Topological Theories*, Kinokuniya, Tokyo, pp. 391–404.
- [11] Priddy, S. 1990, 'On characterizing summands in the classifying space of a group, II,' in *Lecture Notes in Mathematics 1418: Homotopy theory and related topics (Kinosaki)*, Springer, Berlin.

Jill Dietz dietz@stolaf.edu

Department of Mathematics, Statistics, and Computer Science
 St. Olaf College
 1520 St. Olaf Ave.
 Northfield, MN 55057
 USA

This article is available at <http://intlpress.com/HHA/v8/n2/a4>