Commun. Math. Phys. 177, 157–171 (1996)

Communications in Mathematical Physics

On the Universal R-Matrix of $U_q\widehat{sl}_2$ at Roots of Unity

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Abstract: We show that the action of the universal R-matrix of the affine $U_q\widehat{sl}_2$ quantum algebra, when q is a root of unity, can be renormalized by some scalar factor to give a well-defined nonsingular expression, satisfying the Yang–Baxter equation. It can be reduced to intertwining operators of representations, corresponding to Chiral Potts, if the parameters of these representations lie on the well-known algebraic curve.

We also show that the affine $U_q\widehat{sl}_2$ for q is a root of unity from the autoquasi-triangular Hopf algebra in the sense of Reshetikhin.

1. Introduction

The intertwining operators of quantum groups ([1–4]) lead to solutions of the Yang–Baxter equation, which play the crucial role in two dimensional field theory and integrable statistical systems ([4, 5]). It is well-known that most of them can be obtained from the universal R-matrix ([1]) for a given quantum group: the solutions of the spectral parameter dependent Yang–Baxter equation can be obtained from the universal R-matrix of affine quantum groups ([6]) and the solutions of the non-spectral parameter dependent Yang–Baxter equations can be obtained from the universal R-matrix of finite quantum groups.

The situation is not the same for the case when the parameter q of the quantum group is a root of unity.

In this case the center of the quantum group is larger and a new type of representations appear, which have no classical analog ([5, 7, 8, 9]). It was shown in [10, 11] that the cyclic representations lead to solutions of the Yang-Baxter equation with a spectral parameter, lying on some algebraic curve. These solutions correspond to Chiral Potts Model([12–14]) and its generalizations (for quantum groups $U_q\widehat{sl}_n$).

The formal expression of the universal R-matrix fails in this case: it has singularities when q is a root of unity. Recently in [15] Reshetikhin introduced the notion

This work is supported by NATO linkage grant LG 9303057.

of the autoquasitriangular Hopf algebra to avoid these singularities. He treated the $U_q sl_2$ case.

The main goal of this paper is to show that after suitable renormalization by a scalar factor the universal R-matrix produces R-matrices for concrete representations.

In Sect. 2, we consider the universal R-matrix on Verma modules of $U_q sl_2$ when q is a root of unity. We prove that it is well defined and make a connection with the R-matrix of the autoquasitriangular Hopf algebra, founded by Reshetikhin.

In Sect. 3 we consider the algebra $U_q\widehat{sl_2}$ at roots of unity. We found the central elements of its Poincaré–Birkoff–Witt (PBW) basis, generalizing the results of [9] for the affine case. It appears that a new type of central elements appear for some imaginary roots, which have no analog for finite quantum groups. After this we prove the autoquasitriangularity of $U_q\widehat{sl_2}$, generalizing the results of [15] for the affine case. Then we consider the action of the affine universal R-matrix on $U_q\widehat{sl_2}$- and $U_q sl_2$-Verma modules. On $U_q\widehat{sl_2}$-Verma modules it is well defined. For $U_q sl_2$-Verma modules we renormalize its expression by a scalar factor to exclude the singularities. The remaining part leads to solutions of the infinite dimensional spectral parameter dependent Yang-Baxter equation. We showed that under a certain condition this R-matrix can be restricted to semicyclic representations, giving the Boltzmann weights of the Chiral Potts model, corresponding to such a type of representations, which was considered in [16–19]. The condition, mentioned above, is on the parameters of representations: they must lie on the well-known algebraic curve. It is the integrability condition of Chiral Potts model.

In the last section we made the same type of suggestion for cyclic representations.

2. The $U_q sl_2$ Case

2.1. The Universal R-matrix on Verma Modules at Root of Unity. The quantum group $U_q sl_2$ is a $[q,q^{-1}]$-algebra, generated by the elements E, F, K with the following relations between them:

$$[K,K^{-1}]=0\ , \qquad [E,F]=\tfrac{K-K^{-1}}{q-q^{-1}}\ , \tag{1}$$
$$KEK^{-1}=q^2E\ , \qquad KFK^{-1}=q^{-2}F\ .$$

On $U_q sl_2$ there is a Hopf algebra structure with the comultiplication $\Delta:\ U_q sl_2 \rightarrow U_q sl_2 \otimes U_q sl_2$ defined by

$$\Delta(K)=K\otimes K\ , \qquad \Delta(E)=E\otimes 1+K^{-1}\otimes E\ , \qquad \Delta(F)=F\otimes K+1\otimes F\ .$$

We denote $K=q^H$, $q=e^{\hbar}$, as usual, and consider the $[[\hbar]]$-algebra $U_\hbar sl_2$ with the same defining relations. $U_\hbar sl_2$ is a quasitriangular Hopf algebra, i.e. it possesses the universal R-matrix $R\in U_\hbar sl_2\otimes U_\hbar sl_2$ connecting the comultiplication Δ with the opposite comultiplication $\Delta'=\sigma\circ\Delta$, where $\sigma(x\otimes y):=y\otimes x$:

$$\Delta'(a)=R\Delta(a)R^{-1}, \quad \forall a\in U_\hbar sl_2\ . \tag{2}$$

It satisfies the quasitriangularity relations

$$(\varDelta \otimes 1)R = R_{13}R_{23} \qquad (1 \otimes \varDelta)R = R_{13}R_{12} \tag{3}$$

and the Yang–Baxter equation ([1])

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ . \tag{4}$$

Here we used the usual notation: if $R = \sum_i a_i \otimes b_i$, $a_i, b_i \in U_q sl_2$, then

$$R_{12} = \sum_i a_i \otimes b_i \otimes 1 \ , \qquad R_{13} = \sum_i a_i \otimes 1 \otimes b_i \ , \qquad R_{23} = \sum_i 1 \otimes a_i \otimes b_i \ .$$

The explicit expression of R in terms of the formal power series is

$$R = \exp_{q^{-2}}((q - q^{-1})(E \otimes F))q^{\frac{1}{2}H \otimes H}, \tag{5}$$

where the q-exponent is defined by $\exp_q(z) = \sum_{n \geqq 0} \frac{z^n}{(z)_q!}$, $(z)_q := \frac{1-q^n}{1-q}$.

Note, that to be precise, $U_q sl_2$ is not a quasitriangular Hopf algebra, because the term $q^{H \otimes H}$ in (5) does not belong to $U_q sl_2 \otimes U_q sl_2$, but it is an autoquasitriangular Hopf algebra ([15]). The latter is a Hopf algebra A, where the condition (2) is generalised by

$$\varDelta' = \hat{R}(\varDelta) \ ,$$

where $\hat{R}$ is an automorphism of $A \otimes A$ (not inner, in general). So, although (5) is ill defined on $U_q sl_2$, the action

$$\hat{R}(a) = RaR^{-1} \ , \tag{6}$$

where $a \in U_q sl_2 \otimes U_q sl_2$ is still well defined.

For two representations of $U_q sl_2$ V_1 and V_2 one can consider two $U_q sl_2$-actions on $V_1 \otimes V_2$ by means of both comultiplications $\varDelta$ and $\varDelta'$. If R is defined on $V_1 \otimes V_2$, then both $\varDelta$- and $\varDelta'$-actions are equivalent via the intertwining operator $R_{V_1 \otimes V_2} = R|_{V_1 \otimes V_2}$. For general q the restriction of (5) on the tensor product of two irreducible representations (of any highest weight representations) is well defined. And all solutions of the Yang–Baxter equation (4), having $U_q sl_2$- symmetry in the sense of (2) can be obtained from the universal R-matrix (5) in such a way.

The situation is different for q being a root of unity. In this case the singularities appear in the formal expression of R.

Recall that for $q = \exp\left(\frac{2\pi i}{N'}\right)$ the elements F^N, F^N, K^N, where $N = N'$ for odd N' and $N = \frac{N'}{2}$ for even N', belong to the center of $U_q sl_2$. In irreducible representations they are multiples of identity. Recall that every N-dimensional irreducible representation is characterized by the values x, y, z of these central elements (and also by the value of the q-deformed Casimir $c = \frac{Kq + K^{-1}q^{-1}}{q - q^{-1}} + EF$, which for the fixed x, y, z can have in general N discrete values ([9])).

Although the expression of the R-matrix (5) of $U_q sl_2$ has singularities for $q^{N'} \to 1$ in all terms $\frac{1}{(n)_{q^{-2}}!} E^n \otimes F^n$ for $n \geqq N$, its restriction on the tensor product of Verma modules $M_{\lambda_1} \otimes M_{\lambda_2}$ is well defined.

Recall that M_λ is formed by the basic vectors v_m^λ, $m = 0, 1, \dots$, satisfying

$$Ev_0^\lambda = 0, \qquad Fv_m^\lambda = v_{m+1}^\lambda, \qquad Hv_0^\lambda = \lambda v_0^\lambda, \quad \lambda \in C \ .$$

To consider the action of R on $M_{\lambda_1} \otimes M_{\lambda_2}$, we use the formula, which can be obtained from the defining relations (1) ([9]):

$$[E^n, F^s] = \sum_{j=1}^{\min(n,s)} \left[\begin{array}{c} n \\ j \end{array} \right] \left[\begin{array}{c} s \\ j \end{array} \right] [j]! F^{s-j} \left(\prod_{r=1}^{j} [H + j - n - s + r] \right) E^{n-j} ,$$

where

$$\left[\begin{array}{c} a \\ b \end{array} \right] = \frac{[a]!}{[b]![a-b]!} \quad \text{and} \quad [n] = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} .$$

So, for $n > s$ $\frac{E^n}{(n)_{q^{-2}}!} v_s^\lambda = 0$ and for $n \leqq s$:

$$\frac{E^n}{(n)_{q^{-2}}!} v_s^\lambda = q^{\frac{n(n-1)}{2}} \left[\begin{array}{c} s \\ n \end{array} \right] \prod_{r=1}^{n} [\lambda - s + r] v_{s-n}^\lambda .$$

The q-binomial $\left[\begin{smallmatrix} s \\ n \end{smallmatrix} \right]$ has a non-infinity limit for $q^{N'} \to 1$. So,

$$\begin{aligned} R(v_s^{\lambda_1} \otimes v_{s'}^{\lambda_2}) &= \sum_{n=0}^{s} q^{\frac{(\lambda_1 - 2s)(\lambda_2 - 2s')}{2}} q^{\frac{n(n-1)}{2}} (q - q^{-1})^n \left[\begin{array}{c} s \\ n \end{array} \right] \\ &\times \prod_{r=1}^{n} [\lambda_1 - s + r] v_{s-n}^{\lambda_1} \otimes v_{s'+n}^{\lambda_2} \end{aligned} \tag{7}$$

is well defined when q is a root of unity.

2.2. The Connection with Reshetikhin's R-matrix of Autoquasitriangular Hopf Algebra. This R-matrix can be presented in another form by using the recent results of Reshetikhin ([15]). He used an asymptotic formula for the q-exponent in the limit $q^{N'} \to 1$ to bring out multiplicatively singularities from $\exp_q((q - q^{-1}) E \otimes F)$. The expression of the universal R-matrix in this limit then acquires the form:

$$\begin{aligned} R &= \exp\left(\frac{1}{2N^2\hbar} Li_2(E^N \otimes F^N) \right) \left(1 - E^N \otimes F^N\right)^{-\frac{1}{2}} \\ &\times \prod_{m=0}^{N-1} (1 - \varepsilon^m E \otimes F)^{-\frac{m}{N}} q^{\frac{1}{2} H \otimes H} \cdot O(\hbar) . \end{aligned}$$

Here $q = \exp(\hbar)\varepsilon$, $\varepsilon = \exp(\frac{2\pi i}{N'})$ and $Li_2(x) = -\int_0^x \frac{\ln(1-y)}{y} dy$ is a dilogarithmic function.

Recall that although the elements

$$\frac{E^N}{(N)_{q^{-2}}!}, \quad \frac{F^N}{(N)_{q^{-2}}!} \quad \text{and } H$$

do not belong to $U_q sl_2$ for $q^{N'} \to 1$, their adjoint actions

$$ad(x)a = [x, a], \qquad Ad(\exp(x))a = \exp(x) a \exp(-x) = \exp(ad(x))a$$

on $U_q sl_2$ are well defined in this limit and give rise to some derivations ([9]). Let's denote them by e, f and h correspondingly.

The element $\frac{1}{2\hbar N^2}Li_2(E^N\otimes F^N)$ in the exponent of (8) in the adjoint representation also acts on $U_qsl_2\otimes U_qsl_2$ as a derivation in the limit $\hbar\to 0$. It can be expressed by means of the derivations e and f as follows:

$$\lim_{\hbar\to 0} ad\left(\frac{1}{2\hbar N^2}Li_2\left(E^N\otimes F^N\right)\right) = c_{N'}\frac{\ln\left(1-E^N\otimes F^N\right)}{E^N\otimes F^N}\times\left(e\otimes F^N+E^N\otimes f\right),$$

where

$$c_{N'} = \begin{cases} -\left(1-\varepsilon^{-2}\right)^{-N} & \text{for odd } N' \quad (N'=N) \\ (-1)^N\left(1-\varepsilon^{-2}\right)^{-N} & \text{for even } N' \quad (N'=2N)\end{cases}. \tag{8}$$

Note that

$$Ad\left(\varepsilon^{\frac{1}{2}H\otimes H}\right) = \varepsilon^{\frac{1}{2}(h\otimes H_L+H_R\otimes h)} = (1\otimes K_L)^{\frac{1}{2}h\otimes 1}\cdot(K_R\otimes 1)^{1\otimes\frac{1}{2}h},$$

where $X_L\,(X_R)$ is the left (right) multiplication on X, is well defined in the adjoint representation.

So, one can write down the automorphism $\hat{R}$ (6) in the limit $\hbar\to 0$, obtained in [15], in the following form:[1]

$$\begin{aligned}\hat{R} &= \prod_{m=0}^{N-1} Ad\left(\left(1-\varepsilon^m E\otimes F\right)^{-\frac{m}{N}}\right)\\ &\quad\times\exp\left(-\left(1-\varepsilon^{-2}\right)^{-N}\frac{\ln\left(1-E^N\otimes F^N\right)}{E^N\otimes F^N}\left(e\otimes F^N+E^N\otimes f\right)\right)\\ &\quad\times(1\otimes K_L)^{\frac{1}{2}h\otimes 1}\cdot(K_R\otimes 1)^{1\otimes\frac{1}{2}h}.\end{aligned} \tag{9}$$

Let us now consider the restriction of (9) on the quotient algebra obtained from U_qsl_2 by factorisation on the ideal, generated by E^N, i.e. impose $E^N=0$. Although this ideal is not stable with respect to derivations e,f,h, it is easy to see that it is stable with respect to $\hat{R}$.

Moreover, the left $U_qsl_2\otimes U_qsl_2$-module

$$I_{\lambda_1,\lambda_2} = (U_qsl_2\otimes I_{\lambda_2})\bigoplus(I_{\lambda_1}\otimes U_qsl_2)$$

is also stable with respect to $\hat{R}$. Here we denoted by I_λ the left U_qsl_2-module, generated by E and $(K-\varepsilon^\lambda)$. This fact allows to restrict (9) on Verma modules, because we have the left U_qsl_2-module equivalence

$$(U_qsl_2\otimes U_qsl_2)/I_{\lambda_1,\lambda_2}\cong M_{\lambda_1}\otimes M_{\lambda_2}\ .$$

So, one can derive from (9) the restriction of $\hat{R}$ on this factormodule given by the multiplication on[2]

$$R=\prod_{m=0}^{N-1}\left((1-\varepsilon^m E\otimes F)^{-\frac{m}{N}}\right)\exp\left(\left(1-\varepsilon^{-2}\right)^{-N}\left(e\otimes F^N\right)\right)(1\otimes K_L)^{\frac{1}{2}h\otimes 1}\cdot(K_R\otimes 1)^{1\otimes\frac{1}{2}h}. \tag{10}$$

[1] For quantum groups one can introduce 4 equivalent comultiplications: Δ_q, Δ'_q, $\Delta_{q^{-1}}$, $\Delta'_{q^{-1}}$ [6]. In [15] the comultiplication $\Delta'_{q^{-1}}$ had been used as a basic one. So, the R-matrix, used there, is $R^{-1}_{q^{-1}}$ in our notations and differs from the Δ_q-case used here by permutation of q-exponent and $q^{\frac{1}{2}H\otimes H}$.

[2] Note that both h and e are well defined on M_λ in contrast to f.

This expression is another form of expression of universal R-matrix (5) on Verma modules and coincide with (7).

3. The Case of Affine $U_q\widehat{sl}_2$

3.1. The PBW Basis and the Universal R-Matrix. The affine quantum universal enveloping algebra $U_q\widehat{sl}_2$ is a $[q,q^{-1}]$-Hopf algebra, generated by elements $E_i := E_{\alpha_i}$, $F_i = F_{\alpha_i}$, $K_i = q^{H_i}$, $i = 0,1$ and q^d with defining relations ([2]):

$$
\begin{gathered}
[q^{H_i}, q^{H_j}] = 0 \ , \qquad q^d q^{H_i} = q^{H_i} q^d \ , \qquad [E_i, F_j] = \delta_{ij}[H_i]_q \ , \\
q^{H_i} E_j q^{-H_i} = q^{a_{ij}} E_j \ , \qquad q^{H_i} F_j q^{-H_i} = q^{-a_{ij}} \ , \qquad q^d E_1 q^{-d} = q E_1 \ , \qquad (11) \\
q^d F_1 q^{-d} = q^{-1} E_1 \ , \qquad q^d E_0 q^{-d} = E_0 \ , \qquad q^d F_0 q^{-d} = F_0 \ , \\
(ad_q E_i)^{1-a_{ij}} E_j = 0 \ , \qquad (ad_q F_i)^{1-a_{ij}} F_j = 0 \ ,
\end{gathered}
$$

and comultiplication

$$
\begin{aligned}
\Delta(q^{H_i}) &= q^{H_i} \otimes q^{H_i} \ , & \Delta(q^d) &= q^d \otimes q^d \ , \\
\Delta(E_i) &= E_i \otimes 1 + q^{-H_i} \otimes E_i \ , & \Delta(F_i) &= F_i \otimes q^{H_i} + 1 \otimes F_i \ .
\end{aligned}
$$

Here we use the q-deformed adjoint action $(ad_q x)y := \sum_i x_i y s(x^i)$, where $\Delta(x) = \sum_i x_i \otimes x^i$ and $s : U_q\widehat{sl}_2 \to U_q\widehat{sl}_2$ is an antipode of $U_q\widehat{sl}_2$, defined by

$$
s(E_i) = -K_i E_i \ , \qquad s(F_i) = -F_i K_i^{-1} \ , \qquad s(K_i) = K_i^{-1} \ , \qquad s(q^d) = q^{-d}.
$$

Also we denoted by a_{ij} the Cartan matrix of affine $\hat{sl}(2)$ Lie algebra

$$
a_{ij} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.
$$

Let's denote by c the central element $c = H_1 + H_2$.

Define on $U_q\widehat{sl}_2$ an antiinvolution $\imath$ by

$$
\imath(K_i) = K_i^{-1}, \qquad \imath(E_i) = F_i \ , \qquad \imath(F_i) = E_i \ , \qquad \imath(q) = q^{-1}.
$$

As above, denote by $U_\hbar\widehat{sl}_2$ the $[[\hbar]]$-algebra with the same relations but with the elements H_i instead of q^{H_i}.

The PBW basis of $U_\hbar\widehat{sl}_2$ is formed by elements H_i, d, $E_{\alpha_i+n\delta}$, $F_{\alpha_i+n\delta}$, $E'_{n\delta}$ and $F'_{n\delta}$, which are inductively defined by the relations

$$
\begin{gathered}
E_{\alpha_0+n\delta} = (-1)^n (ad_{E'_\delta})^n E_0 \ , \qquad E_{\alpha_1+n\delta} = (ad_{E'_\delta})^n E_1 \ , \\
E'_{n\delta} = [2]^{-1}(E_{\alpha_0+(n-1)\delta} E_1 - q^{-2} E_1 E_{\alpha_0+(n-1)\delta}) \ , \qquad (12) \\
F_{\alpha_i+n\delta} = \imath(E_{\alpha_i+n\delta}) \ , \qquad F'_{n\delta} = \imath(E'_{n\delta}) \ .
\end{gathered}
$$

The expression of the universal R-matrix of $U_\hbar\widehat{sl}_2$ is simpler if one redefines $E'_{n\delta}$ and $F'_{n\delta}$ by means of Schur polynomials ([6]):

$$E'_{n\delta} = \sum_{\substack{0<k_1<\dots<k_m \\ k_1p_1+\dots+k_mp_m=n}} \frac{(q-q^{-1})^{\sum p_i-1}}{p_1!\dots p_m!}(E_{k_1\delta})^{p_1}\cdots(E_{k_m\delta})^{p_m} \ ,$$

$$F'_{n\delta} = \iota(E'_{n\delta}) \ .$$

In order to rewrite all the relations between (12) in compact form it is suitable to change slightly the basis as follows:

$$K_1^{\pm 1} = k^{\pm 1}, \qquad K_0^{\pm 1} = k^{\mp 1}q^{\pm c},$$

$$E_{\alpha_0+n\delta} = (-1)^n q^{-2n} x_{n+1}^- k^{-1}, \qquad E_{\alpha_1+n\delta} = (-1)^n q^{-(c+2)n} x_n^+, \tag{13}$$

$$E'_{n\delta} = (-1)^n \frac{q^{-\frac{c}{2}n-2n}}{q^2-q^{-2}}\psi_n k^{-1}, \qquad E_{n\delta} = (-1)^n \frac{q^{-(\frac{c}{2}+2)n}}{[2]}a_n.$$

Then the elements $x_n^\pm, (n \in Z)$, $a_k, (k \in Z, k \neq 0)$, ψ_m, φ_{-m}, $(m \geqq 1)$ and $\psi_0 = \varphi_0^{-1} = k$ satisfy the following relations:

$$[a_m, a_n] = \delta_{m,-n}\tfrac{[2m][mc]}{m} \ , \qquad [a_m, k] = 0,$$

$$kx_m^\pm k^{-1} = q^{\pm 2}x_m^\pm \ , \qquad [a_m, x_n^\pm] = \pm\tfrac{[2m]}{m}q^{\mp\frac{|m|}{2}c}x_{m+n}^\pm \ ,$$

$$x_{m+1}^\pm x_n^\pm - q^{\pm 2}x_n^\pm x_{m+1}^\pm = q^{\pm 2}x_m^\pm x_{n+1}^\pm - x_{n+1}^\pm x_m^\pm \ , \tag{14}$$

$$[x_m^+, x_n^-] = \tfrac{1}{q-q^{-1}}(q^{\frac{c}{2}(m-n)}\psi_{m+n} - q^{-\frac{c}{2}(m-n)}\varphi_{m+n}) \ ,$$

$$\sum_{m=0}^{\infty}\psi_m z^{-m} = k\exp\left((q-q^{-1})\sum_{m=1}^{\infty}a_m z^{-m}\right),$$

$$\sum_{m=0}^{\infty}\varphi_{-m}z^m = k^{-1}\exp\left(-(q-q^{-1})\sum_{m=1}^{\infty}a_{-m}z^m\right).$$

These relations had been introduced by Drinfeld in [20] and define another realization of affine algebra $U_q\widehat{sl}_2$. The antiinvolution ι in this notation is

$$\iota(x_n^\pm) = x_{-n}^\mp \ , \qquad \iota(\psi_n) = \varphi_{-n} \ , \qquad \iota(a_n) = a_{-n} \ , \qquad \iota(q) = q^{-1} \ .$$

We choose the normal ordering of the positive root system Δ_+ of $U_q\widehat{sl}_2$ as follows:

$$\alpha_0, \alpha_0+\delta, \dots, \alpha_0+n\delta, \dots \delta, 2\delta, \dots, n\delta, \dots, \alpha_1+n\delta, \dots, \alpha_1+\delta, \dots, \alpha_1 \ . \tag{15}$$

Then the universal R-matrix has the form [6]:

$$R = \left(\prod_{n\geqq 0}\exp_{q^{-2}}((q-q^{-1})(E_{\alpha_0+n\delta}\otimes F_{\alpha_0+n\delta}))\right)\exp\left(\sum_{n>0}\frac{nE_{n\delta}\otimes F_{n\delta}}{q^{2n}-q^{-2n}}\right)$$
$$\times\left(\prod_{n\geqq 0}\exp_{q^{-2}}((q-q^{-1})(E_{\alpha_1+n\delta}\otimes F_{\alpha_1+n\delta}))\right)q^{\frac{1}{2}H_0\otimes H_0+c\otimes d+d\otimes c}, \tag{16}$$

where the product is given according to the normal order (15).

3.2. $U_q\widehat{sl}_2$ at roots of Unity. For q being a root of unity ($q = \varepsilon$, $\varepsilon = e^{\frac{2\pi i}{N'}}$), the center of $U_q\widehat{sl}_2$ is enlarged by the N^{th} power of the root vectors, as for finite quantum groups:

$$[E_\gamma^N, x] = 0 \ , \qquad [F_\gamma^N, x] = 0 \ , \qquad [K_i^N, x] = 0 \ , \tag{17}$$

where $\gamma \in \Delta_+ := \{\alpha_i + n\delta, m\delta | n \geqq 0, m > 0\}$ and $x \in U_\varepsilon\widehat{sl}_2$.

These conditions for the simple roots $\gamma = \alpha_i$ can be proven by using the defining relations of the Cartan–Weyl basis (11) as for finite quantum algebras it had been done in [9]. Indeed, using

$$\Delta(E_i^N) = K_i^{-N} \otimes E_i^N + E_i^N \otimes 1 \ ,$$

recalling that the q-deformed adjoint action ad_q is a $U_q\widehat{sl}_2$-representation:

$$ad_q(ab)c = ad_q(a)ad_q(b)c \ , \quad \forall a,b,c \in U_q\widehat{sl}_2 \ ,$$

and using Serre relations in (12), we obtain for $i \neq j$, $N \geqq 3$:

$$[E_i^N, E_j] = ad_q(E_i^N)E_j = (ad_q(E_i))^N E_j = 0 \ .$$

Other commutations in (17) for $\gamma = \alpha_i$ can be verified easily.

To carry out (17) for other roots one can try to use the isomorphism, induced by the q-deformed Weyl group. In the affine case it had been considered in [21]. But it is easier to use the symmetries of Drinfeld realization of $U_q\widehat{sl}_2$ directly. It is easy to see from (14) that the operation $\omega_\pm$ on $U_q\widehat{sl}_2$ defined by

$$\omega_\pm(x_m^\pm) = x_{m\pm1}^\pm \ , \qquad \omega_\pm(a_m) = a_m, \qquad \omega_\pm(q) = q \ ,$$
$$\omega_\pm(\psi_n) = q^c\psi_n \ , \quad \omega_\pm(\varphi_n) = q^{-c}\varphi_n \ , \quad \omega_\pm(c) = c \ . \tag{18}$$

is an algebra automorphism. As the roots can be obtained by applying $\omega_\pm$ from the simple ones, we finished the proof.

In addition to this, the elements $E_{kN\delta}$, $F_{kN\delta}$ are central for $k \in N_+$. This can be seen from (14) and (13). These central elements have no analog for finite algebras.

The adjoint action of $\frac{E_\gamma^N}{(N)_{q^{-2}}!}$, $\frac{F_\gamma^N}{(N)_{q^{-2}}!}$, $\gamma \in \Delta_+$ and $\frac{kNE_{kN\delta}}{q^{2kN}-q^{-2kN}}$, $\frac{kNF_{kN\delta}}{q^{2kN}-q^{-2kN}}$ lead in the limit $\hbar \to 0$ to derivations of $U_\varepsilon\widehat{sl}_2$, which we denote by e_γ, f_γ, $\hat{e}_k$, $\hat{f}_k$ correspondingly. The action of the automorphism ω on these derivations inherits its action from corresponding root vectors.

3.3. The Universal R-Matrix at Roots of 1. Now let's consider the expression of the universal R-matrix (16) in the limit $\hbar \to 0$. The singularities, which appear in all q-exponents, are the same type as in the expression of the universal R-matrix of $U_\hbar sl_2$. A new type of singularities appear due to the factor $\frac{kN}{q^{2kN}-q^{-2kN}}$ in the exponent before all terms $E_{kN\delta} \otimes F_{kN\delta}$ for any natural k.

But as in the $U_\varepsilon sl_2$ case, the adjoint action $\hat{R}$ of R on $U_\varepsilon\widehat{sl}_2 \otimes U_\varepsilon\widehat{sl}_2$ is well defined.

Indeed, the adjoint action of every q-exponent term

$$R_\gamma = \exp_{q^{-2}}((q - q^{-1})(E_\gamma \otimes F_\gamma)), \quad \gamma = \alpha_i + n\delta$$

in (16) can be treated as it has been done in the $U_\varepsilon sl_2$ case:

$$\begin{aligned}\lim_{\hbar\to 0} Ad(R_\gamma) &= \prod_{m=0}^{N-1} Ad((1 - \varepsilon^m E_\gamma \otimes F_\gamma)^{-\frac{m}{N}}) \\ &\times \exp\left(c_{N'} \frac{\ln(1 - E_\gamma^N \otimes F_\gamma^N)}{E_\gamma^N \otimes F_\gamma^N}(e_\gamma \otimes F_\gamma^N + E_\gamma^N \otimes f_\gamma)\right),\end{aligned}$$

where $c_{N'}$ is defined by (8).

From (13) and (14) it follows that the operations

$$\hat{e}_k = \lim_{\hbar\to 0} ad\left(\frac{kNE_{kN\delta}}{q^{kN} - q^{-kN}}\right), \qquad \hat{f}_k = \lim_{\hbar\to 0} ad\left(\frac{kNF_{kN\delta}}{q^{kN} - q^{-kN}}\right)$$

also are the derivations on $U_\varepsilon\widehat{sl}_2$, as it was mentioned above. So,

$$\begin{aligned}\hat{R}_{kN\delta} &= \lim_{\hbar\to 0} Ad(R_{kN\delta}) = \lim_{\hbar\to 0} Ad\left(\exp\left(\frac{kN}{q^{kN} - q^{-kN}} E_{kN\delta} \otimes F_{kN\delta}\right)\right) \\ &= \exp(\hat{e}_k \otimes F_{kN\delta} + E_{kN\delta} \otimes \hat{f}_k),\end{aligned}$$

gives rise to an outer automorphism of $U_\varepsilon\widehat{sl}_2$.

Finally, the right term in (16) has the following adjoint action:

$$\hat{\mathscr{K}} = Ad(\varepsilon^{\frac{1}{2}H_0\otimes H_0 + c\otimes d + d\otimes c}) = (1 \otimes (K_0)_L)^{\frac{1}{2}h_0\otimes 1} \cdot ((K_0)_R \otimes 1)^{1\otimes\frac{1}{2}h_0} \varepsilon^{c\otimes ad(d) + ad(d)\otimes c}.$$

Here $h_0 = ad(H_0)$ is a derivation on $U_q\widehat{sl}_2$.

So, we proved that the quantum algebra $U_\varepsilon\widehat{sl}_2$ is an autoquasitriangular Hopf algebra with the automorphism

$$\hat{R} = \left(\prod_{\gamma\in\Delta_+} \hat{R}_\gamma\right) \hat{\mathscr{K}}, \tag{19}$$

where the product over positive roots is ordered according to the normal order (15).

3.4. The Universal R-matrix on Verma Modules. Consider now the Verma module $M_{\hat{\lambda}}$ over $U_\varepsilon\widehat{sl}_2$ with highest weight $\hat{\lambda}$. It is generated by vectors

$$v^{\hat{\lambda}}_{k_1\cdots k_n} = F^{k_n}_{\gamma_n} \cdots F^{k_1}_{\gamma_1} v^{\hat{\lambda}}_0, \quad k_1,\dots,k_n = 0,1,\dots, \quad \gamma \in \Delta_+ \quad \gamma_1 < \cdots < \gamma_n,$$

where $v^{\hat{\lambda}}_0$ is a highest weight vector:

$$E_\gamma v^{\hat{\lambda}}_0 = 0 \quad , \; H v^{\hat{\lambda}}_0 = \hat{\lambda}(H) v^{\hat{\lambda}}_0.$$

As for $U_q sl_2$-case all terms R_γ and K in the product of the universal R-matrix (16) are well defined in the limit $\hbar \to 0$. Indeed, there is a well defined action of derivations e_i, $\hat{e}_i$ on $M_{\hat{\lambda}}$ by

$$e_i g v_0^{\hat{\lambda}} := e_i(g) v_0^{\hat{\lambda}} , \qquad \hat{e}_i g v_0^{\hat{\lambda}} := \hat{e}_i(g) v_0^{\hat{\lambda}} \quad \forall g \in U_\varepsilon \widehat{sl}_2 .$$

Moreover, in the action of (16), on any vector $x \in M_{\hat{\lambda}_1} \otimes M_{\hat{\lambda}_1}$ the term R_γ with sufficiently large γ give rise to the identity and the only finite number of R_γ survive. In the decomposition of each such R_γ only finitely many terms also survive. So, the action of R on $x \in M_{\hat{\lambda}_1} \otimes M_{\hat{\lambda}_1}$ is well defined.

To define the action of the Universal R-matrix (16) on $U_q sl_2$-Verma modules, the spectral parameter dependent homomorphism ρ_x: $U_q \widehat{sl}_2 \to U_q sl_2$ must be introduced [3]:

$$\rho_x(E_{\alpha_0}) = E , \qquad \rho_x(F_{\alpha_0}) = F , \qquad \rho_x(H_0) = H ,$$
$$\rho_x(E_{\alpha_1}) = xF , \qquad \rho_x(F_{\alpha_1}) = x^{-1}E , \qquad \rho_x(H_1) = -H .$$

Note that in this representation the central charge c is zero. Under the action of ρ_x the root vectors acquire the form ([22]):

$$\begin{aligned}
E_{\alpha_0+n\delta} &= (-1)^n x^n q^{-nh} E , \qquad F_{\alpha_0+n\delta} = (-1)^n x^{-n} F q^{nh} ,\\
E_{\alpha_1+n\delta} &= (-1)^n x^{n+1} F q^{-nh} , \qquad F_{\alpha_1+n\delta} = (-1)^n x^{-n-1} q^{nh} E ,\\
E'_{n\delta} &= \tfrac{(-1)^{n-1}}{[2]_q} x^n q^{-(n-1)h} (EF - q^{-2} FE) ,\\
F'_{n\delta} &= \tfrac{(-1)^{n-1}}{[2]_q} x^{-n} q^{(n-1)h} (FE - q^{-2} EF) .
\end{aligned} \tag{20}$$

Substituting this in the expression of the affine universal R-matrix following [22], one can obtain the spectral parameter R-matrix:

$$R\left(\frac{x}{y}\right) = (\rho_x \otimes \rho_y) R = R^+\left(\frac{x}{y}\right) R^0\left(\frac{x}{y}\right) R^-\left(\frac{x}{y}\right) \mathscr{K}, \tag{21}$$

where

$$\begin{aligned}
R^+(z) &= \prod_{n\geqq 0} \exp_{q^{-2}} \left((q - q^{-1}) z^n (q^{-nH} E \otimes F q^{nH})\right),\\
R^0(z) &= \exp\left(\sum_{n>0} \frac{n}{q^{2n} - q^{-2n}} z^n E_{n\delta} \otimes F_{n\delta}\right),\\
R^-(z) &= \prod_{n\geqq 0} \exp_{q^{-2}} \left((q - q^{-1}) z^{n+1} (F q^{-nH} \otimes q^{nH} E)\right),\\
\mathscr{K} &= q^{\frac{1}{2} H \otimes H}.
\end{aligned} \tag{22}$$

Now we consider (22) on Verma modules M_λ of $U_q sl_2$ and its behavior at roots of unity.

Note that one can represent the terms $R^{\pm}, R^0$ of the universal R-matrix in a more suitable way by performing the infinite sum and infinite product in (22). So, we have ([23]):

$$\begin{aligned} R^+(z) &= 1 + (E \otimes F)\frac{(q-q^{-1})}{1 - zq^{-2}K^{-1} \otimes K} \\ &+ \frac{(E\otimes F)^2}{(2)_{q^{-2}}!}\frac{(q-q^{-1})^2}{(1-zq^{-2}K^{-1}\otimes K)(1-zq^{-4}K^{-1}\otimes K)} \\ &+ \cdots + \frac{(E\otimes F)^n}{(n)_{q^{-2}}!}\frac{(q-q^{-1})^n}{(1-zq^{-2}K^{-1}\otimes K)\cdots(1-zq^{-2n}K^{-1}\otimes K)} + \cdots, \end{aligned} \tag{23}$$

$$\begin{aligned} R^-(z) &= 1 + \frac{z(q-q^{-1})}{1 - zq^{-2}K^{-1} \otimes K} F \otimes E \\ &+ \frac{1}{(2)_{q^{-2}}!}\frac{z^2(q-q^{-1})^2}{(1-zq^{-2}K^{-1}\otimes K)(1-zq^{-4}K^{-1}\otimes K)}(F\otimes E)^2 + \cdots \\ &+ \frac{1}{(n)_{q^{-2}}!}\frac{z^n(q-q^{-1})^n}{(1-zq^{-2}K^{-1}\otimes K)\cdots(1-zq^{-2n}K^{-1}\otimes K)}(F\otimes E)^n + \cdots, \end{aligned} \tag{24}$$

and

$$R^0(z) = f(z)\bar{R}^0(z), \tag{25}$$

where

$$\begin{aligned} f(z) &= \exp\sum_{n\geqq 1}\left((q-q^{-1})\frac{[\lambda_1 n]_q[\lambda_2 n]_q}{[2n]_q}\right)\frac{z^n}{n} \\ &= \frac{(zq^{\lambda_1-\lambda_2-2};q^{-4})_\infty(zq^{\lambda_1-\lambda_2-2};q^{-4})_\infty}{(zq^{\lambda_1+\lambda_2-2};q^{-4})_\infty(zq^{-\lambda_1-\lambda_2-2};q^{-4})_\infty}, \end{aligned} \tag{26}$$

$$(z;q)_\infty = \prod_{i=0}^{\infty}(1-zq^k),$$

$$\begin{aligned} \bar{R}^0(z) &= \exp\sum_{n\geqq 1}\left(\frac{q^n+q^{-n}}{(q^n-q^{-n})}\left(q^{-\lambda_1 n}-K^{-n}\right)\otimes\left(K^n - q^{\lambda_2 n}\right)\right)\frac{z^n}{n} \\ &\times \exp\sum_{n\geqq 1}\left((q^{-\lambda_1 n}-K^{-n})\otimes q^{-n}\frac{[\lambda_2 n]_q}{[n]_q} + q^n\frac{[\lambda_1 n]_q}{[n]_q}\otimes(K^n - q^{\lambda_2 n})\right)\frac{z^n}{n}. \end{aligned} \tag{27}$$

By performing the infinite sum in (27) one can easily show that the term $\bar{R}^0(z)$ acting on $v_i^{\lambda_1}\otimes v_j^{\lambda_2}$ gives rise to the following expression, which is well defined in the limit $q^N \to 1$:

$$\bar{R}^0(z)v_i^{\lambda_1}\otimes v_j^{\lambda_2} = \frac{\prod_{l=j-i+1}^{j}(1-q^{-2l}q^{\lambda_2-\lambda_1}z)\prod_{l=0}^{j-1}(1-q^{-2l}q^{\lambda_2+\lambda_1}z)}{\prod_{l=i-j+1}^{i}(1-q^{2l}q^{\lambda_2-\lambda_1}z)\prod_{l=0}^{i-1}(1-q^{2l}q^{-\lambda_2-\lambda_1}z)}v_i^{\lambda_1}\otimes v_j^{\lambda_2}. \tag{28}$$

The scalar factor $f(z)$ (26) is singular for $q^{N'} = 1$. It can be omitted from the expression of the R-matrix. So, the regular expression of the R-matrix for $q^N = 1$ on $M_{\lambda_1} \otimes M_{\lambda_2}$ has the form

$$R_{\lambda_1,\lambda_2}(z) = R^+(z)\bar{R}^0(z)R^-(z)\mathscr{K} \; . \tag{29}$$

Note that it satisfies $R_{\lambda_1,\lambda_2}(z)v_0^{\lambda_1} \otimes v_0^{\lambda_2} = q^{\frac{1}{2}\lambda_1\lambda_2}v_0^{\lambda_1} \otimes v_0^{\lambda_2}$. This renormalized expression of the R-matrix doesn't satisfy the quasitriangularity condition (3). The intertwining property (2) and the spectral parameter dependent Yang–Baxter equation

$$R_{\lambda_1,\lambda_2}\left(\frac{x_1}{x_2}\right) R_{\lambda_1,\lambda_3}\left(\frac{x_1}{x_3}\right) R_{\lambda_2,\lambda_3}\left(\frac{x_2}{x_3}\right) = R_{\lambda_2,\lambda_3}\left(\frac{x_2}{x_3}\right) R_{\lambda_1,\lambda_3}\left(\frac{x_1}{x_3}\right) R_{\lambda_1,\lambda_2}\left(\frac{x_1}{x_2}\right) \tag{30}$$

are satisfied.

Let us consider now the possibility to restrict (29) on finite dimensional semicyclic modules. Recall that the semicyclic module $V_{\alpha,\lambda}$ is obtained by factorisation of M_λ on $I_{\alpha,\lambda} = (F^N - \alpha)M_\lambda$ for some $\alpha \in C$:

$$V_{\alpha,\lambda} = M_\lambda / I_{\alpha,\lambda} \; .$$

The R-matrix (29) is well defined on $V_{\alpha_1,\lambda_1} \otimes V_{\alpha_2,\lambda_2}$ if it preserves this factorization, i.e.

$$R_{\lambda_1,\lambda_2}(z)(M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \subset (M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \bigoplus (I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2}) \tag{31}$$

and

$$R_{\lambda_1,\lambda_2}(z)(I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2}) \subset (M_{\lambda_1} \otimes I_{\alpha_2,\lambda_2}) \bigoplus (I_{\alpha_1,\lambda_1} \otimes M_{\lambda_2}) \; . \tag{32}$$

The conditions above follow from

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)({\lambda_2}^N \cdot F^N \otimes 1 + 1 \otimes F^N) = ({\lambda_1}^N \cdot 1 \otimes F^N + F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right),$$

$$\begin{aligned} R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)&(x^N \cdot F^N \otimes 1 + y^N {\lambda_1}^N \cdot 1 \otimes F^N) \\ &= (y^N \cdot 1 \otimes F^N + x^N {\lambda_2}^N \cdot F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right). \end{aligned}$$

Here we used the intertwining property (2) for

$$\Delta(E_i^N) = E_i^N \otimes 1 + K_1^{-N} \otimes E_1 \; , \qquad \Delta(F_i^N) = F_i^N \otimes K_i^N + 1 \otimes F_i^N .$$

So, one can express the operators

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)(F^N \otimes 1) \quad \text{and} \quad R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)(1 \otimes F^N)$$

as a linear combination of the operators

$$(F^N \otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) \quad \text{and} \quad (1 \otimes F^N)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)$$

(if $\frac{x^N}{y^N} \neq {\lambda_1}^N {\lambda_2}^N$). In the same way,

$$R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)((F^N-\lambda_1)\otimes 1) \quad \text{and} \quad R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)(1\otimes(F^N-\lambda_2))$$

are a linear combination of terms

$$((F^N-\lambda_1)\otimes 1)R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right) \quad \text{and} \quad (1\otimes(F^N-\lambda_2))R_{\lambda_1,\lambda_2}\left(\frac{x}{y}\right)$$

with the same coefficients if parameters $x, y, \lambda_1, \lambda_2, \alpha_1, \alpha_2$ lie on the algebraic curve

$$\frac{\alpha_1}{1-{\lambda_1}^N} = \frac{\alpha_2}{1-{\lambda_2}^N}\,, \qquad z^N = \left(\frac{x}{y}\right)^N = 1\,. \tag{33}$$

In this case the factorisation conditions (31), (32) are fulfilled and the R-matrix (29) can be reduced to the R-matrix $R_{V_{\alpha_1,\lambda_1}\otimes V_{\alpha_2,\lambda_2}}$ of semicyclic representations of $U_\varepsilon\widehat{sl}_2$, considered in [17–19].The condition (33) on parameters of representations appears naturally as a consistency of factorisation $V_{\alpha,\lambda} = M_\lambda/I_{\alpha,\lambda}$ with the intertwining property (2) of R-matrix.

Note that the formulae (23), (24), (27), (29) can be applied directly to semicyclic modules, using the constraint $F^N = \alpha \cdot id$ on $V_{\alpha,\lambda}$.

4. Discussions

Let's consider now the possibility of restriction of the automorphism (19) in the evaluation representation (20) to cyclic modules. Recall that their intertwining operators are the Boltzmann weight of the Chiral Potts model ([10]). The cyclic modules are representations of the quotient algebra $Q_\xi = Q_{\beta,\alpha,\lambda}$, $\xi = (\beta,\alpha,\lambda)$, which is obtained from $U_\varepsilon sl_2$ by factorisation on the ideal $I_{\beta,\alpha,\lambda}$, generated by $(F^N-\alpha), (E^N-\beta), (K^N-\lambda^N)$, $(\beta,\alpha,\lambda \in C)$ ([9]):

$$Q_{\beta,\alpha,\lambda} = U_\varepsilon sl_2/I_{\beta,\alpha,\lambda}\,.$$

The necessary condition for restriction of $\hat{R}(z)$ to Q_ξ is the constraint on the parameters of the representation to lie on the algebraic curve, defined by

$$\frac{\alpha_1}{1-{\lambda_1}^N} = \frac{\alpha_2}{1-{\lambda_2}^N}\,, \qquad \left(\frac{x}{y}\right)^N = 1\,,$$
$$\frac{\beta_1}{1-{\lambda_1}^{-N}} = \frac{\beta_2}{1-{\lambda_2}^{-N}}\,. \tag{34}$$

We expect that this condition is also sufficient and the automorphism $\hat{R}$ can be restricted on some automorphism (outer, in general) of the quotient algebra $Q_{\xi_1}\otimes Q_{\xi_2}$, which we denote by $\hat{R}^{Q_{\xi_1}\otimes Q_{\xi_2}}$.

Consider now its action on the tensor product of cyclic modules $V_{\xi_1}\otimes V_{\xi_2}$. $\hat{R}^{Q_{\xi_1}\otimes Q_{\xi_2}}$ is reduced here to the matrix algebra automorphism. Recall that every automorphism of the matrix algebra is inner. So,

$$\hat{R}^{Q_{\xi_1}\otimes Q_{\xi_2}}_{V_{\xi_1}\otimes V_{\xi_2}} = Ad(R_{\xi_1,\xi_2})$$

with some matrix R_{ξ_1,ξ_2}. This R-matrix is nothing but the Boltzmann weights of the Chiral Potts model.

For quotients $Q_{0,\alpha,\lambda}$, corresponding to semicyclic irreps, this suggestion is true.

Note that in the case of $q^4 = 1$ there is a Hopf algebra homomorphism between different quotients, as was observed in [24]. This fact was used there to construct R-matrices of quotient algebras for $q^4 = 1$ from the R-matrix of $Q_{0,0,\lambda_1} \otimes Q_{0,0,\lambda_1}$, which corresponds to nilpotent irreps.

Another question is to extend these results in the case of other quantum algebras.

When we had finished this work, we saw the paper [25, 26] where the center of the quantum Kac–Moody algebras was studied also. As was observed there the automorphisms $\omega_\pm$ (18) correspond to translations of the quantum Weyl group.

Acknowledgements. The authors thank V. Pasquier and J.-B. Zuber for helpful discussions and the Saclay Theory Division for hospitality, where this work was started.

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Communicated by G. Felder