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CATEGORICAL HOMOTOPY THEORY

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(*communicated by Gunnar Carlsson*)

Abstract

This paper is an exposition of the ideas and methods of Cisinksi, in the context of $\mathcal{A}$-presheaves on a small Grothendieck site, where $\mathcal{A}$ is an arbitrary test category in the sense of Grothendieck. The homotopy theory for the category of simplicial presheaves and each of its localizations can be modelled by $\mathcal{A}$-presheaves in the sense that there is a corresponding model structure for $\mathcal{A}$-presheaves with an equivalent homotopy category. The theory specializes, for example, to the homotopy theories of cubical sets and cubical presheaves, and gives a cubical model for motivic homotopy theory. The applications of Cisinski's ideas are explained in some detail for cubical sets.

1. Introduction

Traditionally, categorical homotopy theory is a small collection of simple ideas and definitions, combined with a rather subtle skill set.

In broad outline, one associates to each small category C a simplicial set BC, variously called its nerve or classifying space, whose n-simplices are strings of composable arrows of length n in C. This is a functorial construction: given a functor $f\colon C \to D$, applying f to strings of arrows of length n in C produces a corresponding string in D, and one obtains an induced simplicial set map $f_*\colon BC \to BD$.

The classifying space functor $C \mapsto BC$ preserves products, and it is almost a tautology that if $\mathbf{n} = \{0, \dots, n\}$ is a finite ordinal number, viewed as a poset and hence as a small category, then $B\mathbf{n}$ is the standard n-simplex Δ^n. It follows that any natural transformation $C \times \mathbf{1} \to D$ of functors $f, g\colon C \to D$ induces a simplicial homotopy $BC \times \Delta^1 \to BD$ between the induced simplicial set maps $f_*, g_*\colon BC \to BD$. Thus, if C and D are equivalent categories or if a functor $C \to D$ has an adjoint, then the associated classifying spaces are homotopy equivalent. A further consequence is that the classifying space BC is contractible if a category C has either an initial or terminal object.

The subtlety of the theory lies in the analysis of the homotopy fibres of the map $f_*\colon BC \to BD$ which is induced by a functor $f\colon C \to D$. Every object $d \in D$ has an

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associated slice (or, in older language, "comma") category f/d whose objects consist of all morphisms $f(c) \to d$ in D; the morphisms of this category are commutative diagrams

$$\begin{tikzcd} f(c) \arrow[rr, "f(\alpha)"] \arrow[dr, "\tau"'] & & F(c') \arrow[dl, "\tau'"] \\ & d & \end{tikzcd}$$

where $\alpha\colon c \to c'$ is a morphism of C. There is an obvious forgetful functor $f/c \to C$ which takes the diagram above to the morphism α in C, and any morphism $\beta\colon d \to d'$ of D induces a functor $\beta_*\colon f/d \to f/d'$ by composition with β.

It is a basic observation of Quillen that the forgetful functors $f/d \to C$ assemble to define an isomorphism

$$\underset{\substack{\longrightarrow \\ d \in D}}{\operatorname{holim}}\, B(f/d) \to BC$$

in the homotopy category. Quillen's Theorem B asserts that if all induced maps $\beta_*\colon B(f/d) \to B(f/d')$ are weak equivalences, then all diagrams of simplicial set maps

$$\begin{tikzcd} B(f/d) \arrow[r] \arrow[d] & BC \arrow[d, "f_*"] \\ B(D/d) \arrow[r] & BD \end{tikzcd}$$

are homotopy cartesian. It follows, in this case, that $B(f/d)$ is weakly equivalent to the homotopy fibre of $f_*\colon BC \to BD$ over the vertex corresponding to the object d.

Quillen's Theorem A says that if all of the simplicial sets $B(f/d)$ are weakly equivalent to a point, then the map $f_*\colon BC \to BD$ is a weak equivalence. This result is a consequence of Theorem B, but it is more effectively proved with a comparison of homotopy colimits — part of the appeal of the result lies in the simplicity of that proof.

All of this has been known since the early 1970s, when Quillen [**17**] introduced these concepts and results as a foundation for his description of higher algebraic K-theory. This set of techniques is still fundamental for algebraic K-theory, and Theorem B is now one of the most important theorems in the foundations of homotopy theory, although recognizing when it can be applied can be something of a black art.

The homotopy theory of simplicial sheaves and presheaves is a direct extension of the homotopy theory of simplicial sets. The main techniques and results of the theory are geometric in the sense that they come from ordinary homotopy theory, but they are expressed in the categorical context of sheaves and presheaves on a Grothendieck site and derive much of their power in applications from holding at that level of generality. The development of these theories and their applications was initiated in the 1980s [**7**], [**12**], [**19**] and continues to the present [**8**], [**16**]. The homotopy theory of stacks [**9**], [**13**] is a vital and important subindustry of this work.

Thomason's work on the model structure for the category of small categories [**18**] is also part of a history of the subject, but its impact has so far been rather muted. It is strongly related to but not necessary for the ideas exposed in this paper.

The thesis of Denis-Charles Cisinski [**3**] represents the next leap forward for the subject. Cisinski's thesis is primarily concerned with the proof of some conjectures of Grothendieck [**6**], [**15**] concerning diagram categories that model homotopy theory, but the techniques that he has developed are arguably more important than the conjectures themselves.

The theory begins with Grothendieck's concept of a test category $\mathcal{A}$ and the corresponding category of $\mathcal{A}$-sets, which consists of contravariant set-valued functors on $\mathcal{A}$, or functors $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. In general, if $\mathcal{A}$ is a test category, then the corresponding category of $\mathcal{A}$-sets is a model for the standard homotopy category.

The standard examples arise in the following ways:

1) The category of ordinal numbers $\mathbf{\Delta}$ is a standard example of a test category (Example 6 below): the corresponding category of $\mathbf{\Delta}$-sets is the category of simplicial sets.

2) The category $\square$ of abstract hypercubes, here called the box category, is a test category (Lemma 3.11): the corresponding category of $\square$-sets is the category of cubical sets.

3) If $\mathcal{A}$ is a test category and $\mathcal{C}$ is a small category such that the classifying space $B\mathcal{C}$ is contractible, then the product $\mathcal{A} \times \mathcal{C}$ is a test category (Corollary 9). It follows that the class of test categories is closed under finite products, so that bisimplicial sets, multi-simplicial sets, bicubical sets, simplicial cubical sets and so on, all give models for the homotopy category.

So, when is a small category $\mathcal{A}$ a test category? Each object a of the category $\mathcal{A}$ determines a representable functor $\Delta^a = \hom(a, \ )$, and there is a cell category $i_{\mathcal{A}}X$ for each $\mathcal{A}$-set X: the objects of $i_{\mathcal{A}}X$ are the $\mathcal{A}$-set morphisms $\Delta^a \to X$ (or elements of $X(a)$, $a \in \mathcal{A}$) and the morphisms of the cell category are the diagrams

$$\begin{tikzcd} \Delta^a \arrow[rr] \arrow[dr] & & \Delta^b \arrow[dl] \\ & X & \end{tikzcd}$$

of $\mathcal{A}$-set morphisms, which can be interpreted as incidence relations in the $\mathcal{A}$-set X. If Y is a simplicial set, then the corresponding cell category $i_{\mathbf{\Delta}}Y$ is the usual simplex category of Y, which has often been denoted in the literature (see, for example, [**5**]) by $\mathbf{\Delta} \downarrow Y$.

The functor $X \mapsto i_{\mathcal{A}}X$ has a right adjoint $C \mapsto i_{\mathcal{A}}^*C$, and we say that the small category $\mathcal{A}$ is a test category if the space $B\mathcal{A}$ is contractible, and the canonical functor $\epsilon\colon i_{\mathcal{A}}i_{\mathcal{A}}^*C \to C$ is aspherical in the sense that all spaces $B(\epsilon/c)$ are contractible for all small categories C.

We can now display another example of a class of test categories which is important in applications:

4) If $\mathcal{A}$ is a test category and X is an $\mathcal{A}$-set such that the nerve $B(i_{\mathcal{A}}X)$ is a contractible simplicial set, then the cell category $i_{\mathcal{A}}X$ is a test category

(Lemma 2.15). The category of $i_{\mathcal{A}}X$-sets is, in general, equivalent to the category of $\mathcal{A}$-sets $Y \to X$ fibred over X.

We can also be more precise about the homotopy theory: if $\mathcal{A}$ is a test category, then there is a closed model structure on the category of $\mathcal{A}$-sets with cofibrations defined to be inclusions of diagrams and for which the weak equivalences are those $\mathcal{A}$-set maps $f\colon X \to Y$ such that the induced simplicial set map $f_*\colon Bi_{\mathcal{A}}X \to Bi_{\mathcal{A}}Y$ is a weak equivalence. Then it is relatively easy to show that the functor $X \mapsto Bi_{\mathcal{A}}X$ induces an equivalence

$$\mathrm{Ho}(\mathcal{A}\text{-}\mathbf{Set}) \simeq \mathrm{Ho}(\mathbf{S})$$

between the homotopy category of $\mathcal{A}$-sets and the homotopy category of simplicial sets. One can go further, and formally invert a set S of cofibrations in the model structure of $\mathcal{A}$-sets to produce a Bousfield localization of the homotopy category of $\mathcal{A}$-sets in an essentially standard way. These model structures are given by Cisinski in his thesis [**3**].

Grothendieck introduced the notion of test category, and he knew that $\mathcal{A}$-sets would model the ordinary homotopy category for all test categories $\mathcal{A}$ — indeed, the equivalence of homotopy categories is just a formal consequence of the definition of test category. Grothendieck also introduced the study of good classes of functors between small categories, which could potentially serve as classes of weak equivalences for homotopy theories. He called such classes “fundamental localisers”, and the terminology persists in [**3**].

These classes are called “weak equivalence classes” in this paper. A weak equivalence class is a class $\mathcal{W}$ of functors between small categories which satisfy the conditions that one would expect: informally speaking, the class satisfies the analog of the closed model axiom **CM2** (the two out of three axiom), contains all strong deformation retractions, and contains the functor $C \to *$ if C has a terminal object. The “total space” of a functor $f\colon C \to D$ is a formal homotopy colimit of the slice categories f/c in such a theory.

The standard features of categorical homotopy theory imply that the class $\mathcal{W}_\infty$ of all functors $C \to D$ such that the induced map $BC \to BD$ is a weak equivalence of simplicial sets satisfies the requirements for a weak equivalence class of functors.

Grothendieck made two conjectures about these objects:

Conjecture A. *Suppose that $\mathcal{W}$ is a weak equivalence class and that $f\colon C \to D$ is a functor such that $f_*\colon BC \to BD$ is a weak equivalence of simplicial sets. Then f is a member of $\mathcal{W}$.*

In other words, $\mathcal{W}_\infty$ is the smallest weak equivalence class.

Conjecture B. *Suppose that $\mathcal{W}$ is a weak equivalence class and that $\mathcal{A}$ is a (local) test category. Then the class of all maps $f\colon X \to Y$ of $\mathcal{A}$-sets such that the functor $i_{\mathcal{A}}X \to i_{\mathcal{A}}Y$ is a member of $\mathcal{W}$ is the class of weak equivalences for a model structure on the category of $\mathcal{A}$-sets for which the cofibrations are the monomorphisms.*

In [**3**], Cisinski proves the first conjecture in its entirety and the second conjecture in the case where $\mathcal{W}$ is generated over $\mathcal{W}_\infty$ by a set of functors — this is the “accessible” case (Cisinski has told me that Conjecture B is, in fact, false in general).

Conjecture A, at least so far, appears to be much more important for applications than Conjecture B.

This paper was written to express this collection of ideas and their proofs in something like standard homotopy theoretic language and notation, and to begin to describe their applications.

I was initially attracted to Cisinksi's thesis as a result of my own work on the homotopy theory of cubical sets — see [**10**]. This paper displays a model structure on the category of cubical sets whose associated homotopy category was equivalent to that of simplicial sets. The cofibrations are the monomorphisms and the weak equivalences are those maps which induce weak equivalences of topological realizations. The verification of this model structure is achieved with some bounded cofibration tricks from localization theory, and the equivalence of homotopy categories depends on a cubical set excision theorem which is proved with a somewhat involved subdivision argument.

Cisinski displays the same model structure on cubical sets as an example of his theory, and then the equivalence of homotopy categories arises from formal nonsense, since the box category $\square$ is a test category. He also proves much more, namely, that the model structure on the category of cubical sets is proper and that the fibrations are the expected analogs of Kan fibrations.

The techniques of [**10**] cannot begin to reach these last results, and their proofs involve some of the most delicate aspects of Cisinski's work. These include an internal description of homotopy colimits in a set-based cofibrantly generated model structure and a general notion of regularity, which amounts the assertion that an $\mathcal{A}$-set X is a homotopy colimit of its cells. Regularity holds in contexts, like cubical sets, where an $\mathcal{A}$-set can be constructed inductively by attaching cells. The subtlety of the theory for cubical sets is this: properness and the identification of fibrations are proved here by displaying three ostensibly different model structures for the category of cubical sets, which are then shown to be identical as a consequence of Grothendieck's Conjecture A and regularity.

The main results for cubical sets are proved in the final section of this paper: properness is proved in Theorem 8.2, and Theorem 8.6 gives the good classification of cubical set fibrations. After the fact, cubical set excision (Theorem 8.9) turns out to be a direct consequence of the formal techniques displayed here, along with the excision theorem for simplicial sets [**11**].

Much of the rest of the paper is an exposition of the basic theory. I have chosen to emphasize presheaves of $\mathcal{A}$-sets (here called $\mathcal{A}$-presheaves) on an arbitrary small Grothendieck site $\mathcal{C}$ because I believe that this is an important context for potential applications. That said, $\mathcal{A}$-presheaves are $(\mathcal{A} \times \mathcal{C})$-sets, and much of the elementary theory for $\mathcal{A}$-presheaves arises from formal manipulations of the homotopy theory of $\mathcal{A}$-sets.

That homotopy theory arises in part from a "Swiss army knife" result (Theorem 4.17 in Section 4) which establishes a model structure for the category of $\mathcal{A}$-sets in which some set S of monomorphisms become weak equivalences, and which depends on a suitable theory of intervals. An interval theory is expressed here as a monoidal action

$$\otimes\colon (\mathcal{A}\text{-}\mathbf{Set}) \times \Box \to \mathcal{A}\text{-}\mathbf{Set}$$

of the box category on the category of $\mathcal{A}$-sets, satisfying a list of expected properties, and the purpose of which is to define some notion of naive homotopy of morphisms. The main examples of such theories arise either from taking iterated products $X \times I^{\times n}$ with objects I having two distinct global sections, or from Kan's tensor product operation [**14**] for cubical sets. The affine line $\mathbb{A}^1$ and the global sections $0, 1\colon * \to \mathbb{A}^1$ generate an interval theory in the motivic context.

The construction of the resulting $(\otimes, S)$-model structure on the category of $\mathcal{A}$-sets follows the general outlines that one finds in localization theory, except that one is not localizing another model structure to construct it. It is general nonsense that an injective replacement of a map or object can always be constructed, and then one defines a weak equiivalence to be a map $f\colon X \to Y$ which induces an isomorphism $\pi(Y, Z) \cong \pi(X, Z)$ of naive homotopy classes (defined by intervals) for all injective objects Z. It is one of the innovations of Cisinksi's thesis that naive homotopy equivalences alone can be used to prove a bounded cofibration property (Lemma 4.9), and then Theorem 4.17 comes out in the usual way, modulo some fussing with pushouts of trivial cofibrations (Lemma 4.12). This model structure is proper if the set S of cofibrations is decently behaved and the interval theory is defined by an actual interval I (Theorem 4.18). One of the interesting aspects of Theorems 4.17 and 4.18 is that the set S can be empty, so that there is always a "primitive" model structure defined by an interval theory, and this model structure is proper.

Theorem 6.2 of Section 6 says that if $\mathcal{A}$ is a test category, then any localized model structure on the category of simplicial presheaves induces a model structure on the category of $\mathcal{A}$-presheaves in such a way that the associated homotopy categories are equivalent. This result holds over any small Grothendieck site, and says that all simplicial presheaf homotopy theories (including the motivic homotopy theories) have $\mathcal{A}$-models.

Theorem 6.2 specializes to the existence of a model structure on the category of $\mathcal{A}$-sets with corresponding homotopy category equivalent to any localized homotopy theory of simplicial sets, if $\mathcal{A}$ is a test category. In particular, the proof of Grothendieck's Conjecture B (in the accessible case) is therefore reduced to the statement for simplicial sets. Theorem 6.2 does not follow from Theorem 4.17 — it is a subsidiary structure, but the model structures that these results generate coincide in a wide variety of interesting cases, including cubical sets.

One of the morals of this stream of ideas is that cubical sets are everywhere. The definition and formal properties of the box category $\Box$ and the category $\Box$-**Set** of cubical sets are summarized in Section 3 of this paper. The basic properties of test categories are treated in Section 2. The proof of the assertion that the box category is a test category turns out to be a bit subtle. In fact, the category of cubical sets seems to be the delicate case throughout the theory. It can be a rather disconcerting fact that products behave very badly in the homotopy theory of cubical sets: in particular that the product $\Box^1 \times \Box^1$ of a pair of copies of the standard interval in cubical sets has the homotopy type of the wedge of circles $S^2 \vee S^1$ (Remark 3.5).

This forces one to be careful with interval theories everywhere, and prompts the discussion of aspherical $\mathcal{A}$-sets.

Section 5 contains a general discussion of homotopy colimits, internally defined nerves, and the relation with the Grothendieck construction in an $(\otimes, S)$-model structure on a category of $\mathcal{A}$-sets. Homotopy colimits are defined internally, by taking colimits of projective cofibrant resolutions. From this point of view, the internal nerve B_hC of a small category C is the homotopy colimit for the diagram which assigns a point to each object of C. This generalizes the observation that the ordinary nerve BC is the homotopy colimit of a diagram of points in the category of simplicial sets. The standard properties of the ordinary nerve BC also hold for the internal nerve B_hC. In particular, there is a weak equivalence

$$\underrightarrow{\mathrm{holim}}_{d \in D} B_h(f/d) \to B_hC$$

for any functor $f\colon C \to D$, which, in turn, means that the internal nerve of the Grothendieck construction models a homotopy colimit in this sense.

Section 7 contains an exposition of the basic aspects of the theory of weak equivalence classes of functors, along with proofs of Conjecture A (Corollary 7.7) and the case of Conjecture B corresponding to test categories and accessible weak equivalence classes (Theorem 7.8). As one might expect, Grothendieck's Conjecture B can be proved with a localization argument in the presence of Conjecture A, but that is not the way that it is done here. I prefer instead to follow Cisinski's lead in using an omnibus result (i.e. Theorem 6.2) which subsumes all localization arguments. Theorem 4.17 has a similar flavour.

There is yet another innovation of Cisinski which is displayed in Section 7: the cell category functor $X \mapsto i_{\mathcal{A}}X$ preserves homotopy cocartesian diagrams in striking generality (Corollary 7.5). This was certainly not well known, even for simplex categories, and it is a central feature of this theory.

Contents

2. Homotopy theory of categories

Suppose that X is a simplicial set. The simplex category

$$i_{\mathbf{\Delta}} X = \mathbf{\Delta} \downarrow X$$

has objects consisting of all simplices $\Delta^n \to X$ and morphisms consisting of commutative triangles of simplicial set maps

$$\begin{tikzcd} \Delta^n \arrow[rr] \arrow[dr] & & \Delta^m \arrow[dl] \\ & X & \end{tikzcd}$$

Write **cat** for the category of small categories, and consider the functors

$$\mathbf{S} \xrightarrow{i_{\mathbf{\Delta}}} \mathbf{cat} \xrightarrow{B} \mathbf{S}$$

Say that a functor $f\colon C \to D$ between small categories is a *weak equivalence* if the induced map $f_*\colon BC \to BD$ is a weak equivalence of simplicial sets.

For each simplicial set X there is a functor $Q_X\colon i_{\mathbf{\Delta}} X \to \mathbf{S}$ which takes an object $\sigma\colon \Delta^n \to X$ to the simplicial set Δ^n. Then it is well known that the maps $\sigma\colon \Delta^n \to X$ define a natural weak equivalence $f_X\colon \underrightarrow{\mathrm{holim}}\ Q_X \to X$, and that the canonical projection $\pi_X\colon \underrightarrow{\mathrm{holim}}\ Q_X \to B(i_{\mathbf{\Delta}} X)$ is also a natural weak equivalence.

It follows that the nerve functor B and the simplex category functor induce an equivalence of categories

$$\mathrm{Ho}(\mathbf{cat}) \simeq \mathrm{Ho}(\mathbf{S})$$

after formally inverting the weak equivalences in **cat** and **S**, respectively.

Suppose that $\mathcal{A}$ is a small category, and write $\mathcal{A}$-**Set** (written $\hat{\mathcal{A}}$ in [**3**]) for the category of set-valued contravariant functors defined on $\mathcal{A}$; these functors will be called $\mathcal{A}$-*sets*. Write $\Delta^a = \hom(\ ,a)$ for the representable contravariant functor associated to an object $a \in \mathcal{A}$. The $\mathcal{A}$-set Δ^a will often be called the *standard a-cell*. Similarly, if X is an $\mathcal{A}$-set, the elements of set $X(a)$ will be called the *a-cells* of X. The a-cells of X are classified by A-set maps $\Delta^a \to X$, by the usual Yoneda Lemma argument.

Suppose that X is an $\mathcal{A}$-set, and write $i_{\mathcal{A}} X$ for the category whose objects are the natural transformations $\Delta^a \to X$ and whose morphisms are the commutative triangles

$$\begin{tikzcd} \Delta^a \arrow[rr] \arrow[dr] & & \Delta^b \arrow[dl] \\ & X & \end{tikzcd}$$

The assignment $X \mapsto i_{\mathcal{A}} X$ is functorial in X, and defines a functor $i_{\mathcal{A}}\colon \mathcal{A}\text{-}\mathbf{Set} \to \mathbf{cat}$. The category $i_{\mathcal{A}} X$ will often be called the *cell category* of X.

Say that a map $f\colon X \to Y$ is a *weak equivalence of $\mathcal{A}$-sets* if the induced map $f_*\colon B(i_{\mathcal{A}} X) \to B(i_{\mathcal{A}} Y)$ is a weak equivalence of simplicial sets, or equivalently if the induced functor $f_*\colon i_{\mathcal{A}} X \to i_{\mathcal{A}} Y$ is a weak equivalence in **cat**.

According to these definitions, the functor $i_{\mathcal{A}}$ induces a "functor"

$$i_{\mathcal{A}*}\colon \mathrm{Ho}(\mathcal{A}\text{-}\mathbf{Set}) \to \mathrm{Ho}(\mathbf{cat}).$$

A basic question of Grothendieck [**6**], [**15**] is the following: when is $i_{\mathcal{A}*}$ an equivalence of categories?

The functor $i_{\mathcal{A}}\colon \mathcal{A}\text{-}\mathbf{Set} \to \mathbf{cat}$ has a right adjoint $i_{\mathcal{A}}^*\colon \mathbf{cat} \to \mathcal{A}\text{-}\mathbf{Set}$ which is defined by

$$i_{\mathcal{A}}^*(C)(a) = \hom(\mathcal{A}/a, C).$$

This follows from the fact that every $\mathcal{A}$-set (being a contravariant functor) is a colimit of representables.

More explicitly, the natural map

$$\hom(i_{\mathcal{A}}X, C) \to \hom(X, i_{\mathcal{A}}^*(C))$$

is easy to describe: if $\sigma\colon \Delta^a \to X$ is an element of $X(a)$ and $f\colon i_{\mathcal{A}}X \to C$ is a functor, then the composite functor

$$\mathcal{A}/a \cong i_{\mathcal{A}}\Delta^a \xrightarrow{\sigma_*} i_{\mathcal{A}}X \xrightarrow{f} C$$

is an element $f_*(\sigma) \in i_{\mathcal{A}}^*C(a)$. An $\mathcal{A}$-set morphism $g\colon X \to i_{\mathcal{A}}^*C$ is determined by functors $g(\sigma)\colon \mathcal{A}/a \to C$, one for each element $\sigma\colon \Delta^a \to X$, which make the obvious diagrams of functors commute. Given such a g, define a functor $g_*\colon i_{\mathcal{A}}X \to C$ by associating to an object $\sigma\colon \Delta^a \to X$ the object $g(\sigma)(1_a) \in C$. One can show that these two natural maps are inverse to each other, and there is a corresponding bijection

$$\hom(i_{\mathcal{A}}X, C) \cong \hom(X, i_{\mathcal{A}}^*C),$$

so that that $i_{\mathcal{A}}^*$ is right adjoint to $i_{\mathcal{A}}$.

Note that the category $i_{\mathcal{A}}i_{\mathcal{A}}^*C$ has objects all functors $f\colon \mathcal{A}/a \to C$ and has morphisms given by all commutative diagrams

$$\begin{tikzcd} \mathcal{A}/a \arrow[rr, "\theta_*"] \arrow[dr, "f"'] & & \mathcal{A}/b \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

where $\theta\colon a \to b$ is a morphism of $\mathcal{A}$. The adjunction map

$$\epsilon\colon i_{\mathcal{A}}i_{\mathcal{A}}^*C \to C$$

is the functor which associates to each functor $f\colon \mathcal{A}/a \to C$ the object $f(1_a) \in C$.

Lemma 2.1. *There is an isomorphism of categories*

$$\epsilon/c \cong i_{\mathcal{A}}i_{\mathcal{A}}^*(C/c)$$

for all categories C.

Proof. An object of the category $i_{\mathcal{A}}i^*_{\mathcal{A}}(C/c)$ is a functor $f\colon \mathcal{A}/a \to C/c$, and a morphism of this category is a commutative diagram

$$\begin{tikzcd} \mathcal{A}/a \arrow[rr, "\theta_*"] \arrow[dr, "f"'] & & \mathcal{A}/b \arrow[dl, "g"] \\ & C/c & \end{tikzcd}$$

as above. A functor $f\colon \mathcal{A}/a \to C/c$ can be identified uniquely with a pair (f', f'') consisting of a functor $f'\colon \mathcal{A}/a \to C$ and a morphism $f''\colon f'(1_a) \to c$. This identification induces the required isomorphism of categories, since an object of ϵ/c consists of a functor $f\colon \mathcal{A}/a \to C$ and a morphism $f(1_a) \to c$. $\square$

The essential idea is to come up with conditions on $\mathcal{A}$ so that the adjunction maps $\epsilon\colon i_{\mathcal{A}}i^*_{\mathcal{A}}(C) \to C$ are weak equivalences for all categories C. Observe that if all counit maps ϵ are weak equivalences, then all unit maps $\eta\colon X \to i^*_{\mathcal{A}}i_{\mathcal{A}}X$ are weak equivalences of $\mathcal{A}$-sets, by a triangle identity. It also follows easily that a functor $f\colon C \to D$ is a weak equivalence if and only if $f_*\colon i^*_{\mathcal{A}}C \to i^*_{\mathcal{A}}D$ is a weak equivalence of $\mathcal{A}$-sets in this case.

A functor $f\colon C \to D$ is said to be *aspherical* if the simplicial set $B(f/d)$ is weakly equivalent to a point for all $d \in D$. If f is aspherical, then it is a weak equivalence by Quillen's Theorem A. Say that a category A is aspherical if the canonical map $\pi\colon A \to *$ is aspherical. In view of the fact that $\pi/* \cong A$, A is aspherical if and only if A is weakly equivalent to a point.

Say that a map $f\colon X \to Y$ of $\mathcal{A}$-sets is *aspherical* if the induced functor $f_*\colon i_{\mathcal{A}}X \to i_{\mathcal{A}}Y$ is aspherical. In general, there is an isomorphism

$$f_*/(\Delta^a \to Y) \cong i_A(\Delta^a \times_G F) \tag{1}$$

so that $f\colon X \to Y$ is aspherical if and only if all pullbacks $\Delta^a \times_G F$ are weakly equivalent to a point. Every aspherical map of $\mathcal{A}$-sets is a weak equivalence, by Quillen's Theorem A. The class of aspherical maps of $\mathcal{A}$-sets is closed under pullback.

¿From this point of view, an $\mathcal{A}$-set F is aspherical if the map $F \to *$ is aspherical. This means precisely that the induced functor $i_{\mathcal{A}}F \to \mathcal{A}$ is aspherical. The isomorphism

$$i_{\mathcal{A}}(F)/a \cong i_{\mathcal{A}}(F \times \Delta^a) \tag{2}$$

is of central use in analyzing objects of this sort.

Say that $\mathcal{A}$ is a *weak test category* if the adjunction map $\epsilon\colon i_{\mathcal{A}}i^*_{\mathcal{A}}(C) \to C$ is a weak equivalence for all small categories C.

It follows from Lemma 2.1 and Quillen's Theorem A that $\mathcal{A}$ is a weak test category if and only if the functor $D \mapsto i^*_{\mathcal{A}}D$ takes categories having a terminal object to $\mathcal{A}$-sets which are weakly equivalent to a point.

Suppose that the functor $C \mapsto i^*_{\mathcal{A}}(C)$ takes aspherical categories to $\mathcal{A}$-sets which are weakly equivalent to a point. Then categories having terminal objects are examples of aspherical categories, so that $\mathcal{A}$ is a weak test category. Suppose that $\mathcal{A}$ is a weak test category and that C is an aspherical category. Then the adjunction map

$\epsilon \colon i_{\mathcal{A}} i_{\mathcal{A}}^*(C) \to C$ is a weak equivalence, so that the $\mathcal{A}$-set $i_{\mathcal{A}}^*(C)$ is weakly equivalent to a point.

We have proved the following:

Lemma 2.2. *The following statements are equivalent:*

1) *$\mathcal{A}$ is a weak test category, i.e. all adjunction maps $\epsilon \colon i_{\mathcal{A}} i_{\mathcal{A}}^*(C) \to C$ are weak equivalences.*
2) *if D is a category with terminal object, then the $\mathcal{A}$-set $i_{\mathcal{A}}^*(D)$ is weakly equivalent to a point*
3) *if C is aspherical, then the $\mathcal{A}$-set $i_{\mathcal{A}}^*(C)$ is weakly equivalent to a point.*

Say that $\mathcal{A}$ is *local test category* if all categories $\mathcal{A}/a$ are weak test categories.

Lemma 2.3. *The following are equivalent:*

1) *$\mathcal{A}$ is a local test category;*
2) *if D is a category with a terminal object, then the $\mathcal{A}$-set $i_{\mathcal{A}}^*(D)$ is aspherical, or equivalently the canonical functor $\pi \colon i_{\mathcal{A}} i_{\mathcal{A}}^*(D) \to \mathcal{A}$ is aspherical;*
3) *if C is an aspherical category, then the $\mathcal{A}$-set $i_{\mathcal{A}}^*(C)$ is aspherical, or equivalently the canonical functor $\pi \colon i_{\mathcal{A}} i_{\mathcal{A}}^*(C) \to \mathcal{A}$ is aspherical.*

Proof. The $\mathcal{A}$-set $i_{\mathcal{A}}^*(C)$ is aspherical if and only if all categories

$$i_{\mathcal{A}} i_{\mathcal{A}}^*(C)/a \cong i_{\mathcal{A}/a} i_{\mathcal{A}/a}^*(C)$$

are weakly equivalent to a point. Now use Lemma 2.2. □

Say that $\mathcal{A}$ is a *test category* if it is both a local test category and a weak test category. This, however, is not the right definition to use in practice, in view of the following:

Lemma 2.4. *A category $\mathcal{A}$ is a test category if and only if it is a local test category and is aspherical.*

Proof. Suppose that $\mathcal{A}$ is a local test category and that $\mathcal{A}$ is aspherical. Suppose that D is a category with terminal object. We want to show that the $\mathcal{A}$-set $i_{\mathcal{A}}^*(D)$ is weakly equivalent to a point. But the functor $i_{\mathcal{A}} i_{\mathcal{A}}^*(D) \to \mathcal{A}$ is aspherical by Lemma 2.3 and $\mathcal{A}$ is aspherical, so that the $\mathcal{A}$-set $i_{\mathcal{A}}^*(D)$ is weakly equivalent to a point. It follows that $\mathcal{A}$ is a weak test category as well as a local test category.

Suppose that $\mathcal{A}$ is a test category. Then the functor $i_{\mathcal{A}}^*$ has a left adjoint and therefore preserves terminal objects. The terminal object of the category of $\mathcal{A}$-sets is the one point $\mathcal{A}$-set $*$, and there is an isomorphism $i_{\mathcal{A}}(*) \cong \mathcal{A}$. Since $\mathcal{A}$ is a weak test category, the adjunction map $\epsilon \colon i_{\mathcal{A}} i_{\mathcal{A}}^*(*) \to *$ is a weak equivalence. It follows that $\mathcal{A}$ is aspherical. □

Remark 2.5. One can show by using the argument in the proof of Lemma 2.4 that if $\mathcal{A}$ is a weak test category, then $\mathcal{A}$ is aspherical.

Example 2.6. Suppose that $\mathcal{A}$ is the category $\mathbf{\Delta}$ of finite ordinal numbers, so that $\mathbf{\Delta}$-**Set** is the category $\mathbf{S}$ of simplicial sets.

If C is a category $i_{\mathbf{\Delta}}^*(C)$ is the simplicial set with n-simplices specified by

$$i_{\mathbf{\Delta}}^*(C)_n = \hom(\mathbf{\Delta}/\mathbf{n}, C)$$

If D is a category with terminal object t, then there is a contracting homotopy $h: D \times \mathbf{1} \to D$. The functor $i_{\mathbf{\Delta}}^*$ preserves products, so that h induces the composite

$$i_{\mathbf{\Delta}}^*(D) \times i_{\mathbf{\Delta}}^*(\mathbf{1}) \cong i_{\mathbf{\Delta}}^*(D \times \mathbf{1}) \xrightarrow{h_*} i_{\mathbf{\Delta}}^*(D).$$

There is a natural functor $\alpha: \mathbf{\Delta}/\mathbf{n} \to \mathbf{n}$. This functor is essentially a last vertex map, and is specified on objects by $\alpha(\theta: \mathbf{m} \to \mathbf{n}) = \theta(m)$. The particular example $\alpha: i_{\mathbf{\Delta}}\mathbf{1} \to \mathbf{1}$ of this functor defines a 1-simplex $\alpha: \Delta^1 \to i_{\mathbf{\Delta}}^*(\mathbf{1})$, and there is a composite

$$i_{\mathbf{\Delta}}^*(D) \times \Delta^1 \xrightarrow{1\times\alpha} i_{\mathbf{\Delta}}^*(D) \times i_{\mathbf{\Delta}}^*(\mathbf{1}) \cong i_{\mathbf{\Delta}}^*(D \times \mathbf{1}) \xrightarrow{h_*} i_{\mathbf{\Delta}}^*(D)$$

which gives a contracting homotopy for $i_{\mathbf{\Delta}}^*(D)$.

It follows that all maps $\epsilon: i_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^*(C) \to C$ are weak equivalences. We know [**5**, p. 236] that every simplicial set X is a homotopy colimit of its simplices in the sense that there is a weak equivalence

$$\underset{\Delta^n \to X}{\underrightarrow{\operatorname{holim}}}\ \Delta^n \to X,$$

and that the homotopy colimit is weakly equivalent to $B(i_{\mathbf{\Delta}}X)$. It follows that the simplicial set $i_{\mathbf{\Delta}}^*(C)$ is naturally weakly equivalent to BC, and there are natural weak equivalences

$$i_{\mathbf{\Delta}}^*(C) \simeq Bi_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^*(C) \xrightarrow{\epsilon_*} BC. \tag{3}$$

Suppose that D is a category with a terminal object. In order to show that $\mathbf{\Delta}$ is a local test category (and hence a test category), we must show that the canonical functor $\pi: i_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^*(D) \to \mathbf{\Delta}$ is aspherical. The identification (2) restricts to an isomorphism

$$i_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^*(D)/\mathbf{n} \cong i_{\mathbf{\Delta}}(i_{\mathbf{\Delta}}^*(D) \times \Delta^n).$$

But then $i_{\mathbf{\Delta}}^*(D)$ is a contractible simplicial set, so that $i_{\mathbf{\Delta}}^*(D) \times \Delta^n$ is weakly equivalent to a point, and it follows that the category $i_{\mathbf{\Delta}}(i_{\mathbf{\Delta}}^*(D) \times \Delta^n)$ is aspherical.

Lemma 2.7. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are small categories and that $f: X \to Y$ is a morphism of $(\mathcal{A} \times \mathcal{B})$-***Set***. If f induces weak equivalences of $\mathcal{B}$-sets $X(a,\ ) \to Y(a,\ )$ for all objects $a \in \mathcal{A}$, then f is a weak equivalence of $(\mathcal{A} \times \mathcal{B})$-sets.*

Proof. Consider the functors

$$i_{\mathcal{A}\times\mathcal{B}}X \xrightarrow{\pi_X} \mathcal{A} \times \mathcal{B} \xrightarrow{p} \mathcal{A},$$

where q is a projection.

An element of the category $a/p\pi_X$ can be identified with a pair

$$(a \xrightarrow{\gamma} a_1, x \in X(a_1, b_1)),$$

and a morphism $(\gamma, x) \to (\gamma', y)$ consists of a morphism

$$(a_1 \xrightarrow{\theta} a_2, b_1 \xrightarrow{\tau} b_2)$$

of $\mathcal{A} \times \mathcal{B}$ such that $\theta\gamma = \gamma'$ and $(\theta, \gamma)^*(y) = x$.

There is a functor ω_a: $i_{\mathcal{B}}X(a,\ ) \to a/p\pi_X$ which is defined by sending the object $x \in X(a, b)$ of $i_{\mathcal{B}}X(a,\ )$ to the object

$$(a \xrightarrow{1_a} a, x \in X(a, b))$$

There is a functor γ_a: $a/p\pi_X \to i_{\mathcal{B}}X(a,\ )$ which is defined by sending the element

$$(a \xrightarrow{\gamma} a_1, x \in X(a_1, b_1))$$

to the element $(\gamma, 1)^*(x) \in X(a, b_1)$. Then $\gamma_a\omega_a = 1$ and the morphisms

$$(\gamma, 1)\colon (1_a, (\gamma, 1)^*(x)) \to (a, x)$$

define a natural transformation $\omega_a\gamma_a \to 1$.

The functors ω_a and γ_a therefore define a homotopy equivalence

$$Bi_{\mathcal{B}}X(a,\ ) \simeq B(a/p\pi_X)$$

which is natural in presheaves X. The assumptions imply that the map f: $X \to Y$ induces a weak equivalence

$$B(a/p\pi_X) \xrightarrow{f_*} B(a/p\pi_Y)$$

for all objects $a \in \mathcal{A}$. It follows that f induces a weak equivalence

$$Bi_{\mathcal{A}\times\mathcal{B}}(X) \to Bi_{\mathcal{A}\times\mathcal{B}}(Y)$$

of the respective homotopy colimits over $\mathcal{A}$. $\square$

Lemma 2.8. *Suppose that $\mathcal{A}$ is a local test category and that $\mathcal{B}$ is a small category. Then $\mathcal{A} \times \mathcal{B}$ is a local test category.*

Proof. Suppose that C is a small category with terminal object t. It suffices to show that the object

$$i^*_{\mathcal{A}\times\mathcal{B}/(a,b)}(C)$$

is weakly equivalent to a point (see the proof of Lemma 2.3).

There is an isomorphism of categories

$$\mathcal{A} \times \mathcal{B}/(a, b) \cong \mathcal{A}/a \times \mathcal{B}/b.$$

Write $\mathcal{A}' = \mathcal{A}/a$ and $\mathcal{B}' = \mathcal{B}/b$. Then, in this notation, we must show that the $(\mathcal{A}' \times \mathcal{B}')$-set $i^*_{\mathcal{A}'\times\mathcal{B}'}C$ is weakly equivalent to a point when we know that the $\mathcal{A}'$-set $i^*_{\mathcal{A}'}C$ is weakly equivalent to a point.

There are identifications

$$i^*_{\mathcal{A}'\times\mathcal{B}'}C(a',b') = \hom(\mathcal{A}'/a' \times \mathcal{B}'/b', C) = \hom(\mathcal{A}'/a', \mathbf{hom}(\mathcal{B}'/b', C)),$$

where $\mathbf{hom}(\mathcal{B}'/b', C)$ is the obvious category of functors and natural transformations. This category has a terminal object, namely the functor $\mathcal{B}'/b' \to C$ which takes all objects to the terminal point. It follows that all $\mathcal{A}'$-sets

$$\hom(\mathcal{A}'/? \times \mathcal{B}'/b, C) \cong i^*_{\mathcal{A}'}\,\mathbf{hom}(\mathcal{B}'/b, C)$$

are weakly equivalent to a point. It therefore follows from Lemma 2.7 that the $(\mathcal{A}' \times \mathcal{B}')$-set morphism

$$\hom(\mathcal{A}'/a' \times \mathcal{B}'/b', C) \to *$$

is a weak equivalence. □

Corollary 2.9. *Suppose that $\mathcal{A}$ is a test category and that the small category $\mathcal{B}$ is aspherical. Then the product $\mathcal{A} \times \mathcal{B}$ is a test category.*

Other useful tools include the following:

Lemma 2.10. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are small categories, and suppose that $\mathcal{B}$ is aspherical. Let*

$$p^*\colon \mathcal{A}\text{-}\mathbf{Set} \to (\mathcal{A} \times \mathcal{B})\text{-}\mathbf{Set}$$

which is induced by composition with the projection functor $p\colon \mathcal{A} \times \mathcal{B} \to \mathcal{A}$. Then a map $f\colon X \to Y$ is a weak equivalence of $\mathcal{A}$-sets if and only if the induced map $f_*\colon p^*X \to p^*Y$ is a weak equivalence of $(\mathcal{A} \times \mathcal{B})$-sets.

Proof. There is an isomorphism $i_{(\mathcal{A}\times\mathcal{B})}p^*X \cong i_{\mathcal{A}}X \times \mathcal{B}$. □

Let $q^*\colon \mathbf{S} \to (\mathcal{A} \times \mathbf{\Delta})\text{-}\mathbf{Set}$ be the functor which is defined by composition with the projection $\mathcal{A} \times \mathbf{\Delta} \to \mathbf{\Delta}$. The functor $i\colon \mathcal{A} \to \mathbf{S}$ defined by $a \mapsto B(\mathcal{A}/a)$ induces a functor $i^*\colon \mathbf{S} \to \mathcal{A}\text{-}\mathbf{Set}$ where $i^*X(a) = \hom(B(\mathcal{A}/a), X)$. Similarly the functor $j\colon \mathcal{A} \times \mathbf{\Delta} \to \mathbf{S}$ defined by $(a, \mathbf{n}) \mapsto B(\mathcal{A}/a) \times \Delta^n$ defines a functor $j^*\colon \mathbf{S} \to (\mathcal{A} \times \mathbf{\Delta})\text{-}\mathbf{Set}$ with

$$j^*X(a, \mathbf{n}) = \hom(B(\mathcal{A}/a) \times \Delta^n, X).$$

Lemma 2.11. *Suppose that $\mathcal{A}$ is a local test category. Then with the definitions above, there are natural weak equivalences of $(\mathcal{A} \times \mathbf{\Delta})$-sets*

$$p^*i^*X \to j^*X \leftarrow q^*X$$

for all simplicial sets X.

Proof. The map $q^*X(a,*) \to j^*X(a,*)$ is the simplicial set map

$$X \to \mathbf{hom}(B(\mathcal{A}/a), X).$$

The contracting homotopy $B(\mathcal{A}/a) \times \Delta^1 \to B(\mathcal{A}/a)$ induces a homotopy equivalence $X \to \mathbf{hom}(B(\mathcal{A}/a), X)$ for all simplicial sets X. It follows that all maps $q^*X(a,*) \to j^*X(a,*)$ are weak equivalences of simplicial sets, so that the induced map $q^*X \to j^*X$ is a weak equivalence of $(\mathcal{A} \times \mathbf{\Delta})$-sets for all simplicial sets X by Lemma 2.7.

The map $p^*i^*X \to j^*X$ can be identified in simplicial degree n with the $\mathcal{A}$-set map

$$\hom(B(\mathcal{A}/a), X) \to \hom(B(\mathcal{A}/a), \mathbf{hom}(\Delta^n, X)).$$

The contracting homotopy $\Delta^n \times \Delta^1 \to \Delta^n$ induces a contracting homotopy of $\mathbf{hom}(\Delta^n, X)$ onto X, and hence induces a contracting $i^*_{\mathcal{A}}(\mathbf{1})$-homotopy of $j^*X(*, \mathbf{n})$ onto $p^*i^*X(*, \mathbf{n})$ (we need to know that $\mathcal{A}$ is a local test category, so that $i^*_{\mathcal{A}}(\mathbf{1})$ is aspherical, exactly at this point). The category $\mathcal{A}$ is a local test category, so all maps $p^*i^*X(*, \mathbf{n}) \to j^*X(*, \mathbf{n})$ are weak equivalences of $\mathcal{A}$-sets. □

Corollary 2.12. *The functor $i^*\colon \mathbf{S} \to \mathcal{A}$-**Set** preserves weak equivalences if $\mathcal{A}$ is a local test category.*

Proof. The functor q^* preserves weak equivalences by Lemma 2.7, so that p^*i^* preserves weak equivalences by Lemma 2.11. The functor p^* reflects weak equivalences by Lemma 2.10, so i^* preserves weak equivalences as claimed. □

Lemma 2.11 admits a more general formulation, which will be of some use later. Suppose that $i\colon \mathcal{A} \to \mathbf{cat}$ is an arbitrary functor. Then i induces a functor $i^*\colon \mathbf{S} \to \mathcal{A}$-**Set** which is defined by $a \mapsto \mathbf{hom}(Bi(a), X)$. Then the functor $j\colon \mathcal{A} \times \mathbf{\Delta} \to \mathbf{S}$ defined by $(a, \mathbf{n}) \mapsto Bi(a) \times \Delta^n$ induces

$$j^*\colon \mathbf{S} \to (\mathcal{A} \times \mathbf{\Delta})\text{-}\mathbf{Set},$$

with

$$j^*X(a, \mathbf{n}) = \hom(Bi(a) \times \Delta^n, X).$$

Then the proof of the following is an abstraction of the proof of Lemma 2.11.

Lemma 2.13. *Suppose that $\mathcal{A}$ is a small category. Suppose that all categories $i(a)$ have terminal objects, and that the $\mathcal{A}$-set $i^*\Delta^1$ is aspherical. Then with the definitions above, there are natural weak equivalences of $(\mathcal{A} \times \mathbf{\Delta})$-sets*

$$p^*i^*X \to j^*X \leftarrow q^*X$$

for all simplicial sets X.

Corollary 2.14. *Suppose, in addition to the assumptions of Lemma 2.13, that the category $\mathcal{A}$ is aspherical. Then the functor $i^*\colon \mathbf{S} \to \mathcal{A}$-**Set** preserves and reflects weak equivalences.*

Proof. The functors p^* preserves and reflects weak equivalences by Lemma 2.10. Now argue as in the proof of Corollary 2.12. □

Here is a source of local test categories:

Lemma 2.15.
1) *Suppose that $\mathcal{A}$ is a local test category and that X is an $\mathcal{A}$-set. The category $i_{\mathcal{A}}X$ is a local test category.*
2) *The category of $i_{\mathcal{A}}X$-sets is equivalent to the category $\mathcal{A}$-$\mathbf{Set}/X$ of $\mathcal{A}$-set morphisms $Y \to X$ over X.*

Proof. Suppose that $\sigma \colon \Delta^a \to X$ is an object of $i_{\mathcal{A}}X$. Then there is an isomorphism of categories

$$i_{\mathcal{A}}X/\sigma \cong \mathcal{A}/a$$

by the Yoneda Lemma. All categories $\mathcal{A}/a$ are weak test categories since $\mathcal{A}$ is a local test category. It follows that $i_{\mathcal{A}}X$ is a local test category.

Suppose that $Y \colon (i_{\mathcal{A}}X)^{op} \to \mathbf{Set}$ is an $i_{\mathcal{A}}X$-set. There is an $\mathcal{A}$-set $\tilde{Y}$ with

$$\tilde{Y}(a) = \bigsqcup_{\sigma \in X(a)} Y(\sigma),$$

and there is plainly an induced $\mathcal{A}$-set morphism $\pi_Y \colon \tilde{Y} \to X$. The assignment $Y \mapsto \pi_Y$ is functorial in Y. Conversely, if $p \colon Z \to X$ is a morphism of $\mathcal{A}$-sets, then the assignment $\sigma \mapsto p^{-1}(\sigma) \subset Z(a)$ for $\sigma \colon \Delta^a \to X$ defines a presheaf p^{-1} on $i_{\mathcal{A}}X$. These two functors are inverse to each other up to isomorphism, so that there is an equivalence of categories

$$i_{\mathcal{A}}X - \mathbf{Set} \simeq \mathcal{A}\text{-}\mathbf{Set}/X. \qquad \square$$

The following is a relative version of Corollary 2.12:

Corollary 2.16. *Suppose that $\mathcal{A}$ is a local test category and that Y is an $\mathcal{A}$-set. Then the functor $\mathbf{S} \to \mathcal{A}$-$\mathbf{Set}$ defined by $X \mapsto Y \times i^*X$ preserves weak equivalences.*

Proof. Write $i^*_{[\mathcal{A}]}X = i^*X$, where $i^*X(a) = \hom(B(\mathcal{A}/a), X)$ as in Lemma 2.11. Then there is an isomorphism

$$i^*_{[i_{\mathcal{A}}Y]}X \cong Y \times i^*_{[\mathcal{A}]}X$$

in the category of $\mathcal{A}$-sets over Y. The functor $X \mapsto i^*_{[i_{\mathcal{A}}Y]}X$ preserves weak equivalences by Corollary 2.12, since $i_{\mathcal{A}}Y$ is a local test category by Lemma 2.15.

The object $\tilde{\Delta}^\sigma \to Y$ over Y which corresponds to the $i_{\mathcal{A}}Y$-set represented by the cell $\sigma \colon \Delta^a \to Y$ is canonically isomorphic to $\sigma \colon \Delta^a \to Y$. It follows that there is a natural isomorphism

$$i_{i_{\mathcal{A}}Y}(Z \to Y) \cong i_{\mathcal{A}}Z$$

for all objects $Z \to Y$ over Y. In particular, the forgetful functor from $\mathcal{A}$-sets over Y to $\mathcal{A}$-sets defined by sending the object $Z \to Y$ to Z preserves and reflects weak equivalences. $\square$

3. Cubical sets: basic properties

Simplicial sets are contravariant set-valued functors defined on the category of $\mathbf{\Delta}$ of finite sets and order preserving maps, and as such are artifacts of the combinatorics of finite sets. Cubical sets are contravariant set-valued functors which are defined on the "box category" $\square$, and as such depend on the combinatorics of the power sets of finite ordered sets.

Write $\underline{\mathrm{n}} = \{1, 2, \dots, n\}$, and let $\mathbf{1}^n$ be the n-fold product of copies of the category $\mathbf{1}$ defined by the ordinal number $\mathbf{1} = \{0, 1\}$ of the same name. Write $\mathbf{1}^0$ for the category consisting of one object and one morphism.

Write $\mathcal{P}(\underline{\mathrm{n}})$ for the partially ordered set of subsets of the set $\mathbf{n}$, and observe that there is an isomorphism of posets

$$\Omega_n \colon \mathbf{1}^n \xrightarrow{\cong} \mathcal{P}(\underline{\mathrm{n}}),$$

which is defined by associating to the n-tuple $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ the subset $\Omega_n(\epsilon) = \{i \mid \epsilon_i = 1\}$ of the set $\underline{\mathrm{n}} = \{1, \dots, n\}$. The box category $\square$ consists of certain poset maps $\mathbf{1}^m \to \mathbf{1}^n$. Insofar as every finite totally ordered set A has a unique order-preserving bijection $\underline{n} \to A$, it is convenient to represent box category morphisms as poset morphisms $\mathcal{P}(A) \to \mathcal{P}(B)$, where A and B are finite ordered sets.

Suppose that $A \subset B \subset C$ are subsets of a finite ordered set C. The interval $[A, B]$ is the subposet of $\mathcal{P}(C)$ which consists of all subsets D of C such that $A \subset D \subset B$. Note that there is a canonical poset isomorphism $\mathcal{P}(B - A) \to [A, B]$ which is defined by $E \mapsto A \cup E$. The composite

$$\mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(C)$$

is called a *face functor*, and is denoted by $[A, B]$.

Suppose that B is a non-empty subset of a finite ordered set C. Then the assignment $E \mapsto E \cap B$ defines a poset morphism $s_B \colon \mathcal{P}(C) \to \mathcal{P}(B)$ which is called a *degeneracy functor*. The box category $\square$ is the subcategory of the category of all poset morphisms $\mathbf{1}^m \to \mathbf{1}^n$ which is generated by the face and degeneracy functors.

There is a commutative diagram

$$\begin{tikzcd}
\mathcal{P}(B - A) \arrow[r, "{[A,B]}"] \arrow[d, "s_{E \cap (B-A)}"'] & \mathcal{P}(C) \arrow[d, "s_E"] \\
\mathcal{P}((B \cap E) - (A \cap E)) \arrow[r, "{[A \cap E, B \cap E]}"'] & \mathcal{P}(E)
\end{tikzcd}$$

which allows one to show that all morphisms of the box category $\square$ are composites

$$\mathcal{P}(C) \xrightarrow{s_E} \mathcal{P}(E) \xrightarrow{[A, A \cup E]} \mathcal{P}(D).$$

In fact, such decompositions are unique – the proof is left to the reader.

Every element i of the ordered set C determines two intervals in $\mathcal{P}(C)$, namely $[\{i\}, C]$ and $[\emptyset, C - \{i\}]$. If C has n elements, then $[\{i\}, C]$ uniquely determines a functor $d^{(i,1)} \colon \mathbf{1}^{n-1} \to \mathbf{1}^n$, while the interval $[\emptyset, C - \{i\}]$ determines a functor $d^{(i,0)} \colon \mathbf{1}^{n-1} \to \mathbf{1}^n$. Note that for $\epsilon = 0, 1$, the corresponding poset map $d^{(i,\epsilon)} \colon \mathbf{1}^{n-1} \to \mathbf{1}^n$

is defined by

$$d^{(i,\epsilon)}(\gamma_1, \dots, \gamma_{n-1}) = (\gamma_1, \dots, \overset{i}{\epsilon}, \dots, \gamma_{n-1}).$$

Similarly, every $j \in C$ determines a poset map $s_{C-\{j\}} \colon \mathcal{P}(C) \to \mathcal{P}(C - \{j\})$. Write $s^j \colon \mathbf{1}^n \to \mathbf{1}^{n-1}$ for the corresponding induced poset map, and note that s^j is the projection which drops the jth entry in the sense that

$$s^j(\gamma_1, \dots, \gamma_n) = (\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n).$$

Write $s^1 \colon \mathbf{1} \to \mathbf{1}^0$ for the obvious map to the terminal object $\mathbf{1}^0$ in the box category $\square$.

The standard "co-cubical" identities are easy to derive. In particular, if $i < j$, there is a commutative diagram of face functors

$$\begin{tikzcd} \mathbf{1}^{n-2} \arrow[r, "d^{(i,\epsilon_1)}"] \arrow[d, "d^{(j-1,\epsilon_2)}"'] & \mathbf{1}^{n-1} \arrow[d, "d^{(j,\epsilon_2)}"] \\ \mathbf{1}^{n-1} \arrow[r, "d^{(i,\epsilon_1)}"'] & \mathbf{1}^n \end{tikzcd} \tag{4}$$

if $n \geqslant 2$. If $i = j$, there is a diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathbf{1}^{n-1} \arrow[d, "d^{(i,1)}"] \\ \mathbf{1}^{n-1} \arrow[r, "d^{(i,0)}"'] & \mathbf{1}^{n-1} \end{tikzcd} \tag{5}$$

There are relations

$$s^j s^i = s^i s^{j+1}, \qquad \text{if } i \leqslant j. \tag{6}$$

Similarly,

$$s^j d^{(j,\epsilon)} = 1, \tag{7}$$

and there are commutative diagrams

$$\begin{tikzcd} \mathbf{1}^n \arrow[r, "d^{(i,\epsilon)}"] \arrow[d, "s^{j-1}"'] & \mathbf{1}^{n+1} \arrow[d, "s^j"] \\ \mathbf{1}^{n-1} \arrow[r, "d^{(i,\epsilon)}"'] & \mathbf{1}^n \end{tikzcd} \qquad \text{if } i < j \tag{8}$$

and

$$\begin{tikzcd} \mathbf{1}^n \arrow[r, "d^{(i+1,\epsilon)}"] \arrow[d, "s^j"'] & \mathbf{1}^{n+1} \arrow[d, "s^j"] \\ \mathbf{1}^{n-1} \arrow[r, "d^{(i,\epsilon)}"'] & \mathbf{1}^n \end{tikzcd} \qquad \text{if } i \geqslant j. \tag{9}$$

A *cubical set* X is a contravariant set-valued functor $X \colon \square^{\mathrm{op}} \to \mathbf{Set}$. Write $X_n = X(\mathbf{1}^n)$, and call this set the set of *n-cells* of X. A *morphism* $f \colon X \to Y$ *of cubical* *sets* is a natural transformation of functors, and we have a category $\square$-**Set** of cubical sets.

The *standard n-cell* $\square^n$ is the contravariant functor on the box category $\square$ which is represented by $\mathbf{1}^n$. All box category morphisms $\theta\colon \mathbf{1}^m \to \mathbf{1}^n$ induce cubical set maps $\theta\colon \square^m \to \square^n$ in the obvious way. Among these, the cubical set maps $d^{(i,\epsilon)}\colon \square^{n-1} \to \square^n$ are called *cofaces* and the maps $s^j\colon \square^{n+1} \to \square^n$ are called *codegeneracies.*

Write $X_n = X(\mathbf{1}^n)$ for a cubical set X, and say that an element of this set is an *n-cell* of X; a cell of X is a member of X_n for some n. If x is an n-cell of X, represented by the cubical set map $x\colon \square^n \to X$, then the composite

$$\square^{n-1} \xrightarrow{d^{(i,\epsilon)}} \square^n \xrightarrow{x} X$$

represents the *face* $d_{(i,\epsilon)}(x)$ of x, while the composite

$$\square^{n+1} \xrightarrow{s^j} \square^n \xrightarrow{x} X$$

represents the *degeneracy* $s_j(x)$. A cell y is said to be *degenerate* if it has the form $s_j x$ for some cell x; otherwise, it is *non-degenerate.*

The *cell category* $i_\square X$ for a cubical set X is defined as in Section 2: the objects of $i_\square X$ are the morphisms $\sigma\colon \square^n \to X$ (equivalently, n-cells of X, as n varies), and a morphism is a commutative triangle of cubical set morphisms.

The nerve functor restricts to a covariant simplicial set-valued functor $\square \to \mathbf{S}$ which is defined by $\mathbf{1}^n \mapsto B(\mathbf{1}^n) = (\Delta^1)^{\times n}$ This functor can be used to define a *cubical singular functor* $S\colon \mathbf{S} \to \square$-**Set**, where

$$S(Y)_n = \hom_{\mathbf{S}}((\Delta^1)^{\times n}, Y).$$

This functor has a left adjoint (called *triangulation*) $X \mapsto |X|$, where

$$|X| = \varinjlim_{\square^n \to X} (\Delta^1)^{\times n}.$$

Here, the colimit is indexed by members of the cell category $i_\square X$.

There are similarly defined realization and singular functors

$$|\ |\colon \square\text{-}\mathbf{Set} \leftrightarrows \mathbf{Top}\colon S$$

relating cubical sets and topological spaces, and of course realization is left adjoint to the singular functor in that context as well.

Example 3.1. Suppose that $\mathcal{C}$ is a small category. The *cubical nerve* $B_\square(\mathcal{C})$ is the cubical set whose n-cells are all functors of the form $\mathbf{1}^n \to \mathcal{C}$, and whose structure maps $B_\square(\mathcal{C})_n \to B_\square(\mathcal{C})_m$ are induced by precomposition with box category morphisms $\mathbf{1}^m \to \mathbf{1}^n$. Observe that there is a natural isomorphism

$$B_\square(\mathcal{C}) \cong S(B\mathcal{C}),$$

where $B\mathcal{C}$ is the standard nerve for the category $\mathcal{C}$ in the category of simplicial sets.

Define the *n-skeleton* $\mathrm{sk}_n X$ for a cubical set X to be the subcomplex which is generated by the k-cells X_k for $0 \leqslant k \leqslant n$.

Lemma 3.2. *Suppose that x and y are degenerate n-cells of a cubical set X which have the same boundary in the sense that $d_{(i,\epsilon)}x = d_{(i,\epsilon)}y$ for all i and ϵ. Then $x = y$.*

Proof. We shall use the classical cubical set identities arising from the diagrams (4) to (9).

Suppose that the degenerate cells $s_i x$ and $s_j y$ have the same boundary. If $i = j$, then $x = d_{(i,0)} s_i x = d_{(i,0)} s_i y = y$, and so the cells $s_i x$ and $s_i y$ coincide. Suppose, therefore, that $i < j$.

Then

$$x = d_{(i,0)} s_i x = d_{(i,0)} s_j y = s_{j-1} d_{(i,0)} y,$$

so that

$$s_j s_i (d_{(i,0)} y) = s_i s_{j-1} (d_{(i,0)} y) = s_i x.$$

This means that $s_j s_i d_{(i,0)} y$ and $s_j y$ have a common boundary, and then one applies $d_{(j,0)}$ to show that $s_i d_{(i,0)} y = y$. Finally, one sees that

$$s_i x = s_i s_{j-1} d_{(i,0)} y = s_j s_i d_{(i,0)} y = s_j y. \qquad \square$$

Corollary 3.3. *A map $f\colon \operatorname{sk}_n X \to Y$ of cubical sets is completely determined by the restrictions $f\colon X_k \to Y_k$ for $0 \leqslant k \leqslant n$.*

Proof. We want to show that the maps $f\colon X_k \to Y_k$ extend uniquely to a morphism $f_*\colon \operatorname{sk}_n X \to Y$. Suppose that $z \in \operatorname{sk}_n X_{n+1}$. Then z is degenerate, so that $z = s_i x$ for some $x \in X_n$, and it must be that $f_*(z) = s_i f(x)$, if the extension exists. Suppose that z is degenerate in two ways, so that also $z = s_j y$ for some $i < j$ and $y \in X_n$. Then the cells $s_i f(x)$ and $s_j f(y)$ have a common boundary, and therefore coincide by Lemma 3.2. $\square$

It follows that there are pushout diagrams

$$\begin{tikzcd} \bigsqcup_{x \in NX_n} \partial\square^n \arrow[r] \arrow[d] & \operatorname{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{x \in NX_n} \square^n \arrow[r] & \operatorname{sk}_n X \end{tikzcd}$$

where NX_n denotes the non-degenerate part of X_n, and $\partial\square^n = \operatorname{sk}_{n-1}\square^n$. In other words, there is a good notion of skeletal decomposition for cubical sets.

The object $\partial\square^n$ is the subcomplex of the standard n-cell which is generated by the faces $d^{(i,\epsilon)}\colon \square^{n-1} \to \square^n$. It follows from the fact that diagram (4) is a pullback in the box category that there is a coequalizer

$$\bigsqcup_{\substack{(\epsilon_1,\epsilon_2) \\ 0 \leqslant i < j \leqslant n}} \square^{n-2} \rightrightarrows \bigsqcup_{(i,\epsilon)} \square^{n-1} \to \partial\square^n$$

where $\epsilon_i \in \{0, 1\}$.

Example 3.4. The cubical set $\sqcap^n_{(i,\epsilon)}$ is the subobject of $\square^n$ which is generated by all faces $d^{(j,\gamma)}\colon \square^{n-1} \subset \square^n$ except for $d^{(i,\epsilon)}\colon \square^{n-1} \to \square^n$. From diagram (4), it again follows that there is a coequalizer diagram

$$\bigsqcup \square^{n-2} \rightrightarrows \bigsqcup_{(j,\gamma)\neq(i,\epsilon)} \square^{n-1} \to \sqcap^n_{(\epsilon,i)}$$

where the first disjoint union is indexed over all pairs (j_1,γ_1), (j_2,γ_2) with $0 \leqslant j_1 < j_2 \leqslant n$ and $(j_k,\gamma_k) \neq (i,\epsilon)$, $k = 1,2$.

Remark 3.5. The triangulation functor $|\ |\colon \square\text{-}\mathbf{Set} \to \mathbf{S}$ does not preserve products. The product $\square^1 \times \square^1$ has two non-degenerate 2-cells given by the identity and twist isomorphism $\mathbf{1} \times \mathbf{1} \to \mathbf{1} \times \mathbf{1}$, and it has an "extra" non-degenerate 1-cell which corresponds to the diagonal map $\mathbf{1} \to \mathbf{1} \times \mathbf{1}$. It follows that $|\square^1 \times \square^1|$ has the homotopy type of the wedge of circles $S^2 \vee S^1$.

Lemma 3.6. *Suppose that $x, y\colon \square^n \to X$ are n-cells of a cubical set X such that the induced simplicial set maps $x_*, y_*\colon |\square^n| \to |X|$ coincide. Then $x = y$.*

Proof. The inclusion $\mathrm{sk}_n X \subset X$ induces a monomorphism $|\,\mathrm{sk}_n X| \to |X|$, so that we can assume that $X = \mathrm{sk}_n X$. We may further suppose that X is generated by the subcomplex $\mathrm{sk}_{n-1} X$ together with the n-cells x and y.

The proof is by induction on n. The assumption that $x_* = y_*$ therefore guarantees that x and y have the same boundary in the sense that $d_{(i,\epsilon)}x = d_{(i,\epsilon)}y$ for all i and ϵ. Thus, if x and y are both degenerate, then $x = y$ by Lemma 3.2.

Suppose that y is non-degenerate and is distinct from x. Write X_0 for the smallest subcomplex of X containing $\mathrm{sk}_{n-1} X$ and x, and let $i\colon X_0 \to X$ be the inclusion of the subcomplex X_0 in X. There is a pushout diagram

$$\begin{tikzcd}
\partial\square^n \arrow[r] \arrow[d] & X_0 \arrow[d] \\
\square^n \arrow[r, "y"'] & X
\end{tikzcd}$$

On applying the triangulation functor, the assumption that $x_* = y_*$ implies that the dotted arrow lifting exists in the solid arrow pushout diagram

$$\begin{tikzcd}
{|\partial\square^n|} \arrow[r] \arrow[d] & {|X_0|} \arrow[d, "i_*"] \\
{|\square^n|} \arrow[r, "y_*"'] \arrow[ur, dotted, "x_*"] & {|X|}
\end{tikzcd}$$

making it commute. The map i_* is an inclusion which is not surjective, since the solid arrow diagram is a pushout. But the existence of the dotted arrow forces i_* to be surjective. This is a contradiction, so $x = y$. $\square$

Corollary 3.7. *Suppose that $f\colon X \to Y$ is a map of cubical sets such that the induced simplicial set map $f_*\colon |X| \to |Y|$ is a monomorphism. Then f is a monomorphism of cubical sets.*

Corollary 3.8. *Suppose that $f\colon X \to Y$ is a map of cubical sets such that the induced simplicial set map $f_*\colon |X| \to |Y|$ is an isomorphism. Then f is an isomorphism of cubical sets.*

Proof. The map f is a monomorphism by Corollary 3.7. If Y has non-degenerate cells outside of X, then $|Y|$ has non-degenerate simplices outside of $|X|$. $\square$

The problem with realizations of products which is displayed in Remark 3.5 can be fixed (following Kan [**14**]) as follows. The object $\mathbf{1}^{n+m}$ is not the product $\mathbf{1}^n \times \mathbf{1}^m$ in the box category, but there is nevertheless a functor $\tilde{\times}\colon \square \times \square \to \square$ which is defined on objects by

$$\mathbf{1}^n \tilde{\times} \mathbf{1}^m = \mathbf{1}^{n+m},$$

(equivalently, $\mathcal{P}(A)\tilde{\times}\mathcal{P}(B) = \mathcal{P}(A \sqcup B)$) and is defined on morphisms by $\theta\tilde{\times}\gamma = \theta \times \gamma$.

If X and Y are cubical sets, define

$$X \otimes Y = \varinjlim_{\sigma\square^n \to X,\ \tau\square^m \to Y} \square^{n+m}.$$

Here, if the morphisms $\theta\colon \mathbf{1}^n \to \mathbf{1}^r$ and $\gamma\colon \mathbf{1}^m \to \mathbf{1}^s$ define morphisms $\theta\colon \sigma \to \sigma'$ and $\gamma\colon \tau \to \tau'$ in the box categories for X and Y, respectively, then the corresponding map $\mathbf{1}^{n+m} \to \mathbf{1}^{r+s}$ is induced by $\theta\tilde{\times}\gamma$.

There is an isomorphism

$$K \otimes \square^n \cong \varinjlim_{\square^m \to K} \square^{m+n}.$$

It follows that the functor $K \mapsto K \otimes \square^n$ has a right adjoint $Z \mapsto Z^{(n)}$, where $Z^{(n)}_r = Z_{r+n}$ and has cubical structure map $\gamma^*\colon Z^{(n)}_r \to Z^{(n)}_s$ defined by $(\gamma\tilde{\times}1)^*\colon Z_{r+n} \to Z_{s+n}$.

The functor $K \mapsto K \otimes \square^n$ therefore preserves colimits. Thus, if $K \subset \square^n$ is the subcomplex which is generated by some list of faces $d^{(i,\epsilon)}\colon \square^{n-1} \to \square^n$, then $K \otimes \square^m$ is isomorphic to the subcomplex of $\square^{n+m}$ which is generated by the list of faces $d^{(i,\epsilon)}\colon \square^{n+m-1} \to \square^{n+m}$. A similar statement holds for all objects $\square^n \otimes L$.

It follows that the induced maps $\partial\square^n \otimes \square^m \to \square^n \otimes \square^m$ and $\square^n \otimes \partial\square^m \to \square^n \otimes \square^m$ are monomorphisms of cubical sets, and there are isomorphisms

$$\begin{aligned}
(\partial\square^n \otimes \square^m) \cup (\square^n \otimes \partial\square^m) &\cong \partial\square^{n+m}\\
(\sqcap^n_{(i,\epsilon)} \otimes \square^m) \cup (\square^n \otimes \partial\square^m) &\cong \sqcap^{n+m}_{(i,\epsilon)}\\
(\partial\square^n \otimes \square^m) \cup (\square^n \otimes \sqcap^m_{(i,\epsilon)}) &\cong \sqcap^{n+m}_{(n+i,\epsilon)}.
\end{aligned} \tag{10}$$

There are isomorphisms

$$\begin{aligned}
|X \otimes Y| &\cong \varinjlim_{\square^n \to X,\ \square^m \to Y} |\square^{n+m}|\\
&\cong \varinjlim_{\square^n \to X,\ \square^m \to Y} |\square^n| \times |\square^m|\\
&\cong |X| \times |Y|.
\end{aligned} \tag{11}$$

It is an easy consequence that the functor $X \mapsto X \otimes Y$ preserves monomorphisms of cubical sets. To see this, use the natural isomorphism (11) in conjunction with Corollary 3.7.

Lemma 3.9. *Suppose that $A \to B$ and $K \to L$ are monomorphisms of cubical sets. Then the induced diagram*

$$\begin{tikzcd} A \otimes K \arrow[r] \arrow[d] & B \otimes K \arrow[d] \\ A \otimes L \arrow[r] & B \otimes L \end{tikzcd} \tag{12}$$

is a pushout and a pullback.

Proof. The induced diagram of triangulations is isomorphic to the diagram

$$\begin{tikzcd} |A| \times |K| \arrow[r] \arrow[d] & |B| \times |K| \arrow[d] \\ |A| \times |L| \arrow[r] & |B| \times |L| \end{tikzcd}$$

which is a pushout. The triangulation functor $K \mapsto |K|$ preserves pushouts. It follows that the cubical set map

$$(A \otimes L) \cup (B \otimes K) \to B \otimes L$$

induces an isomorphism of realizations, and is therefore an isomorphism by Corollary 3.8.

We have thus proved that diagram (12) is a pushout in the category of cubical sets. It is a diagram of monomorphisms by the observation preceding the statement of the lemma, and must therefore be a pullback by elementary set theory. $\square$

The canonical forgetful functor $\pi\colon i_\square X \to \square$ for a cubical set X specializes to a forgetful functor

$$\pi\colon i_\square B_\square(C) \to \square,$$

where $B_\square C$ is the cubical nerve for a small category C — see Example 3.1.

Lemma 3.10. *Suppose that C has a terminal object t. Then the functor*

$$i_\square B_\square(C) \to \square$$

is aspherical. In particular, the cubical set $B_\square(C)$ is aspherical, and the category $i_\square B_\square(C)$ is weakly equivalent to a point.

Proof. We must show that all categories $i_\square(B_\square(C) \times \square^n)$ are aspherical (see (1)).

The objects of the category $i_\square(B_\square(C) \times \square^n)$ consist of pairs of functors

$$(f\colon \mathbf{1}^k \to C, \mathbf{1}^k \xrightarrow{\sigma} \mathbf{1}^n),$$

and morphisms are defined in the obvious way.

The category C has terminal object t, so there are natural diagrams

$$\begin{tikzcd} \mathbf{1}^k \arrow[d, "d^{(k+1,0)}"'] \arrow[dr, "f"] & \\ \mathbf{1}^{k+1} \arrow[r, "f_*"] & C \\ \mathbf{1}^k \arrow[u, "d^{(k+1,1)}"] \arrow[ur, "t"'] & \end{tikzcd}$$

Suppose that $s\colon \mathbf{1}^{k+1} \to \mathbf{1}^k$ is the degeneracy defined by projection onto the first k factors. Then the assignment

$$(f\colon \mathbf{1}^k \to C, \mathbf{1}^k \xrightarrow{\sigma} \mathbf{1}^n) \mapsto (f_*, \sigma \cdot s)$$

defines a functor $h\colon i_\square(B_\square(C) \times \square^n) \to i_\square(B_\square(C) \times \square^n)$, and the coface maps $d^{(k+1,0)}$ define a homotopy $d^{(k+1,0)}\colon (f,\sigma) \to (f_*, \sigma \cdot s)$ from the identity on $i_\square B_\square(C)$ to h. The coface maps $d^{(k+1,1)}$ define a homotopy from the endofunctor $(f,\sigma) \mapsto (t,\sigma)$ to the functor h. It follows that the category $i_\square(B_\square C \times \square^n)$ is equivalent to $i_\square \square^n$, and is therefore aspherical. $\square$

Lemma 3.11. *The box category $\square$ is a test category.*

Proof. Suppose that D is a category with terminal object. By Lemma 2.3, in order to show that $\square$ is a local test category, we must show that all cell categories $i_\square(i_\square^* D \times \square^n)$ are aspherical.

Every poset $\mathbf{1}^n$ has a terminal object $t_n = (1, \dots, 1)$. There is a functor $\square/\mathbf{1}^n \to \mathbf{1}^n$ which is defined by sending an object $\theta\colon \mathbf{1}^m \to \mathbf{1}^n$ to $\theta(t_m)$. This functor is natural in morphisms of the box category $\square$, and induces a cubical set map α: $B_\square C \to i_\square^* C$ which is natural in small categories C.

We know from Lemma 3.10 that the cubical set $B_\square(\mathbf{1})$ is aspherical.

Let $h\colon D \times \mathbf{1} \to D$ be the contracting homotopy for the category D, and consider the induced composite

$$i_\square^* D \times B_\square(\mathbf{1}) \xrightarrow{1\times\alpha} i_\square^* D \times i_\square^*(\mathbf{1}) \cong i_\square^*(D \times \mathbf{1}) \to i_\square^*(D).$$

Then the projection $i_\square^* D \times B_\square(\mathbf{1}) \to i_\square^* D$ is aspherical since $B_\square(\mathbf{1})$ is aspherical. The displayed homotopy also implies that the projection $i_\square^* D \times \square^n \to \square^n$ induces a weak equivalence $i_\square(i_\square^* D \times \square^n) \to i_\square \square^n$.

Thus, the box category $\square$ is a local test category. The category $\square$ is also plainly aspherical because it has a terminal object, so Lemma 2.4 shows that it is a test category. $\square$

Remark 3.12. Lemma 3.11 and its proof are part of a general yoga. Suppose that $i\colon \mathcal{A} \to \mathbf{cat}$ is a functor which is defined on a small category $\mathcal{A}$. Then the $\mathcal{A}$-set $i^*(C)$ is defined for a small category C by $a \mapsto \hom(i(a), C)$. Suppose that the following conditions hold:

1) all categories $i(a)$ have terminal objects.
2) if D has a terminal object, then the $\mathcal{A}$-set $i^* D$ is aspherical.
3) the category $\mathcal{A}$ is aspherical.

Then $\mathcal{A}$ is a test category. The argument is the same as that given for Lemma 3.11.

This argument appears in the context of the discussion of aspherical functors in [**3**]. Note that the cubical nerve $B_{\square}C$ is i^*C for the inclusion functor $i\colon \square \to \mathbf{cat}$, as in Lemma 2.13.

Remark 3.13. Lemma 3.10 implies that $B_{\square}\mathbf{1}$ is aspherical, and the proof of Lemma 3.11 implies that $I = i_{\square}^*\mathbf{1}$ is aspherical, but the cubical set $\square^1$ is not aspherical by Remark 3.5 and Lemma 8.7.

4. Fundamental model structures

Suppose throughout this section that $\mathcal{A}$ is a small category, and recall that $\mathcal{A}$-**Set** denotes the category of $\mathcal{A}$-sets, consisting of functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ with natural transformations as morphisms.

The box category $\square$ is a monoidal category with multiplication

$$\otimes\colon \square \times \square \to \square$$

induced by the product functor

$$(\mathbf{1}^n, \mathbf{1}^m) \mapsto \mathbf{1}^{n+m},$$

and with unit object the terminal object $* = \mathbf{1}^0$. Note that a monoidal functor $\square \to M$ taking values in a monoidal category M is completely determined by the image of the maps $0, 1\colon * \to \mathbf{1}$ in M, so that monoidal functors $\square \to M$ can be identified with interval objects in M.

An *interval theory* in the category of $\mathcal{A}$-sets is a coherent action

$$\otimes\colon \mathcal{A} - \mathrm{Pre}\ \mathcal{C} \times \square \to \mathcal{A}\text{-}\mathbf{Set}$$

of the box category on the category of $\mathcal{A}$-sets, written as

$$(X, \mathbf{1}^n) \mapsto X \otimes \square^n,$$

and which is subject to the following conditions:

DH1 The functor $X \mapsto X \otimes \square^1$ preserves filtered colimits and monomorphisms.

DH2 For every monomorphism $i\colon X \to Y$ and every coface $d^{(i,\epsilon)}\colon \square^{n-1} \to \square^n$, the square

$$\begin{tikzcd} X \otimes \square^{n-1} \arrow[r, "i\otimes 1"] \arrow[d, "1\otimes d^{(i,\epsilon)}"'] & Y \otimes \square^{n-1} \arrow[d, "1\otimes d^{(i,\epsilon)}"] \\ X \otimes \square^n \arrow[r, "i\otimes 1"'] & Y \otimes \square^n \end{tikzcd}$$

is a pullback.

DH3 For $1 \leqslant i \leqslant n$, the square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \otimes \square^{n-1} \arrow[d, "d^{(i,0)}"] \\ X \otimes \square^{n-1} \arrow[r, "d^{(i,1)}"'] & X \otimes \square^{n} \end{tikzcd}$$

is a pullback.

Example 4.1. If I is any $\mathcal{A}$-set equipped with a monomorphism $(d_0, d_1)\colon * \sqcup * \to I$, then the assignment $(X, \mathbf{1}^n) \mapsto X \times I^{\times n}$ defines a coherent action

$$I\colon \mathcal{A}\text{-}\mathbf{Set} \times \square \to \mathcal{A}\text{-}\mathbf{Set}$$

of the box category on the category of $\mathcal{A}$-sets, and this action satisfies the conditions **DH1**–**DH3**. Note that **DH3** follows from the condition that (d_0, d_1) is a monomorphism.

Example 4.2. The assignment $(X, Y) \mapsto X \otimes Y$ defines a monoidal structure on the category of cubical sets, and this monoidal structure induces a coherent action

$$\otimes\colon \square\text{-}\mathbf{Set} \times \square \to \square\text{-}\mathbf{Set}$$

of the box category on the category of cubical sets, given by $(X, \mathbf{1}^n) \mapsto X \otimes \square^n$ by the obvious restriction of structure.

The axiom **DH1** is a consequence of the fact that the functor $X \otimes \square^n$ has a right adjoint and and the separate knowledge (see the paragraph before the statement of Lemma 3.9) that it preserves monomorphisms. **DH2** is a consequence of Lemma 3.9, while **DH3** is most effectively proved by a separate triangulation argument, as in the proof of Lemma 3.9.

Suppose that X is an $\mathcal{A}$-set, and write

$$X \otimes K = \varinjlim_{\square^n \to K} X \otimes \square^n.$$

Define a cubical function space $\mathbf{hom}_\square(X, Y)$ for $\mathcal{A}$-sets X and Y by

$$\mathbf{hom}_\square(X, Y)_n = \hom(X \otimes \square^n, Y).$$

Then there is a natural bijection

$$\hom(X \otimes K, Y) \cong \hom(K,\ \mathbf{hom}_\square(X, Y))$$

relating morphisms in $\mathcal{A}$-sets to cubical set homomorphisms. It follows that the assignment $K \mapsto X \otimes K$ preserves colimits in cubical sets K.

Lemma 4.3. *The cubical set inclusion $\partial\square^n \subset \square^n$ induces a natural inclusion*

$$X \otimes \partial\square^n \to X \otimes \square^n.$$

Proof. The axiom **DH2** implies that all squares

$$\begin{tikzcd}
X \otimes \Box^{n-2} \arrow[r, "1\otimes d^{(i,\epsilon_1)}"] \arrow[d, "1\otimes d^{(j-1,\epsilon_2)}"'] & X \otimes \Box^{n-1} \arrow[d, "1\otimes d^{(j,\epsilon_2)}"] \\
X \otimes \Box^{n-1} \arrow[r, "1\otimes d^{(i,\epsilon_1)}"'] & X \otimes \Box^{n}
\end{tikzcd}$$

are pullbacks for $i < j$. In effect,this diagram is isomorphic to the diagram

$$\begin{tikzcd}
(X \otimes \Box^{j-2}) \otimes \Box^{n-j} \arrow[r, "(1\otimes d^{(i,\epsilon_2)})\otimes 1"] \arrow[d, "1\otimes d^{(1,\epsilon_2)}"'] & (X \otimes \Box^{j-1}) \otimes \Box^{n-j} \arrow[d, "1\otimes d^{(1,\epsilon_2)}"] \\
(X \otimes \Box^{j-2}) \otimes \Box^{n-j+1} \arrow[r, "(1\otimes d^{(i,\epsilon_1)})\otimes 1"'] & (X \otimes \Box^{j-1}) \otimes \Box^{n-j+1}
\end{tikzcd}$$

In the presence of axiom **DH3** (which takes care of the intersections of faces not covered by instances of the square above), it follows that the canonical map

$$X \otimes \partial\Box^n \to \cup_{(i,\epsilon)}\ X \otimes \Box^{n-1}$$

is an isomorphism, by comparison of coverings. □

It follows that any cubical set inclusion $K \subset L$ induces a monomorphism $X \otimes K \to X \otimes L$.

Remark 4.4. The axiom **DH2** implies that if $i\colon X \to Y$ is an inclusion of $\mathcal{A}$-sets, then the diagram

$$\begin{tikzcd}
X \otimes \partial\Box^n \arrow[r] \arrow[d, "i\otimes 1"'] & X \otimes \Box^n \arrow[d, "i\otimes 1"] \\
Y \otimes \partial\Box^n \arrow[r] & Y \otimes \Box^n
\end{tikzcd}$$

is a pullback. The morphisms $i \otimes 1$ are monomorphisms by **DH1**, and it follows that the canonical map

$$(Y \otimes \partial\Box^n) \cup_{X\otimes\partial\Box^n} (X \otimes \Box^n) \to Y \otimes \Box^n$$

is a monomorphism. By attaching cells, one can then show that if $j\colon K \to L$ is a monomorphism of cubical sets, then the induced map

$$(Y \otimes K) \cup_{(X\otimes K)} (X \otimes L) \to Y \otimes L$$

is a monomorphism, as is any map $i \otimes 1\colon X \otimes L \to Y \otimes L$. A set theoretic argument finally shows that the diagram

$$\begin{tikzcd}
X \otimes K \arrow[r, "1\otimes j"] \arrow[d, "i\otimes 1"'] & X \otimes L \arrow[d, "i\otimes 1"] \\
Y \otimes K \arrow[r, "1\otimes j"'] & Y \otimes L
\end{tikzcd}$$

is a pullback, and that the map

$$(Y \otimes K) \cup_{(X \otimes K)} (X \otimes L) \to (Y \otimes K) \cup (X \otimes L)$$

is an isomorphism onto a union of subobjects of $Y \otimes L$.

We shall also need the following result.

Lemma 4.5. *There is a cardinal number ζ such that $|X \otimes \square^n| < \lambda$ if $|X| < \lambda$, for all cardinals $\lambda > \zeta$.*

Proof. Choose an infinite cardinal β such that the cardinality of the set of morphisms of $\mathcal{A}$ is bounded above by β. The collection of isomorphism classes of all β-bounded $\mathcal{A}$-sets forms a set. It follows that we can choose a cardinal ζ such that $|A \otimes \square^n| < \zeta$ for all β-bounded objects A.

Choose a cardinal $\lambda > \zeta$, and suppose that X is an $\mathcal{A}$-set such that $|X| < \lambda$. Suppose that $\{x_1, x_2, \ldots\}$ indexed by $\gamma < \lambda$ is a well ordering of the elements x_i appearing in all sections of X. Then X is a filtered colimit of the β-bounded subcomplexes $\langle x_{i_1}, x_{i_2}, \ldots, x_{i_k}\rangle$ which are generated by the finite subsets of the elements x_i, and there are at most λ such subcomplexes. Then $X \otimes \square^n$ is a filtered colimit of the subcomplexes $\langle x_{i_1}, x_{i_2}, \ldots, x_{i_k}\rangle \otimes \square^n$ by **DH1**, and each of these objects is λ-bounded. It follows that $X \otimes \square^n$ is λ-bounded. $\square$

Suppose that S is a set of monomorphisms of $\mathcal{A}$-sets. The set S can be empty.

Define the class of *anodyne $(\otimes, S)$-cofibrations* (or just *anodyne cofibrations*) in the category of $\mathcal{A}$-sets to be the saturation of the set consisting of all inclusions

$$(Y \otimes \square^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)}) \to \Delta^a \otimes \square^n \tag{13}$$

arising from all subobjects $Y \subset \Delta^a$, together with all inclusions

$$(A \otimes \square^n) \cup (B \otimes \partial\square^n) \to B \otimes \square^n \tag{14}$$

arising from all monomorphisms $f\colon A \to B$ in the set S.

The set $\Lambda(\otimes, S)$ consisting of all maps of the form (13) and all maps of the form (14) is a set of generators for the class of $(\otimes, S)$-anodyne cofibrations.

Note that condition (13) implies that any inclusion $C \to D$ of $\mathcal{A}$-sets induces an anodyne cofibration

$$(C \otimes \square^n) \cup (D \otimes \sqcap^n_{(i,\epsilon)}) \to D \otimes \square^n. \tag{15}$$

Lemma 4.6. *If $j\colon C \to D$ is an anodyne cofibration, then the induced map*

$$(D \otimes \partial\square^1) \cup_{(C \otimes \partial\square^1)} (C \otimes \square^1) \to D \otimes \square^1$$

is an anodyne cofibration.

Proof. It is enough to prove the statement of the lemma for maps of the form (13) and (14).

The map

$$(((Y \otimes \square^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)})) \otimes \square^1) \cup ((\Delta^a \otimes \square^n) \otimes \partial\square^1) \to (\Delta^a \otimes \square^n) \otimes \square^1$$

can be identified up to isomorphism with the map

$$(Y \otimes (\square^n \otimes \square^1)) \cup (\Delta^a \otimes ((\sqcap^n_{(i,\epsilon)} \otimes \square^1) \cup (\square^n \otimes \partial\square^1))) \to \Delta^a \otimes (\square^n \otimes \square^1)$$

which is isomorphic to the map

$$(Y \otimes \square^{n+1}) \cup (\Delta^a \otimes \sqcap^{n+1}_{(i,\epsilon)}) \to \Delta^a \otimes \square^{n+1}$$

by a cubical set isomorphism (10).

Similarly, the map

$$(((A \otimes \square^n) \cup (B \otimes \partial\square^n)) \otimes \square^1) \cup ((B \otimes \square^n) \otimes \partial\square^1) \to (B \otimes \square^n) \otimes \square^1$$

is isomorphic to the map

$$(A \otimes \square^{n+1}) \cup (B \otimes \partial\square^{n+1}) \to B \otimes \square^{n+1}. \qquad \square$$

Say that an $\mathcal{A}$-set map $p: X \to Y$ is *injective* if it has the right lifting property with respect to all anodyne cofibrations. An $\mathcal{A}$-set X is said to be injective if the map $X \to *$ is injective.

A *naive homotopy* is a map $X \otimes \square^1 \to Y$. Note that naive homotopy of maps $f: X \to Z$ is an equivalence relation if Z is injective, by extension arguments along anodyne cofibrations of the form $X \otimes \sqcap^n_{(i,\epsilon)} \to X \otimes \square^n$.

Say that a map $g: X \to Y$ is an $(\otimes, S)$-*equivalence* (or just an *equivalence*) if it induces an isomorphism

$$g^*: \pi(Y,Z) \xrightarrow{\cong} \pi(X,Z)$$

in naive homotopy classes for all injective objects Z.

A *cofibration* is a monomorphism. An $(\otimes, S)$-*fibration* (or a *fibration*) is a map which has the right lifting property with respect to all maps which are simultaneously cofibrations and $(\otimes, S)$-equivalences, a.k.a. trivial cofibrations.

Lemma 4.7. *All anodyne cofibrations are $(\otimes, S)$-equivalences.*

Proof. If $i: C \to D$ is an anodyne cofibration, and $f: C \to Z$ is a map, where Z is injective, then the dotted lifting exists in the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "i"'] & Z \\ D \arrow[ur, dotted] & \end{tikzcd}$$

so that the function $i^*: \pi(D,Z) \to \pi(C,Z)$ is surjective. If $g_1, g_2: D \to Z$ are maps which become homotopic when restricted to C, then g_1, g_2 and the homotopy define a map

$$(C \otimes \square^1) \cup (D \otimes \partial\square^1) \to Z$$

which extends to a map $D \otimes \square^1 \to Z$ by Lemma 4.6, so g_1 and g_2 are homotopic. Thus, the function i^* is injective as well as surjective. $\square$

Here is a general set of tricks that applies to any set T of monomorphisms $g\colon C \to D$ of $\mathcal{A}$-sets.

Suppose that α is a cardinal such that $\alpha > \zeta$, where ζ is a cardinal as in Lemma 4.5. Suppose further that $\alpha > |\mathcal{A}|$. Suppose that $\alpha > |D|$ for all morphisms $g\colon C \to D$ appearing in the set T and that $\alpha > |T|$. Choose a cardinal λ such that $\lambda > 2^\alpha$.

Suppose that $f\colon X \to Y$ is a morphism of $\mathcal{A}$-sets. Define a functorial system of factorizations

$$\begin{tikzcd} X \arrow[r, "i_s"] \arrow[dr, "f"'] & E_s(f) \arrow[d, "f_s"] \\ & Y \end{tikzcd}$$

of the map f indexed on all ordinal numbers $s < \lambda$ as follows:

1) Given the factorization (f_s, i_s) define the factorization (f_{s+1}, i_{s+1}) by requiring that the diagram

$$\begin{tikzcd} \bigsqcup_{\mathbf{D}} C \arrow[r, "\alpha_{\mathbf{D}}"] \arrow[d, "g_*"'] & E_s(f) \arrow[d] \\ \bigsqcup_{\mathbf{D}} D \arrow[r] & E_{s+1}(f) \end{tikzcd}$$

is a pushout, where the disjoint union is indexed over all diagrams $\mathbf{D}$ of the form

$$\begin{tikzcd} C \arrow[r, "\alpha_{\mathbf{D}}"] \arrow[d, "g"'] & E_s(f) \arrow[d, "f_s"] \\ D \arrow[r, "\beta_{\mathbf{D}}"'] & Y \end{tikzcd}$$

with $g\colon C \to D$ in the set T. Then the map i_{s+1} is the composite

$$X \xrightarrow{i_s} E_s(f) \xrightarrow{g_*} E_{s+1}(f)$$

2) If s is a limit ordinal, set $E_s(f) = \varinjlim_{t<s} E_s(f)$.

Set $E_\lambda(f) = \varinjlim_{s<\lambda} E_s(f)$. Then there is an induced factorization

$$\begin{tikzcd} X \arrow[r, "i_\lambda"] \arrow[dr, "f"'] & E_\lambda(f) \arrow[d, "f_\lambda"] \\ & Y \end{tikzcd}$$

of the map f. Then i_λ is a cofibration. The map f_λ has the right lifting property with respect to the maps $g\colon C \to D$ in T by a standard argument, since any map $\alpha\colon C \to E_\lambda(f)$ must factor through some $E_s(f)$ by the choice of cardinal λ.

Write $\mathcal{L}(X) = E_\lambda(c)$ for the result of this construction when applied to the canonical map $c\colon X \to *$. Then we have the following:

Lemma 4.8.

1) *Suppose that* $t \mapsto X_t$ *is a diagram of cofibrations indexed by the cardinal* $\omega > 2^\alpha$. *Then the natural map*
$$\varinjlim_{t<\omega} \mathcal{L}(X_t) \to \mathcal{L}(\varinjlim_{t<\omega} X_t)$$
is an isomorphism.
2) *The functor* $X \mapsto \mathcal{L}(X)$ *preserves cofibrations.*
3) *Suppose that* γ *is a cardinal with* $\gamma > \alpha$, *and let* $\mathcal{F}_\gamma(X)$ *denote the filtered system of subobjects of* X *having cardinality less than* γ. *Then the natural map*
$$\varinjlim_{Y \in \mathcal{F}_\gamma(X)} \mathcal{L}(Y) \to \mathcal{L}(X)$$
is an isomorphism.
4) *If* $|X| < 2^\omega$, *where* $\omega \geqslant \lambda$, *then* $|\mathcal{L}(X)| < 2^\omega$.
5) *Suppose that* U, V *are subobjects of an* $\mathcal{A}$*-set* X. *Then the natural map*
$$\mathcal{L}(U \cap V) \to \mathcal{L}(U) \cap \mathcal{L}(V)$$
is an isomorphism.

Proof. It suffices to prove all of these statements for the functor $X \to E_1(X)$. Note as well that $E_1(X)$ is defined by the pushout diagram
$$\begin{array}{ccc} \bigsqcup_{g\in T} C \times \hom(C,X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bigsqcup_{g\in T} D \times \hom(C,X) & \longrightarrow & E_1(X) \end{array}$$

Statements 1) and 3) follow, respectively, from the fact that the maps
$$\varinjlim_{t<\omega} \hom(C, X_t) \to \hom(C, \varinjlim_{t<\omega} X_t)$$
and
$$\varinjlim_{Y \in \mathcal{F}_\gamma(X)} \hom(C, Y) \to \hom(C, \varinjlim_{Y \in \mathcal{F}_\gamma(X)} Y) = \hom(C, X)$$
are bijections on account of the size of C relative to the chosen cardinals.

Observe that, in sections,
$$E_1(X) = \left(\bigsqcup_{g\in T} (D - C) \times \hom(C, X) \right) \sqcup X \tag{16}$$
and this construction plainly preserves monomorphisms, giving statement 2). It also follows that, in sections,
$$|E_1(X)| < \alpha \cdot (2^\omega)^\alpha + 2^\omega = 2^\omega,$$
giving statement 4). Statement 5) is also a consequence of the decomposition given in equation (16). $\square$

Now restrict to the special case where T is the generating set $\Lambda(\otimes, S)$, and make the construction $X \mapsto \mathcal{L}(X)$ relative to this choice of T. Then for every $\mathcal{A}$-set X, the object $\mathcal{L}X$ is injective, and every map $f\colon X \to Y$ has a functorial factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & Z \arrow[d, "q"] \\ & Y \end{tikzcd}$$

where the map q is injective and i is anodyne. The map i is therefore a cofibration which has the left lifting property with respect to all injective maps. The functor $X \mapsto \mathcal{L}X$ further satisfies all of the properties described by Lemma 4.8.

Lemma 4.9. *Suppose given a diagram*

$$\begin{tikzcd} & X \arrow[d, "i"] \\ A \arrow[r] & Y \end{tikzcd}$$

of cofibrations of $\mathcal{A}$-sets such that i is an equivalence and $|A| < 2^{\lambda}$. Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^{\lambda}$ and $B \cap X \to B$ is an equivalence.

Proof. Observe that since $i_*\colon \mathcal{L}X \to \mathcal{L}Y$ is an equivalence, it must be a naive homotopy equivalence since $\mathcal{L}X$ and $\mathcal{L}Y$ are injective. Thus, there is a morphism $\sigma\colon \mathcal{L}Y \to \mathcal{L}X$, and homotopies $\sigma \cdot i_* \simeq 1$ and $i_* \cdot \sigma \simeq 1$. Let $h\colon \mathcal{L}X \otimes \square^1 \to \mathcal{L}X$ be a homotopy from $\sigma \cdot i_*$ to the identity on $\mathcal{L}X$. Then the map σ and the homotopy h together determine the map (σ, h) in the diagram

$$\begin{tikzcd} (\mathcal{L}Y \otimes \square^0) \cup (\mathcal{L}X \otimes \square^1) \arrow[r, "{(\sigma,h)}"] \arrow[d] & \mathcal{L}X \\ \mathcal{L}Y \otimes \square^1 \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

which extends to the homotopy H as indicated. Thus, there is a map $\sigma'\colon \mathcal{L}Y \to \mathcal{L}X$ such that $\sigma' \cdot i_* = 1$ and

$$i_* \cdot \sigma' \simeq i_* \cdot \sigma \simeq 1.$$

It follows that we can assume that $\sigma \cdot i_* = 1$. Let $K\colon \mathcal{L}Y \otimes \square^1 \to \mathcal{L}Y$ be a homotopy from $i_* \cdot \sigma$ to the identity.

Suppose that A_i is a subobject of Y such that $|A_i| < 2^{\lambda}$. Then $|\mathcal{L}A_i \otimes \square^1| < 2^{\lambda}$ by Lemma 4.5, so there is a 2^{λ}-bounded subobject A_{i+1} of Y such that $A_i \subset A_{i+1}$ and such that the composite

$$\mathcal{L}A_i \otimes \square^1 \to \mathcal{L}Y \otimes \square^1 \xrightarrow{K} \mathcal{L}Y$$

factors (uniquely) through $\mathcal{L}A_{i+1}$ in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathcal{L}A_i \otimes \square^1 \arrow[r] \arrow[d] & \mathcal{L}A_{i+1} \arrow[d] \\ \mathcal{L}Y \otimes \square^1 \arrow[r, "K"'] & \mathcal{L}Y \end{tikzcd}$$

This is the successor ordinal step in the construction of a system $i \mapsto A_i$ with $i < \lambda$ (recall that $\lambda > 2^\alpha$) and $A = A_0$. Let $B = \varinjlim_i A_i$. Then, by construction, B is 2^λ-bounded and the restriction of the homotopy K to $\mathcal{L}B \otimes \square^1$ factors through the inclusion $j\colon \mathcal{L}B \to \mathcal{L}Y$.

The diagram

$$\begin{tikzcd} \mathcal{L}(B \cap X) \arrow[r, "\tilde{j}"] \arrow[d, "\tilde{i}"'] & \mathcal{L}X \arrow[d, "i_*"] \\ \mathcal{L}B \arrow[r, "j"'] & \mathcal{L}Y \end{tikzcd}$$

is a pullback, and $i_*\sigma(\mathcal{L}B) \subset \mathcal{L}B$. It follows that σ restricts to a map $\sigma'\colon \mathcal{L}B \to \mathcal{L}(B \cap X)$. Then

$$\tilde{j}\sigma'\tilde{i} = \sigma j\tilde{i} = \sigma i_*\tilde{j} = \tilde{j}$$

so that $\sigma'\tilde{i} = 1$. Finally, $j\tilde{i}\sigma' = i_*\sigma j$ by construction, so the restricted homotopy $\mathcal{L}B \otimes \square^1 \to \mathcal{L}B$ is a homotopy from $\tilde{i}\sigma'$ to the identity. In particular, the induced map $B \cap X \to B$ is an equivalence. $\square$

We need to know that the class of trivial cofibrations is closed under pushout, and for that we need to prove the following:

Lemma 4.10. *Suppose given a diagram*

$$\begin{tikzcd} C \arrow[r, "f{,}g"] \arrow[d, "i"'] & E \\ D \end{tikzcd}$$

where i is a cofibration, and suppose that there is a naive homotopy $h\colon C \otimes \square^1 \to E$ from f to g. Then the induced map $g_*\colon D \to D \cup_g E$ is an equivalence if and only if $f_*\colon D \to D \cup_f E$ is an equivalence.

Proof. There are pushout diagrams

$$\begin{tikzcd} C \arrow[r, "d_0"] \arrow[d, "i"'] & C \otimes \square^1 \arrow[r, "h"] \arrow[d, "i_*"] & E \arrow[d, "i_*"] \\ D \arrow[r, "d_{0*}"'] & D \cup_C (C \otimes \square^1) \arrow[r, "h'"'] \arrow[d, "j"'] & D \cup_f E \arrow[d, "j_*"] \\ & D \otimes \square^1 \arrow[r, "h_*"'] & (D \otimes \square^1) \cup_h E \end{tikzcd}$$

where the top composite is f. The maps d_{0*}, j and j_* are anodyne cofibrations. Thus $f_* = h' \cdot d_{0*}$ is equivalent to h', and h' is equivalent to h_*, so f_* is an equivalence if and only if h_* is an equivalence.

A similar analysis holds for the induced map $g_*\colon D \to D \cup_g E$. Thus f_* is an equivalence if and only if g_* is an equivalence. $\square$

Lemma 4.11. *Suppose that* $i\colon C \to D$ *is a trivial cofibration. Then the cofibration*

$$(C \otimes \square^1) \cup (D \otimes \partial\square^1) \to D \otimes \square^1$$

is an equivalence.

Proof. Consider the diagram

$$\begin{tikzcd} C \otimes \partial\square^1 \arrow[r] \arrow[d] & D \otimes \partial\square^1 \arrow[r] \arrow[d] & \mathcal{L}D \otimes \partial\square^1 \arrow[d] \\ C \otimes \square^1 \arrow[r] & D \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1 \end{tikzcd}$$

Then there is an induced diagram

$$\begin{tikzcd} (C \otimes \square^1) \cup (D \otimes \partial\square^1) \arrow[r] \arrow[d] & (C \otimes \square^1) \cup (\mathcal{L}D \otimes \partial\square^1) \arrow[d] \\ D \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1 \end{tikzcd}$$

in which the horizontal maps are anodyne extensions, and hence equivalences.

There is a factorization

$$\begin{tikzcd} C \arrow[r, "i'"] \arrow[dr, "i"'] & D' \arrow[d, "p"] \\ & D \end{tikzcd}$$

where i' is anodyne and p is both injective and an equivalence. In the induced diagram

$$\begin{tikzcd} (C \otimes \square^1) \cup (\mathcal{L}D' \otimes \partial\square^1) \arrow[r] \arrow[d] & (C \otimes \square^1) \cup (\mathcal{L}D \otimes \partial\square^1) \arrow[d] \\ \mathcal{L}D' \otimes \square^1 \arrow[r] & \mathcal{L}D \otimes \square^1 \end{tikzcd}$$

the top horizontal map is induced by the homotopy equivalence

$$\mathcal{L}D' \otimes \partial\square^1 \to \mathcal{L}D \otimes \partial\square^1,$$

and is therefore an equivalence by Lemma 4.10. The bottom horizontal map is also a homotopy equivalence. The left-hand vertical map is an equivalence by comparison with the map

$$(C \otimes \square^1) \cup (D' \otimes \partial\square^1) \to D' \otimes \square^1$$

and Lemma 4.6. $\square$

Lemma 4.12. *The class of trivial cofibrations is closed under pushout.*

Proof. First of all, if $j\colon C \to D$ is a cofibration and an equivalence, then every map $\alpha\colon C \to Z$ taking values in an injective object Z extends to a map $D \to Z$. In effect, there is a homotopy $h\colon C \otimes \square^1 \to Z$ from α to a map $\beta \cdot j$ for some map $\beta\colon D \to Z$.

Then the lifting H exists in the diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (D \otimes \{1\}) \arrow[r, "{(h,\beta)}"] \arrow[d] & Z \\
D \otimes \square^1 \arrow[ur, dotted, "H"'] &
\end{tikzcd}$$

since the vertical map is an anodyne cofibration, so α extends to the morphism $H|_{D\otimes\{0\}}$.

Suppose that the diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "j"'] & C' \arrow[d, "j'"] \\
D \arrow[r] & D'
\end{tikzcd}$$

is a pushout, where j is a trivial cofibration. Suppose that the $\mathcal{A}$-set Z is injective. Then every map $\alpha'\colon C' \to Z$ extends to a map $\beta'\colon D' \to Z$ since the diagram is a pushout and j has this extension property. The diagram

$$\begin{tikzcd}
(C \otimes \square^1) \cup (D \otimes \partial\square^1) \arrow[r] \arrow[d] & (C' \otimes \square^1) \cup (D' \otimes \partial\square^1) \arrow[d] \\
D \otimes \square^1 \arrow[r] & D' \otimes \square^1
\end{tikzcd}$$

is a pushout. The left-hand vertical map is a trivial cofibration by Lemma 4.11 and therefore has the left lifting property with respect to the map $Z \to *$ by the argument above.

It follows that the induced map

$$j'^*\colon \pi(D', Z) \to \pi(C', Z)$$

is a bijection, so that j' is an equivalence. $\square$

Lemma 4.13. *Suppose that the map $p\colon X \to Y$ is injective and that the object Y is injective. Then p has the right lifting property with respect to all trivial cofibrations, so that p is a fibration.*

Proof. Suppose given a diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "\beta"'] & Y
\end{tikzcd} \tag{17}$$

where i is a trivial cofibration. Then there is a map $\theta\colon B \to X$ such that $\theta \cdot i = \alpha$ since X is injective. The extension h exists in the diagram

$$\begin{tikzcd}
(A \otimes \square^1) \cup (B \otimes \partial\square^1) \arrow[r, "{(p\alpha pr_A,(\beta,p\theta))}"] \arrow[d] & Y \\
B \otimes \square^1 \arrow[ur, dotted, "h"'] &
\end{tikzcd}$$

since the vertical map is a trivial cofibration and Y is injective. Here, $\mathrm{pr}_A\colon A \otimes \square^1 \to A$ is the map induced by the cubical set map $\square^1 \to *$.

In particular, diagram (17) is homotopic to the diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "p\theta"'] \arrow[ur, dotted, "\theta"] & Y
\end{tikzcd}$$

for which the indicated lifting exists, via the diagram

$$\begin{tikzcd}
A \otimes \square^1 \arrow[r, "\alpha \mathrm{pr}_A"] \arrow[d, "i \times i"'] & X \arrow[d, "p"] \\
B \otimes \square^1 \arrow[r, "h"'] & Y
\end{tikzcd}$$

Form the diagram

$$\begin{tikzcd}
(A \otimes \square^1) \cup B \arrow[r, "(\alpha \mathrm{pr}_A, \theta)"] \arrow[d] & X \arrow[d, "p"] \\
B \otimes \square^1 \arrow[r, "h"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

to show that the required lifting exists for diagram (17). $\square$

Corollary 4.14. *Every injective object is fibrant, so that the classes of fibrant objects and injective objects coincide.*

Lemma 4.15. *Suppose that $p\colon X \to Y$ is a fibration and an equivalence. Then p has the right lifting property with respect to all cofibrations.*

Proof. Suppose, first of all, that Y is injective.

The map p is a naive homotopy equivalence, so there is a map $g : Y \to X$ and a homotopy $h : Y \otimes \square^1 \to Y$ from $p \cdot g$ to 1_Y. The lifting h' exists in the diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "d_0"'] & X \arrow[d, "p"] \\
Y \otimes \square^1 \arrow[r, "h"'] \arrow[ur, dotted, "h'"] & Y
\end{tikzcd}$$

since d_0 is an anodyne cofibration and p is injective. Let $\sigma = h' \cdot d_1$. Then $p \cdot \sigma = 1_Y$.

The map σ is a trivial cofibration. Thus, the lifting exists in the diagram

$$\begin{tikzcd}
(Y \otimes \square^1) \cup (X \otimes \partial\square^1) \arrow[rr, "(\sigma \cdot \mathrm{pr}, (1_X, \sigma \cdot p))"] \arrow[d] & & X \arrow[d, "p"] \\
X \otimes \square^1 \arrow[r, "p \otimes 1"'] \arrow[urr, dotted, "H"] & Y \otimes \square^1 \arrow[r, "\mathrm{pr}"'] & Y
\end{tikzcd}$$

by Lemma 4.11. It follows that the identity diagram on $p\colon X \to Y$ is homotopic to the diagram

$$\begin{tikzcd} X \arrow[r, "\sigma\cdot p"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ Y \arrow[r, "1"'] \arrow[ur, dotted, "\sigma"] & Y \end{tikzcd}$$

Thus, any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

is homotopic to a diagram which admits a lifting. It follows that p has the right lifting property with respect to all cofibrations.

If Y is not injective, form the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "p"'] & Z \arrow[d, "q"] \\ Y \arrow[r, "j_Y"'] & \mathcal{L}(Y) \end{tikzcd}$$

where j is an anodyne cofibration and q is injective. Then j is an injective model for X and the map p is an equivalence, so the injective map q is an equivalence. Then the map q is a fibration by Lemma 4.13, and has the right lifting property with respect to all cofibrations by the previous paragraphs.

Factorize the map $X \to Y \times_{\mathcal{L}(Y)} Z$ as

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & W \arrow[d, "\pi"] \\ & Y \times_{\mathcal{L}(Y)} Z \end{tikzcd}$$

where π has the right lifting property with respect to all cofibrations and i is a cofibration. Write q_* for the induced map $Y \times_{\mathcal{L}(Y)} Z \to Y$. Then the composite $q_*\pi$ has the right lifting property with respect to all cofibrations and is therefore a homotopy equivalence and hence an equivalence. The cofibration i is also an equivalence, and it follows that the lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q_*\pi"'] \arrow[ur] & Y \end{tikzcd}$$

so that p is a retract of a map which has the right lifting property with respect to all cofibrations. □

Corollary 4.16. *A map $p\colon X \to Y$ is a fibration and an equivalence if and only if it has the right lifting property with respect to all cofibrations.*

Proof. Suppose that p has the right lifting property with respect to all cofibrations. Then p is a fibration. It is also a homotopy equivalence by a standard argument, so it is an equivalence. $\square$

Theorem 4.17. *Suppose that $\mathcal{A}$ is a small category. Suppose that the morphism*

$$\otimes\colon \mathcal{A}\text{-}\mathbf{Set} \times \square \to \mathcal{A}\text{-}\mathbf{Set}$$

*is an interval theory for the category of $\mathcal{A}$-sets. Suppose that S is a set of monomorphisms of $\mathcal{A}$-sets. Then the category $\mathcal{A}$-**Set** of $\mathcal{A}$-sets and the classes of $(\otimes, S)$-equivalences, $(\otimes, S)$-fibrations and cofibrations together satisfy the axioms for a closed model category.*

Proof. Corollary 4.16 and a small object argument based on all inclusions $Y \subset \Delta^a$ together imply that every map $f\colon X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & W \arrow[ur, "p"'] & \end{tikzcd}$$

where i is a cofibration and p is a fibration and an equivalence.

Lemmas 4.9 and 4.12 together imply that there is a set of trivial cofibrations $A \to B$ which generates the class of all trivial cofibrations. It follows that every map $f\colon X \to Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$$

where j is a trivial cofibration and p is a fibration.

We have therefore verified the factorization axiom **CM5**. The lifting axiom **CM4** is a consequence of Corollary 4.16. All other axioms are trivial. $\square$

Theorem 4.18. *Suppose that $\mathcal{A}$ is a small category. Suppose that the interval theory*

$$\otimes\colon \mathcal{A}\text{-}\mathbf{Set} \times \square \to \mathcal{A}\text{-}\mathbf{Set}$$

is defined by an interval I in the sense that

$$Z \otimes \square^n = Z \times I^{\times n}.$$

Suppose further that all cofibrations in the set S pull back to weak equivalences along all fibrations $p\colon X \to Y$ with Y fibrant. Then the corresponding $(\otimes, S)$-model structure on the category of $\mathcal{A}$-sets is proper.

Proof. Write $\mathcal{W}$ for the class of all maps $f\colon U \to V$ such that the induced map f_* is an equivalence in all diagrams

$$\begin{tikzcd} U \times_Y X \arrow[r, "f_*"] \arrow[d] & V \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ U \arrow[r, "f"'] & V \arrow[r] & Y \end{tikzcd}$$

for all fibrations p with Y fibrant. The class includes all projections

$$K \otimes \square^n = K \times I^{\times n} \to K.$$

The class of cofibrations in $\mathcal{W}$ is closed under pushout, and an iterated pushout argument therefore implies that all projections $K \otimes \sqcap^n_{(i,\epsilon)} \to K \otimes \square^n$ are members of $\mathcal{W}$. It follows that all generating anodyne cofibrations

$$(Y \otimes \square^n) \cup (\Delta^a \otimes \sqcap^n_{(i,\epsilon)}) \to \Delta^a \otimes \square^n$$

are in $\mathcal{W}$.

All maps $f\colon A \to B$ in the set S are in $\mathcal{W}$ by assumption. It follows by induction on n using comparisons of pushout diagrams

$$\begin{tikzcd} C \otimes \partial\square^{n-1} \arrow[r] \arrow[d] & C \otimes \sqcap^n_{(i,\epsilon)} \arrow[d] \\ C \otimes \square^{n-1} \arrow[r] & C \otimes \partial\square^n \end{tikzcd}$$

that all morphisms $f \otimes 1\colon A \otimes \partial\square^n \to B \otimes \partial\square^n$ are in $\mathcal{W}$, and hence that all morphisms $f \otimes 1\colon A \otimes K \to B \otimes K$ are in $\mathcal{W}$ for all cofibrations $f \in S$.

The class $\mathcal{W}$ is closed under retractions and transfinite compositions as well as pushout, so all anodyne cofibrations are in $\mathcal{W}$.

Suppose that $p\colon X \to Y$ is a fibration with Y fibrant, and consider a diagram

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ A \arrow[r, "i"'] & B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

where i is a trivial cofibration. Then there is a diagram

$$\begin{tikzcd} & & & X \arrow[d, "p"] \\ & A \arrow[r, "i"] \arrow[d, "j_A"'] & B \arrow[r, "\beta"] \arrow[d, "j_B"'] & Y \\ \mathcal{L}(B) \arrow[r, "r"'] & \mathcal{L}(A) \arrow[r, "i_*"'] & \mathcal{L}(B) \arrow[ur] & \end{tikzcd}$$

where j_A and j_B are anodyne cofibrations, and i_* is a section of a homotopy equivalence $r\colon \mathcal{L}(B) \to \mathcal{L}(A)$. To show that i pulls back to an equivalence along p, it suffices to show that i_* pulls back to an equivalence along p. But i_*r is homotopic to a map which pulls back to an equivalence along p and $ri = 1$, so i_* pulls back to an equivalence along p; the point is that r must pull back to a weak equivalence along p, so that i_* pulls back to a weak equivalence.

We have shown that all trivial cofibrations pull back to weak equivalences along all fibrations $p\colon X \to Y$ for which Y is fibrant. Trivial cofibrations also pull back to weak equivalences along all trivial fibrations. An arbitrary fibration $q\colon Z \to W$ is a retract of a fibration $q'\colon V \to W$ having the property that all trivial cofibrations pull back to weak equivalences along q'.

This follows from the usual argument: form the diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[d, "q"'] & W' \arrow[d, "p"] \\ W \arrow[r, "j_W"'] & \mathcal{L}W \end{tikzcd}$$

where j_W and j' are anodyne cofibrations, $\mathcal{L}W$ is fibrant and p is a fibration (see Lemma 4.13). Then the induced map $\theta\colon Z \to W \times_{\mathcal{L}W} W'$ is a weak equivalence since the map $j_{W*}\colon W \times_{\mathcal{L}W} W' \to W'$ is a weak equivalence by the previous paragraphs. The map θ has a factorization

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "\theta"'] & & V \arrow[dl, "\pi"] \\ & W \times_{\mathcal{L}W} W' & \end{tikzcd}$$

where i is a trivial cofibration and π is a trivial fibration. Set $q' = p_* \cdot \pi$, where $p_*\colon W \times_{\mathcal{L}W} W' \to W$. Then all trivial cofibrations pull back to weak equivalences along q', and q is a retract of q' since the lifting exists in the diagram

$$\begin{tikzcd} Z \arrow[r, "1"] \arrow[d, "i"'] & Z \arrow[d, "q"] \\ V' \arrow[r, "p_*\pi"'] \arrow[ur, dotted] & W \end{tikzcd}$$

Every equivalence $f\colon X \to Y$ has a factorization $f = q \cdot j$, where q is a trivial fibration and j is a trivial cofibration. It follows that every equivalence pulls back to an equivalence along all fibrations. □

We shall also commonly say that the $(\otimes, S)$-model structure of Theorem 4.18, which is defined by an interval I, is the (I, S) model structure.

Example 4.19. Suppose that $\mathcal{A} = \mathcal{C} \times \mathbf{\Delta}$, where $\mathcal{C}$ is a small Grothendieck site and $\mathbf{\Delta}$ is the ordinal number category. Then the category of $\mathcal{A}$-sets is the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$, which is well known [**7**] to have a cofibrantly generated simplicial model structure for which the weak equivalences are the local weak equivalences and the cofibrations are the monomorphisms. Pick a generating set S of trivial cofibrations for this theory. Let Δ^1 denote the interval theory associated to the constant simplicial presheaf on the simplicial set Δ^1 with the inclusions of its vertices.

The associated (Δ^1, S)-model stucture is the standard model structure on $s\,\mathrm{Pre}(C)$. In effect, every injective object for this theory is globally fibrant in the usual sense, and the injective model construction $j\colon X \to \mathcal{L}X$ is a local weak equivalence as well as a cofibration. A map $f\colon X \to Y$ of simplicial presheaves is a local weak equivalence if and only if the induced map $\mathcal{L}X \to \mathcal{L}Y$ of fibrant models is a naive homotopy equivalence, and this is equivalent to f being a (Δ^1, S)-equivalence.

Example 4.20. If $\mathcal{A} = \mathbf{\Delta}$ ($\mathcal{C} = *$ in the previous example), then the category of $\mathcal{A}$-sets is the category $\mathbf{S}$ of simplicial sets, which has a standard definition of weak equivalence. The generating set S of trivial cofibrations can be taken to be the set of inclusions $\Lambda^n_k \subset \Delta^n$, and the interval is Δ^1 with the inclusion of its two vertices. The associated (Δ^1, S)-model structure on the simplicial set category $\mathbf{S}$ is the standard model structure.

Example 4.21. Suppose that $\mathcal{C}$ is a small Grothendieck site and $f\colon A \to B$ is a monomorphism of simplicial presheaves on $\mathcal{C}$. One formally inverts f in the homotopy category [**5**] by enlarging the generating set S of local trivial cofibrations (Example 4.19) to also include the set of cofibrations

$$(Y \times B) \cup (A \times L_U\Delta^n) \to B \times L_U\Delta^n$$

arising from all inclusions $Y \subset L_U\Delta^n$ of simplicial presheaves which are freely generated by simplices in sections, where U ranges over the objects of $\mathcal{C}$. The resulting set S_f, together with the interval structure Δ^1, gives the (Δ^1, S_f)-model structure on $s\operatorname{Pre}(\mathcal{C})$. This model structure is the f-local model structure for $s\operatorname{Pre}(\mathcal{C})$, since every injective model for the (Δ^1, S_f)-model structure is a fibrant model for the f-local model structure.

It is a consequence of Theorem 4.18 that the f-local structure on $s\operatorname{Pre}(\mathcal{C})$ is proper, if f is a map of the form $f\colon * \to I$ for some simplicial presheaf I.

Example 4.22. Suppose that X is a scheme of finite dimension, and let $\mathcal{C}$ be the site $(Sm|_X)_{\mathrm{Nis}}$ of smooth schemes over X with the Nisnevich topology [**4**], [**8**], [**16**]. The motivic model structure for the category of simplicial presheaves on the smooth Nisnevich site $(Sm|_X)_{\mathrm{Nis}}$ is the f-local theory, where $f\colon * \to \mathbb{A}^1$ is some choice of rational point. It follows from Example 4.21 that the motivic model structure on $s\operatorname{Pre}(Sm|_X)_{\mathrm{Nis}}$ is the (Δ^1, S_f)-model structure.

One can take a different approach, by specifying the interval theory $\mathbb{A}^1$ to be the theory arising from the presheaf represented by the X-scheme $\mathbb{A}^1$, with the rational points $0, 1\colon * \to \mathbb{A}^1$ as endpoints. Let S be the generating set of trivial cofibrations for the ordinary local model structure on $s\operatorname{Pre}(Sm|_X)_{\mathrm{Nis}}$ as in Example 4.19. Then the $(\mathbb{A}^1, S)$-model structure on $s\operatorname{Pre}(Sm|_X)_{\mathrm{Nis}}$ is the motivic model structure on that category.

The motivic model structure on $s\operatorname{Pre}(Sm|_X)_{\mathrm{Nis}}$ is proper, by the general remark at the end of Example 4.21.

Example 4.23. The cubical set category $\square$-**Set** corresponds to the case $\mathcal{A} = \square$. There is an interval theory

$$\otimes\colon \square\text{-}\mathbf{Set} \times \square \to \square\text{-}\mathbf{Set},$$

which is specified by $(X, \mathbf{1}^n) \mapsto X \otimes \square^n$ — see Example 4.2 in Section 3. In this case, take $S = \emptyset$.

The monomorphisms in the category of cubical sets are generated by all inclusions $\partial\square^n \subset \square^n$ (these take the place of the inclusions $Y \subset L_U\Delta^a$ for this theory). It follows that the injective maps in the $(\otimes, \emptyset)$-model structure for cubical sets are those maps $p\colon X \to Y$ which have the right lifting property with respect to all inclusions $\sqcap^n_{(i,\epsilon)} \subset \square^n$. Every weak equivalence $f\colon X \to Y$ in this model structure induces a weak equivalence $f_*\colon |X| \to |Y|$ of the associated topological realizations.

We shall see later (as a consquence of Theorem 8.8) that maps which induce weak equivalences of topological realizations are exactly the weak equivalences for this model structure. It will also come from a more sophisticated analysis that the model structure for cubical sets is proper (Theorem 8.2) and that the fibrations are exactly the injective maps (Theorem 8.6).

5. Homotopy colimits

Suppose that $\mathcal{A}$ is a small category and that

$$\otimes\colon \mathcal{A}\text{-}\mathbf{Sets} \times \square \to \mathcal{A}\text{-}\mathbf{Sets}$$

is an interval theory on the category of $\mathcal{A}$-sets. Let S be a set of cofibrations of $\mathcal{A}$-sets.

We shall be primarily interested in $(\otimes, S)$-model structures $\mathbf{M}$ on the category of $\mathcal{A}$-sets which arise from Theorem 4.17, and for which the following assumption is satisfied:

M1 Every map $\Delta^a \to *$ is a weak equivalence of $\mathbf{M}$.

That said, much of what follows does not depend on this assumption. It will be specifically invoked as needed, starting in Corollary 5.10.

Recall that the cofibrations in all such model structures $\mathbf{M}$ are the inclusions of $\mathcal{A}$-sets. Here is a general observation that is quite useful, and I would like to thank Denis-Charles Cisinksi for pointing it out:

Lemma 5.1. *Suppose that $\mathcal{A}$ is a small category with an interval theory $\otimes$. Suppose that S is a set of cofibrations of $\mathcal{A}$-sets, and let C be a small category having a terminal obect t. Then the map $i_{\mathcal{A}}^*(C) \to *$ has the right lifting property with respect to all cofibrations.*

Proof. Suppose that $A \subset B$ is an inclusion of $\mathcal{A}$-sets. Then the induced functor $i_{\mathcal{A}}(A) \to i_{\mathcal{A}}(B)$ identifies $i_{\mathcal{A}}(A)$ with a subcategory of $i_{\mathcal{A}}(B)$ which has a very strong closure property: if $\tau \to \sigma$ is a map of $i_{\mathcal{A}}(B)$ such that σ is an object of $i_{\mathcal{A}}(A)$, then the morphism $\tau \to \sigma$ is in $i_{\mathcal{A}}(A)$. It follows that any functor $i_{\mathcal{A}}(A) \to C$ can be extended to a functor $i_{\mathcal{A}}(B) \to C$ which sends every object γ outside $i_{\mathcal{A}}(A)$ to the terminal object t. $\square$

An obvious consequence of Lemma 5.1 is that every projection map $X \times i_{\mathcal{A}}^*(C) \to X$ is a weak equivalence in the $(\otimes, S)$-model structure on the category of $\mathcal{A}$-sets.

We say that the model structure $\mathbf{M}$ is *regular* (or that its class of weak equivalences is regular), if the map

$$\underrightarrow{\operatorname{holim}}_{\Delta^a \to X} \Delta^a \to X$$

is a weak equivalence of $\mathbf{M}$ for all $\mathcal{A}$-sets X.

Homotopy colimits are constructed internally in the model structure $\mathbf{M}$. This construction is perhaps not yet widely known and will be summarized here. It is also related to the "internal nerve" $B_h(C)$ for a small category C in the model category $\mathbf{M}$.

If $X\colon I \to \mathcal{A}$-**Set** is a functor defined on a small category I, then one defines the *homotopy colimit* $\underrightarrow{\mathrm{holim}}_{i\in I} X(i)$ *in* **M** by setting

$$\underrightarrow{\mathrm{holim}}_{i\in I} X(i) = \underrightarrow{\lim}_{i} Z(i),$$

where $\pi\colon Z \to X$ is a pointwise trivial fibration and Z is a projective cofibrant I-diagram.

To explain, when we say that a map $f\colon X \to Y$ of I-diagrams has the property $\mathcal{P}$ *pointwise*, we mean that all constitutent maps $f\colon X(i) \to Y(i)$ of $\mathcal{A}$-sets have the property $\mathcal{P}$. In particular, a map $\pi\colon Z \to W$ is a pointwise trivial fibration if and only if all maps $\pi\colon Z(i) \to Y(i)$ are trivial fibrations of **M**.

Recall (see, for example, [**1**]) that, since **M** is cofibrantly generated, there is a model structure on the category of I-diagrams $I \to \mathbf{M}$ for which the weak equivalences and fibrations are defined pointwise. The cofibrations for the theory are called *projective cofibrations*, and the model structure on the category of I-diagrams is called the *projective model structure*.

Observe that if $f\colon Z \to Z'$ is a pointwise weak equivalence of projective cofibrant I-diagrams, then it has a factorization

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "i"'] & & Z' \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where i is a trivial projective cofibration and q is left inverse to a trivial projective cofibration. The colimit functor takes trivial projective cofibrations i to trivial cofibrations of **M**; in effect, the colimit functor is left adjoint to the constant functor from $\mathcal{A}$-**Set** to I-diagrams in $\mathcal{A}$-sets, and the constant functor preserves fibrations and trivial fibrations.

It follows that the homotopy type in **M** of the homotopy colimit $\underrightarrow{\mathrm{holim}}_{i} X(i)$ is independent of the choice of projective cofibrant model $\pi\colon Z \to X$. It also follows that any pointwise weak equivalence $f\colon X \to Y$ of I-diagrams induces a weak equivalence

$$f_*\colon \underrightarrow{\mathrm{holim}}_{i} X(i) \to \underrightarrow{\mathrm{holim}}_{i} Y(i)$$

in **M**.

Example 5.2. The construction just given specializes to the standard description of homotopy colimit for simplicial sets, up to natural weak equivalence.

To see this, recall [**2**, Section XI.3.2] that the homotopy inverse limit $\underleftarrow{\mathrm{holim}}\ X$ of a small diagram $X\colon I \to \mathbf{S}$ of Kan complexes can be defined by

$$\underleftarrow{\mathrm{holim}}\ X = \mathbf{hom}(B(I/?), X),$$

where the function complex construction takes place in the category $\mathbf{S}^I$ of I-diagrams in simplicial sets.

It is also shown in [**5**] that if all objects $X(i)$ of the diagram X are Kan complexes and if $j\colon X \to Z$ is a (globally) fibrant model for X in the model category of

I-diagrams with pointwise weak equivalences and pointwise cofibrations, then there is a weak equivalence

$$\varprojlim{}\!\!\text{holim}\, X \cong \varprojlim Z.$$

It is worth repeating the proof here: the map j induces a weak equivalence

$$j_*\colon \underleftarrow{\text{holim}}\, X \to \underleftarrow{\text{holim}}\, Z$$

by a comparison of towers of fibrations, and the induced map

$$\mathbf{hom}(*, Z) \to \mathbf{hom}(B(I/?), Z)$$

is a weak equivalence since the map $B(I/?) \to *$ is a pointwise weak equivalence of I-diagrams (all I-diagrams are cofibrant) and Z is globally fibrant.

The homotopy colimit $\underrightarrow{\text{holim}}_I\, X$ is defined dually, so that there is a natural isomorphism of function spaces

$$\mathbf{hom}(\underrightarrow{\text{holim}}_I\, X, Y) \cong \underleftarrow{\text{holim}}_{I^{\text{op}}}\, \mathbf{hom}(X, Y)$$

for all simplicial sets Y, as in [**2**, Proposition XII.4.1]. This isomorphism forces $\underrightarrow{\text{holim}}\, X$ to be the co-end of the diagrams

$$\begin{array}{ccc} B(j/I) \times X(i) & \xrightarrow{\alpha^* \times 1} & B(i/I) \times X(i) \\ {\scriptstyle 1\times\alpha_*}\downarrow & & \\ B(j/I) \times X(j) & & \end{array}$$

arising from all morphisms $\alpha\colon i \to j$ of I. It is then an exercise to show that the object $\underrightarrow{\text{holim}}_I\, X$ is isomorphic to the diagonal of the bisimplicial set

$$\bigsqcup_{i_0 \to \cdots \to i_n} X(i_0),$$

which is the standard description.

Now suppose that $\pi\colon Z \to X$ is a projective cofibrant model for the I-diagram X. Then I claim that there is a weak equivalence

$$\varinjlim Z \simeq \underrightarrow{\text{holim}}\, X,$$

where $\underrightarrow{\text{holim}}\, X$ has the standard definition [**2**],[**5**].

In effect, the map π induces a weak equivalence

$$\pi_*\colon \underrightarrow{\text{holim}}\, Z \to \underrightarrow{\text{holim}}\, X$$

by standard results about bisimplicial sets. If Y is a Kan complex, then the function complex $\mathbf{hom}(Z, Y)$ is a globally fibrant I^{op}-diagram, by an adjunction argument. There is a commmutative diagram

$$\begin{array}{ccc} \mathbf{hom}(\varinjlim_I Z, Y) & \xrightarrow{\cong} & \varprojlim_{I^{\text{op}}} \mathbf{hom}(Z, Y) \\ \downarrow{\scriptstyle \rho_Z^*} & & \downarrow{\scriptstyle \rho^*} \\ \mathbf{hom}(\underrightarrow{\text{holim}}_I\, Z, Y) & \xrightarrow[\cong]{} & \underleftarrow{\text{holim}}_{I^{\text{op}}}\, \mathbf{hom}(Z, Y) \end{array}$$

and the map ρ^* is a weak equivalence since $\mathbf{hom}(Z,Y)$ is a globally fibrant I^{op}-diagram. The induced map ρ_Z^* is a weak equivalence for all Kan complexes Y, so that the canonical map ρ_Z is a weak equivalence of simplicial sets.

Remark 5.3. The usual model structure on the category $\mathbf{S}$ of simplicial sets is the primary example of a regular model structure $\mathbf{M}$ on a category of $\mathcal{A}$-sets. In this case, $\mathcal{A}$ is the category $\mathbf{\Delta}$ of finite ordinal numbers — see Example 4.20. The fact that a simplicial set Y is a homotopy colimit of its simplices in the sense that the map

$$\underrightarrow{\mathrm{holim}}_{\Delta^n \to Y} \Delta^n \to Y$$

is a weak equivalence is standard, and is usually seen [**5**, Lemma IV.5.2] as a consequence of a result of Quillen which asserts that if $f : X \to Y$ is a map of simplicial sets then the induced map

$$\underrightarrow{\mathrm{holim}}_{\Delta^n \to Y} \Delta^n \times_Y X \to X$$

is a weak equivalence. This result is in fact equivalent to regularity for the standard model structure on simplicial sets — see Corollary 5.10 below.

Lemma 5.4.

1) *Suppose that the diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout in the category of $\mathcal{A}$-sets, where i is a cofibration. Then the canonical map from the homotopy colimit of the diagram

$$B \xleftarrow{i} A \xrightarrow{\alpha} X$$

to Y is a weak equivalence of $\mathbf{M}$.

2) *Suppose that a diagram*

$$X_0 \to X_1 \to \cdots$$

indexed by some ordinal number consist of cofibrations. Then the canonical map from the homotopy colimit of this diagram to $\varinjlim_i X_i$ is a weak equivalence of $\mathbf{M}$.

Proof. For part 1), find a factorization

$$\begin{tikzcd} & X' \arrow[d, "p"] \\ A \arrow[ur, "j"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

where p is a trivial fibration and j is a cofibration. Then the diagram

$$B \xleftarrow{i} A \xrightarrow{j} X'$$

is projective cofibrant, and one can use a standard patching lemma argument since all $\mathcal{A}$-sets are cofibrant in $\mathbf{M}$.

For part 2), observe that the given diagram is projective cofibrant. □

Suppose that $f\colon I \to J$ is a functor between small categories, and that $X\colon I \to \mathcal{A}\text{-}\mathbf{Set}$ is a functor on I. One defines the *homotopy left Kan extension* $Lf^*X\colon J \to \mathcal{A}\text{-}\mathbf{Set}$ by setting $Lf^*X = f^*Z$, where $\pi\colon Z \to X$ is a pointwise trivial fibration and Z is projective cofibrant. Here f^*Z denotes the left Kan extension of Z along f; it is defined for $j \in J$ by setting

$$f^*Z(j) = \varinjlim_{f(i)\to j} Z(i).$$

Note that the functor f^* is left adjoint to restriction along the functor f, which is denoted by f_*. The restriction functor f_* clearly preserves pointwise fibrations and pointwise weak equivalences, so that the functor f^* preserves projective cofibrations and trivial projective cofibrations. It follows in particular that the object $Lf^*X = f^*Z$ is cofibrant, and that the homotopy type of Lf^*X in the projective model structure of J-diagrams in $\mathbf{M}$ is independent of the choice of cofibrant resolution Z up to pointwise weak equivalence. Once again, if $\alpha\colon X \to Y$ is a pointwise equivalence of I-diagrams in $\mathbf{M}$, then the induced map $\alpha_*\colon Lf^*X \to Lf^*Y$ of J-diagrams in $\mathbf{M}$.

Note that left Kan extension along the functor $I \to *$ is just the colimit, and that left Kan extensions compose up to natural isomorphism. The latter statement means that if

$$I \xrightarrow{f} J \xrightarrow{g} K$$

are composable functors between small categories, then there is a natural isomorphism of functors

$$g^*f^* \cong (gf)^*. \tag{18}$$

It follows that if $\pi\colon Z \to X$ is a projective cofibrant resolution of a diagram $X\colon I \to \mathbf{M}$, then there are identifications

$$Lg^*(Lf^*X) = g^*(Lf^*X) = g^*(f^*Z) \cong (gf)^*Z = L(gf)^*X, \tag{19}$$

where the first identification follows from the fact that $Lf^*X = f^*Z$ is projective cofibrant.

Suppose that C is a small category, and define the *internal nerve* B_hC in $\mathbf{M}$ by setting

$$B_h(C) = \underrightarrow{\mathrm{holim}}_{c\in C} * .$$

In other words,

$$B_h(C) = \varinjlim_{c\in C} Z(c),$$

where $Z \to *$ is a cofibrant resolution of the functor $*\colon C \to \mathcal{A}\text{-}\mathbf{Set}$ which takes all objects of C to a point.

Any functor $f\colon C \to D$ induces a map $f_*\colon B_h(C) \to B_h(D)$, albeit somewhat non-canonically. Suppose that $\pi_C\colon Z_C \to *$ and $\pi_D\colon Z_D \to *$ are projective cofibrant resolutions in the categories of C-diagrams and D-diagrams, respectively. Then π_D is a pointwise trivial fibration, so that the restriction $f_*\pi_D\colon f_*Z_D \to *$ is a pointwise trivial fibration. It follows that the lifting θ exists in the diagram

$$\begin{tikzcd} & f_*Z_D \arrow[d, "f_*\pi_D"] \\ Z_C \arrow[ur, "\theta"] \arrow[r, "\pi_C"'] & * \end{tikzcd}$$

and any two such lifts are homotopic. The composite

$$\varinjlim_c Z_C(c) \to \varinjlim_c f_*Z_D(c) \to \varinjlim_d Z_D(d)$$

defines a map $B_h(C) \to B_h(D)$, and this map is well defined in the homotopy category. In this way, the association $C \mapsto B_h(C)$ defines a functor $\mathbf{cat} \to \mathrm{Ho}(\mathbf{M})$.

Lemma 5.5. *Suppose that the small category D has a terminal object, and that $X\colon D \to \mathcal{A}$-**Set** is a functor. Then there is a weak equivalence $\underrightarrow{\mathrm{holim}}\, X \to X(t)$ in $\mathbf{M}$. In particular, the map $B_hD \to *$ is a weak equivalence of $\mathbf{M}$.*

Proof. Suppose that t is the terminal object of D and let $Z \to X$ be a projective cofibrant resolution of the D-diagram X. Then there is an isomorphism $\varinjlim Z \cong Z(t)$, and there is a weak equivalence $Z(t) \to X(t)$, which is part of the structure of the projective cofibrant resolution. □

Lemma 5.6. *Suppose that X is a set which is identified with a discrete category. Then there is a weak equivalence $B_hX \to X$ in $\mathbf{M}$.*

Proof. An X-diagram in $\mathcal{A}$-sets is a collection $\{Z_x\}$ of $\mathcal{A}$-sets Z_x indexed by the elements $x \in X$, and there is an isomorphism

$$\varinjlim \{Z_x\} \cong \bigsqcup_{x \in X} Z_x.$$

If $\{Z_x\} \to \{*\}$ is a projective cofibrant resolution, then each of the trivial fibrations $Z_x \to *$ has a section $* \to Z_x$ which is a trivial cofibration. The induced map $X \to \bigsqcup_x Z_x$ is a trivial cofibration of $\mathbf{M}$, so the map

$$B_hX = \bigsqcup_x Z_x \to X$$

is a weak equivalence of $\mathbf{M}$. □

Lemma 5.7. *Suppose that $f\colon C \to D$ is a functor between small categories. Then there is a canonical weak equivalence*

$$\underrightarrow{\mathrm{holim}}_{d \in D}\, B_h(f/c) \to B_h(C)$$

in the model structure $\mathbf{M}$.

Proof. Consider the functors

$$C \xrightarrow{f} D \to *$$

and choose a projective cofibrant resolution $\pi\colon Z \to *$ in the category of C-diagrams. There is an identification

$$\varinjlim_{c\in C} Z(c) \cong \varinjlim_{d\in D} \varinjlim_{f(c)\to d} Z(c)$$

on account of the isomorphism (18). The restriction functor Q_* defined by the forgetful functor $Q\colon f/d \to C$ has a right adjoint $Q^!$ defined by

$$Q^! F(c) = \varprojlim_{c\to c', \alpha'\colon f(c')\to d} F(\alpha'),$$

where the inverse limit is computed over all pairs of diagrams

$$\begin{tikzcd} & c' \arrow[dd, "\beta"] \\ c \arrow[ur] \arrow[dr] & \\ & c'' \end{tikzcd} \qquad \begin{tikzcd} f(c') \arrow[dd, "f(\beta)"'] \arrow[dr, "\alpha'"] & \\ & d \\ f(c'') \arrow[ur, "\alpha''"'] & \end{tikzcd}$$

The index category has one component for each morphism $\omega\colon f(c) \to d$ in D, and each such component contains an initial object defined by the pair of arrows

$$c \xrightarrow{1} c, \quad f(c) \xrightarrow{\omega} d.$$

It follows that

$$Q^! F(c) = \prod_{\omega\colon f(c)\to d} F(\omega)$$

In particular, the functor $Q^!$ preserves pointwise fibrations and pointwise trivial fibrations, and so the restriction functor Q_* preserves projective cofibrations as well as pointwise trivial fibrations. It follows that

$$\varinjlim_{\omega\colon f(c)\to d} Z(c) = \varinjlim_{\omega\colon f(c)\to d} Q_* Z(\omega) \simeq B_h(f/d).$$

for all objects d of D. □

Lemma 5.8. *Suppose that the functors $f\colon C \to D$ and $g\colon D \to C$ define a homotopy equivalence of categories. Then the induced maps $f_*\colon B_hC \to B_hD$ and $g_*\colon B_hD \to B_hC$ are weak equivalences of* $\mathbf{M}$.

Proof. The assumption that the functors f and g define a homotopy equivalence in **cat** means that there are natural transformations between both $f \cdot g$ and $g \cdot f$ and the respective identity functors.

Suppose that a category E has a terminal object and consider a projection $\mathrm{pr}\colon C \times E \to C$. Then there is an isomorphism $\mathrm{pr}/c \cong C/c \times E$ for each object $c \in C$. The category $C/c \times E$ has a terminal object, so that the projection $C/c \times$

$E \to C/c$ induces a weak equivalence

$$B_h(C/c \times E) \to B_h(C/c)$$

for each $c \in C$ by Lemma 5.5. It follows from Lemma 5.7 that the map

$$B_h(C \times E) \to B_h C$$

is a weak equivalence.

Suppose that the functor $h\colon C \times \mathbf{1} \to D$ is a homotopy of functors $f_1, f_2\colon C \to D$. The projection functor pr: $C \times \mathbf{1} \to C$ induces a weak equivalence $B_h(C \times \mathbf{1}) \to B_h C$, so that the two canonical inclusions $C \to C \times \mathbf{1}$ induce the same map $B_h C \to B_h(C \times \mathbf{1})$ in the homotopy category. It follows that f_1 and f_2 induce the same map in the homotopy category.

The composites fg and gf are both homotopic to identity functors. It follows that the induced functors $(fg)_*\colon B_h D \to B_h D$ and $(gf)_*\colon B_h C \to B_h C$ are isomorphisms in the homotopy category $\mathrm{Ho}(\mathbf{M})$, so f_* is an isomorphism in the homotopy category. $\square$

Corollary 5.9. *Suppose that $f : X \to Y$ is an $\mathcal{A}$-set morphism. Then there is a cononical weak equivalence*

$$\underset{\sigma:\Delta^a \to Y}{\underrightarrow{\mathrm{holim}}}\ B_h(i_{\mathcal{A}}(\Delta^a \times_Y X)) \to B_h(i_{\mathcal{A}} X)$$

in the model structure $\mathbf{M}$.

Proof. Apply Lemma 5.7 to the induced functor $f_*\colon i_{\mathcal{A}} X \to i_{\mathcal{A}} Y$ and observe that there is an isomorphism

$$f_*/\sigma \cong i_{\mathcal{A}}(\Delta^a \times_Y X)$$

for each $\sigma\colon \Delta^a \to Y$. $\square$

Corollary 5.10. *Suppose that the model structure* $\mathbf{M}$ *on the category of $\mathcal{A}$-sets satisfies the property* $\mathbf{M1}$ *and is regular. Suppose that $f\colon X \to Y$ is a map of $\mathcal{A}$-sets. Then the canonical maps $\Delta^a \times_Y X \to X$ induce a weak equivalence*

$$\underset{\Delta^a \to Y}{\underrightarrow{\mathrm{holim}}}\ (\Delta^a \times_Y X) \to X.$$

in $\mathbf{M}$.

Proof. Apply Corollary 5.9, and observe that the regularity condition and $\mathbf{M1}$ together imply that there are natural weak equivalences

$$B_h(i_{\mathcal{A}}(Z)) \simeq Z$$

for all $\mathcal{A}$-sets Z. $\square$

Corollary 5.11. *Suppose that the model structure* $\mathbf{M}$ *on the category of $\mathcal{A}$-sets satisfies the condition* $\mathbf{M1}$ *and is regular. Then there are natural weak equivalences*

$$i_{\mathcal{A}}^* C \leftarrow \underset{\Delta^a \to i_{\mathcal{A}}^* C}{\underrightarrow{\mathrm{holim}}}\ \Delta^a \to B_h(i_{\mathcal{A}} i_{\mathcal{A}}^* C) \to B_h C$$

in the model structure $\mathbf{M}$*, for all small categories C.*

Proof. The fibres ϵ/c of the functor $\epsilon\colon i_{\mathcal{A}}i_{\mathcal{A}}^{*}C \to C$ have the form $\epsilon/c \cong i_{\mathcal{A}}i_{\mathcal{A}}^{*}(C/c)$, by Lemma 2.1. The maps $i_{\mathcal{A}}^{*}(C/c) \to *$ are weak equivalences of $\mathbf{M}$ by Lemma 5.1. Thus, the map ϵ induces a weak equivalence $B_h i_{\mathcal{A}}i_{\mathcal{A}}^{*}C \to B_h C$. The other two displayed morphisms are weak equivalences by, respectively, the regularity assumption and a comparison of homotopy colimits. $\square$

Suppose given a small diagram $F\colon I \to \mathbf{cat}$ taking values in small categories. Recall that the *Grothendieck construction* $\int_I F$ (also denoted by some variant of $\int_{i\in I} F(i)$) is a category having all pairs (x, i) with $i \in I$ and $x \in F(i)$ as objects. The morphisms of this category are the pairs $(f, \alpha)\colon (x, i) \to (y, j)$ such that $\alpha\colon i \to j$ is a morphism of I and $f\colon \alpha_*(x) \to y$ is a morphism of $F(j)$.

There are a few general things to know about Grothendieck constructions:

Lemma 5.12. *Suppose that $f\colon C \to D$ is a functor between small categories. Then there is a natural homotopy equivalence*

$$Q\colon \int_{d\in D} (f/d) \to C$$

in the category **cat** *of small categories.*

Proof. The functor Q is the forgetful functor which sends a pair $(f(c) \to d, d)$ to the object $c \in C$.

We shall display a functor

$$i\colon C \to \int_{d\in D} (f/d)$$

such that the composite $Q \cdot i$ is the identity. We shall also show that there is a natural transformation (or homotopy) from $i \cdot Q$ to the identity functor on $\int_d (f/d)$.

The Grothendieck construction $\int_d (f/d)$ can be identified with the category which has as objects all morphisms $\beta\colon f(c) \to d$ of D, and the morphisms are commutative diagrams

$$\begin{tikzcd} f(c) \arrow[r, "f(\alpha)"] \arrow[d, "\beta"'] & f(c') \arrow[d, "\beta'"] \\ d \arrow[r, "\theta"'] & d' \end{tikzcd} \tag{20}$$

where $\alpha\colon c \to c'$ is a morphism of C and $\theta\colon d \to d'$ is a morphism of D. From this point of view, the functor Q is defined by sending the morphism (20) to the arrow $\alpha\colon c \to c'$ of C. There is a functor $i\colon C \to \int_d (f/d)$ which sends the morphism α to the diagram

$$\begin{tikzcd} f(c) \arrow[r, "f(\alpha)"] \arrow[d, "1"'] & f(c') \arrow[d, "1"] \\ f(c) \arrow[r, "f(\alpha)"'] & f(c') \end{tikzcd}$$

The composite $Q \cdot i$ is the identity, and the diagrams

$$\begin{tikzcd} f(c) \arrow[r, "1"] \arrow[d, "1"'] & f(c) \arrow[d, "\theta"] \\ f(c) \arrow[r, "\theta"'] & d \end{tikzcd}$$

define a natural transformation $i \cdot Q \to 1$ of functors from $\int_d (f/d)$ to itself. $\square$

There is a canonical functor $\pi\colon \int_I F \to I$ for any diagram $F\colon I \to \mathbf{cat}$ of small categories, which is idefined by $\pi(x, i) = i$.

There is a functor $f_i\colon F(i) \to \pi/i$ which is defined by the assignment $x \mapsto 1_i\colon \pi(x, i) \to i$. There is a functor $g_i\colon \pi/i \to F(i)$ which is defined by sending the morphism $\alpha\colon \pi(j, y) \to i$ to $\alpha_*(y) \in F(i)$. The functors g_i are natural in i, and one sees that $g_i \cdot f_i = 1$ for all $i \in I$. For each object $\alpha\colon \pi(j, y) \to i$, there is commutative diagram

$$\begin{tikzcd} \pi(j,x) \arrow[rr, "(\alpha,1)"] \arrow[dr, "\alpha"'] & & \pi(i, \alpha_*(x)) \arrow[dl, "1"] \\ & i & \end{tikzcd}$$

and the collection of all such diagrams defines a homotopy from the identity on π/i to $f_i \cdot g_i$. We have proved the following:

Lemma 5.13. *For any small diagram $F\colon I \to \mathbf{cat}$, there is a natural homotopy equivalence $g_i\colon F(i) \to \pi/i$, where $\pi\colon \int_I F \to I$ is the canonical functor.*

Recall that $\mathbf{M}$ denotes the $(\otimes, S)$-model structure on the category of $\mathcal{A}$-sets, where $\otimes$ is an interval theory and S is a set of cofibrations of $\mathcal{A}$-sets which become weak equivalences in $\mathbf{M}$.

Corollary 5.14. *There is a weak equivalence*

$$\underset{i\in I}{\underrightarrow{\operatorname{holim}}}\, B_h F(i) \to B_h(\textstyle\int_I F)$$

in $\mathbf{M}$ for any small diagram $F\colon I \to \mathbf{cat}$ taking values in small categories.

Proof. There is a weak equivalence

$$\underset{i\in I}{\underrightarrow{\operatorname{holim}}}\, B_h(\pi/i) \to B_h(\textstyle\int_I F)$$

by Lemma 5.7. Now use Lemma 5.8 and Lemma 5.13 to identify $B_h(\pi/i)$ with $B_h F(i)$. $\square$

Corollary 5.15. *Suppose that $f\colon F \to G$ is a natural transformation of I-diagrams of small categories such that each induced map $B_h F(i) \to B_h G(i)$ is a weak equivalence of $\mathbf{M}$. Then the induced map*

$$B_h(\textstyle\int_I F) \to B_h(\textstyle\int_I G)$$

is a weak equivalence of $\mathbf{M}$.

Suppose that $F\colon I \to \mathcal{A}\text{-}\mathbf{Set}$ is a small diagram of $\mathcal{A}$-sets. Then $i \mapsto i_{\mathcal{A}}F(i)$ is a diagram of categories. The corresponding Grothendieck construction $\int_I i_{\mathcal{A}}F$ is isomorphic to the category whose objects are all morphisms $\Delta^a \to F(i)$, and whose morphisms are all commutative diagrams

$$\begin{tikzcd} \Delta^a \arrow[r, "\theta"] \arrow[d, "x"'] & \Delta^b \arrow[d, "y"] \\ F(i) \arrow[r, "\alpha_*"'] & F(j) \end{tikzcd}$$

where $\alpha\colon i \to j$ is a morphism of I. Note that this category also coincides up to isomorphism with the Grothendieck construction

$$\int_{a\in\mathcal{A}} \hom(\Delta^a, F)$$

associated to the $\mathcal{A}$-set of categories $a \mapsto \hom(\Delta^a, F)$, where $\hom(\Delta^a, F)$ is the category with objects $x\colon \Delta^a \to F(i)$ and with morphisms $\alpha\colon x \to y$ defined by morphisms $\alpha\colon i \to j$ in I such that the diagram

$$\begin{tikzcd} \Delta^a \arrow[d, "x"'] \arrow[dr, "y"] & \\ F(i) \arrow[r, "\alpha_*"'] & F(j) \end{tikzcd}$$

commutes.

The canonical $\mathcal{A}$-set maps $F(i) \to \varinjlim_i F$ induce a functor of $\mathcal{A}$-diagrams of categories

$$\Psi\colon \hom(\Delta^a, F) \to \varinjlim_i F(i)(a),$$

where the set $\varinjlim_i F(i)(a)$ has been identified with a discrete category. In general, if X is an $\mathcal{A}$-set which is identified with a presheaf $a \mapsto X(a)$ taking values in discrete categories, then the Grothendieck construction $\int_a X(a)$ is isomorphic to the category $i_{\mathcal{A}}X$. It follows that the functor Ψ induces a functor

$$\psi\colon \int_{i\in I} i_{\mathcal{A}}F(i) \to i_{\mathcal{A}}(\varinjlim_i F(i))$$

Note that the category $\hom(\Delta^a, F)$ is isomorphic to the Grothendieck construction $\int_i F(i)(a)$ of the functor taking values in discrete categories given by $i \mapsto F(i)(a)$.

Lemma 5.16. *The functor ψ induces a weak equivalence*

$$\psi_*\colon B_h(\int_{i\in I} i_{\mathcal{A}}F(i)) \to B_h(i_{\mathcal{A}}(\varinjlim_i F(i)))$$

of $\mathbf{M}$ *in the following cases:*

1) I is the category

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

and $F(0) \to F(1)$ is a monomorphism.

2) I is an ordinal number poset and all maps $F(s) \to F(t)$ are monomorphisms.

Proof. By Lemma 5.6 and Corollary 5.15, it suffices to show that the natural transformation

$$F(i)(a) \to \varinjlim_i F(i)(a)$$

of I-diagrams in discrete categories induces a weak equivalence

$$B_h(\int_i F(i)(a)) \to B_h(\varinjlim_i F(i)(a)) \cong \varinjlim_i F(i)(a)$$

in both cases under consideration. We know from Corollary 5.14 that there is an equivalence

$$\underrightarrow{\operatorname{holim}}_{i \in I} B_h F(i)(a) \to B_h(\int_i F(i)(a)),$$

and each $B_h F(i)(a)$ is equivalent to the discrete $\mathcal{A}$-set $F(i)(a)$ by Lemma 5.6. Finally, the canonical map

$$\underrightarrow{\operatorname{holim}}_{i \in I} F(i)(a) \to \varinjlim_{i \in I} F(i)(a)$$

is a weak equivalence in cases 1) and 2), by Lemma 5.4. □

Remark 5.17. Lemma 5.7 suggests a way to avoid the problem of the non-functoriality of the assignment $C \mapsto B_h C$. Suppose given a small diagram $C\colon I \to \mathbf{cat}$ of small categories, and form the Grothendieck construction $\int_I C$. Let $\pi\colon \int_I C \to I$ be the canonical functor, and suppose that $Z \to *$ is a projective cofibrant resolution of the point over $\int_i C_i$. Then the restriction $Q_{i*}Z \to *$ is a projective cofibrant resolution of the point over π/i, so that $\varinjlim Q_{i*}Z$ represents $B_h(\pi/i)$ and thus has the homotopy type of $B_h C_i$. The diagram

$$i \mapsto \varinjlim Q_{i*}Z$$

is functorial in i and thus represents a diagram $i \mapsto B_h(C_i)$ up to weak equivalence.

If $\alpha\colon J \to I$ is a functor and C is the same I-diagram of small categories, then there is an induced commutative diagram of functors

$$\begin{tikzcd} \int_j C_{\alpha(j)} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \int_i C_i \arrow[d, "\pi"] \\ J \arrow[r, "\alpha"'] & I \end{tikzcd}$$

Choose a cofibrant resolution $Z \to *$ over $\int_i C_i$ as above and choose a cofibrant resolution $Z' \to *$ over $\int_j C_{\alpha(j)}$. Choose also a map $\theta_\alpha\colon Z' \to \alpha_* Z$. Then the maps $\pi/j \to \pi/\alpha(j)$ induce natural weak equivalences

$$\varinjlim Q_{j*}Z' \to \varinjlim Q_{\alpha(j)*}Z,$$

so the B_h construction is insensitive to the "change of universes" given by restriction along $\alpha\colon J \to I$.

We shall need a more precise approach to regularity in applications. Say that an $\mathcal{A}$-set is *regular in* $\mathbf{M}$ if the map

$$\underrightarrow{\operatorname{holim}}_{\Delta^a \to X} \Delta^a \to X$$

is a weak equivalence of $\mathbf{M}$. From this point of view, the model structure $\mathbf{M}$ is regular if and only if all $\mathcal{A}$-sets are regular in $\mathbf{M}$.

Lemma 5.18.

1) Suppose that the diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d, "i"'] & X_3 \arrow[d] \\ X_2 \arrow[r] & X_4 \end{tikzcd}$$

is a pushout and that i is a cofibration. Then if X_1, X_2 and X_3 are regular in $\mathbf{M}$ *so is X_4.*

2) If

$$X_0 \to X_1 \to \cdots$$

is a totally ordered system of cofibrations between objects which are regular in $\mathbf{M}$*, then $\varinjlim X_i$ is regular in* $\mathbf{M}$*.*

Proof. The diagram

$$\begin{tikzcd} B_h i_{\mathcal{A}} X_1 \arrow[r] \arrow[d] & B_h i_{\mathcal{A}} X_3 \arrow[d] \\ B_h i_{\mathcal{A}} X_2 \arrow[r] & B_h i_{\mathcal{A}} X_4 \end{tikzcd}$$

is homotopy cocartesian in $\mathbf{M}$: this follows from Corollary 5.14 and Lemma 5.16. It follows that the corresponding diagram of homotopy colimits

$$\begin{tikzcd} \underrightarrow{\operatorname{holim}}_{\Delta^a \to X_1} \Delta^a \arrow[r] \arrow[d] & \underrightarrow{\operatorname{holim}}_{\Delta^a \to X_3} \Delta^a \arrow[d] \\ \underrightarrow{\operatorname{holim}}_{\Delta^a \to X_2} \Delta^a \arrow[r] & \underrightarrow{\operatorname{holim}}_{\Delta^a \to X_4} \Delta^a \end{tikzcd}$$

is also homotopy cocartesian, and then the map

$$\underrightarrow{\operatorname{holim}}_{\Delta^a \to X_4} \Delta^a \to X_4$$

is a weak equivalence of $\mathbf{M}$ by a patching lemma argument.

Statement 2) has a similar proof. □

6. Homotopy theories for test categories

Suppose that $\mathcal{C}$ is a small Grothendieck site, and that $\mathcal{A}$ is a small category. Write $\mathcal{A}\text{-}\operatorname{Pre}(\mathcal{C})$ for the category of presheaves of $\mathcal{A}$-sets on the site $\mathcal{C}$. Note that

$\mathcal{A}$-presheaves on $\mathcal{C}$ are $\mathcal{B}$-sets for $\mathcal{B} = \mathcal{A} \times \mathcal{C}$, so that all of the results of the last two sections apply.

Suppose in particular that $\mathcal{A}$ is a test category, and let the interval $I = i_{\mathcal{A}}^*(\mathbf{1})$ define an interval theory

$$I\colon \mathcal{A}\text{-}\operatorname{Pre}(\mathcal{C}) \times \square \to \mathcal{A}\text{-}\operatorname{Pre}(\mathcal{C})$$

on the category of $\mathcal{A}$-presheaves on $\mathcal{C}$.

Let Δ^1 denote the interval theory on the category $s\ \operatorname{Pre}(\mathcal{C})$ which is associated to the simplicial set Δ^1 and its inclusions of vertices. Suppose that S is a set of cofibrations of simplicial presheaves such that the class of weak equivalences for the associated (Δ^1, S)-model structure on $s\ \operatorname{Pre}(\mathcal{C})$ contains all ordinary local equivalences — see Examples 4.19 and 4.21.

Say that a map $f\colon X \to Y$ of $\mathcal{A}$-presheaves is an *S-equivalence* if the induced map $i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(X) \to i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(Y)$ is a (Δ^1, S)-equivalence of simplicial presheaves. Since there are natural weak equivalences of simplicial sets

$$i_{\mathbf{\Delta}}^*(C) \simeq B i_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^*(C) \xrightarrow{\epsilon_*} BC$$

for any small category C, one sees that $f\colon X \to Y$ is an S-equivalence of $\mathcal{A}$-presheaves if and only if the induced map $Bi_{\mathcal{A}}X \to Bi_{\mathcal{A}}Y$ is an (Δ^1, S)-equivalence of simplicial presheaves. It is a consequence of Lemma 2.2 that the maps

$$i_{\mathcal{A}}^* i_{\mathbf{\Delta}} i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(X) \xrightarrow{i_{\mathcal{A}}^*\epsilon} i_{\mathcal{A}}^* i_{\mathcal{A}}(X) \xleftarrow{\eta} X \tag{21}$$

are S-equivalences for all $\mathcal{A}$-presheaves X. Similarly, for each simplicial presheaf Y the natural morphisms

$$i_{\mathbf{\Delta}}^* i_{\mathcal{A}} i_{\mathcal{A}}^* i_{\mathbf{\Delta}}(Y) \xrightarrow{i_{\mathbf{\Delta}}^*(\epsilon)} i_{\mathbf{\Delta}}^* i_{\mathbf{\Delta}}(Y) \xleftarrow{\eta} Y \tag{22}$$

are local weak equivalences of simplicial presheaves.

Choose an infinite cardinal ζ such that $|i_{\mathcal{A}}^*(\mathbf{1})| < \zeta$. Choose a cardinal α such that $\alpha > \zeta$ and α is larger than $|\mathcal{C}|$ and $|\mathcal{A}|$. Suppose further that $\alpha > |D|$ for all morphisms $C \to D$ in the set of cofibrations of simplicial presheaves S and that $\alpha > |S|$. Finally, choose a cardinal λ such that $\lambda > 2^\alpha$.

The "bounded cofibration" statement Lemma 4.9 says in the case at hand that given a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

of cofibrations of simplicial presheaves such that i is an (Δ^1, S)-equivalence and $|A| < 2^\lambda$, there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^\lambda$ and $B \cap X \to B$ is an (Δ^1, S)-equivalence. We shall prove the corresponding statement for cofibrations and S-equivalences in the category of $\mathcal{A}$-presheaves, subject to the choices of cardinals made above.

Lemma 6.1. *Suppose given a diagram*

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle i} \\ A & \longrightarrow & Y \end{array}$$

of cofibrations of $\mathcal{A}$-presheaves such that i is an S-equivalence and $|A| < 2^\lambda$. Then there is a subobject $B \subset Y$ with $A \subset B$ such that $|B| < 2^\lambda$ and $B \cap X \to B$ is an S-equivalence.

Proof. The induced diagram

$$\begin{array}{ccc} & & i^*_{\mathbf{\Delta}} i_{\mathcal{A}} X \\ & & \big\downarrow{\scriptstyle i_*} \\ i^*_{\mathbf{\Delta}} i_{\mathcal{A}} A & \longrightarrow & i^*_{\mathbf{\Delta}} i_{\mathcal{A}} Y \end{array}$$

of cofibrations of simplicial presheaves satisfies the conditions of Lemma 4.9. Thus, there is a subobject α: $B_1 \subset i^*_{\mathbf{\Delta}} i_{\mathcal{A}} Y$ with $|B_1| < 2^\lambda$ such that $i^*_{\mathbf{\Delta}} i_{\mathcal{A}} A \subset B_1$ and such that the restricted map

$$B_1 \cap i^*_{\mathbf{\Delta}} i_{\mathcal{A}} X \to B_1$$

is an equivalence of simplicial presheaves. Write

$$C_1 = i_{\mathcal{A}} A \cup i_{\mathcal{A}} B_1$$

for the smallest subobject of $i_{\mathcal{A}} Y$ which contains $i_{\mathbf{\Delta}} A$ and the image of the adjoint map α_*: $i_{\mathbf{\Delta}} B_1 \to i_{\mathcal{A}} Y$ in the category of presheaves of categories. The presheaf of categories C_1 is 2^λ-bounded in the sense that its presheaves of morphisms and arrows both have cardinality bounded above by 2^λ.

The subobject $C_1 \subset i_{\mathcal{A}} Y$ is contained in a (smallest) subobject C_2 which is a sieve in the sense that whenever $\Delta^a \to X(U)$ is an object of $C_2(U)$ and θ: $b \to a$ is a morphism of $\mathcal{A}$, then the morphism

$$\begin{tikzcd} \Delta^b \arrow[rr, "\theta"] \arrow[dr] & & \Delta^a \arrow[dl, "\sigma"] \\ & X(U) & \end{tikzcd}$$

is in $D_1(U)$. The subobject D_1 is 2^λ-bounded. Furthermore, there is a subobject $A_1 \subset Y$ such that $i_{\mathbf{\Delta}} A_1 = D_1$. In effect,

$$A_1(U)(a) = \{\ \sigma(1_a) \mid \sigma\text{: } \Delta^a \to Y(U) \text{ is an object of } D_1(U)\}.$$

Note that $|A_1| < 2^\lambda$.

We have therefore found a 2^λ-bounded subobject $A_1 \subset Y$ such that $A \subset A_1$, $B_1 \subset i^*_{\mathcal{A}} i_{\mathbf{\Delta}} A_1$, and such that the cofibration $i^*_{\mathbf{\Delta}} i_{\mathcal{A}} A \to i^*_{\mathbf{\Delta}} i_{\mathcal{A}} Y$ has a factorization

$$i^*_{\mathbf{\Delta}} i_{\mathcal{A}} A \subset B_1 \subset i^*_{\mathbf{\Delta}} i_{\mathcal{A}} A_1 \to i^*_{\mathbf{\Delta}} i_{\mathcal{A}} Y.$$

Continue inductively to produce families of subobjects

$$B_i \subset B_{i+1} \subset i^*_{\mathbf{\Delta}} i_{\mathcal{A}} Y$$

and subobjects

$$A \subset A_i \subset A_{i+1} \subset Y$$

such that

$$i_{\mathbf{\Delta}}^* i_{\mathcal{A}} A_i \subset B_{i+1} \subset i_{\mathbf{\Delta}}^* i_{\mathcal{A}} A_{i+1},$$

where $i < \gamma$ and γ is a cardinal with $2^\alpha < \gamma$.

Write $B = \varinjlim A_i$. The functor $i_{\mathbf{\Delta}}^* i_{\mathcal{A}}$ preserves filtered colimits of size γ by the assumptions on the cardinal γ, as well as monomorphisms and pullbacks. It follows that the induced map

$$i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(B \cap X) \to i_{\mathbf{\Delta}}^* i_{\mathcal{A}} B$$

is a filtered colimit of the maps

$$B_i \cap i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(X) \to i_{\mathbf{\Delta}}^* i_{\mathcal{A}} B_i$$

and is therefore a trivial cofibration of simplicial presheaves. Note as well that $|B| < 2^\lambda$ by construction. □

Theorem 6.2. *Suppose that $\mathcal{A}$ is a test category and let $\mathcal{C}$ be a small Grothendieck site. Suppose that S is a set of cofibrations of simplicial presheaves on $\mathcal{C}$ such that the class of all weak equivalences in the resulting (Δ^1, S)-model structure on the category of simplicial presheaves contains all local equivalences. Then there is model structure on the category of $\mathcal{A}$-$\operatorname{Pre}(\mathcal{C})$ for which the weak equivalences are the S-equivalences and the cofibrations are the monomorphisms. There is an equivalence*

$$\operatorname{Ho}(s\operatorname{Pre}(\mathcal{C}))_{(\Delta^1,S)} \simeq \operatorname{Ho}(\mathcal{A}\text{-}\operatorname{Pre}(\mathcal{C}))_S$$

of the associated homotopy categories.

Proof. Say that a map $p: X \to Y$ of $\mathcal{A}$-presheaves is an S-fibration if it has the right lifting property with respect to all maps which are cofibrations and S-equivalences.

Choose a cardinal λ as in the preamble to statement of Lemma 6.1. Let T_S be the set of all cofibrations $C \to D$ of $\mathcal{A}$-presheaves which are S-equivalences and such that $|D| < 2^\lambda$.

It follows from Lemma 5.16 that if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "i_*"] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout diagram of $\mathcal{A}$-presheaves with i a cofibration, then the induced diagram

$$\begin{tikzcd} i_{\mathbf{\Delta}}^* i_{\mathcal{A}} A \arrow[r] \arrow[d] & i_{\mathbf{\Delta}}^* i_{\mathcal{A}} X \arrow[d] \\ i_{\mathbf{\Delta}}^* i_{\mathcal{A}} B \arrow[r] & i_{\mathbf{\Delta}}^* i_{\mathcal{A}} Y \end{tikzcd}$$

is a homotopy co-cartesian diagram of simplicial presheaves. The functor $X \mapsto i_{\mathbf{\Delta}}^* i_{\mathcal{A}} X$ preserves filtered colimits indexed over sufficiently large infinite ordinals

γ. It is a standard consequence of Lemma 6.1 that a small object argument of size γ produces a factorization

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

for every map $f\colon X \to Y$ of $\mathcal{A}$-presheaves, where p is an S-fibration and j is a filtered colimit of size γ of pushouts of coproducts of maps appearing in T_S. The map j is a cofibration and an S-equivalence.

The codiagonal $\nabla\colon X \sqcup X \to X$ has a factorization

$$\begin{tikzcd} X \sqcup X \arrow[rr, "\nabla"] \arrow[dr, "(i_0,i_1)"'] & & X \\ & X \times I \arrow[ur, "\mathrm{pr}"'] & \end{tikzcd}$$

where (i_0, i_1) is a cofibration and pr is a weak equivalence, since $I = i_{\mathcal{A}}^*(\mathbf{1})$ is aspherical. It follows (since all $\mathcal{A}$-presheaves are cofibrant) that each of the maps i_0 and i_1 is an S-equivalence as well as a cofibration.

Suppose that a map $p\colon X \to Y$ has the right lifting property with respect to all cofibrations. Then there are commutative diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ Y \arrow[r, "1"'] \arrow[ur, "\sigma"] & Y \end{tikzcd}$$

and

$$\begin{tikzcd} X \sqcup X \arrow[r, "(1,\sigma p)"] \arrow[d, "(i_0,i_1)"'] & X \arrow[d, "p"] \\ X \times I \arrow[r, "p\cdot\mathrm{pr}"'] \arrow[ur] & Y \end{tikzcd}$$

It follows that the induced map $p_*\colon i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(X) \to i_{\mathbf{\Delta}}^* i_{\mathcal{A}}(Y)$ is a (Δ^1, S)-equivalence of simplicial presheaves, so that p is an S-equivalence of $\mathcal{A}$-presheaves.

Conversely, suppose that $p\colon X \to Y$ is a fibration and an S-equivalence. Then p has a factorization

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where j is a cofibration and q has the right lifting property with respect to all cofibrations — this follows from a transfinite small object argument based on the inclusions $Y \subset L_U\Delta^a$. But then q is an S-equivalence so j is a cofibration and an

S-equivalence, and there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ W \arrow[r, "q"'] \arrow[ur, "r"] & Y \end{tikzcd}$$

so that p is a retract of q. The map p therefore has the right lifting property with respect to all cofibrations.

We have shown that a map $p\colon X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to the set of inclusions $Y \subset L_U \Delta^a$. It follows that every map $f\colon X \to Y$ has a factorization $f = q \cdot j$, where j is a cofibration and q is an S-fibration and an S-equivalence.

The factorization axiom **CM5** and the lifting axiom **CM4** have therefore both been established. The rest of the closed model axioms are trivial to verify.

The demonstration of the equivalence of homotopy categories

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}))_{(\Delta^1,S)} \simeq \mathrm{Ho}(\mathcal{A}\text{-}\,\mathrm{Pre}(\mathcal{C}))_S$$

uses the weak equivalences displayed in (21) and (22). □

Say that the model structure on the category of $\mathcal{A}$-presheaves given by Theorem 6.2 is the S-model structure.

Example 6.3. Suppose that S is a generating set for the class of locally trivial cofibrations of simplicial presheaves, as in Example 4.19. Let $\mathcal{A}$ be an arbitrary test category. Then the S-model structure on the category of $\mathcal{A}$-presheaves gives a homotopy category which is equivalent to the homotopy category of the standard model structure on simplicial presheaves.

This result specializes to the case $\mathcal{C} = *$, giving a model structure on the category of $\mathcal{A}$-sets with associated homotopy category equivalent to the homotopy category of simplicial sets. This homotopy category is therefore equivalent to the standard homotopy theory of topological spaces and continuous maps. This result applies, in particular, to cubical sets, bisimplicial sets, cubical simplicial sets and so on.

In the broader context, we obtain sensible homotopy theories of cubical presheaves, bisimplicial presheaves and so on, all of which have homotopy categories equivalent to the homotopy category of simplicial presheaves.

Example 6.4. All localized simplicial presheaf homotopy theories (Example 4.21) have analogues over any test category, by Theorem 6.2. In particular, the motivic homotopy theory of simplicial presheaves on the smooth Nisnevich site $(Sm|_X)_{\mathrm{Nis}}$ on a scheme X (Example 4.22) has an equivalent counterpart over any test category. Thus, for example, all motivic homotopy types have cubical and bisimplicial representatives.

7. Weak equivalence classes of functors

A *weak equivalence class* is a class $\mathcal{W}$ of functors between small categories such that the following conditions are satisfied:

LF1 The class $\mathcal{W}$ is *weakly saturated* in the sense that the following hold:

a) Every identity morphism is in $\mathcal{W}$.

b) Given functors

$$C \xrightarrow{f} D \xrightarrow{g} E$$

if any two of f, g and $g \cdot f$ are in $\mathcal{W}$, then so is the third.

c) Given functors

$$A \xrightarrow{i} B \xrightarrow{r} A$$

such that $r \cdot i = 1$, if $i \cdot r$ is a member of $\mathcal{W}$ then r is a member of $\mathcal{W}$.

LF2 If C has a terminal object, then the functor $C \to *$ is in $\mathcal{W}$.

LF3 Given a commutative triangle of functors

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$

if all induced functors $\alpha/c \to \beta/c$ are in $\mathcal{W}$, then the functor u is in $\mathcal{W}$.

A weak equivalence class is called a fundamental localiser in [**3**]; the terminology was introduced by Grothendieck.

Example 7.1. Let $\mathcal{W}_\infty$ denote the class of all functors $f\colon C \to D$ such that the induced map $f_*\colon BC \to BD$ is a weak equivalence of simplicial sets. Since there is a natural weak equivalence $BC \simeq i^*_{\mathbf{\Delta}} C$, we could equally well specify the members of $\mathcal{W}_\infty$ to be those functors $f\colon C \to D$ which induce weak equivalences $i^*_{\mathbf{\Delta}} C \to i^*_{\mathbf{\Delta}} D$. The class $\mathcal{W}_\infty$ is a weak equivalence class of functors in the sense described above. The proof of **LF3** uses the fact that if $\pi\colon D \to C$ is a functor, then there is a weak equivalence

$$\underrightarrow{\operatorname{holim}}_{c \in C} B(\pi/c) \to BD$$

This is an old result of Quillen. Alternatively, it follows from Lemma 5.7.

Remark 7.2. Consider the projection functor pr: $C \times D \to C$, where D has a terminal object. For each $c \in C$, the induced functor $\mathrm{pr}/c \to C/c$ may be identified up to isomorphism with the projection $(C/c) \times D \to C/c$. The categories $(C/c) \times D$ and C/c both have terminal objects, so the projection $(C/c) \times D \to C/c$ is in $\mathcal{W}$. It follows that the projection pr: $C \times D \to C$ is $\mathcal{W}$-aspherical and hence is a member of the weak equivalence class $\mathcal{W}$.

It follows that if $h\colon C \times \mathbf{1} \to D$ is a homotopy (a.k.a. natural transformation) between functors $f, g\colon C \to D$, then f is a member of the class $\mathcal{W}$ if and only if g is a member of $\mathcal{W}$.

Lemma 7.3. *Suppose that*

$$\begin{tikzcd} C_0 \arrow[r, "f"] \arrow[d, "g"'] & C_2 \\ C_1 & \end{tikzcd}$$

is a diagram of functors of small categories. Then if g is in $\mathcal{W}$, then so is the canonical map $j\colon C_2 \to \int_i C_i$

Proof. It suffices to assume that g is the identity functor. In effect, there is a map of diagrams

$$\begin{tikzcd} C_0 \arrow[d, "g"'] & C_0 \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "1"'] & C_2 \arrow[d, "1"] \\ C_1 & C_0 \arrow[l, "g"] \arrow[r, "f"'] & C_2 \end{tikzcd}$$

such that all the (vertical) transition functors are members of $\mathcal{W}$. It follows that the induced functor on Grothendieck constructions is a member of $\mathcal{W}$, by **LF3**.

Suppose that $i\colon C_0 \to C_1$ is the identity functor. The canonical functor $\int_i C_i \to C_1 \cup_{C_0} C_2$ can be indentified with a functor $r\colon \int_i C_i \to C_2$ which is specified by the assignments $(x, 2) \mapsto x$, $(y, 0) \mapsto f(y)$ and $(y, 1) \mapsto f(y)$. The canonical functor $j\colon C_2 \to \int_i C_i$ is specified by $x \mapsto (x, 2)$, so obviously $r \cdot j = 1$. The sets of morphisms

$$(f(y), 2) \leftarrow (y, 0) \to (y, 1)$$

$$(f(y), 2) \leftarrow (y, 0) \to (y, 0)$$

$$(x, 2) \leftarrow (x, 2) \to (x, 2)$$

specify a string of homotopies from the identity on $\int_i C_i$ to the composite $j \cdot r$.

It follows that the composite $j \cdot r$ is a member of $\mathcal{W}$, so that the morphisms r and j are members of $\mathcal{W}$ by **LF1**. □

Note that there is an isomorphism

$$\int_i C_i \cong \bigsqcup_i C_i$$

for all diagrams indexed by discrete categories. It follows that the class $\mathcal{W}$ is closed under small disjoint unions.

In what follows suppose that $\mathcal{A}$ is a fixed choice of test category.

Lemma 7.4. *1) Suppose given a diagram of $\mathcal{A}$-sets*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "i"'] & X_2 \\ X_1 \end{tikzcd}$$

where the map i is a monomorphism. Then the induced map

$$\int_i i_{\mathcal{A}} X_i \to i_{\mathcal{A}}(X_1 \cup_{X_0} X_2)$$

is in $\mathcal{W}$.

*2) Suppose given a diagram Y in $\mathcal{A}$-**Set** which is indexed by some ordinal number α and such that all morphisms $Y_i \to Y_j$ are monomorphisms. Then the induced map*

$$\int_i i_{\mathcal{A}} Y_i \to i_{\mathcal{A}}(\varinjlim_i Y(i))$$

is a member of $\mathcal{W}$.

Proof. According to the method of proof of Lemma 5.16, it suffices to prove part 1) in the case where all X_i are sets (i.e. discrete $\mathcal{A}$-sets) and $X_1 \cup_{X_0} X_2$ is a singleton set. Then the pushout diagram has one of the forms

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \emptyset \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r] \arrow[d, "\cong"'] & * \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd}$$

In either case, there is a canonical functor $* \to \int_i X_i$ which is a member of $\mathcal{W}$, by Lemma 7.3.

For 2) it suffices again to assume that all $\mathcal{A}$-sets Y_i are discrete. Given $y \in \varinjlim Y_i$, there is a smallest $i < \alpha$ such that $y \in Y_i$, and the fibre of the functor π: $\int_i Y_i \to \varinjlim Y_i$ over y is isomorphic to the subcategory of α consisting of all t such that $i \leqslant t$. This fibre has an initial object and is therefore $\mathcal{W}$-aspherical. This is true of all fibres, and the fibres coincide with the categories π/y since $\varinjlim Y_i$ is discrete, so that the functor $\int_i Y_i \to \varinjlim Y_i$ is in $\mathcal{W}$. □

The argument for the proof of part 2) of Lemma 7.4 came from [**15**]. The following is now a direct consequence of Lemmas 7.3 and 7.4.

Corollary 7.5. *Suppose given a pushout diagram of $\mathcal{A}$-sets*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "i"'] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_1 \cup_{X_0} X_2 \end{tikzcd}$$

where i is a monomorphism. If the functor $i_{\mathcal{A}} X_0 \to i_{\mathcal{A}} X_2$ is a member of $\mathcal{W}$, then the functor $i_{\mathcal{A}} X_1 \to i_{\mathcal{A}}(X_1 \cup_{X_0} X_2)$ is in the class $\mathcal{W}$. If $i_{\mathcal{A}} X_0 \to i_{\mathcal{A}} X_1$ is a member of $\mathcal{W}$, then $i_{\mathcal{A}} X_2 \to i_{\mathcal{A}}(X_1 \cup_{X_0} X_2)$ is in $\mathcal{W}$.

Weak equivalences of simplicial sets are essentially initial in the collection of all weak equivalence classes of functors, according to the following result:

Theorem 7.6. *Suppose that $\mathcal{W}$ is a weak equivalence class of functors. Suppose that $f\colon X \to Y$ is a weak equivalence of simplicial sets. Then the induced functor $i_{\mathbf{\Delta}}X \to i_{\mathbf{\Delta}}Y$ of simplex categories is a member of $\mathcal{W}$.*

Proof. First of all, note that $i_{\mathbf{\Delta}}(\Delta^n) \cong \mathbf{\Delta}/\mathbf{n}$ and therefore has a terminal object, so that $i_{\mathbf{\Delta}}\Delta^n$ is $\mathcal{W}$-aspherical. All maps of simplices $\Delta^n \to \Delta^m$ therefore induce functors $i_{\mathbf{\Delta}}\Delta^n \to i_{\mathbf{\Delta}}\Delta^n$ which are members of $\mathcal{W}$.

Suppose that $0 \leqslant s_0 < s_1 < \cdots < s_r \leqslant n$ and let $\Delta^n\langle s_0, \dots, s_r\rangle$ be the subcomplex of the boundary $\partial\Delta^n$ which is generated by the faces $d^{s_j}\colon \Delta^{n-1} \to \Delta^n$. Then there is a pushout diagram

$$\begin{tikzcd}
\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle \arrow[r, "d^{s_r-1}"] \arrow[d] & \Delta^n\langle s_0, \dots, s_{r-1}\rangle \arrow[d] \\
\Delta^{n-1} \arrow[r, "d^{s_r}"'] & \Delta^n\langle s_0, \dots, s_r\rangle
\end{tikzcd}$$

in which the vertical maps are inclusions (see [**5**, p. 218]). Note that if a face is missing from $\Delta^n\langle s_0, \dots, s_r\rangle$, then a face is missing from $\Delta^{n-1}\langle s_0, \dots, s_{r-1}\rangle$. Thus, one can use Corollary 7.5 and Lemma 7.4 to show that the induced functor

$$i_{\mathbf{\Delta}}\Delta^n\langle s_0, \dots, s_r\rangle \to i_{\mathbf{\Delta}}\Delta^n$$

is a member of $\mathcal{W}$ provided that some face is missing from $\Delta^n\langle s_0, \dots, s_r\rangle$. It follows, in particular, that all inclusions $\Lambda^n_k \subset \Delta^n$ induce functors $i_{\mathbf{\Delta}}\Lambda^n_k \to i_{\mathbf{\Delta}}\Delta^n$ which are members of $\mathcal{W}$.

It suffices to show that every trivial cofibration $i\colon A \to B$ induces a functor $i_{\mathbf{\Delta}}A \to i_{\mathbf{\Delta}}B$ which is a member of $\mathcal{W}$, by a standard factorization argument.

If $i\colon A \to B$ is a trivial cofibration, it has a factorization

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[dr, "i"'] & X \arrow[d, "p"] \\
& B
\end{tikzcd}$$

where p is a Kan fibration and j is a filtered colimit of pushouts of disjoint unions of inclusions $\Lambda^n_k \subset \Delta^n$. It follows from Corollary 7.5 and Lemma 7.4 that the induced functor $j_*\colon i_{\mathbf{\Delta}}A \to i_{\mathbf{\Delta}}X$ is a member of $\mathcal{W}$. The fibration p is a weak equivalence, so the lifting σ exists in the diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "1"'] \arrow[ur, "\sigma"] & B
\end{tikzcd}$$

¿From the commutative diagram

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "j"'] & A \arrow[r, "1"] \arrow[d, "i"'] & A \arrow[d, "j"] \\ X \arrow[r, "p"'] & B \arrow[r, "\sigma"'] & X \end{tikzcd}$$

we see that the composite $\sigma \cdot p$ induces a functor $i_{\mathbf{\Delta}} X \to i_{\mathbf{\Delta}} X$ which is a member of the class $\mathcal{W}$. It follows from **LF1** that σ and hence i induce functors which are members of $\mathcal{W}$. $\square$

The following result, which, in other words, asserts that $\mathcal{W}_\infty$ is the minimal weak equivalence class (see Example 7.1), is Grothendieck's Conjecture A. The first proof of this result appeared in Cisinski's thesis [**3**].

Corollary 7.7. *Suppose that $\mathcal{W}$ is a weak equivalence class of functors and that $f\colon C \to D$ is a functor between small categories such that the induced map $f_*\colon BC \to BD$ is a weak equivalence of simplicial sets. Then f is a member of $\mathcal{W}$.*

Proof. The map $i^*_{\mathbf{\Delta}} C \to i^*_{\mathbf{\Delta}} D$ is a weak equivalence, since there is a natural weak equivalence $BC \simeq i^*_{\mathbf{\Delta}} C$. The natural map $\epsilon\colon i_{\mathbf{\Delta}} i^*_{\mathbf{\Delta}} C \to C$ is a member of $\mathcal{W}$ by **LF2** and **LF3**. Theorem 7.6 implies that the induced map $i_{\mathbf{\Delta}} i^*_{\mathbf{\Delta}} C \to i_{\mathbf{\Delta}} i^*_{\mathbf{\Delta}} D$ is a member of $\mathcal{W}$. It therefore follows from the commutativity of the diagram

$$\begin{tikzcd} i_{\mathbf{\Delta}} i^*_{\mathbf{\Delta}} C \arrow[r] \arrow[d, "\epsilon"'] & i_{\mathbf{\Delta}} i^*_{\mathbf{\Delta}} D \arrow[d, "\epsilon"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

that the functor f is in the class $\mathcal{W}$. $\square$

The following is a special case of Grothendieck's Conjecture B. This result was first proved by Cisinski [**3**], and the proof given here is essentially his.

Theorem 7.8. *Suppose that $\mathcal{W}(T)$ is the smallest weak equivalence class containing a set of functors T and that $\mathcal{A}$ is a test category. Then the class of all maps $f\colon X \to Y$ of $\mathcal{A}$-sets such that the functor $i_{\mathcal{A}} X \to i_{\mathcal{A}} Y$ is a member of $\mathcal{W}(T)$ is the class of weak equivalences for a model structure on the category of $\mathcal{A}$-sets for which the cofibrations are the monomorphisms.*

Proof. Suppose, first of all, that $\mathcal{A}$ is the category of ordinal numbers, so that the $\mathcal{A}$-set category is the category of simplicial sets. It is enough to establish the result in this case, since the general statement is then a consequence of Theorem 6.2.

The class $\mathcal{W}_\infty$ of functors $C \to D$ which induce ordinary weak equivalences $BC \to BD$ is contained in $\mathcal{W}(T)$ by Corollary 7.7. Each functor $f\colon C \to D$ in the set T induces a simplicial set map $f_*\colon BC \to BD$, which can be replaced by a cofibration $i(f)\colon BC \to Y$ up to weak equivalence. Let S denote the union of the set of all cofibrations $i(f)$, $f \in T$, along with the set of all anodyne extensions $\Lambda^n_k \subset \Delta^n$. The (Δ^1, S)-model structure on simplicial sets is the localization of the standard model on $\mathbf{S}$ at the set of cofibrations $i(f)$, $f \in T$.

Let $\mathcal{W}'$ be the class of all functors $g\colon C \to D$ such that $g_*\colon BC \to BD$ is a weak equivalence in the (Δ^1, S) model structure. I claim that $\mathcal{W}'$ coincides with the weak equivalence class $\mathcal{W}(T)$.

Note that $\mathcal{W}'$ is a weak equivalence class which contains all elements of T, so that $\mathcal{W}(T) \subset \mathcal{W}'$.

All simplicial set maps $i(f)\colon BC \to Y$ are weakly equivalent to maps $f_*\colon BC \to BD$ induced by generators $f\colon C \to D$ of T, and the functors $i_{\mathbf{\Delta}}BC \to i_{\mathbf{\Delta}}BD$ are equivalent to the functors $f\colon C \to D$ on account of the natural weak equivalences

$$i_{\mathbf{\Delta}}BC \simeq i_{\mathbf{\Delta}}i^*_{\mathbf{\Delta}}C \xrightarrow{\epsilon} C$$

displayed first in (3). It follows from Lemma 7.4 that all (Δ^1, S)-weak equivalences $X \to Y$ induce functors $i_{\mathbf{\Delta}}X \to i_{\mathbf{\Delta}}Y$ which are members of $\mathcal{W}(T)$. Thus, if the functor $g\colon E \to F$ is a member of $\mathcal{W}'$, then the functor $i_{\mathbf{\Delta}}BE \to i_{\mathbf{\Delta}}BF$ is a member of $\mathcal{W}(T)$, and so g is a member of $\mathcal{W}(T)$. □

A map $f\colon X \to Y$ of $\mathcal{A}$-sets is said to be a *simplicial weak equivalence* if the induced map $Bi_{\mathcal{A}}X \to Bi_{\mathcal{A}}Y$ is a weak equivalence of simplicial sets. Recall further (Theorem 6.2, Example 6.3) that the weak equivalences, so defined, are the weak equivalences for a model structure $\mathbf{M}_s$ on the category of $\mathcal{A}$-sets. This model structure satisfies the condition **M1** of Section 5, since $i_{\mathcal{A}}\Delta^a = \mathcal{A}/a$ has a terminal object.

Theorem 7.9. *Suppose that $\mathcal{A}$ is a test category. Suppose that $\mathbf{M}$ is an $(\otimes, S)$-model structure on the category of $\mathcal{A}$-sets which satisfies* **M1** *and is regular. Then every weak equivalence of $\mathbf{M}_s$ is a weak equivalence of $\mathbf{M}$.*

Proof. The class $F(\mathbf{M})$ of all functors $f\colon C \to D$ which induce a weak equivalence $B_hC \to B_hD$ of $\mathbf{M}$ is a weak equivalence class. In particular, the axiom **LF1** follows from the model axioms for $\mathbf{M}$, the axiom **LF2** follows from Lemma 5.5 and **LF3** is a consequence of Lemma 4.7.

If $g\colon C \to D$ is a functor such that $BC \to BD$ is a weak equivalence of simplicial sets, then the induced map $B_hC \to B_hD$ is a weak equivalence of $\mathbf{M}$ by Corollary 7.7.

If $f\colon X \to Y$ is a weak equivalence of $\mathbf{M}_s$, then $Bi_{\mathcal{A}}X \to Bi_{\mathcal{A}}Y$ is a weak equivalence of simplical sets. Thus, $B_hi_{\mathcal{A}}X \to B_hi_{\mathcal{A}}Y$ is a weak equivalence of $\mathbf{M}$ by the previous paragraphs, so that $f\colon X \to Y$ is a weak equivalence of $\mathbf{M}$ by the regularity assumption. □

Lemma 7.10. *Suppose that $\mathcal{A}$ is a test category. Suppose that Y is a fibrant object in the model structure $\mathbf{M}_s$ on the category of $\mathcal{A}$-sets. Then the functor $X \mapsto X \times Y$ preserves weak equivalences.*

Proof. Let $i^*\colon \mathbf{S} \to \mathcal{A}$-**Set** be the functor which is defined by

$$i^*X(a) = \hom(B(\mathcal{A}/a), X),$$

as in the preamble to Lemma 2.11, and recall that i^* is right adjoint to the functor $Z \mapsto Bi_{\mathcal{A}}Z$. Then the canonical morphism $\eta\colon Z \to i^*Bi_{\mathcal{A}}Z$ is isomorphic to the map $\eta\colon Z \to i^*_{\mathcal{A}}i_{\mathcal{A}}Z$, and is therefore a weak equivalence of $\mathbf{M}_s$. The functor

$Z \mapsto Bi_{\mathcal{A}}Z$ preserves trivial cofibrations by Corollary 7.5, so that the functor i^* preserves fibrations.

Let $j\colon Bi_{\mathcal{A}}Y \to Z$ be a trivial cofibration with Z a fibrant simplicial set. Then the composite

$$X \times Y \xrightarrow{1\times\eta} X \times i^* Bi_{\mathcal{A}}Y \xrightarrow{1\times i^* j} X \times i^* Z$$

is the product of the identity on X with a homotopy equivalence $Y \to i^*Z$ of fibrant objects. It follows that Y may be replaced by i^*Z.

The functor $Z \mapsto X \times i^*W$ preserves weak equivalences of simplicial sets W by Corollary 2.16. It follows that the simplicial set Z may be replaced up to weak equivalence by the nerve BC of a small category C.

Observe that $i^*BC = i^*_{\mathcal{A}}C$. Write π for the composite

$$i_{\mathcal{A}}(X \times i^*_{\mathcal{A}}C) \to i_{\mathcal{A}} i^*_{\mathcal{A}}C \xrightarrow{\epsilon} C,$$

which is induced by the projection $X \times i^*_{\mathcal{A}}c \to i^*_{\mathcal{A}}C$. Then there are isomorphisms

$$\pi/c \cong i_{\mathcal{A}}X \times (\epsilon/c) \cong i_{\mathcal{A}}X \times i_{\mathcal{A}} i^*_{\mathcal{A}}(C/c)$$

by Lemma 2.1. The functor $X \mapsto X \times i^*_{\mathcal{A}}(D)$ preserves weak equivalences if the category D has a terminal object, since $i^*_{\mathcal{A}}D$ is aspherical. Also, there is a natural weak equivalence

$$\underrightarrow{\operatorname{holim}}_{c\in C}\, B(\pi/c) \to Bi_{\mathcal{A}}(X \times i^*_{\mathcal{A}}C).$$

It follows that the functor $X \mapsto X \times i^*_{\mathcal{A}}C$ preserves weak equivalences. □

8. Homotopy theory of cubical sets

Let the object $I = i^*_{\square}(\mathbf{1})$ define an interval theory for the category $\square$-**Set** of cubical sets. Let S be the set of vertex maps $* \to \square^n$ of the standard n-cells. Then there is an (I,S)-model structure on the category of cubical sets, as a result of Theorem 6.2.

We shall say that the model structure $\mathbf{M}_s$ on the category of cubical sets arising from Theorem 6.2 and Example 6.3 is the *standard structure*. This is the model structure on $\square$-**Set** whose weak equivalences are those maps $f\colon X \to Y$ which induce weak equivalences $Bi_{\square}X \to Bi_{\square}Y$ of simplicial sets.

A priori, the standard and the (I,S)-model structures on the category of cubical sets are potentially distinct, but we have the following result:

Theorem 8.1. *The class of weak equivalences of the (I,S)-model structure on the category of cubical sets coincides with the class of weak equivalences of the standard model structure $\mathbf{M}_s$ on $\square$-**Set**, and so the two model structures coincide.*

Proof. Every weak equivalence of the (I,S)-model structure is a weak equivalence of $\mathbf{M}_s$.

The (I,S)-model structure on $\square$-**Set** is constructed to satisfy the axiom **M1**. Thus, according to Theorem 7.9, we only need to show that the (I,S)-model structure on the category of cubical sets is regular.

This, however, is a consequence of Lemma 5.18, together with the observation that the cofibrations of the category cubical sets are generated by the inclusions $\partial\square^n \subset \square^n$, provided we can show that all maps

$$\underset{\square^k \to \square^n}{\underrightarrow{\mathrm{holim}}}\ \square^k \to \square^n$$

are (I,S)-equivalences.

We know that $\square^n \to *$ is an (I,S)-equivalence, by construction. It follows that the map

$$\underset{\square^k \to \square^n}{\underrightarrow{\mathrm{holim}}}\ \square^k \to B_h i_\square \square^n$$

is an (I,S)-equivalence. But finally, the category

$$i_\square \square^n \cong \square/\mathbf{1}^{\times n}$$

has a terminal object, and so the cubical set map $B_h i_\square \square^n \to *$ is an (I,S)-equivalence by Lemma 5.5. $\square$

Theorem 8.2. *The standard model structure* $\mathbf{M}_s$ *on the category of cubical sets is proper.*

Proof. On account of Theorem 4.18, it is enough to show that all vertex maps $* \to \square^n$ pull back to weak equivalences along all fibrations $p\colon X \to Y$ for which the base Y is fibrant.

Suppose given a diagram

$$\begin{tikzcd} & & X \arrow[d, "p"] \\ * \arrow[r, "v"'] & \square^n \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

The map v is an anodyne cofibration for the (I,S)-structure and Y is fibrant, so there is a map $x\colon * \to Y$ and a naive homotopy $\square^n \times I \to Y$ from α to the composite

$$\square^n \to * \xrightarrow{x} Y.$$

The standard anodyne cofibrations $d_0, d_1\colon U \to U \times I$ pull back to weak equivalences along p (see the argument for Theorem 4.18), so it follows that the pullback along p of the composite

$$* \xrightarrow{v} \square^n \xrightarrow{\alpha} Y$$

may be replaced by the pullback of the composite

$$* \xrightarrow{v} \square^n \to * \xrightarrow{x} Y.$$

Let F be the fibre of p over the vertex x. Then there are pullback diagrams

$$\begin{tikzcd} F \arrow[r, "v_*"] \arrow[d] & F \times \square^n \arrow[r, "pr"] \arrow[d] & F \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ * \arrow[r, "v"'] & \square^n \arrow[r] & * \arrow[r, "x"'] & Y \end{tikzcd}$$

and the map v_* is a weak equivalence by Lemma 7.10. $\square$

Suppose that $\square^k$ is a fixed standard cell in the category of cubical sets. Recall from Lemma 2.4 and Lemma 2.15 the associated cell category $i_\square\square^k$ is a test category, and that the category of $i_\square\square^k$-sets can be identified with the category of $(\square\text{-}\mathbf{Set})/\square^k$ of cubical sets $\tau\colon X \to \square^k$. The tensor product pairing $\otimes$ for the category of cubical sets determines an interval theory $(\tau, \mathbf{1}^n) \mapsto \tau \otimes \square^n$, where $\tau \otimes \square^n$ is the composite

$$X \otimes \square^n \xrightarrow{pr} X \xrightarrow{\tau} \square^k.$$

Theorem 4.17 determines a $(\otimes, \emptyset)$-model structure on the category of cubical sets over $\square^k$.

The $(\otimes, \emptyset)$-structure for cubical sets over the standard cells $\square^k$ specializes to the $(\otimes, \emptyset)$-structure of Example 4.23 for the full category of cubical sets by taking $k = 0$. It is a consequence of the next result (which is relative to all cells $\square^k$) that the standard structure coincides with the $(\otimes, \emptyset)$-structure for the category of cubical sets.

Lemma 8.3. *Suppose that $\mathcal{A}$ is the test category $i_\square\square^k$, and that $\mathbf{M}$ is the corresponding $(\otimes, \emptyset)$-model structure on the category $(\square\text{-}\mathbf{Set})/\square^k$ of $\mathcal{A}$-sets. Then every weak equivalence of the standard model structure $\mathbf{M}_s$ is a weak equivalence of $\mathbf{M}$.*

Proof. All vertex maps $* \to \square^n \to \square^k$ are trivial cofibrations, so that all morphisms

$$\square^n \to \square^m \to \square^k$$

are weak equivalences of $\mathbf{M}$. In particular, the map

$$\square^n \to \square^k \xrightarrow{1} \square^k$$

to the terminal object is an equivalence of $\mathbf{M}$, so that the condition **M1** is verified for this model structure.

In the diagram

$$\begin{tikzcd} \underrightarrow{\operatorname{holim}}_{\square^r \to \square^n} \square^r \arrow[r] \arrow[d, "\simeq"'] & \square^n \arrow[d, "\simeq"] \\ \underrightarrow{\operatorname{holim}}_{\square^r \to \square^n} \square^k \arrow[r, "\simeq"'] & \square^k \end{tikzcd}$$

of cubical sets over $\square^k$, the indicated maps are weak equivalences of $\mathbf{M}$, so that the map

$$\underrightarrow{\operatorname{holim}}_{\square^r \to \square^n} \square^r \to \square^n$$

is also an equivalence of $\mathbf{M}$, for all standard cells $\square^n \to \square^k$ of $(\square\text{-}\mathbf{Set})/\square^k$. The inclusions in this category are generated by morphisms of the form

$$\partial\square^n \subset \square^n \to \square^k,$$

and it follows from Lemma 6.1 that the model structure $\mathbf{M}$ is regular.

The Lemma is now a consequence of Theorem 7.9. $\square$

Corollary 8.4. *The standard model structure coincides with the* $(\otimes, \emptyset)$*-structure for the category* $(\square - \mathbf{Set})/\square^k$ *of cubical sets over* $\square^k$*, for all* $k \geqslant 0$*.*

Proof. The two theories have the same weak equivalences (and cofibrations), by Lemma 8.3 and the observation that every anodyne weak equivalence is a standard weak equivalence. $\square$

A map of cubical sets $f\colon X \to Y$ is an injective fibration (for the $(\otimes, \emptyset)$-structure) if it has the right lifting property with respect to all inclusions $\sqcap^n_{(i,\epsilon)} \subset \square^n$. A fibration of cubical sets, in the standard structure, is a map which has the right lifting property with respect to all trivial cofibrations $A \subset B$. Every fibration is an injective fibration.

Lemma 8.5. *Every injective fibration* $f\colon X \to \square^k$ *of cubical sets is a fibration.*

Proof. A map

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl] \\ & \square^k & \end{tikzcd}$$

is a standard weak equivalence of cubical sets over $\square^k$ if and only if the map $X \to Y$ of cubical sets is a standard weak equivalence. This is a consequence of the isomorphism

$$i_{\mathcal{A}}(X \to \square^k) \cong i_{\square} X$$

for $\mathcal{A} = i_{\square}\square^k$. It follows that the map g is a standard fibration of cubical sets over $\square^k$ if and only if the map $X \to Y$ is a standard fibration of cubical sets.

The standard and $(\otimes, \emptyset)$-model structures for the category of cubical sets over $\square^k$ coincide (Corollary 8.4), so the two theories have the same fibrant objects. In particular, every injective object of $(\square\text{-}\mathbf{Set})/\square^k$ is fibrant for the standard theory, by Lemma 4.13. $\square$

Theorem 8.6. *Every injective fibration of cubical sets is a fibration.*

Proof. Suppose we know that if a map $q\colon V \to W$ is an injective fibration and a standard weak equivalence, then it is a trivial fibration.

Suppose further that $f\colon X \to Y$ is an injective fibration, and form the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & U \arrow[d, "p"] \\ Y \arrow[r, "j_Y"'] & \mathcal{L}(Y) \end{tikzcd}$$

where the horizontal maps are trivial cofibrations, $\mathcal{L}(Y)$ is fibrant and p is an injective fibration (this can be done in the $(\otimes, \emptyset)$-model structure). Then Lemma 4.13 implies that p is a fibration. It follows that the induced map $p_*\colon Y \times_{\mathcal{L}(Y)} U \to Y$ is a fibration.

The map $X \to Y \times_{\mathcal{L}(Y)} U$ is a weak equivalence by properness (Theorem 8.2), and it has a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & W \arrow[d, "q"] \\ & Y \times_{\mathcal{L}(Y)} U \end{tikzcd}$$

where i is an $(\otimes, \emptyset)$-anodyne cofibration and q is an injective fibration. Then the map q is also a weak equivalence, and so it is a trivial fibration by our assumption. One sees easily that f is a retract of the composite $p_* q$, and so f is a fibration.

Suppose now that the cubical set map $q\colon V \to W$ is an injective fibration and a weak equivalence. Then in all diagrams

$$\begin{tikzcd} \square^m \times_W V \arrow[r, "\tau_*"] \arrow[d, "q_*"'] & \square^n \times_W V \arrow[r] \arrow[d, "q_*"] & V \arrow[d, "q"] \\ \square^m \arrow[r, "\tau"'] & \square^n \arrow[r, "\sigma"'] & W \end{tikzcd}$$

the maps labelled q_* are fibrations (Lemma 8.5), and the map τ_* is a weak equivalence by properness. It is therefore a consequence of Quillen's Theorem B that all diagrams of simplicial set maps

$$\begin{tikzcd} Bi_{\square}(\square^n \times_W V) \arrow[r] \arrow[d] & Bi_{\square} V \arrow[d] \\ Bi_{\square}(\square^n) \arrow[r, "\sigma_*"'] & Bi_{\square} W \end{tikzcd}$$

are homotopy cartesian: recall that there are isomorphisms

$$i_{\square}(\square^n \times_W V) \cong f_*/\sigma$$

for all cells $\sigma\colon \square^n \to W$ of W. The map $Bi_{\square}(V) \to Bi_{\square}(W)$ is a weak equivalence by assumption, so that all induced maps $\square^n \times_W V \to \square^n$ are weak equivalences, and hence trivial fibrations. It follows that $q\colon V \to W$ has the right lifting property with respect to all inclusions $\partial\square^n \subset \square^n$, and is therefore a trivial fibration, as claimed. $\square$

Recall from Section 3 that the triangulation $|X|$ of a cubical set X is the simplicial set defined by

$$|X| = \varinjlim_{\square^n \to X} B(\mathbf{1}^n).$$

The cells $\sigma\colon \square^n \to X$ of a cubical set X induce simplicial set maps

$$B(\mathbf{1}^n) \cong |\square^n| \xrightarrow{\sigma_*} |X|,$$

and these maps together determine a map

$$f_X\colon \underset{\sigma:\square^n \to X}{\underrightarrow{\mathrm{holim}}} |\square^n| \to |X|$$

in the obvious way. Observe that the canonical map

$$\pi_X\colon \underset{\sigma:\square^n\to X}{\underrightarrow{\mathrm{holim}}} |\square^n| \to Bi_\square X$$

is a weak equivalence for all cubical sets X. This is a consequence of the fact that all triangulations $|\square^n|$ are contractible simplicial sets.

Lemma 8.7. *The map*

$$f_X\colon \underset{\sigma:\square^n\to X}{\underrightarrow{\mathrm{holim}}} |\square^n| \to |X|$$

is a weak equivalence of simplicial sets.

Proof. In fact, the realization functor preserves projective cofibrations and pointwise weak equivalences for all diagrams of cubical sets, and of course preserves all colimits. It therefore preserves all homotopy colimits.

The standard model structure on the category of cubical sets is regular — this is demonstrated in the proof of Lemma 8.3. The regularity property means that the canonical map

$$\underset{\sigma:\square^n\to X}{\underrightarrow{\mathrm{holim}}} \square^n \to X$$

is a weak equivalence of cubical sets. The desired result follows, by applying the realization functor. □

As a consequence, the description of standard weak equivalence of cubical sets given here, via the functor $X \mapsto Bi_\square X$, coincides with the geometric description of weak equivalence defined by the functor $X \mapsto |X|$. The standard model structure $\mathbf{M}_s$ for cubical sets therefore coincides with the geometric model structure given in [**10**]. The triangulation functor $|\ |\colon \square\text{-}\mathbf{Set} \to \mathbf{S}$ also preserves and reflects weak equivalences of cubical sets.

The right adjoint $S\colon \mathbf{S} \to \square\text{-}\mathbf{Set}$ of the triangulation functor is defined by

$$S(X)_n = \hom(B(\mathbf{1}^n), X).$$

This functor is also (see Lemma 2.13) the functor $i^*\colon \mathbf{S} \to \square\text{-}\mathbf{Set}$ induced by the inclusion $i\colon \square \to \mathbf{cat}$. It is plainly the case that all of the categories $\mathbf{1}^n$ have terminal objects, and we know from Corollary 3.8 that the cubical set $i^*(\Delta^1) = B_\square(\mathbf{1})$ is aspherical. It follows from Corollary 2.14 and Lemma 3.11 that the functor S preserves and reflects weak equivalences of simplicial sets.

Theorem 8.8. *The triangulation functor $|\ |$ and its right adjoint S induce an adjoint equivalence of homotopy categories*

$$\mathrm{Ho}(\square\text{-}\mathbf{Set}) \simeq \mathrm{Ho}(\mathbf{S}).$$

The adjunction maps $\eta\colon X \to S|X|$ and $\epsilon\colon |SY| \to Y$ are natural weak equivalences.

Proof. There are natural weak equivalences

$$i_{\mathbf{\Delta}}^* i_{\square} X \simeq B i_{\square} X \simeq |X|$$

for all simplicial sets X: the first comes from Equation (3) and the second is a consequence of Lemma 8.7. The functor $i_{\mathbf{\Delta}}^* i_{\square}$ induces an equivalence

$$i_{\mathbf{\Delta}}^* i_{\square}\colon \operatorname{Ho}(\square\text{-}\mathbf{Set}) \xrightarrow{\simeq} \operatorname{Ho}(\mathbf{S})$$

by Theorem 6.2 (Example 6.3). This functor is, in particular, fully faithful. It follows that the triangulation functor $|\ |$ induces a fully faithful functor $|\ |$ on the level of homotopy categories. The functor S also preserves weak equivalences, and therefore induces a functor

$$S\colon \operatorname{Ho}(\mathbf{S}) \to \operatorname{Ho}(\square\text{-}\mathbf{Set})$$

which is right adjoint to $|\ |$. From the collection of pictures

$$\begin{tikzcd} {[Y,Z]} \arrow[r, "\cong"] \arrow[d, "\eta_*"'] & {[|Y|,|Z|]} \\ {[Y,S|Z|]} \arrow[ur, "\cong"'] & \end{tikzcd}$$

one sees that composition with the natural cubical set morphism $\eta\colon Z \to S|Z|$ is an isomorphism for all maps $Y \to Z$ in the homotopy category. It follows that η is an isomorphism in $\operatorname{Ho}(\square\text{-}\mathbf{Set})$, and hence that η is a weak equivalence of cubical sets — see [**5**, I.1.14]. It follows that $S\epsilon$ is a weak equivalence for all natural simplicial set maps $\epsilon\colon |S(Y)| \to Y$. The functor S reflects weak equivalences, so all canonical maps ϵ are weak equivalences of simplicial sets. $\square$

At the risk of adding a final bit of notational confusion, I shall define the *topological realization* $|X|$ of a cubical set X by setting

$$|X| = \varinjlim_{\square^n \to X} |B(\mathbf{1}^n)|,$$

where $|B(\mathbf{1}^n)|$ is the topological realization of the simplicial set $B(\mathbf{1}^n)$. The object $|B(\mathbf{1}^n)|$ is, in other words, an ordinary topological hypercube. The topological realization functor has a right adjoint

$$S_{\square}\colon \mathbf{Top} \to \square\text{-}\mathbf{Set}$$

which is defined for a topological space Y by

$$S_{\square}(Y)_n = \hom(|B(\mathbf{1}^n)|, Y).$$

Write $S_{\Delta}\colon \mathbf{Top} \to \mathbf{S}$ for the ordinary singular functor taking values in simplicial sets. The topological realization of a cubical set X is naturally isomorphic to the topological realization of the triangulation $|X| \in \mathbf{S}$, so there is a corresponding natural isomorphism

$$S_{\square}(Y) \cong S(S_{\Delta}(Y))$$

relating the right adjoints.

The following result is the excision statement for cubical sets:

Theorem 8.9. *Suppose that a topological space Y is covered by open subsets U_1 and U_2. Then the canonical map*

$$S_\Box U_1 \cup S_\Box U_2 \to S_\Box Y$$

is a weak equivalence of cubical sets.

Proof. The idea of proof is to show that the induced map of triangulations

$$|S_\Box U_1| \cup |S_\Box U_2| \cong |S_\Box U_1 \cup S_\Box U_2| \to |S_\Box Y|$$

is a weak equivalence of simplicial sets. There is a natural isomorphism $S_\Box Z \cong S(S_\Delta Z)$ for all topological spaces Z, and it follows from Theorem 8.8 that there is a natural weak equivalence

$$|S_\Box Z| \cong |S(S_\Delta Z)| \xrightarrow{\epsilon} S_\Delta Z,$$

which will be denoted by ϵ. It follows that there is a commutative diagram

$$\begin{tikzcd} {|S_\Box U_1| \cup |S_\Box U_2|} \arrow[r] \arrow[d, "\epsilon_*"'] & {|S_\Box Y|} \arrow[d, "\epsilon"] \\ S_\Delta U_1 \cup S_\Delta U_2 \arrow[r] & S_\Delta Y \end{tikzcd}$$

in which the vertical maps are weak equivalences of simplicial sets by a patching lemma argument. The map

$$S_\Delta U_1 \cup S_\Delta U_2 \to S_\Delta Y$$

is a weak equivalence of simplicial sets, by excision for simplicial sets [**11**, Theorem 20]. □

Theorems 8.8 and 8.9 also appear in [**10**]. In particular, Theorem 8.9 appears as Theorem 2.7 in that paper, and is the central device given there for establishing the Theorem 8.9. The proof of Theorem 8.9 which is given in [**10**] is a direct and somewhat dirty subdivision argument.

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