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The Construction of the d + 1-Dimensional Gaussian Droplet

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Dedicated to the memory of Roland Dobrushin, who passed away on 13 November 1995

Abstract: The aim of this note is to study the asymptotic behavior of a gaussian random field, under the condition that the variables are positive and the total volume under the variables converges to some fixed number $v > 0$. In the context of Statistical Mechanics, this corresponds to the problem of constructing a droplet on a hard wall with a given volume. We show that, properly rescaled, the profile of a gaussian configuration converges to a smooth hypersurface, which solves a quadratic variational problem. Our main tool is a scaling dependent large deviation principle for random hypersurfaces.

1. Introduction

What is the most probable shape for a random interface on a wall under the constraint that this interface stays above the wall, is pinned down at its boundary, and moreover that the volume between the wall and the interface is fixed? This question about the shape of a droplet has been treated in one dimension when the random interface (i.e., here a random curve) is a classical random walk, see [3, 4]. Dynamics of such droplets are also of interest and are studied in [1], see also [9] for questions dealing with the fluctuations around this most probable shape.

In dimension larger than one, the problem is much harder; we treat here a simple model: the case of a gaussian interface. We show that, under appropriate scaling, a large deviation principle holds which enables us to find this limiting shape as the solution of an elliptic PDE. One of the major difficulties is to control the positivity condition at the boundary of the droplet. We present a scaling-dependent result which relies essentially upon the "entropic repulsion" phenomenon as exhibited in [6, 7].

More precisely, let $\Lambda = (0,1)^d$ be the unit cube in $\mathbb{R}^d$, $V_N = N\Lambda \cap \mathbb{Z}^d$, $d \geqq 2$, be the (discrete) box of side $(N-1)$ and set $\Omega_N = \mathbb{R}^{V_N}$. Our a priori distribution is the centered gaussian field P_N^0 on Ω_N, with density with respect to the Lebesgue measure $\lambda_N(dX) = \prod_{i \in V_N} dX(i)$, of the form

$$P_N^0(dX) = \frac{1}{Z_N} \exp\left(-\frac{1}{2} \sum_{\{i,j\} \cap V_N \neq \emptyset} Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2 \right) \lambda_N(dX),$$

where Z_N is a normalizing constant, $Q_{\mathrm{d}}(i,j) = \frac{1}{2d} 1_{|i-j|=1}$ is the transition matrix of the simple random walk on $\mathbb{Z}^d$, and we set $X(j) = 0$ for $j \notin V_N$. Thus the spins are "tied down" at the boundary of V_N. P_N^0, sometimes called the *Euclidean massless free field*, can be viewed as the Gibbs distribution to the nearest neighbor quadratic interaction $\mathscr{J} = \{J_{\{i,j\}}(X) = Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2,\ \{i,j\} \subseteq \mathbb{Z}^d\}$ with 0-boundary conditions.

For a given configuration $X \in \Omega_N$, we introduce the continuous profile $t : \Lambda \to X_N(t) \in \mathbb{R}$:

$$X_N(t) = \sum_{i \in V_N} h(Nt - i)X(i), \quad \text{where } h(t) = \prod_{i=1}^{d} (1 - |t_i|)1_{|t_i| \leqq 1}.$$

X_N is the linear interpolation of X along the bonds of $\frac{1}{N}\mathbb{Z}^d$; in particular $X_N(i/N) = X(i)$, $i \in V_N$. We will use different scalings $\varepsilon_N \searrow 0$ and set $\bar{X}_N = \varepsilon_N X_N$.

In constructing our droplet, we need two kinds of conditions. First we assume that the droplet lies on a hard wall, that is, the variables are restricted to the *positive configurations*:

$$\Omega_N^+ = \{X \in \Omega_N : X(i) \geqq 0,\ i \in V_N\} = \{\bar{X}_N : \bar{X}_N(t) \geqq 0,\ t \in \Lambda\}.$$

Moreover, the total volume under the variables converges as $N \to \infty$ to a given $v > 0$:

$$A_N(v, \delta_N) = \left\{ X \in \Omega_N : \left| \frac{\varepsilon_N}{N^d} \sum_{k \in V_N} X(k) - v \right| \leqq \delta_N \right\}$$

$$= \{\bar{X}_N : |\langle \bar{X}_N, 1_\Lambda \rangle - v| \leqq \delta_N\}, \quad \text{for some } \delta_N \searrow 0,$$

where $\langle f, g \rangle \equiv \int_\Lambda f(t)g(t)\,dt$ denotes the scalar product in $L^2(\Lambda)$.

Our purpose is to show that $\mathscr{P}_N^+$,

$$\mathscr{P}_N^+(\,\cdot\,) \equiv P_N^0(\,\cdot\,\mid \Omega_N^+ \cap A_N(v, \delta_N)) \circ \bar{X}_N^{-1},$$

the law of the profile conditioned on $\Omega_N^+ \cap A_N(v, \delta_N)$, concentrates to a limiting hypersurface ψ_v, solution of the variational problem

$$\inf\left\{ \frac{1}{2d} \|\nabla \psi\|_{L^2(\Lambda)}^2 :\ \psi \in H_0^1(\Lambda),\ \psi \geqq 0,\ \langle \psi, 1_\Lambda \rangle = v \right\}, \tag{1.1}$$

where $H_0^1(\Lambda)$ denotes the usual Sobolev space, i.e. the closure of $C_0^\infty(\Lambda)$ with respect to $\|\nabla \psi\|_{L^2(\Lambda)}$. In Sect. 3 below, we give the following explicit expression for ψ_v: let $\mathfrak{G}_\Lambda$ be the Green function of the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$, then

$$\psi_v(t) = c_v\, \mathfrak{G}_\Lambda 1_\Lambda(t) = c_v \int_\Lambda \mathfrak{g}_\Lambda(t,s)\,ds, \quad t \in \Lambda,$$

with $c_v = v/\langle 1_\Lambda, \mathfrak{G}_\Lambda 1_\Lambda \rangle$. That is, ψ_v solves the Poisson equation

$$-\Delta\psi_v = c_v 1_\Lambda \quad \text{with} \lim_{\substack{t \to s \\ t \in \Lambda}} \psi_v(t) = 0 \quad \text{for } s \in \partial\Lambda \,.$$

Our main tool will be a large deviation principle for $\{P_N^0 \circ \bar{X}_N^{-1}, N \in \mathbb{N}\}$ derived in Sect. 2. This large deviation principle has the particularity that its topology improves with ε_N converging faster to 0, cf. Theorem 2.5. The main problem, due to the 0 boundary condition, is the positivity condition Ω_N^+ *at the boundary* of V_N. The convergence, based on the a priori estimate

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log P_N^0(\Omega_N^+) = -\kappa_d > -\infty \,, \tag{1.2}$$

derived in [7], depends critically on the rate at which $\varepsilon_N \searrow 0$. For *fast scaling* $\varepsilon_N = o(N^{-1/2})$ and $\delta_N \searrow 0$ with $\delta_N \geqq O(\varepsilon_N^2 N)$, our main result, Proposition 3.6, states

$$\lim_{N\to\infty} \mathscr{P}_N^+(B_2(\psi_v;\varepsilon)) = 1, \quad \text{for all } \varepsilon > 0 \,, \tag{1.3}$$

where $B_2(\psi_v;\varepsilon) \equiv \{\phi \in L^2(\Lambda) : \|\psi_v - \phi\|_{L^2(\Lambda)} < \varepsilon\}$ is the open ball in $L^2(\Lambda)$.

On the other hand, for *slow scaling* of the form $\varepsilon_N = o(1/\log N)$ for $d = 2$ and $\varepsilon_N = o(1/\log^{1/2}(N))$ for $d \geqq 3$, the positivity condition Ω_N^+ has to be approximated by

$$\Omega_N^+(\eta_N) = \{X \in \Omega_N : X(i) \geqq 0,\ i \in V_N,\ \mathrm{dist}(i, V_N^{\mathrm{C}}) \geqq N\eta_N\}, \quad \text{for some } \eta_N \searrow 0 \,.$$

More precisely, let

$$\tilde{\mathscr{P}}_N^+ \equiv P_N^0(\,\cdot\,|\,\Omega_N^+(\eta_N) \cap A_N(v,\delta_N)) \circ \bar{X}_N^{-1}$$

be the corresponding conditional distribution, then using the large deviation principle and some estimates for $P_N^0(\Omega_N^+(\eta_N))$ of [7], we show in Proposition 3.9 the convergence

$$\lim_{N\to\infty} \tilde{\mathscr{P}}_N^+(B_2(\psi_v;\varepsilon)) = 1 \,, \tag{1.4}$$

for each $\varepsilon > 0$ and $\eta_N, \delta_N \searrow 0$ with

$$\delta_N \geqq \begin{cases} O(\varepsilon_N^2 \log^2(N\eta_N)/\eta_N) & d = 2 \\ O(\varepsilon_N^2 \log(N\eta_N)/\eta_N) & d \geqq 3 \,. \end{cases}$$

This type of results has been proved for the one-dimensional droplet using local central limit theorem techniques, in the gaussian setting in [3], and in [4], for general interactions. The one-dimensional case is very special, since the spins can be represented as sums of independent variables. In particular, this allows to improve the convergence in (1.3) to the supremum norm.

The difference between the two regimes in (1.4) and (1.3) can be intuitively explained in the following way: choosing a volume v/ε_N large enough in (1.3) pushes the interface far away from the wall, so that the variables remain positive up to the boundary. For very slow scalings $\varepsilon_N \geqq O(1/\log N)$ $d = 2$, and $\varepsilon_N = O(1/\log^{1/2}(N))$ $d \geqq 3$, we expect that the positivity condition Ω_N^+, rather than the volume condition $A_N(v,\delta_N)$, characterizes the limiting hypersurface.

Our method, based on the random walk representation of P_N^0, cf. [5], allows us to treat slightly more general finite range quadratic interaction potentials $\mathscr{J}$ and non-linear conditions of the type

$$\left\{\bar{X}_N : \left|\int_\Lambda F(t,\bar{X}_N(t))\,dt - v\right| \leqq \delta\right\}$$

for some $F \in C(\Lambda \times \mathbb{R})$. However a generalization to non-quadratic interactions, i.e. to non-gaussian models, cf. [2], remains open.

In Sect. 2 we derive the large deviation principle for $\{P_N^0 \circ \bar{X}_N^{-1},\ N \in \mathbb{N}\}$. In Sect. 3 we give a proof of (1.3) and (1.8) and present some further convergence results of the approximate microcanonical distribution. Finally Sect. 4 contains a fluctuation result for the exact microcanonical distribution, which is the d-dimensional pendant of [9].

2. The Large Deviation Principle

Let $\{Q(i,j),\ i,j \in \mathbb{Z}^d\}$ be the stationary, symmetric, irreducible transition matrix of a random walk $\{\xi_n,\ n \in \mathbb{N}\}$ on $\mathbb{Z}^d$. We will assume that Q is of finite range $R > 0$. Let $\tau_N \equiv \inf\{n \geqq 0 : \xi_n \notin V_N\}$ be the first exit time of V_N and G_N,

$$G_N(i,j) \equiv \mathbb{E}_i\left[\sum_{n=0}^{\tau_N} 1_j(\xi_n)\right], \quad i,j \in V_N\,,$$

be the corresponding Green function. Here and below $\mathbb{P}_i$ and $\mathbb{E}_i$ denote the probability and expectation with respect to the random walk starting at $i \in \mathbb{Z}^d$. Next let P_N^0 be the centered Gaussian field on Ω_N with covariances

$$\mathrm{cov}_{P_N^0}(i,j) = E_N^0[X(i)X(j)] = G_N(i,j)\,, \quad i,j \in V_N\,.$$

We give a Gibbsian representation of P_N^0: consider the interaction potential $\mathscr{J} = \{J_F : F \subseteq \mathbb{Z}^d\}$,

$$J_F(X) = \begin{cases} Q(i,j)(X(i)-X(j))^2 & F = \{i,j\} \\ 0 & \text{otherwise}\,, \end{cases}$$

and let

$$\mathscr{H}_N^0(X) = \sum_{F:F\cap V_N \neq \emptyset} J_F(X) = \frac{1}{2}\sum_{\{i,j\}\cap V_N \neq \emptyset} Q(i,j)(X(i)-X(j))^2\,,$$

(we set $X(j) = 0$ for $j \notin V_N$) be the Hamiltonian of the box V_N. Then P_N^0 is the corresponding 0-boundary Gibbs distribution on V_N, i.e.,

$$P_N^0(dX) = \frac{1}{Z_N}\exp(-\mathscr{H}_N^0(X))\lambda_N(dX)\,,$$

with $Z_N = \int_{\Omega_N}\exp(-\mathscr{H}_N^0(X))\lambda_N(dX)$, cf. [5]. Let A be the symmetric $d \times d$ matrix associated with the covariances of Q:

$$|y|_A^2 = y \cdot Ay = \sum_{k\in\mathbb{Z}^d}(y \cdot k)^2 Q(k,0), \quad y \in \mathbb{R}^d\,,$$

and define the good rate function $I : L^2(\Lambda) \to [0,\infty]$:

$$I(\phi) = \begin{cases} \frac{1}{2}\| |\nabla\phi|_\Lambda \|^2_{L^2(\Lambda)}, & \phi \in H_0^1(\Lambda) \\ \infty & \text{otherwise}. \end{cases}$$

Next, let $h : [-1,1]^d \to [0,1]$ be a piecewise continuous function with compact support in $[-1,1]^d$ such that $h(0) = 1$, $\int_{[-1,1]^d} h(t)dt = 1$, and set

$$\bar{X}_N(t) = \varepsilon_N \sum_{k \in V_N} h(Nt - k)X(k), \quad t \in \Lambda,$$

where $\{\varepsilon_N\}$ is a positive sequence converging to 0 as $N \to \infty$. The aim of this section is to derive a large deviation principle for $\{P_N^0 \circ \bar{X}_N^{-1},\ N \in \mathbb{N}\}$. The quality of the result depends critically upon the rate at which ε_N converges to 0. More precisely, we consider 4 regimes:

$$\varepsilon_N = \begin{cases} o(1/\log^{1/2}(N)) & d = 2 \\ o(1) & d \geqq 3, \end{cases} \quad \text{very slow regime}, \tag{2.1}$$

$$\varepsilon_N = \begin{cases} o(1/\log N) & d = 2 \\ o(1/\log^{1/2}(N)) & d \geqq 3, \end{cases} \quad \text{slow regime}, \tag{2.2}$$

$$\varepsilon_N = o(N^{-1/2}) \quad d \geqq 2, \quad \text{fast regime}, \tag{2.3}$$

$$\varepsilon_N = N^{-1} \quad d \geqq 2, \quad \text{very fast regime}. \tag{2.4}$$

In case (2.4), we assume that the function h is of the form $h(t) = \prod_{i=1}^d \rho(t_i)$, where ρ is piecewise differentiable on $[-1,1]$ with $\rho(0) = 1$, $\rho(1) = 0$, and

$$\rho(t_i) = \rho(-t_i) = 1 - \rho(1 - t_i), \quad t_i \in [0,1].$$

Finally for a Borel measurable set $\Gamma \in \mathscr{B}(L^p(\Lambda))$, $p \in [2,\infty]$, we denote by $\Gamma^{\circ,p}$ and $\overline{\Gamma}^p$, the interior, respectively the closure, of Γ in $L^p(\Lambda)$. Also $B_p(\psi;\varepsilon)$ is the open ball in $L^p(\Lambda)$.

The main result of this section is the following large deviation principle:

Theorem 2.5. *Set* $\varepsilon_N' = \varepsilon_N^2 N^{2-d}$ *and let* $\Gamma \in \mathscr{B}(L^2(\Lambda))$. *Then in the very slow regime* (2.1), *we have a full large deviation principle in the* L^2*-norm*:

$$\begin{aligned} -\inf_{\Gamma^{\circ,2}} I &\leqq \liminf_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in \Gamma) \\ &\leqq \limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in \Gamma) \leqq -\inf_{\overline{\Gamma}^2} I. \end{aligned} \tag{2.6}$$

Next, in the slow regime (2.2), *we get a lower bound in* L^∞*-norm*:

$$\liminf_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in \Gamma) \geqq -\inf_{\Gamma^{\circ,\infty}} I. \tag{2.7}$$

Finally, in the very fast regime (2.4), *for each* $2 \leqq p < 2d/(d-2)$, *we get an upper bound in* L^p*-norm*:

$$\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in \Gamma) \leqq -\inf_{\overline{\Gamma}^p} I. \tag{2.8}$$

Note that (2.8) is optimal, since, by Sobolev's embedding theorem, I has compact level sets in $L^p(\Lambda)$ for $2 \leqq p < 2d/(d-2)$. The proof of Theorem 2.5 is given in several lemmas, following the usual pattern of large deviations. We first give a Legrendre transform representation of our rate function I: let $\{W_s : s \geqq 0\}$ be the diffusion process on $\mathbb{R}^d$ generated by

$$\Delta_A = \sum_{i,j=1}^{d} A_{i,j} \frac{\partial^2}{\partial x_i \partial x_j},$$

and let $\mathfrak{G}_\Lambda$ be the Green operator corresponding to Dirichlet boundary conditions:

$$\mathfrak{G}_\Lambda f(t) = \int_\Lambda g_\Lambda(t,s) f(s)\, ds = \mathbb{E}_t \left[\int_0^{\tau(W)} f(W_s)\, ds \right] \quad t \in \Lambda,\ f \in C_b(\Lambda),$$

where $\tau(W) = \inf\{s \geqq 0 : W_s \notin \Lambda\}$ is the first exit time of Λ.

Lemma 2.9. *I is the Legendre transformation of $\frac{1}{2}\mathfrak{G}_\Lambda$:*

$$I(\phi) = \sup_{f \in C_b(\Lambda)} \left\{ \langle f, \phi \rangle - \frac{1}{2} \langle f, \mathfrak{G}_\Lambda f \rangle \right\} \quad \phi \in L^2(\Lambda).$$

Proof. Let $\{l_n : n \in \mathbb{N}^d\} \subseteq (0, \infty)$ and $\{e_n : n \in \mathbb{N}^d\} \in C_b^\infty(\Lambda)$ be the eigenvalues and eigenfunctions of the self-adjoint extension of $-\Delta_A$ corresponding to Dirichlet boundary conditions:

$$-\Delta_A e_n = l_n e_n \quad \text{and} \quad \lim_{x \to a,\, x \in \Lambda} e_n(x) = 0 \quad \text{for } a \in \partial\Lambda,$$

cf. Sect. 8.1 of [15]. Then

$$\langle f, \mathfrak{G}_\Lambda f \rangle = \sum_{n \in \mathbb{N}^d} \frac{1}{l_n} \langle f, e_n \rangle^2,$$

and therefore

$$\sup_{f \in C_b(\Lambda)} \left\{ \langle f, \phi \rangle - \frac{1}{2} \langle f, \mathfrak{G}_\Lambda f \rangle \right\} = \frac{1}{2} \sum_{n \in \mathbb{N}^d} l_n \langle f, e_n \rangle^2 = I(\phi). \qquad \square$$

Lemma 2.10. *Let $\phi \in C_b(\Lambda)$, then*

$$\lim_{N \to \infty} \frac{1}{\varepsilon_N'} E_N^0[\langle \bar{X}_N, \phi \rangle^2] = \langle \phi, \mathfrak{G}_\Lambda \phi \rangle. \tag{2.11}$$

Proof. Note that $\langle \bar{X}_N, \phi \rangle = \varepsilon_N N^{-d} \sum_{k \in V_N} \bar{\phi}_N(k) X(k)$, where

$$\bar{\phi}_N(k) = N^d \int_\Lambda h(Nt - k) \phi(t)\, dt = \int_{[-1,1]^d} h(t) \phi(t/N + k/N)\, dt, \quad k \in V_N.$$

Thus

$$\begin{aligned} \frac{1}{\varepsilon_N'} E_N^0[\langle \bar{X}_N, \phi \rangle^2] &= N^{-2-d} \sum_{k,j \in V_N} \bar{\phi}_N(k) G_N(k,j) \bar{\phi}_N(j) \\ &= N^{-2-d} \sum_{k \in V_N} \bar{\phi}_N(k) \mathbb{E}_k \left[\sum_{n=0}^{\tau_N} \bar{\phi}_N(\xi_n) \right]. \end{aligned}$$

Let $\{\xi_s^N,\ s \geqq 0\}$ be the rescaled random walk given by $\xi_s^N = \frac{\xi_{[N^2 s]}}{N}$ and set $\tau(\xi^N) = \inf\{\xi_s^N : s \geqq 0,\ \xi_s^N \notin \Lambda\}$, then

$$\begin{aligned} N^{-2} \sum_{k \in V_N} \bar{\phi}_N(k) \mathbb{E}_k \left[\sum_{n=0}^{\tau_N} \bar{\phi}_N(\xi_n) \right] &= \sum_{k \in V_N} \bar{\phi}_N(k) \int_{[-1,1]^d} h(t) \mathbb{E}_{k/N} \left[\int_0^{\tau(\xi^N)} \phi(\xi_s^N + t/N)\, ds \right] dt \\ &= N^d \mathbb{E}_{\mu_N}[F(\xi^N)]\,, \end{aligned}$$

where $F(\xi^N) = \int_0^{\tau(\xi^N)} \phi(\xi_s^N)\, ds$ and μ_N is the distribution on Λ given by

$$\int_\Lambda f(t) \mu_N(dt) = N^{-d} \int_{[-1,1]^d} h(t) \sum_{k \in V_N} f(t/N + k/N) \bar{\phi}_N(k)\, dt\,.$$

We may assume that $\phi \geqq 0$ and $\mu_N(\Lambda) = 1$ (otherwise write $\phi = \phi_+ - \phi_-$ and rescale). Then, with respect to the weak convergence on Λ,

$$\mu_N(dt) \Rightarrow \mu(dt) = \phi(t)\, dt\,. \tag{2.12}$$

By the invariance principle, we know that, with respect to the weak convergence on the Skorohod space $D_{\mathbb{R}^d}[0,\infty)$,

$$\mathbb{P} \circ \{\xi_s^N,\ s \geqq 0\}^{-1} \Rightarrow \mathbb{P} \circ \{W_s,\ s \geqq 0\}^{-1}\,, \tag{2.13}$$

cf. Theorem 1.2, p. 278 of [10]. We would like to apply (2.12) and (2.13) in order to prove the convergence

$$\lim_{N \to \infty} \mathbb{E}_{\mu_N}[F(\xi^N)] = \mathbb{E}_\mu[F(W)] = \int_\Lambda \mathbb{E}_t \left[\int_0^{\tau(W)} \phi(W_s)\, ds \right] \phi(t)\, dt = \langle \phi, \mathfrak{G}_\Lambda \phi \rangle\,.$$

However, $F \notin C_b(D_{\mathbb{R}^d}[0,\infty))$, since τ is neither bounded, nor continuous. The unboundedness is easily taken care of with $F_T(W) = \int_0^{\tau(W) \wedge T} \phi(W_s)\, ds$ and letting $T \to \infty$. Next, due to the regularity of $\partial\Lambda$, we have

$$\lim_{W' \to W} \tau(W') = \tau(W) \quad \text{for } \mathbb{P}_\mu \text{ a.a. } W\,.$$

This implies

$$\lim_{N \to \infty} \mathbb{E}_{\mu_N}[F_T(\xi^N)] = \mathbb{E}_\mu[F_T(W)]$$

for each $T \geqq 0$, cf. Exercise 8.2.38 of [15], and concludes the proof. □

The crucial step in our proof of Theorem 2.5 will be the exponential tightness. We first deal with the simpler case (2.4). Note that, by Sobolev's embedding Theorem, for each $L > 0$, the ball $\{\phi \in H_0^1(\Lambda) : \|\nabla\phi\|_{L^2(\Lambda)} \leqq L\}$ is compact in $L^p(\Lambda)$, $2 \leqq p < 2d/(d-2)$.

Lemma 2.14. *Assume* (2.4). *There exists* $\alpha > 0$, *such that*

$$\varlimsup_{N \to \infty} N^{-d} \log E_N^0[\exp(\alpha N^d \|\nabla \bar{X}_N\|_{L^2(\Lambda)}^2)] < \infty\,. \tag{2.15}$$

In particular

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{-d}\log P_N^0(\|\nabla \bar{X}_N\|_{L^2(\Lambda)} > L) = -\infty \,. \tag{2.16}$$

Proof. Let us first verify, that, for some constant $c_1 < \infty$,

$$N^d\|\nabla \bar{X}_N\|^2_{L^2(\Lambda)} \leqq c_1 \sum_{\{i,j\}\cap V_N \neq \emptyset} Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2, \tag{2.17}$$

where we set $X(j)=0$ for $j\notin V_N$. Write $h_{d-1}(t)=\prod_{i=2}^d \rho(t_i)$ and $e(1)=(1,0,\dots,0)$, then

$$\begin{aligned}\frac{\partial}{\partial t_1}\bar{X}_N(t) &= \frac{1}{N}\sum_{i\in V_N}\frac{\partial}{\partial t_1}h(Nt-i)X(i) = \sum_{i\in V_N}\rho'(Nt_1-i_1)h_{d-1}(Nt-i)X(i)\\ &= \sum_{i\in V_N-e(1)}\rho'(Nt_1-i_1-1)h_{d-1}(Nt-i)X(i+e(1))\\ &= -\sum_{i\in V_N-e(1)}\rho'(Nt_1-i_1)h_{d-1}(Nt-i)X(i+e(1))\,,\end{aligned}$$

since $\rho'(s)=-\rho'(s-1)$ under (2.4). Thus adding the two last lines yields

$$\begin{aligned}\left|\frac{\partial}{\partial t_1}\bar{X}_N(t)\right| &\leqq \frac{c_2}{2}\sum_{i\in V_{N+1}} h_{d-1}(Nt-i)\,|X(i+e(1))-X(i)|\\ &= \frac{c_2}{2}\sum_{i\in V^{\mathrm{e}}_{N+1}} h_{d-1}(Nt-i)\,|X(i+e(1))-X(i)|\\ &\quad+\frac{c_2}{2}\sum_{i\in V^{\mathrm{o}}_{N+1}} h_{d-1}(Nt-i)\,|X(i+e(1))-X(i)|\;,\end{aligned}$$

for some $c_2<\infty$, where V^{e}_{N+1}, respectively V^{o}_{N+1}, denote the odd and even points of V_{N+1}. Note that the supports of $h(N\,\cdot\,-i)$ and $h(N\,\cdot\,-j)$ are disjoints for $i\neq j$ with $i,j\in V^{\mathrm{e}}_{N+1}$ or $i,j\in V^{\mathrm{o}}_{N+1}$. This yields

$$N^d\int_\Lambda \left|\frac{\partial}{\partial t_1}\bar{X}_N(t)\right|^2 dt \leqq c_2'\sum_{i\in V_{N+1}}(X(i+e(1))-X(i))^2\;,$$

for some constant $c_2'<\infty$. Of course the same type of equality holds for t_j, $j=2,\dots,d$. This shows (2.17). Next, as a consequence of the *irreducibility* of the transition matrix Q, we can find a constant $c_3<\infty$, such that

$$\begin{aligned}\sum_{\{i,j\}\cap V_N\neq\emptyset} Q_{\mathrm{d}}(i,j)(X(i)-X(j))^2 &\leqq c_3\sum_{\{i,j\}\cap V_N\neq\emptyset} Q(i,j)(X(i)-X(j))^2\\ &= 2c_3\mathscr{H}_N^0(X)\,,\end{aligned} \tag{2.18}$$

cf. P5, p. 70 of [14]. Finally, let us view $\mathscr{H}_N^0$ as a quadratic form on Ω_N with positive eigenvalues $\{l_n^N,\ n\in V_N\}$, then

$$Z_N(\beta)\equiv\int_{\Omega_N}\exp(-\beta\mathscr{H}_N^0(x))\,\lambda_N(dx)=\prod_{n\in V_N}\left(\frac{2\pi}{\beta l_n^N}\right)^{1/2},\quad \beta>0\,.$$

Thus, for each $\alpha' < 1$,

$$\limsup_{N\to\infty} N^{-d} \log E_N^0[\exp(\alpha' \mathscr{H}_N^0(X))] = \limsup_{N\to\infty} N^{-d} \log \frac{Z_N(1-\alpha')}{Z_N(1)}$$
$$= -\frac{1}{2}\log(1-\alpha') < \infty ,$$

which, in view of (2.17) and (2.18) shows (2.15). Equation (2.16) follows immediately. $\square$

We now turn to the proof of the exponential tightness under (2.1). For simplicity, we restrict ourselves to the nearest neighbor interaction Q_{d} and denote by $-\Delta_{\mathrm{d}}$ the discrete Laplacian

$$-\Delta_{\mathrm{d}}\phi(k) = \sum_{j\in\mathbb{Z}^d} Q_{\mathrm{d}}(k,j)(\phi(k)-\phi(j)) = \frac{1}{2d}\sum_{|j-k|=1}(\phi(k)-\phi(j)), \quad k\in\mathbb{Z}^d .$$

In this case we can give explicitly the eigenvectors $\{e_n^N,\ n\in V_N\}\subset\Omega_N$ and eigenvalues $\{l_n^N,\ n\in V_N\}\subset\mathbb{R}^+$ of $-\Delta_{\mathrm{d}}$ with Dirichlet boundary conditions on ∂V_N: Let $\{e_n,\ n\in\mathbb{N}^d\}\subset C_b(\Lambda)$ and $\{l_n,\ n\in\mathbb{N}^d\}\subset\mathbb{R}^+$ be the eigenfunctions and eigenvalues of the (continuous) Laplacian $-\Delta_\Lambda = -\frac{1}{d}\Delta$ with Dirichlet boundary conditions on $\partial\Lambda$:

$$e_n(t) = 2^{d/2}\prod_{i=1}^d \sin(\pi n_i t_i), \quad t\in\Lambda, \qquad l_n = \frac{\pi^2|n|^2}{d} = \frac{\pi^2}{d}\sum_{i=1}^d n_i^2 , \tag{2.19}$$

then

$$e_n^N(k) = e_n\left(\frac{k}{N}\right), \quad k\in V_N, \qquad l_n^N = \frac{4}{d}\sum_{i=1}^d \sin^2\left(\frac{\pi n_i}{2N}\right) ,$$

cf. Proposition 9.5.3 of [11]. In particular $\{e_n^N,\ n\in V_N\}$ forms an orthonormal basis of Ω_N equipped with the scalar product $\langle x,y\rangle_{V_N} \equiv N^{-d}\sum_{k\in V_N} x(k)y(k)$.

Next, let $\mathscr{L}_N \equiv \{\phi\in L^2(\Lambda) : \phi(t) = \sum_{k\in V_N} h(Nt-k)\phi(k/N),\ t\in\Lambda\}$, be the set of "interpolated" functions. For given $\{\alpha(n),\ n\in\mathbb{N}^d\}\subset\mathbb{R}^+$ with $\lim_{|n|\to\infty}\alpha(n) = \infty$, we define an Hilbertian norm $\|\cdot\|_{\alpha,N}$ on $\mathscr{L}_N$:

$$\|\phi\|_{\alpha,N}^2 \equiv \sum_{n\in V_N}\alpha(n)y_n^2 ,$$

where $y = \{y_n,\ n\in\mathbb{N}^d\}$ is such that

$$\phi(k/N) = \sum_{n\in V_N} y_n e_n(k/N), \quad k\in V_N . \tag{2.20}$$

Lemma 2.21. *For each $L,\delta > 0$, there exist $\phi^1,\dots,\phi^M\in L^2(\Lambda)$, such that*

$$K_{L,N} \equiv \{\phi\in\mathscr{L}_N : \|\phi\|_{\alpha,N} \leqq L\} \subseteq \bigcup_{i=1}^M B_2(\phi^i;\delta) \quad \textit{for large enough } N .$$

Proof. First note that $\{e_n^N,\ n\in V_N\}$ is an orthonormal basis, thus if ϕ is given by (2.20), we have

$$N^{-d}\sum_{k\in V_N}\phi(k/N)^2 = \sum_{n\in V_N} y_n^2 .$$

Also, there exists a constant $c > 0$, such that

$$\|\phi\|^2_{L^2(\Lambda)} \leqq cN^{-d} \sum_{k\in V_N} \phi(k/N)^2 \quad \phi \in \mathscr{L}_N . \tag{2.22}$$

Next, $K_L \equiv \{y \in l^2(\mathbb{Z}^d) : \sum_{n\in\mathbb{N}^d} \alpha(n)y_n^2 \leqq L\}$ is a compact subset of $l^2(\mathbb{N}^d)$. Thus, for each δ', $L > 0$, there exists $y^1,\dots,y^M$, such that

$$K_L \subset \bigcup_{j=1}^{M} \{y : \|y - y^j\|_{l^2(\mathbb{N}^d)} \leqq \sqrt{\delta'}\} .$$

Without loss of generality, we may assume that $\max_{j=1,\ \ ,M} |y_n^j| = 0$, if $|n| \geqq n_0$, for some $n_0 = n_0(\delta', L) > 0$. Set

$$t : \Lambda \to \phi^i(t) = \sum_{n\in\mathbb{N}^d} y_n^i e_n(t), \qquad t : \Lambda \to \phi_N^i(t) = \sum_{k\in V_N} h(Nt-k)\phi^i(k/N) \in \mathscr{L}_N .$$

Then, for $N \geqq n_0$, by (2.22),

$$\begin{aligned}\mathscr{K}_{N,L} &\subseteq \bigcup_{i=1}^{M} \left\{\phi \in \mathscr{L}_N : \sum_{n\in V_N} |y_n - y_n^i|^2 \leqq \delta'\right\} \\ &= \bigcup_{i=1}^{M} \left\{\phi \in \mathscr{L}_N : N^{-d} \sum_{k\in V_N} |\phi(k/N) - \phi^i(k/N)|^2 \leqq \delta'\right\} \\ &\subseteq \bigcup_{i=1}^{M} \{\phi \in \mathscr{L}_N : \|\phi - \phi_N^i\|^2_{L^2(\Lambda)} \leqq c\delta'\} .\end{aligned}$$

Now the result follows, since

$$\lim_{N\to\infty} \max_{i=1,\ \ ,M} \|\phi_N^i - \phi^i\|_{L^2(\Lambda)} = 0 . \qquad \square$$

Lemma 2.23. *Assume* (2.1). *Let* $\{\alpha(n) : n \in \mathbb{N}^d\}$ *satisfy* $\alpha(n) \leqq O(|n|^2)$ *and*

$$\limsup_{N\to\infty} \varepsilon_N' \sum_{n\in V_N} \alpha(n)|n|^{-2} < \infty , \tag{2.24}$$

then, $\limsup_{N\to\infty} \varepsilon_N' \log E[\exp(\frac{\beta}{\varepsilon_N'}\|\bar{X}_N\|^2_{\alpha,N})] < \infty$, *for some* $\beta > 0$. *In particular*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin K_{L,N}) = -\infty .$$

Proof. We know that $X \in \Omega_N$ satisfies

$$X(k) = \sum_{n\in V_N} \langle X, e_n^N\rangle_{V_N} e_n^N(k) = \sum_{n\in V_N} \Xi_n^N e_n(k/N), \quad k \in V_N ,$$

where, under P_N^0, $\{\Xi_n^N \equiv \langle X, e_n^N\rangle_{V_N},\ n \in V_N\}$ are independent centered gaussian with variance $E_N^0[(\Xi_n^N)^2] = \frac{N^{-d}}{l_n^N}$. Also, $\inf_{N\geqq 0} N^2 l_n^N \geqq c|n|^2$, for some constant $c > 0$, cf. (9.5.14) [11]. Thus

$$\|\bar{X}_N\|^2_{\alpha,N} = \varepsilon_N^2 \sum_{n\in V_N} \alpha(n)(\Xi_n^N)^2 ,$$

and, for $0 < \beta < \sup_{n\in\mathbb{N}^d} \frac{c|n|^2}{2\alpha(n)}$, using the independence of $\{\Xi_n^N\}$,

$$\log E_N^0[\exp(\beta\varepsilon_N^{-2}N^{d-2}\|\bar{X}_N\|_{\alpha,N}^2)] = \log E_N^0\left[\exp\left(\beta N^{d-2}\sum_{n\in V_N}(\Xi_n^N)^2\right)\right]$$
$$= \sum_{n\in V_N} -\frac{1}{2}\log\left(1-\frac{2\beta\alpha(n)}{N^2 l_n^N}\right) \leqq \sum_{n\in V_N} -\frac{1}{2}\log\left(1-\beta\frac{2\alpha(n)}{c|n|^2}\right),$$

which implies the result by (2.24). □

Proof of Theorem 2.5

The upper bound under (2.1). Note that Lemma 2.10 yields

$$\lim_{N\to\infty} \varepsilon_N' \log E_N^0\left[\exp\left(\frac{1}{\varepsilon_N'}\langle\bar{X}_N,\phi\rangle\right)\right] = \frac{1}{2}\langle\phi,\mathfrak{G}_\Lambda\phi\rangle, \quad \phi\in C_b(\Lambda).$$

Since $C_b(\Lambda)$ is dense in $L^2(\Lambda)$, we get by Lemma 2.9 and standard large deviation results the following weak upper bound:

$$\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in B_2(\phi;\delta)) \leqq -\inf_{\bar{B}_2(\phi;\delta)} I, \quad \phi\in L^2(\Lambda),\ \delta>0,$$

cf. Sect. 5.1 of [8]. The strong upper bound follows from the exponential tightness Lemma 2.23.

The upper bound under (2.4). The only change is the exponential tightness (2.16) which holds in the stronger $L^p(\Lambda)$-topology, $2 \leqq p < 2d/(d-2)$.

The lower bound under (2.1). It suffices to show that, for each $\varepsilon > 0$ and $\phi \in C_0^1(\Lambda)$,

$$\liminf_{N\to\infty} \varepsilon_N' \log P_0^N(\bar{X}_N \in B_2(\phi;\varepsilon)) \geqq -I(\phi).$$

Let Q_N^0 be the gaussian measure on Ω_N with covariances G_N and mean $E_{Q_N^0}[X(k)] = \varepsilon_N^{-1}\phi(k/N)$, $k\in V_N$. Then

$$\mathbf{H}(Q_N^0|P_N^0) = \frac{\varepsilon_N^{-2}}{4}\sum_{\{i,j\}\cap V_N\neq\emptyset} Q(i,j)(\phi(i/N)-\phi(j/N))^2 = \frac{\varepsilon_N^{-2}}{2}\mathscr{H}_N^0(\phi(\,\cdot\,/N)),$$

where $\mathbf{H}(Q_N^0|P_N^0) = E_N^0\left[\frac{dQ_N^0}{dP_N^0}\log\frac{dQ_N^0}{dP_N^0}\right]$ denotes the relative entropy, and therefore

$$\lim_{N\to\infty} \varepsilon_N'\mathbf{H}(Q_N^0|P_N^0) = \frac{1}{2}\|\,|\nabla\phi|_A\|_{L^2(\Lambda)}^2 = I(\phi), \tag{2.25}$$

cf. [7]. On the other hand, let $t:\Lambda\to\phi_N(t)\equiv\sum_{i\in V_N} h(Nt-k)\phi(k/N)$, then $\lim_{N\to\infty}\|\phi_N-\phi\|_{L^2(\Lambda)}=0$, and, for each $\delta>0$,

$$\lim_{N\to\infty} Q_N^0(\bar{X}_N\in B_2(\phi_N;\delta)) = \lim_{N\to\infty} P_N^0(\bar{X}_N\in B_2(0;\delta)) = 1, \tag{2.26}$$

since, by (2.22),

$$\lim_{N\to\infty} P_N^0(\|\bar{X}_N\|_{L^2(\Lambda)} \geqq \delta) \leqq \lim_{N\to\infty} \frac{E_N^0[\|\bar{X}_N\|_{L^2(\Lambda)}^2]}{\delta^2} \leqq \lim_{N\to\infty} \frac{c\varepsilon_N^2 \max_{k\in V_N} G^N(k,k)}{\delta^2} = 0,$$

with

$$\max_{k\in V_N} G^N(k,k) = \max_{k\in V_N} E_N^0[|X(k)|^2] = \begin{cases} O(1) & d \geqq 3 \\ O(\log N) & d = 2\,, \end{cases}$$

cf. [7]. Now the lower bound follows from (2.25) and (2.26) by the usual change of measure argument, cf. Lemma 5.4.21 of [8].

The lower bound under (2.2). The only difference is the last step in the $L^\infty(\Lambda)$-topology:

$$\lim_{N\to\infty} Q_N^0(\bar{X}_N \in B_\infty(\phi_N;\delta)) = \lim_{N\to\infty} P_N^0(\bar{X}_N \in B_\infty(0;\delta)) = 1\,,$$

with

$$P_N^0(\|\bar{X}_N\|_\infty \geqq \delta) \leqq P_N^0\left(\max_{k\in V_N} |X(k)| \geqq \delta\varepsilon_N^{-1}\right) \leqq N^d \max_{k\in V_N} P_N^0(|X(k)| \geqq \delta\varepsilon_N^{-1})$$

$$\leqq 2N^d \exp\left(-\frac{\delta^2\varepsilon_N^{-2}}{2\max_{k\in V_N} G^N(k,k)}\right) \to 0 \quad \text{as } N \to \infty. \qquad \square$$

Remark 2.27. The large deviation principle allows us to change the a priori measure P_N^0 via Varadhan's lemma, cf. 2.1.24 of [8]. More precisely, let

$F \in C_p(\Lambda \times \mathbb{R}) \equiv \{F \in C(\Lambda\times\mathbb{R}) \text{ with } |F(t,x)| \leqq A + B|x|^p, \text{ for some } A,B < \infty\}$,

and set $\mathbb{F}(\phi) = \int_\Lambda F(t,\phi(t))\,dt$. Next, for $1 \leqq p < 2$ and $\beta \in \mathbb{R}$, define

$$P_N^{0,\beta}(dX) = \frac{1}{Z_N(\beta\mathbb{F})} \exp\left(-\frac{\beta}{\varepsilon'_N}\mathbb{F}(\bar{X}_N)\right) P_N^0(dX)\,,$$

where $Z_N(\beta\mathbb{F}) = E_N^0[\exp(-\frac{\beta}{\varepsilon'_N}\mathbb{F}(\bar{X}_N))]$. For example, for $h(t) = 1_{[0,1)^d}(t)$ and $F(t,x)$ $= F(x)$, the new measure is of the form

$$P_N^{0,\beta}(dX) = \frac{1}{Z_N(\beta\mathbb{F})} \exp\left(-\frac{\beta}{\varepsilon_N^2 N^2}\sum_{k\in V_N} F(\varepsilon_N X(k))\right) P_N^0(dX)\,.$$

Varadhan's lemma implies

$$\lim_{N\to\infty} \varepsilon'_N \log Z_N(\beta\mathbb{F}) = -\inf_{\phi\in L^2(\Lambda)} \{I(\phi) + \beta\mathbb{F}(\phi)\} \equiv -\lambda(\beta\mathbb{F})\,,$$

and $\{P_N^{0,\beta} \circ \bar{X}_N^{-1},\ N \in \mathbb{N}\}$ satisfies a large deviation principle with rate function

$$I^\beta(\phi) = I(\phi) + \beta\mathbb{F}(\phi) - \lambda(\beta\mathbb{F})\,.$$

In particular, if $K(\beta\mathbb{F}) \equiv \{\psi \in H_0^1(\Lambda) : I(\psi) + \beta\mathbb{F}(\psi) = \lambda(\beta\mathbb{F})\}$, then for each $\varepsilon > 0$,

$$\limsup_{N\to\infty} \varepsilon'_N \log P_N^{0,\beta}(\bar{X}_N \notin B_2(K(\beta\mathbb{F});\varepsilon)) < 0\,. \tag{2.28}$$

Consider the special case of a linear functional $\mathbb{F}(\phi) = \langle \phi, f \rangle$, for some $f \in C_b(\Lambda)$. Then $P_N^{0,\beta} \circ \bar{X}_N^{-1}$ is gaussian with mean

$$m_N^{\beta}(t) = E_N^{0,\beta}[\bar{X}_N(t)] = \frac{\beta}{\varepsilon_N'} E_N^0[\bar{X}_N(t) \langle X_N, f \rangle], \quad t \in \Lambda,$$

and unchanged covariance $\mathrm{cov}_{P_N^{0,\beta}}(\bar{X}_N(t), \bar{X}_N(s)) = \mathrm{cov}_{P_N^0}(\bar{X}_N(t), \bar{X}_N(s))$. If we choose $\beta = \beta(v) = c_v = v / \langle f, \mathfrak{G}_\Lambda f \rangle$, such that $\lim_{N\to\infty} E_N^{0,\beta}[\langle \bar{X}_N, f \rangle] = v$, then $K(\beta\mathbb{F}) = \{\psi_v\}$ and $\limsup_{N\to\infty} \varepsilon_N' \log P_N^{0,\beta}(\bar{X}_N \notin B_2(\psi_v; \varepsilon)) < 0$, (see also Lemma 4.1, below).

3. The Convergence of the Approximate Microcanonical Distribution

As an immediate consequence of Theorem 2.5, we have the following convergence result for approximate microcanonical distributions.

Corollary 3.1. *Consider the slow regime* (2.2). *Let Γ be a closed subset of $L^2(\Lambda)$, such that $\inf_{\Gamma^{\circ,\infty}} I = \inf_\Gamma I < \infty$, and let $K(\Gamma) = \{\phi \in \Gamma : I(\phi) = \inf_\Gamma I\}$, then for all $\varepsilon > 0$,*

$$\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin B_2(K(\Gamma); \varepsilon) \,\big|\, \bar{X}_N \in \Gamma) < 0 .$$

We want to specialize this result to the following situation: for $v \in \mathbb{R}, \delta \geqq 0$ and $F \in C_p(\Lambda \times \mathbb{R})$ set $\mathbb{F}(\phi) = \int_\Lambda F(t, \phi(t))\, dt$,

$$M(v,F,\delta) \equiv \{\phi \in L^2(\Lambda) : |\mathbb{F}(\phi) - v| \leqq \delta\}, \quad M_N(v,F,\delta) \equiv \{\bar{X}_N \in M(v,F,\delta)\} .$$

Note that $M(v,F,\delta)$ is a closed set of $L^2(\Lambda)$ for $1 \leqq p < 2$. Next recall

$$\Omega^+(\eta) \equiv \{\phi \in L^2(\Lambda) : \phi(t) \geqq 0, \ t \in \Lambda_\eta\}, \quad \Omega_N^+(\eta) \equiv \{\bar{X}_N \in \Omega^+(\eta)\} ,$$

where $\Lambda_\eta = (\eta, 1-\eta)^d$. In case $\eta = 0$, we write $\Omega^+ = \Omega^+(0)$ and $\Omega_N^+ = \Omega_N^+(0)$.

Next, let $K(v,F)$ and $K^+(v,F) \subseteq H_0^1(\Lambda)$, be the set of solutions to the variational problem

$$\inf\{I(\phi) : \phi \in M(v,F,0)\} \equiv \lambda(v) ,$$

$$\inf\{I(\phi) : \phi \in M(v,F,0) \cap \Omega^+\} \equiv \lambda^+(v) . \tag{3.2}$$

In case $F \in C^1(\Lambda \times \mathbb{R})$ with $F_x(\,\cdot\,, \phi) \in L^2(\Lambda)$ for $\phi \in H_0^1(\Lambda)$, $\psi \in K(v,F)$ solves the Euler equation

$$-\Delta_\Lambda \psi = l\, F_x(t, \psi), \quad \text{for some } l = l(v) \in \mathbb{R} ,$$

with Dirichlet boundary conditions $\lim_{t\to a, t\in\Lambda} \psi(t) = 0$ for $a \in \partial\Lambda$. In non-degenerate situations, that is $F_x(\,\cdot\,, \psi) \not\equiv 0$, there exists $\beta = \beta(v) \in \mathbb{R}$, such that $\psi \in K(\beta\mathbb{F})$, cf. Remark 2.27.

Corollary 3.3. *Assume* (2.2) *and take $F \in C_p(\Lambda \times R)$ with $p < 2$. Let $v \in \mathbb{R}$ be such that $\lambda(v) < \infty$, then for each $\varepsilon > 0$,*

$$\lim_{\delta \searrow 0} \limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin B_2(K(v,F); \varepsilon) \,|\, M_N(v,F,\delta)) < 0 .$$

Next, if $\lambda^+(v) < \infty$, then for each $\varepsilon > 0$,

$$\lim_{\delta,\eta\searrow 0}\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin B_2(K^+(v,F);\varepsilon) \,|\, M_N(v,F,\delta)\cap \Omega_N^+(\eta)) < 0 .$$

Example 3.4. Consider the quadratic case $F(t,x) = x^2$. In this case $M(v,F,\delta)$ is closed in $L^2(\Lambda)$ and we can apply the above corollary. For each $v > 0$, $\psi \in K(v,F)$ solves $-\Delta_\Lambda \psi = l_1\psi$, where l_1 is the first eigenvalue associated with $-\Delta_\Lambda$. Thus $K(v,F) = \{\sqrt{v}e_1^+, \sqrt{v}e_1^-\}$ and $K^+(v,F) = \{\sqrt{v}e_1^+\}$, where e_1^+ is the positive $L^2(\Lambda)$-normalized eigenfunction and $e_1^- = -e_1^+$. In particular, with respect to the weak convergence on $\mathscr{M}_1(L^2(\Lambda))$, letting first $N \to \infty$ and then $\eta, \delta \searrow 0$,

$$P_N^0(\,\cdot\,|M_N(v,F,\delta)) \circ \bar{X}_N^{-1} \Rightarrow \frac{1}{2}\delta_{\sqrt{v}e_1^+} + \frac{1}{2}\delta_{\sqrt{v}e_1^-}$$

and

$$P_N^0(\,\cdot\,|M_N(v,F,\delta)\cap\Omega_N^+(\eta)) \circ \bar{X}_N^{-1} \Rightarrow \delta_{\sqrt{v}e_1^+} . \qquad \square$$

We would like to introduce the exact positivity condition

$$\Omega_N^+ \equiv \{\bar{X}_N : \bar{X}_N(t) \geqq 0, \ t \in \Lambda\}$$

in our conditioning. The major difficulty is that our large deviation principle is too rough for an accurate estimation of $P_N^0(\Omega_N^+)$. In fact, one has $\inf_{\Omega^+} I = 0 \neq \inf_{(\Omega^+)^{\circ,\infty}} I = \infty$, but

$$\lim_{N\to\infty} N^{-d+1} \log P_N^0(\Omega_N^+) = -\kappa_d , \tag{1.2}$$

cf. [7]. We will work with *monotonicity* arguments. More precisely, let $C_p^\uparrow(\Lambda \times \mathbb{R})$ be the set of $F \in C_p(\Lambda \times \mathbb{R})$ such that $x \to F(t,x)$ is strictly monotone increasing, for a.a. $t \in \Lambda$. We will assume that F is "normalized," i.e. $\mathbb{F}(0) = \int_\Lambda F(t,0)\,dt = 0$.

Proposition 3.5. *Consider the fast regime $\varepsilon_N = o(N^{-1/2})$. Take $F \in C_p^\uparrow(\Lambda \times \mathbb{R})$ with $p < 2$ and $v > 0$ such that $\lambda^+(v) < \infty$. Then, for each $\varepsilon > 0$,*

$$\lim_{\delta\searrow 0}\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin B_2(K^+(v,F);\varepsilon) \,|\, M_N(v,F,\delta)\cap \Omega_N^+) < 0 .$$

In the very fast regime $\varepsilon_N = \frac{1}{N}$, we may choose $F \in C_p^\uparrow(\Lambda \times \mathbb{R})$ for $2 \leqq p < 2d/(d-2)$ and replace $B_2(K(v,F);\varepsilon)$ by $B_p(K(v,F);\varepsilon)$.

Proof. In view of the upper bound in Theorem 2.5 and lsc property of the rate function I, we have

$$\begin{aligned}&\lim_{\delta\searrow 0}\limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \in B_2(K^+(v,F);\varepsilon)^{\mathrm{C}} \cap M_N(v,F,\delta)\cap \Omega_N^+)\\ &\quad\leqq -\inf\{I(\phi) : \phi \in B_2(K^+(v,F);\varepsilon)^{\mathrm{C}} \cap M(v,F,0)\cap\Omega^+\} < -\lambda^+(v) .\end{aligned}$$

Let $\hat{M}(s,F) \equiv \{\phi \in L^2(\Lambda) : \int_\Lambda F(t,\phi(t))\,dt \geqq s\}$ and set $\hat{M}_N(s,F) \equiv \{\bar{X}_N \in \hat{M}(s,F)\}$, $\hat{\lambda}(s) = \inf\{I(\psi) : \psi \in \hat{M}(s,F)\}$, $s \in \mathbb{R}$. Then, by the FKG property of P_N^0, cf. [13],

$$\begin{aligned}P_N^0(M_N(v,F,\delta)\cap\Omega_N^+) &\geqq P_N^0(\hat{M}_N(v-\delta,F)\cap\Omega_N^+) - P_N^0(\hat{M}_N(v+\delta,F)\cap\Omega_N^+)\\ &\geqq P_N^0(\hat{M}_N(v-\delta,F))P_N^0(\Omega_N^+) - P_N^0(\hat{M}_N(v+\delta,F)) .\end{aligned}$$

Now, by the upper bound

$$\limsup_{N\to\infty} \varepsilon'_N \log P_N^0(\bar{X}_N \in \hat{M}_N(v+\delta,F)) \leqq -\hat{\lambda}(v+\delta)\,,$$

and, by the lower bound,

$$\liminf_{N\to\infty} \varepsilon'_N \log P_N^0(\bar{X}_N \in \hat{M}_N(v-\delta,F)) \geqq -\hat{\lambda}(v) \geqq -\lambda^+(v)\,.$$

Since $\varepsilon'_N = \varepsilon_N^2 N^{2-d} = o(N^{1-d})$ by assumption, the result follows from (1.2) as soon as we verify that

$$\hat{\lambda}(v+\delta) > \hat{\lambda}(v), \quad \delta > 0\,.$$

By assumption $0 < \hat{\lambda}(v) \leqq \lambda^+(v) < \infty$, thus we may suppose that $\hat{\lambda}(v+\delta) < \infty$. Take $\phi \in \hat{M}(v+\delta,F)$, such that $I(\phi) = \hat{\lambda}(v+\delta)$. Then $|\phi| \in \hat{M}(v+\delta,F)$ and, since $F \in C_2^\uparrow(\Lambda\times\mathbb{R})$, there is $\beta \in (0,1)$ such that $\int_\Lambda F(t,\beta|\phi(t)|)\,dt = v$. That is, $\beta|\phi| \in \hat{M}(v,F,0)$, and therefore

$$\hat{\lambda}(v) \leqq \lambda(v) \leqq \beta^2 I(|\phi|) \leqq \beta^2 I(\phi) = \beta^2\hat{\lambda}(v+\delta) < \hat{\lambda}(v+\delta). \qquad \square$$

In case of a linear conditioning of the form

$$A_N(v,f,\delta) = \{\bar{X}_N : \left|\langle\bar{X}_N, f\rangle - v\right| \leqq \delta\}\,,$$

where $\delta, v > 0$ and $f \in C_b^+(\Lambda)$, we can let $\delta = \delta_N \searrow 0$ with $N\to\infty$. Recall the definition

$$\mathscr{P}_N^+ \equiv P_N^0(\,\cdot\,|\,A_N(v,f,\delta_N)\cap\Omega_N^+)\circ\bar{X}_N^{-1}\,.$$

Proposition 3.6. *Take* $\varepsilon_N = o(N^{-1/2})$. *Let* $\psi_v : \Lambda \to \mathbb{R}^+, \psi_v(t) = c_v\mathfrak{G}_\Lambda f(t)$, *with* $c_v = v/\langle f, \mathfrak{G}_\Lambda f\rangle$, *then for each* $\varepsilon > 0$, *and* $\delta_N \searrow 0$ *with* $\delta_N \geqq \frac{\kappa_d}{2c_v}\varepsilon_N^2 N = o(1)$,

$$\limsup_{N\to\infty} \varepsilon'_N \log \mathscr{P}_N^+(B_2(\psi_v;\varepsilon)^{\mathrm{C}}) < 0\,.$$

In case $\varepsilon_N = \frac{1}{N}$, *we may take* $\delta_N \geqq \frac{\kappa_d}{2c_v}\frac{1}{N}$ *and replace* $B_2(\psi_v;\varepsilon)$ *by* $B_p(\psi_v;\varepsilon)$ *for* $2 \leqq p < 2d/(d-2)$.

Proof. For $F(t,x) = f(t)x$, $\psi \in K(v,F)$ solves $-\Delta_\Lambda\psi = c_v f(t)$, that is $K(v,F) = \{\psi_v = c_v\,\mathfrak{G}_\Lambda f\}$ with $c_v = v/\langle f, \mathfrak{G}_\Lambda f\rangle$. In view of the preceding proof, all we have to show is

$$\liminf_{N\to\infty} \varepsilon'_N \log P_N^0(A_N(v,f,\delta_N)\cap\Omega_N^+) \geqq -\frac{v^2}{2\langle f,\mathfrak{G}_\Lambda f\rangle} = -I(\psi_v)\,. \tag{3.7}$$

By the FKG property of P_N^0, cf. [13],

$$P_N^0(A_N(v,f,\delta_N)\cap\Omega_N^+) \geqq P_N^0(\langle\bar{X}_N,f\rangle \geqq v-\delta_N)P_N^0(\Omega_N^+) - P_N^0(\langle\bar{X}_N,f\rangle \geqq v+\delta_N)\,.$$

Next, let $\sigma_N^2(f) = E_N^0[\langle\bar{X}_N,f\rangle^2]$, then since $\langle\bar{X}_N,f\rangle$ is gaussian,

$$P_N^0(\langle\bar{X}_N,f\rangle \geqq v-\delta_N) \geqq \frac{3\sigma_N(f)}{\sqrt{2\pi}4(v-\delta_N)}\exp\left(-\frac{(v-\delta_N)^2}{2\sigma_N^2(f)}\right)\,,$$

for $\frac{v-\delta_N}{\sigma_N(f)} \geqq 2$, cf. Sect. 1.3 of [15], and

$$P_N^0(\langle \bar{X}_N, f\rangle \geqq v+\delta_N) \leqq \frac{\sigma_N(f)}{\sqrt{2\pi}(v+\delta_N)} \exp\left(-\frac{(v+\delta_N)^2}{2\sigma_N^2(f)}\right) .$$

Also

$$\lim_{N\to\infty} \frac{1}{\varepsilon_N'} \sigma_N^2(f) = \langle f, \mathfrak{G}_\Lambda f\rangle ,$$

cf. (2.11). Thus for $\delta_N \searrow 0$, $\delta_N \geqq \frac{\kappa_d \langle f, \mathfrak{G}_\Lambda f\rangle}{2v} \varepsilon_N^2 N = \frac{\kappa_d}{2c_v} \varepsilon_N^2 N$, (3.7) follows from (1.2). □

In [7], one shows the existence of a constant $0 < K_d < \infty$, such that for all $\eta_N \searrow 0$, $\eta_N \geqq 2/N$,

$$\liminf_{N\to\infty} \frac{\eta_N}{\log^2(\eta_N N)} \log P_N^0(\Omega_N^+(\eta_N)) \geqq -K_2 > -\infty, \quad d=2 ,$$

$$\liminf_{N\to\infty} \frac{\eta_N}{N^{d-2}\log(\eta_N N)} \log P_N^0(\Omega_N^+(\eta_N)) \geqq -K_d > -\infty, \quad d \geqq 3 . \tag{3.8}$$

Using precisely the same technique as above, we get from (3.8),

Proposition 3.9. *Consider the slow regime* (2.2), *then for each* $\varepsilon > 0$ *and* $\eta_N, \delta_N \searrow 0$ *such that* $\delta_N \geqq \frac{K_2}{2c_V} \frac{\varepsilon_N^2 \log^2(\eta_N N)}{\eta_N}$ *for* $d=2$, *and* $\delta_N \geqq \frac{K_d}{2c_V} \frac{\varepsilon_N^2 \log(\eta_N N)}{\eta_N}$ *for* $d \geqq 3$,

$$\limsup_{N\to\infty} \varepsilon_N' \log \tilde{\mathscr{P}}_N^+(B_2(\psi_v;\varepsilon)^{\mathrm{C}})$$
$$= \limsup_{N\to\infty} \varepsilon_N' \log P_N^0(\bar{X}_N \notin B_2(\psi_v;\varepsilon) \,|A_N(v,f,\delta_N) \cap \Omega_N^+(\eta_N)) < 0 \ .$$

Before concluding this section, let us state a few important remarks:

Remark 3.10. It is interesting to compare our results with [5] in case $d \geqq 3$. Let $\mathscr{M}_1(\Lambda \times \mathbb{R})$ be the set of probability measures on $\Lambda \times \mathbb{R}$, equipped with the *weak topology*. In [5], one shows a large deviation result for the empirical measure

$$\tilde{Y}_N(dt,dz) \equiv \frac{1}{|V_N|} \sum_{k\in V_N} \delta_{k/N}(dt)\delta_{X(k)}(dz) \in \mathscr{M}_1(\Lambda \times \mathbb{R})$$

at speed N^{d-2} with rate function $J : \mathscr{M}_1(\Lambda \times \mathbb{R}) \to [0,\infty]$,

$$J(v) = \begin{cases} I(\phi), & v(dt,dz) = \gamma_{\phi(t)}(dz)\,dt \\ \infty & \text{otherwise} , \end{cases}$$

where $\gamma_{\phi(t)}$ is the gaussian distribution on $\mathbb{R}$ with mean $\phi(t)$ and variance $G(0,0)$. Let

$$\tilde{X}_N(dt) \equiv \int_{\mathbb{R}} z\, \tilde{Y}_N(dt,dz) = \frac{1}{|V_N|} \sum_{i\in V_N} X(i)\delta_{i/N}(dt) ,$$

viewed as a signed measure on Λ. Set $h_N(t) = |V_N| h(Nt)$, then

$$\int_\Lambda h_N(t-x)\tilde{X}_N(dx) = \sum_{i\in V_N} h(Nt-i)X(i) = X_N(t), \quad t \in \Lambda .$$

Thus, $\tilde{X}_N$ corresponds to the constant scaling $\varepsilon_N = 1$ which is beyond the techniques of Theorem 2.5 and yields different results. For example, consider a non-linear conditioning $\tilde{M}_N(v,F,\delta)$ of the form

$$\tilde{M}_N(v,F,\delta) = \left\{ X \in \Omega_N : \left| \frac{1}{|V_N|} \sum_{k \in V_N} F(k/N, X(k)) - v \right| \leqq \delta \right\}$$
$$= \left\{ \tilde{Y}_N : \left| \int_\Lambda \int_{\mathbb{R}} F(t,z)\, \tilde{Y}_N(dt,dz) - v \right| \leqq \delta \right\} ,$$

where $F \in C_1(\Lambda \times \mathbb{R})$. Next let $\tilde{K}(v,F)$ be the set of signed measures on Λ of the form $\mu(dt) = \psi(t)\, dt$, where ψ solves

$$\inf \left\{ \frac{1}{2} \| |\nabla \psi|_\Lambda \|^2_{L^2(\Lambda)} : \psi \in H_1^0(\Lambda), \int_\Lambda \int_{\mathbb{R}} F(t, z + \psi(t)) \gamma_0(dz)\, dt = v \right\} \equiv \tilde{\lambda}(v)\, . \quad (3.11)$$

For $F \in C^1(\Lambda \times \mathbb{R})$, ψ solves the Euler equation

$$-\Delta_\Lambda \psi(t) = l \int_{\mathbb{R}} F_x(t, z + \psi(t)) \gamma_0(dz), \quad \text{for some } l = l(v) \in \mathbb{R}\, ,$$

with Dirichlet boundary conditions. Assuming $\tilde{\lambda}(v) < \infty$, for any open neighborhood $\mathscr{U}$ of $\tilde{K}(v,F)$ with respect to the weak topology, we have

$$\lim_{\delta \searrow 0} \limsup_{N \to \infty} N^{-d+2} \log P_N^0(\tilde{X}_N \notin \mathscr{U} \mid \tilde{M}_N(v,F,\delta)) < 0\, .$$

For non-linear F, the solutions to (3.11) are quite different from the solutions of (3.2). Thus, the scaling $\varepsilon_N = o(1)$ improves the topology in Theorem 2.5, but trivializes the variational problem. In some sense we could speak of *moderate deviations* in this setting.

Remark 3 12. In dimensions $d \geqq 3$, one could remove the 0-boundary condition and consider the infinite Gibbs state as a priori model, i.e. replace P_N^0 with the centered Gaussian field P^0 on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ with covariance G,

$$G(i,j) = \mathbb{P}_i \left[\sum_{n=0}^{\infty} 1_j(\xi_n) \right], \quad i,j \in \mathbb{Z}^d\, ,$$

the Green function of the random walk $\{\xi_n,\ n \in \mathbb{N}\}$ on the whole $\mathbb{Z}^d$. Then, the large deviation result Theorem 2.1 holds with rate function

$$I'(\phi) = \inf \left\{ \frac{1}{2} \| |\nabla h|_\Lambda \|^2_{L^2(\mathbb{R}^d)} : h \in H_0^1(\mathbb{R}^d),\ h = \phi \text{ a.e. on } \Lambda \right\} ,$$

and the convergence result of Proposition 3.6 holds in the weaker slow regime (2.2), since

$$\lim_{N \to \infty} \frac{1}{N^{d-2} \log N} \log P(\Omega_N^+) = -2G(0,0)\ \mathrm{cap}(\Lambda)\, ,$$

where $\mathrm{cap}(\Lambda) = I'(1_\Lambda)$ is the Newtonian capacity of Λ, cf. [6]. In particular the limiting hypersurface is simply $t \to \psi_v(t) = c_v \mathfrak{G} f(t) = c_v \int_\Lambda \mathrm{g}(s,t) f(s)\, ds$, where $\mathrm{g}(s,t) = \mathrm{g}(s-t)$ is the Green function of the Laplacian $-\Delta_\Lambda$ on $\mathbb{R}^d$, cf. [5].

Remark 3.14. We have chosen the unit cube $(0,1)^d$ as a basis for our droplet. This is unnecessarily restrictive and, with some additional work, one could as well consider any open domain $\Lambda \in \mathbb{R}^d$ with piecewise smooth boundary. In this context, the convergence of the covariance functional in Lemma 2.10 follows from the invariance principle. The next step, in the proof of the exponential tightness (Lemmas 2.14 and 2.23), would be to estimate the growth of the eigenvalues of discrete Laplacian with Dirichlet boundary conditions on $V_N^{\mathbf{C}}$. Finally, the corresponding estimate for the repulsion, cf. (1.2) and (3.8), are derived in [7], Remarks 2.4 and 4.16.

4. Fluctuations of the Exact Microcanonical Distribution

Let $f \in C_b^+(\Lambda)$ be fixed. Although P_N^0 is gaussian, $\mathscr{P}_N^+$ is not gaussian. This comes from the positivity condition Ω_N^+ and the approximation $A_N(v,f,\delta_N)$ with $\delta_N > 0$. If one drops the condition Ω_N^+ and sets $\delta_N = 0$, then the exact microcanonical distribution

$$\mathscr{P}_N = P_N^0(\,\cdot\,|\langle \bar{X}_N, f\rangle = v) \circ \bar{X}_N^{-1} \in \mathscr{M}_1(L^2(\Lambda))$$

is Gaussian:

Lemma 4.1. *$\mathscr{P}_N$ is gaussian with mean*

$$m_N(t) = E_{\mathscr{P}_N}[\bar{X}_N(t)] = v\,\frac{E_N^0[\bar{X}_N(t)\langle \bar{X}_N, f\rangle]}{E_N^0[\langle \bar{X}_N, f\rangle^2]}, \quad t \in \Lambda\,,$$

and covariance

$$\operatorname{cov}_{\mathscr{P}_N}(\bar{X}_N(t),\bar{X}_N(s)) = E_0^N[\bar{X}_N(t)\bar{X}_N(s)] - \frac{m_N(t)m_N(s)}{v^2}E_N^0[\langle \bar{X}_N, f\rangle^2], \quad s,t \in \Lambda\,.$$

Proof. Let

$$\alpha_N(t) = \frac{m_N(t)}{v} = \frac{E_N^0[\bar{X}_N(t)\langle \bar{X}_N, f\rangle]}{E_N^0[\langle \bar{X}_N, f\rangle^2]}\,,$$

then one verifies easily that $\{\bar{X}_N(s) - \alpha_N(s)\langle \bar{X}_N, f\rangle,\ s \in \Lambda\}$ is independent of $\{\bar{X}_N(t),\ t \in \Lambda\}$. This implies the result by standard gaussian calculus. □

Note that, in view of Lemma 2.10, for all $\phi \in C_b(\Lambda)$,

$$\lim_{N\to\infty} \langle \phi, m_N\rangle = \lim_{N\to\infty} v\frac{E_N^0[\langle \bar{X}_N, \phi\rangle\langle \bar{X}_N, f\rangle]}{E_N^0[\langle \bar{X}_N, f\rangle^2]} = v\,\frac{\langle \phi, \mathfrak{G}_\Lambda f\rangle}{\langle f, \mathfrak{G}_\Lambda f\rangle} = \langle \phi, \psi_v\rangle\,. \tag{4.2}$$

Thus, m_N converges weakly to $\psi_v = c_v \mathfrak{G}_\Lambda f$.

Proposition 4.3. *Let $\phi \in C_b(\Lambda)$, then the law of*

$$Z_N(\phi) = \frac{1}{\sqrt{\varepsilon_N'}}\langle \bar{X}_N - m_N, \phi\rangle = N^{d/2-1}\left\langle X_N - \frac{1}{\varepsilon_N}m_N, \phi\right\rangle$$

under $\mathscr{P}_N$ converges weakly to a centered gaussian distribution on $\mathbb{R}$ with variance

$$\langle \phi, \mathfrak{G}_\Lambda^f \phi\rangle \equiv \langle \phi, \mathfrak{G}_\Lambda \phi\rangle - \frac{\langle \phi, \mathfrak{G}_\Lambda f\rangle^2}{\langle f, \mathfrak{G}_\Lambda f\rangle}\,.$$

Proof. Since $\mathscr{P}_N$ is gaussian, all we have to verify is the convergence of the variance by (2.11):

$$\lim_{N\to\infty} \operatorname{var}_{\mathscr{P}_N}(Z_N(\phi)) = \lim_{N\to\infty} \frac{1}{\varepsilon_N'} E_{\mathscr{P}_N}[\langle \bar{X}_N - m_N, \phi\rangle^2]$$

$$= \lim_{N\to\infty} \frac{1}{\varepsilon_N'} E_N^0[\langle \bar{X}_N, \phi\rangle^2] - \frac{\langle m_N, \phi\rangle^2}{v^2} \frac{1}{\varepsilon_N'} E_N^0[\langle \bar{X}_N, f\rangle^2]$$

$$= \langle \phi, \mathfrak{G}_\Lambda \phi\rangle - \frac{\langle \psi_v, \phi\rangle^2}{v^2} \langle f, \mathfrak{G}_\Lambda f\rangle$$

$$= \langle \phi, \mathfrak{G}_\Lambda \phi\rangle - \frac{\langle \phi, \mathfrak{G}_\Lambda f\rangle^2}{\langle f, \mathfrak{G}_\Lambda f\rangle} = \langle \phi, \mathfrak{G}_\Lambda^f \phi\rangle . \qquad \square$$

This result is, of course, of little use, since one would rather have a convergence result for the fluctuation around ψ_v, that is, for $\langle \bar{X}_N - \psi_v, \phi\rangle / \sqrt{\varepsilon_N'}$. However it seems quite difficult to estimate the convergence rate in (4.2), see Remark 4.7 below. In order to circumvent this difficulty we introduce a new continuous interpolation $\{X_N^*(t), t \in \Lambda\}$ of X. For simplicity, we restrict ourselves to the nearest neighbor interaction Q_{d}. Let $\{e_n,\ n \in \mathbb{N}^d\} \subset C_b(\Lambda)$ and $\{l_n,\ n \in \mathbb{N}^d\} \subset \mathbb{R}^+$ be the eigenfunctions and eigenvalues of the (continuous) Laplacian $-\Delta_A = -\frac{1}{d}\Delta$ with Dirichlet boundary conditions on $\partial\Lambda$, cf. (2.19), and define the new continuous interpolation X_N^* of X by

$$X_N^*(t) \equiv \sum_{n\in V_N} \langle X, e_n^N\rangle_{V_N}\, e_n(t) = \sum_{n\in V_N} \Xi_n^N e_n(t), \quad t \in \Lambda , \tag{4.4}$$

where, under P_N^0, $\{\Xi_n^N \equiv \langle X, e_n^N\rangle_{V_N} = N^{-d}\sum_{k\in V_N} X(k) e_n(k/N),\ n \in V_N\}$ are centered gaussian with variance $E_N^0[(\Xi_n^N)^2] = \frac{N^{-d}}{l_n^N}$, cf. Sect. 2. As above, we set

$$\bar{X}_N^*(t) = \varepsilon_N X_N^*(t) = \sum_{n\in V_N} \varepsilon_N \langle X, e_n^N\rangle_{V_N}\, e_n(t), \quad t \in \Lambda ,$$

and write $\mathscr{P}_N^* = P_N^0(\,\cdot\,|\langle \bar{X}_N^*, f\rangle = v) \circ (\bar{X}_N^*)^{-1} \in \mathscr{M}_1(L^2(\Lambda))$.

Next, for $\theta \in \mathbb{R}^+$, let us introduce the Hilbertian norm

$$\|\phi\|_{\theta,2}^2 \equiv \sum_{n\in\mathbb{N}^d} |l_n|^\theta \langle \phi, e_n\rangle^2, \quad \phi \in C_0^\infty(\Lambda) ,$$

and let $\mathscr{D}_\theta(\Lambda)$ be the closure of $C_0^\infty(\Lambda)$ with respect to $\|\cdot\|_{\theta,2}$. $\mathscr{D}_\theta'(\Lambda) = \mathscr{D}_{-\theta}(\Lambda)$, the adjoint of $\mathscr{D}_\theta(\Lambda)$, is the set of distributions on Λ. Note that $\mathscr{D}_1(\Lambda) = H_0^1(\Lambda)$ with $\|\phi\|_{1,2} = \|\nabla\phi\|_{L^2(\Lambda)}$, and $\langle \phi, \mathfrak{G}_\Lambda\phi\rangle = \|\phi\|_{-1,2}^2$.

Proposition 4.5. *Let $d = 2, 3$ and assume that $\varepsilon_N^{-1} = o(N^{3-d/2})$. Set*

$$Z_N^* = \frac{1}{\sqrt{\varepsilon_N'}}(\bar{X}_N^* - \psi_v) = N^{d/2-1}\left(X_N^* - \frac{1}{\varepsilon_N}\psi_v\right) ,$$

then with respect to the weak convergence on $\mathscr{M}_1(\mathscr{D}_1'(\Lambda))$,

$$\mathscr{P}_N^* \circ Z_N^{*-1} \Rightarrow \mathscr{Q}^f, \quad as\ N \to \infty\,,$$

where $\mathscr{Q}^f \in \mathscr{M}_1(\mathscr{D}_1'(\Lambda))$ *is the law of the centered generalized gaussian field* Z *with covariance operator*

$$\mathrm{cov}_{\mathscr{Q}}(Z(\phi), Z(\psi)) = \langle \phi, \mathfrak{G}_\Lambda^f \psi \rangle \quad \phi, \psi \in \mathscr{D}_1(\Lambda)\,.$$

Proof. First note that $\langle \phi, \mathfrak{G}_\Lambda^f \phi \rangle \leqq 2\langle \phi, \mathfrak{G}_\Lambda \phi \rangle = 2\|\phi\|_{2,-1}^2$. Also, in dimensions $d = 2, 3$, the injection $\mathscr{D}_1(\Lambda) \to \mathscr{D}_{-1}(\Lambda)$ is of Hilbert–Schmidt type. This implies that $\mathscr{Q}^f$ can be constructed on $\mathscr{D}_1'(\Lambda)$, cf. Theorem 3.1 of [12]. In order to prove the weak convergence, it suffices to show that

$$\lim_{N\to\infty} E_{\mathscr{P}_N^*}[\exp(iZ_N^*(\phi))] = E_{\mathscr{Q}^f}[\exp(iZ(\phi))] = \exp\left(-\frac{1}{2}\langle \phi, \mathfrak{G}_\Lambda^f \phi \rangle\right), \quad \phi \in \mathscr{D}_1(\Lambda)\,,$$

where $i = \sqrt{-1}$. Since $\mathscr{P}_N^*$ is Gaussian,

$$E_{\mathscr{P}_N^*}[\exp(iZ_N^*(\phi))] = \exp\left(i\frac{1}{\sqrt{\varepsilon_N'}}\langle m_N^* - \psi_v, \phi \rangle - \frac{1}{2}\frac{1}{\varepsilon_N'}\mathrm{var}_{\mathscr{P}_N^*}(\langle \bar{X}_N^*, \phi \rangle)\right),$$

where

$$m_N^*(t) = E_{\mathscr{P}_N^*}[\bar{X}_N^*(t)] = v\,\frac{E_N^0[\bar{X}_N^*(t)\langle \bar{X}_N^*, f\rangle]}{E_N^0[\langle \bar{X}_N^*, f\rangle^2]}, \quad t \in \Lambda\,,$$

and

$$\frac{1}{\varepsilon_N'}\mathrm{var}_{\mathscr{P}_N^*}(\langle \bar{X}_N^*, \phi \rangle) = \frac{1}{\varepsilon_N'}E_N^0[\langle \bar{X}_N^*, \phi\rangle^2] - \frac{\langle m_N^*, \phi\rangle^2}{v^2}\frac{1}{\varepsilon_N'}E_N^0[\langle \bar{X}_N^*, f\rangle^2]\,.$$

The main advantage of working with $\bar{X}_N^*$ instead of $\bar{X}_N$ is the following estimate:

$$\frac{1}{\varepsilon_N'}E_N^0[\langle \bar{X}_N^*, \phi\rangle^2] = \sum_{n\in V_N}\frac{1}{N^2 l_n^N}\langle \phi, e_n\rangle^2 = \langle \phi, \mathfrak{G}_\Lambda \phi\rangle + O(N^{-2})\|\phi\|_{L^2(\Lambda)}^2\,, \tag{4.6}$$

cf. Proposition 9.5.5 of [11]. This implies

$$\begin{aligned}\langle \phi, m_N^*\rangle &= v\frac{E_N^0[\langle \bar{X}_N^*, \phi\rangle\langle \bar{X}_N^*, f\rangle]}{E_N^0[\langle \bar{X}_N^*, f\rangle^2]} = v\,\frac{\langle \phi, \mathfrak{G}_\Lambda f\rangle}{\langle f, \mathfrak{G}_\Lambda f\rangle} + O(N^{-2})\|\phi\|_{L^2(\Lambda)} \\ &= \langle \phi, \psi_v\rangle + O(N^{-2})\|\phi\|_{L^2(\Lambda)}\,,\end{aligned}$$

thus, for $d = 2,3$, $\lim_{N\to\infty}\frac{1}{\sqrt{\varepsilon_N'}}\langle m_N^* - \psi_v, \phi\rangle = \lim_{N\to\infty}\frac{N^{d/2-1}}{\varepsilon_N}\langle m_N^* - \psi_v, \phi\rangle = 0$. Also, again by (4.6),

$$\lim_{N\to\infty}\frac{1}{\varepsilon_N'}\mathrm{var}_{\mathscr{P}_N^*}(\langle X_N^*, \phi\rangle) = \langle \phi, \mathfrak{G}_\Lambda^f \phi\rangle\,,$$

cf. proof of Proposition 4.3. □

Remark 4.7. We believe that (2.11) can be improved to

$$\frac{1}{\varepsilon_N'} E_N^0[\langle \bar{X}_N, \phi \rangle^2] = \langle \phi, \mathfrak{G}_\Lambda \phi \rangle + O(N^{-1}), \tag{4.8}$$

where $O(N^{-1})$ depends on $\phi \in C_b(\Lambda)$. Actually, this is the case when ϕ is sufficiently smooth, e.g. $\phi \in \mathscr{D}_\theta(\Lambda)$ for $\theta > \frac{d}{2} + 1$. This implies

$$\langle \phi, m_N \rangle = \langle \phi, \psi_v \rangle + O(N^{-1}), \quad \phi \in \mathscr{D}_\theta(\Lambda), \ \theta \geqq \frac{d}{2} + 1 .$$

Using the same argument as above, one deduces from (4.8), for $d = 2, 3$ and $\varepsilon_N^{-1} = o(N^{2-d/2})$, the weak convergence of $\mathscr{P}_N \circ ((\bar{X}_N - \psi_v)/\sqrt{\varepsilon_N'})^{-1}$ to $\mathscr{Q}^f$ in $\mathscr{M}_1(\mathscr{D}_3'(\Lambda))$.

Remark 4.9. Let $\mathscr{Q}^0$ be the centered generalized gaussian field on $\mathscr{D}_1'(\Lambda)$ with covariance

$$\operatorname{cov}(Z(\phi), Z(\psi)) = \langle \phi, \mathfrak{G}_\Lambda \psi \rangle, \quad \phi, \psi \in \mathscr{D}_1(\Lambda) .$$

Then we can identify $\mathscr{Q}^f$ as the law of $\mathscr{Q}^0$, conditioned upon $\{Z(f) = 0\}$, that is

$$\mathscr{Q}^f(\,\cdot\,) = \mathscr{Q}^0(\,\cdot\,|\, Z(f) = 0) .$$

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Note added in the proof Recently, Dima Ioffe told us that he had been working on a similar problem In some sense our results are complementary: his paper deals with a gas of many droplets with random basis, whereas we work with a single droplet with fixed basis However the repulsion problem is not addressed in his work

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