

Geometry of the Transport Equation in Multicomponent WKB Approximations

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Abstract: Although the WKB approximation for multicomponent systems has been intensively studied in the literature, its geometric and global aspects are much less well understood than in the scalar case. In this paper we give a completely geometric derivation of the transport equation, without using local sections and without assuming complete diagonalizability of the matrix valued principal symbol, or triviality of its eigenbundles. The term (unnamed in the previous literature) appearing in the transport equation in addition to the covariant derivative with respect to a natural projected connection is a tensor, independent of the choice of any sections. We give a geometric interpretation of this tensor, involving the contraction of the curvature of the eigenbundle and an analog of the second fundamental form with the Poisson tensor in phase space. In the non-degenerate case this term may be rewritten in an even simpler geometric form. Finally, we discuss obstructions to the existence of WKB states and give a geometric description of the quantization condition for WKB states for a non-degenerate eigenvalue-function.

1. Introduction

In its original analytic form, the so-called WKB method for obtaining asymptotic eigenfunctions for linear partial differential operators involves writing a trial approximate eigenfunction for an operator H in the form $\psi(x) = e^{iS(x)/\hbar} a(x)$. Expanding $H\psi - E\psi$ in powers of \hbar leads first to a nonlinear first order partial differential equation (the eikonal, or Hamilton-Jacobi equation) for the phase function S and then to a linear homogeneous first order partial differential equation for the amplitude a (the transport equation).

A geometric version of the WKB method was developed by Maslov [16] and Hörmander [10], in which the phase function is represented by a lagrangian submanifold L in classical phase space, and the amplitude by a half-density α on L . This geometric approach makes it possible to extend the WKB method to cover in a natural

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way the so-called caustic points, which are inevitable in bound-state problems, and which lead to singularities in the analytic approach. We refer to [1, 7, 9] for extensive treatments of this “geometric WKB” theory.¹

Much of the work described in the preceding paragraphs was carried out originally for equations in a single unknown (complex-valued) function. Since many interesting equations in mathematical physics involve several functions (or even sections of nontrivial vector bundles), it has been of interest to extend the WKB method to such multicomponent equations. Much progress has been made in this direction, both in pure mathematics and in mathematical physics (see for instance [5, 6, 11, 12, 17]). For the physical approach, we refer especially to [14], which contains extensive references to earlier work, and which was the starting point for our own study. The cited authors have extended the analytic version of the WKB method to the case where the symbol matrix of the differential equation at hand has an eigenvalue of constant multiplicity. Their results are general enough to cover the local theory in the presence of caustics, but a completely geometric description has not been achieved, in particular for the transport equation.

The aim of this paper is to present a geometric formulation, with a coordinate-free, gauge-invariant derivation, of the transport equation for multicomponent systems in terms of a linear connection on the eigenvector bundle of the principal symbol matrix of a system of linear differential operators. We compare our results with those of [14], showing in particular that the term which “has no name” in their formulation of the transport equation can be interpreted in terms of the curvature of the eigenvector bundle and its complement. More precisely, the connection involves a contraction of the curvature with the Poisson tensor on phase space. This same contraction appears in Kähler geometry [13], where it is known as “mean curvature” and appears to involve more structure; we note here that it really depends only on the Poisson structure associated to the Kähler form and therefore call it “Poisson curvature.” Its importance in the context of symplectic geometry is only now becoming apparent. (See [20].)

The ultimate goals of our study go beyond the scope of this paper. One is to clarify the semiclassical quantization conditions in the multicomponent setting. Another is to deal with the extremely important “level-crossing” problem in which the eigenvalues of the symbol matrix have variable multiplicity. We hope that our geometric methods will facilitate work on these difficult problems (see [4] and [15] for recent contributions), even though we do not attack them here.

2. Projection Matrices over the Moyal Algebra

Our basic strategy will be the same as that used in much of the previous work—to begin by breaking off from the given operator a piece corresponding to the eigenspaces in question, and then to consider the reduced system, whose principal symbol is a multiple of the identity matrix by a scalar function. In this way, we reduce the problem as far as possible to the scalar case. Interesting geometry arises from the fact that the natural domain of this reduced system is a vector bundle over phase space which is locally “twisted.” When the phase space has non-trivial topology, this bundle can also

¹ We would also like to mention here the paper of Einstein [8], which may be the first publication in which lagrangian submanifolds in phase space are used to represent quantum states. Since this paper was written in the time of the “old quantum theory,” there is no question here of WKB approximations. Nevertheless, the lagrangian submanifolds appear clearly as generalized multiple valued solutions of the Hamilton-Jacobi equation.

be globally nontrivial. (Even when the original phase space is topologically simple, we may have to remove points at which the multiplicity of the eigenvalue increases, leaving behind a topologically complicated space.)

Our treatment will differ from that in [14] in that we do not attempt to put the operator in question into block diagonal form (which requires a choice of eigenvectors which depends smoothly on points in phase space); instead we follow [5, 11, 12] by putting the emphasis on projections onto the eigenspaces. These latter objects are completely canonical.

As in [14], we will use the calculus in which operators are represented by matrices whose entries are formal power series in a small parameter (we use \hbar instead of their ϵ) whose coefficients are C^∞ functions on classical phase space. The operation of these functions on phase space will be by the Weyl ordering, so the appropriate multiplication of the matrix entries is by the Moyal product. We will need to make explicit use of only the initial part of the development of this product,

$$a *_{\hbar} b = ab + (i\hbar/2)\{a, b\} + O(\hbar), \tag{1}$$

so that although the Moyal product itself applies only to the phase space \mathbb{R}^{2n} , the results in this section of our paper will be applicable whenever we are dealing with a phase space carrying a Poisson bracket satisfying the usual axioms [19]. Note that the commutator bracket $[a, b]_* = a *_{\hbar} b - b *_{\hbar} a$ is asymptotic to $i\hbar\{a, b\}$ as $\hbar \rightarrow 0$.

Before going further, let us fix some terminology and notation. We denote by \mathbf{A} the algebra of functions on phase space, with the usual pointwise multiplication. $\mathbf{A}[[\hbar]]$ denotes the algebra of formal power series in \hbar with coefficients in \mathbf{A} , with multiplication given by the Moyal product. \mathbf{M}_N denotes the algebra of $N \times N$ matrices with coefficients in \mathbf{A} . Its elements can also be thought of as matrix-valued functions on phase space, with the multiplication given by pointwise matrix multiplication. We will often write \mathbf{M} for \mathbf{M}_N when it is not important to specify the dimension of the matrices. Finally, $\mathbf{M}_N[[\hbar]]$ (or $\mathbf{M}[[\hbar]]$ for short) denotes the space of formal power series with coefficients in \mathbf{M}_N , with multiplication given by thinking of its elements as matrices with entries in $\mathbf{A}[[\hbar]]$. This multiplication is also given by a formula like (1) above, where the first term is ordinary matrix multiplication, and the Poisson bracket of matrix-valued functions is defined by $\{a, b\}_{ij} = \sum_k \{a_{ik}, b_{kj}\}$.

The hamiltonian H which we will consider will be an element of $\mathbf{M}[[\hbar]]$. Given any such element $A = A_0(x) + \hbar A_1(x) + \dots$, we call the matrix-valued function $A_0(x)$ the **principal symbol** of A . A scalar function $\lambda(x)$ will be called a **regular eigenvalue function** for H if $\lambda(x)$ is an eigenvalue for $H_0(x)$ with multiplicity independent of x , and if the null space and range of $H_0(x) - \lambda(x)I$ are complementary subspaces for each value of x . (The latter condition is satisfied automatically if the values of H_0 are hermitian matrices.) There is a well-defined projection matrix $\pi_0(x)$ onto the $\lambda(x)$ -eigenspace along the range of $H_0(x) - \lambda(x)I$, which depends smoothly on x . The images of the π_0 (the family of eigenspaces of the $H_0(x)$) form a vector bundle over phase space which we denote by E_λ and call the λ -**eigenbundle** of H_0 . E_λ^\perp will denote the family of null spaces of the $\pi_0(x)$. It is a complementary bundle to E_λ ; it really is an orthogonal complement when the $H_0(x)$ are hermitian operators and the $\pi_0(x)$ are consequently orthogonal projections.

For spectral theory, we will need a $\pi \in \mathbf{M}[[\hbar]]$ which is a projection in the sense that $\pi *_{\hbar} \pi = \pi$ and whose principal symbol is π_0 . Such a projection always exists and can even be chosen to commute with H , according to the following proposition:

Proposition 1. *Let λ be a regular eigenvalue function for H , π_0 the corresponding projection onto the eigenbundle E_λ along E_λ^\perp . Then there is a unique projection $\pi \in \mathbf{M}[[\hbar]]$ whose principal symbol is π_0 and which commutes with H .*

Proof. We first show that we can modify π_0 by adding higher order terms such that it becomes a projection with respect to the $*$ -product. To see that, we use an induction argument and suppose that we have chosen elements π_1, \dots, π_k of \mathbf{M} such that $\pi^{(k)} = \pi_0 + \hbar\pi_1 + \dots + \hbar^k\pi_k \in \mathbf{M}[[\hbar]]$ is a projection through order k , i.e. $(\pi^{(k)})^2 - \pi^{(k)} = \hbar^{k+1}a_{k+1} + O(\hbar^{k+2})$. We wish to choose π_{k+1} so that $\pi^{(k+1)} = \pi^{(k)} + \pi_{k+1}$ is a projection through order $k + 1$. This requires us to solve the equation

$$a_{k+1} + \pi_0\pi_{k+1} + \pi_{k+1}\pi_0 = \pi_{k+1},$$

or equivalently

$$\pi_0\pi_{k+1} - \pi_{k+1}(1 - \pi_0) = -a_{k+1}.$$

Now the operator $p \mapsto \pi_0p - p(1 - \pi_0) = \pi_0p\pi_0 - (1 - \pi_0)p(1 - \pi_0)$ maps all matrix valued functions to those which are block diagonal with respect to the splitting determined by π_0 (i.e. those commuting with π_0), and annihilates the matrices which are strictly off-diagonal with respect to this splitting. So it suffices to show that a_{k+1} commutes with π_0 . But a_{k+1} is the principal symbol of $\hbar^{-(k+1)}((\pi^{(k)})^2 - \pi^{(k)})$, which commutes with $\pi^{(k)}$, hence a_{k+1} commutes with the principal symbol π_0 of $\pi^{(k)}$. Thus we can always choose a suitable (non-unique) π_{k+1} , and, by induction, there is always a projection with principal symbol π_0 .

To prove our proposition, we use a second induction argument. Suppose that we have chosen a projection $\pi^{(k)}$ so that $[H, \pi^{(k)}]_* = O(\hbar^{k+1})$. (To start, we take an arbitrary projection with principal symbol π^0 , as constructed above.) Then there is a unique $F \in \mathbf{M}$ such that

$$[H, \pi^{(k)}]_* = \hbar^{k+1}F + O(\hbar^{k+2}).$$

We will choose the next approximation to have the form (exponentials are with respect to the Moyal product)

$$\pi^{(k+1)} = e^{\hbar^{k+1}A} *_\hbar \pi^{(k)} *_\hbar e^{-\hbar^{k+1}A}$$

which is automatically a projection for any $A \in \mathbf{M}$. Expanding the exponentials gives

$$\pi^{(k+1)} = \pi^{(k)} + \hbar^{k+1}[\pi_0, A]_* + O(\hbar^{k+2}),$$

so

$$[H, \pi^{(k+1)}]_* = [H, \pi^{(k)}]_* + \hbar^{k+1}[H, [\pi_0, A]_*]_* + O(\hbar^{k+2}),$$

which equals $\hbar^{k+1}(F + [H_0, [\pi_0, A]]) + O(\hbar^{k+2})$. So we must choose A as a solution of the equation $F + [H_0, [\pi_0, A]] = 0$. This is possible as long as F is off-diagonal with respect to the block decomposition given by π_0 , i.e. if $\pi_0F\pi_0$ and $(1 - \pi_0)F(1 - \pi_0)$ vanish.²

But these matrix functions are nothing but the principal symbols of the operators $\pi^{(k)} *_\hbar [H, \pi^{(k)}]_* *_\hbar \pi^{(k)}$ and $(1 - \pi^{(k)}) *_\hbar [H, \pi^{(k)}]_* *_\hbar (1 - \pi^{(k)})$, which vanish just because $\pi^{(k)}$ is a projection.

² To see this, one needs only to use the invertibility of $Q - \lambda I$, where $Q = (I - \pi_0)H_0(I - \pi_0)$ is the compression of H_0 to the range of $H_0 - \lambda I$.

The uniqueness of π is proven by a similar stepwise argument, using the two requirements that it should be a projection and commute with H . □

We remark that, if we have several regular eigenvalue functions λ_μ , then the corresponding projections $\pi_\mu \in \mathbf{M}[[\hbar]]$ will all commute with one another. In particular, if all the eigenvalues of H_0 are regular, we have a complete decomposition into “polarization sectors.” We would like to stress, though, that this complete decomposition is not as essential for the study of a single eigenvalue function as it may appear to be from some formulas in [5] and [11].

The projection π_0 in \mathbf{M} has been chosen so that $(H_0 - \lambda I)\pi_0$ and $\pi_0(H_0 - \lambda I)$ both vanish. As a result, for the projection π in $\mathbf{M}[[\hbar]]$ constructed in Proposition 1, the elements $(H - \lambda I) *_{\hbar} \pi$ and $\pi *_{\hbar} (H - \lambda I)$ of $\mathbf{M}[[\hbar]]$ are both of order \hbar , as is $\pi *_{\hbar} (H - \lambda I) *_{\hbar} \pi$. Since $\pi *_{\hbar} (H - \lambda I) *_{\hbar} \pi$ commutes with π and is annihilated by left or right multiplication by $(I - \pi)$, its leading order term has the same properties with respect to π_0 .

Let us compute it: the coefficient of \hbar in $\pi *_{\hbar} (H - \lambda I) *_{\hbar} \pi$ is

$$\pi_0 H_1 \pi_0 + \pi_1 (H_0 - \lambda I) \pi_0 + \pi_0 (H_0 - \lambda I) \pi_1 + \frac{i}{2} (\{\pi_0, H_0 - \lambda I\} \pi_0 + \{\pi_0 (H_0 - \lambda), \pi_0\}).$$

All but two of the terms vanish, and we can always add extra factors of π_0 on the outside, so we conclude:

$$\pi *_{\hbar} (H - \lambda I) *_{\hbar} \pi = \hbar \pi_0 (H_1 + \frac{i}{2} \{\pi_0, H_0 - \lambda I\}) \pi_0 + O(\hbar^2).$$

Notice in particular that π_1 has disappeared entirely from this expression.

3. WKB Approximation

We will seek a WKB eigenfunction for H which is in the image of the projection π found above. Specifically, we choose a lagrangian submanifold L in phase space on which the eigenvalue function λ has the constant value E , and a “principal symbol” u on L , which is a section of the tensor product of the half-densities on L with the vector space \mathbb{C}^n . The Maslov procedure associates to this data an \hbar -dependent wave function ψ for which u is called the principal symbol. For instance, if L has the form $p = dS(q)$ for a phase function S on configuration space, we can take q as a coordinate on L and write u in the form $a(q)\sqrt{|dq|}$, where a is a vector-valued function. The associated wave function is then $\psi = e^{\frac{i}{\hbar} S(q)} a(q)\sqrt{|dq|}$.

What is important is not so much the specific form of the WKB ansatz but the fact that, when we apply an operator A to such a ψ , the result is again associated to L , with the principal symbol $A_0 u$. In the special case that $A_0 u = 0$ and A_0 is a scalar function multiple $a_0 I$ of the identity matrix, $A\psi$ is of order \hbar , and $\hbar^{-1} A\psi$ has principal symbol $A_1 u - i \mathcal{L}_{X_{a_0}} u$, where the second term is $-i$ times the Lie derivative of u by the hamiltonian vector field of a_0 . (This vector field is tangent to L because L is a lagrangian submanifold on which the function a_0 vanishes.) We also note that the Moyal product on $\mathbf{M}[[\hbar]]$ is consistent with its operation on wavefunctions: $(A *_{\hbar} B)\psi = A(B\psi)$.

In particular, by applying the projection π to wavefunctions associated with L , we obtain (all the) wavefunctions which are in the image of π and thus candidates for the approximate eigenfunctions which we are seeking.

Suppose then that $\pi\psi = \psi$. Since H commutes with π , we have $H\psi = H\pi\psi = \pi H\psi = \pi H\pi\psi$. Therefore, by the main result of the previous section, $H\psi = (\pi\lambda I\pi + \hbar\pi_0(H_1 + \frac{1}{2}i\{\pi_0, H_0 - \lambda I\})\pi_0)\psi + O(\hbar^2)$. This means that, as far as its action on ψ is concerned, the operator H can be replaced by one whose principal symbol is the scalar multiple λI of the identity, and we can apply the standard analysis in this special case.

Now let E be a candidate for an eigenvalue for H . The order 0 part of $(H - EI)\psi$ is then $(\lambda - E)\psi$, which we can kill by choosing ψ to be associated with a lagrangian submanifold contained in the level surface for the value E of the eigenvalue function λ , which now plays the role of a scalar hamiltonian for our purposes.

The transport equation for the symbol u of ψ is the requirement that the principal symbol of $\hbar^{-1}(H - EI)\psi$ be zero. This principal symbol is

$$\pi_0(H_1 + \frac{1}{2}i\{\pi_0, H_0 - \lambda I\})\pi_0 u - i\mathcal{L}_{X_\lambda} u$$

plus the principal symbol of $\hbar^{-1}\pi *_{\hbar}(\lambda - E)\psi$.

Modulo $O(\hbar^2)$,

$$\pi *_{\hbar}(\lambda - E)\psi = (\pi_0 + \hbar\pi_1) *_{\hbar}(\lambda - E)\psi.$$

Since $(\lambda - E)\psi$ is already of order \hbar , this reduces to $(\pi_0 *_{\hbar}(\lambda - E))\psi$, in which the coefficient of \hbar is $\frac{i}{2}\{\pi_0, \lambda\}\psi$. After further application (always permissible) of the projection π_0 , this becomes zero, since $\pi_0\{\pi_0, \lambda\}\pi_0 = 0$.³

We can now write the transport equation for the symbol u , a half-density on L with values in the λ -eigenbundle:

$$\pi_0\mathcal{L}_{X_\lambda} u + ((-1/2)\pi_0\{\pi_0, H_0 - \lambda I\}\pi_0 + i\pi_0 H_1 \pi_0)u = 0. \tag{2}$$

4. Geometric Interpretation

In this section, we will give a geometric interpretation of the terms in the transport equation in the language of connections on vector bundles and their curvature.

If we write the symbol u as $a \otimes \nu$ for a complex-valued half-density ν on L and a section a in the λ -eigenbundle over L , the first term of the transport equation (2) becomes:

$$\pi_0\mathcal{L}_{X_\lambda} u = a \otimes \mathcal{L}_{X_\lambda} \nu + D_{X_\lambda} a \otimes \nu,$$

where D is the covariant differentiation on sections of the λ -eigenbundle defined by $D\zeta = \pi_0 d\zeta$ for an arbitrary section ζ . D is the covariant differential associated with the connection on E_λ naturally associated with the trivial connection on the trivial \mathbb{C}^N bundle over phase space (having d as its covariant differential) and the projection π_0 from the trivial bundle to the eigenbundle. It was observed by Simon [18] that such projected connections, which are standard in differential geometry, especially the geometry of submanifolds (see for example [3]), are just the ones whose holonomy in certain situations of physical interest is popularly called Berry's phase, after [2]. The corresponding expressions in the transport equation which appear when a local trivialization of the eigenbundle is chosen are named "Berry" terms in

³ The argument is as follows. $\{\pi_0, \lambda\} = \{\pi_0^2, \lambda\} = \{\pi_0, \lambda\}\pi_0 + \pi_0\{\pi_0, \lambda\}$. Multiplying on the left and right by π_0 gives $\pi_0\{\pi_0, \lambda\}\pi_0 = 2\pi_0\{\pi_0, \lambda\}\pi_0$, so $\pi_0\{\pi_0, \lambda\}\pi_0 = 0$.

[14]. In our treatment, these terms are invisible when the projected connection is viewed intrinsically.

We turn next to the matrix-valued function in the second term on the left-hand side of (2). It corresponds to the terms with “no name” in [14], but we will denote it by A_C and call it the **curvature term**, for reasons which will become clear shortly.

The curvature term may be rewritten as follows:

$$\begin{aligned} A_C \stackrel{\text{def}}{=} \pi_0\{\pi_0, H_0 - \lambda I\}\pi_0 &= \pi_0\{\pi_0, \lambda(\pi_0 - I)\}\pi_0 + \pi_0\{\pi_0, H_0 - \lambda\pi_0\}\pi_0 \\ &= \lambda\pi_0\{\pi_0, \pi_0\}\pi_0 + \pi_0\{\pi_0, H_0 - \lambda\pi_0\}\pi_0. \end{aligned} \tag{3}$$

We remark that both terms on the right-hand side of (3) behave tensorially when we multiply $H_0(x)$ (and at the same time its eigenvalue function $\lambda(x)$) by a function $f(x)$. For the first term this is completely obvious; for the second term it follows from $(H_0 - \lambda\pi_0)\pi_0 = 0$.

To give a geometrical interpretation of these tensorial terms, we compute the curvature F of the projected connection D . If we consider D as a covariant exterior derivative, F is the 2-form with values in the endomorphisms of E_λ for which $D^2\psi = F\psi$ for an arbitrary section ψ of E_λ . Since $D^2\psi = \pi_0 d(\pi_0 d(\pi_0\psi)) = \pi_0(d\pi_0) \wedge (d\pi_0)\pi_0\psi$, we find that $F = \pi_0(d\pi_0) \wedge (d\pi_0)\pi_0$. Hence, we see that the first term in (3) is simply $\lambda \langle \Pi, F \rangle$, where $\langle \Pi, F \rangle$ denotes the contraction of the Poisson tensor with the curvature 2-form.

To describe the second tensorial term geometrically, we first introduce an analog of the second fundamental form for embedded submanifolds: It is a 1-form with values in the vector-bundle homomorphisms from the λ -eigenbundle E_λ to the kernel E_λ^\perp of π_0 defined by $S\zeta = (I - \pi_0)d\zeta$ for an arbitrary section of E_λ . Since

$$S\zeta = (I - \pi_0)d\zeta = (I - \pi_0)d(\pi_0\zeta) = (I - \pi_0)\pi_0d\zeta + (I - \pi_0)(d\pi_0)\zeta$$

and $(I - \pi_0)\pi_0 = 0$, we have $S = (I - \pi_0)(d\pi_0)$ which indeed takes values in the vector-bundle homomorphisms. It measures the extent to which the “trivial” parallel transport defined by d tends to move a vector out of the λ -eigenbundle into its complement, i.e. the discrepancy between the connections d and D when applied to sections of E_λ .

Similarly, we can define a 1-form with values in the homomorphisms from E_λ^\perp to E_λ by

$$S^*\eta = -\pi_0d\eta = -\pi_0d(I - \pi_0)\eta$$

for a section η of E_λ^\perp . As the notation suggests, S and S^* with the above choice of sign are adjoint to one another if π_0 is an orthogonal projection on a hermitian vector bundle (e.g., if H_0 is hermitian).

Using S and S^* , we can define a 2-form with values in the endomorphisms of E_λ as $S^* \wedge ((H_0 - \lambda\pi_0)S)$, where $H_0 - \lambda\pi_0$ is considered as an endomorphism of E_λ^\perp (where it is just the restriction of H_0). This 2-form can be contracted with the Poisson tensor to yield the missing term in the transport equation. Indeed:

$$\begin{aligned} \pi_0\{\pi_0, H_0 - \lambda\pi_0\}\pi_0 &= -\pi_0\{I - \pi_0, (H_0 - \lambda\pi_0)(I - \pi_0)\}\pi_0 \\ &= -\langle \Pi, \pi_0d(I - \pi_0) \wedge (H_0 - \lambda\pi_0)d(I - \pi_0)\pi_0 \rangle \\ &= \langle \Pi, \pi_0d(I - \pi_0) \wedge (H_0 - \lambda\pi_0)(I - \pi_0)d\pi_0 \rangle. \end{aligned}$$

Thus the curvature term is a sum:

$$A_C = \lambda \langle \Pi, F \rangle - \langle \Pi, S^* \wedge (H_0 - \lambda\pi_0)S \rangle . \tag{4}$$

In general, the curvature term will be an endomorphism of an m -dimensional vector bundle (represented with respect to a local basis by an $m \times m$ matrix), where m is the dimension of the λ -eigenspace, and it is not possible to simplify further the terms in the transport equation. However, in the special case that the eigenvalue function is non-degenerate (i.e., the multiplicity is 1), we can simplify them by observing that A_C is uniquely determined by its trace (here, we can compute the trace on the whole vector bundle, not just on the eigenbundle, since both give the same result!) and using the invariance of the trace under cyclic permutation of factors. (The trace operation does not act on the form part, so we just have to remember signs when we cyclically change the order of forms.) We find that $A_C = A_C^{(s)}\pi_0$, where the scalar $A_C^{(s)}$ is given by

$$\begin{aligned} A_C^{(s)} &= \langle \Pi, \text{tr} \left(\lambda\pi_0(d\pi_0) \wedge (d\pi_0) - \pi_0 d(I - \pi_0) \wedge (H_0 - \lambda\pi_0)d(I - \pi_0)\pi_0 \right) \rangle \\ &= \langle \Pi, \text{tr} \left(\lambda\pi_0(d\pi_0) \wedge (d\pi_0) - \pi_0(d\pi_0) \wedge (H_0 - \lambda\pi_0)(d\pi_0) \right) \rangle \\ &= \langle \Pi, \text{tr} \left(\lambda\pi_0(d\pi_0) \wedge (d\pi_0) - (d\pi_0) \wedge (H_0 - \lambda\pi_0)(d\pi_0) \right) \rangle \\ &= \langle \Pi, \text{tr}(H_0\tilde{F}) \rangle , \end{aligned}$$

where $\tilde{F} = d\pi_0 \wedge d\pi_0$. (In the second term we first used the cyclicity to get rid of the projection at the end, then $d\pi_0 = \pi_0 d\pi_0 + (d\pi_0)\pi_0$.)

\tilde{F} is simply the curvature of the new connection \tilde{D} on the trivial bundle defined by

$$\tilde{D}\xi = \pi_0 d(\pi_0\xi) + (I - \pi_0)d((I - \pi_0)\xi)$$

for an arbitrary section ξ . This **adapted connection** (see [3]) preserves both the subbundles E_λ and E_λ^\perp , its restriction to E_λ is just D ; in particular F is simply the λ -block of \tilde{F} .

In order to compare our expression with those given in the literature, and in particular that in [14], we assume that our hamiltonian is hermitian so that π_0 is an orthogonal projection onto the one-dimensional eigenbundle E_λ , in which we choose a normalized local section τ . Then $\pi_0 = \tau\tau^\dagger$, and a straightforward calculation⁴ (using $\tau^\dagger d\tau = -d(\tau^\dagger)\tau$, which follows from $\tau^\dagger\tau = 1$) yields

$$\tilde{F} = \tau(d\tau^\dagger \wedge d\tau)\tau^\dagger + d\tau \wedge d\tau^\dagger - [d\tau \wedge d\tau^\dagger, \tau\tau^\dagger]_+ ,$$

where $[\cdot, \cdot]_+$ denotes the anticommutator. If we multiply by H_0 and take the trace, the anticommutator term vanishes, and we finally get

$$A_C^{(s)} = \lambda \{ \tau^\dagger, \tau \} + \sum_{\alpha\beta} (H_0)_{\alpha\beta} \{ \tau_\alpha, \tau_\beta^\dagger \}$$

which is exactly the result in [14].

⁴ We have borrowed here from some notes of Jim Morehead.

5. Existence of Quasiclassical States

In WKB theory for the scalar case, one seeks quasiclassical eigenstates as suitable vector valued half-densities on lagrangian submanifolds L of phase space. In attempting to extend this theory to the multicomponent case, one encounters three difficulties: the presence of the curvature term A_C ; the fact that, even if the curvature term vanishes, the quasiclassical states are required to be covariantly constant along hamiltonian trajectories with respect to a connection which is generally not flat; and finally the fact that the holonomy of this connection, even when it is flat, takes values not in \mathbb{C}^* or $U(1)$ but in $GL(m)$ or $U(m)$ (the latter if the projection π_0 is orthogonal), where m is the multiplicity of λ .

The curvature term presents a problem mainly in the case of a degenerate eigenvalue function λ . In the non-degenerate case it is simply a scalar multiple of π_0 , and hence can be replaced by a scalar, \hbar -dependent part of the scalar hamiltonian; this is obviously not possible in the degenerate case. Even in the non-degenerate case, the presence of the curvature term means that the scalar hamiltonian is \hbar -dependent even if the matrix valued symbol is \hbar -independent, which leads to the necessity of admitting \hbar -dependent lagrangian submanifolds [14].

The non-flatness of the connection makes it impossible to impose a naive analog of the Bohr-Sommerfeld quantization condition, since the parallel transport around cycles depends on the cycles themselves, not just on their homotopy classes.

Whereas the two first problems might be avoidable by a suitable modification of the geometric description of a quasiclassical state (admitting \hbar -dependent lagrangian submanifolds and symplectic structures, and possibly making use of a suitable extended phase space), the third problem is a real obstruction to the existence of quasiclassical states in the case of a degenerate eigenvalue function. If we admit \hbar -dependent lagrangian submanifolds $L(\hbar)$ as in [14], the transport equation (2) will be modified, but only by an additional $U(1)$ phase. Hence, if we write the symbol u as $a \otimes \nu$ for a complex-valued half-density ν and a section a in the λ -eigenbundle over $L = L(0)$, then the transport equation for the corresponding section $[a]$ in the projective λ -eigenbundle will be independent of \hbar . Hence, we have to find a section in the projective eigenbundle which satisfies this transport equation. Due to the $U(m)$ -holonomy, such a section will not always exist, even if the eigenvalue function λ is integrable. If the flow on the corresponding invariant torus is only quasiperiodic, it can come arbitrarily close to a given starting point without the correspondingly transported point in the projective eigenspace being close to its starting value.

This argument shows that the integrability of the eigenvalue function λ is not a sufficiently strong condition for the existence of a global WKB state, and in order to find an analog for the quantization condition for scalar systems one has to formulate a suitable strong notion of integrability for the classical limits of multicomponent systems.

In spite of the problems just listed, quasiclassical states can be shown to exist in certain cases, the easiest one being that where the underlying phase space is only two-dimensional [11] and H is hermitian. In this case, L is one-dimensional, so problems with the non-flatness of the connection do not arise. A suitable section $[a]$ in the projective eigenbundle always exists. To construct it, one simply chooses a point p on L , computes the holonomy around a loop based at p , selects for $[a](p)$ the ray corresponding to one of the eigenvalues of the holonomy (which is always diagonalizable as it is unitary for hermitian H), and defines $[a]$ by the transport equation. Hence, in this case a quasiclassical state exists, and the only effect of the

non-trivial connection and the curvature term is an additional scalar phase which modifies the Bohr-Sommerfeld condition and is of the same order as the Maslov correction.

In [11] the existence of quasiclassical states is shown for certain other examples as well, where the obstructions above are avoided by assuming either that either phase space is two-dimensional, that the fibers of the eigenbundle are only (complex) one-dimensional, or that the curvature term vanishes and that there is an “adiabatic connection”⁵ – i.e., a subbundle of the eigenbundle which is invariant and flat under the projected connection.

In the non-degenerate case, where the λ -eigenbundle is simply a line bundle, there *does* exist a general method for deriving a quantization condition. Such a method is given in [14] for this special case, using local sections, diagonalization, and “non-canonical coordinates.” (In a somewhat different context, a similar result has been obtained in [11].) In purely geometric terms their method for a phase space T^*M with its canonical symplectic structure ω can be described in the following way.

If we include the factor $e^{iS/\hbar}$ in the geometric description, the quasiclassical states are half-densities on a submanifold of phase space with values in the tensor product of the standard trivial prequantum line bundle over a cotangent bundle and the λ -eigenbundle. (We neglect the Maslov correction for the moment.) In the non-degenerate case this bundle is again a line bundle, and we can identify its curvature with a two-form on phase space. Since the connection on the prequantum bundle has curvature $\frac{1}{\hbar}\omega$, the curvature of the tensor product bundle is $\frac{1}{\hbar}\omega + F$. Hence, if we equip phase space with the modified symplectic structure $\omega_{\hbar} = \omega + \hbar F$, then the curvature vanishes on the pullback of the line bundle above to any submanifold L_{\hbar} of T^*M which is lagrangian with respect to ω_{\hbar} . Hence, parallel sections exist at least locally on L_{\hbar} . In particular, WKB states correspond to \hbar -dependent lagrangian submanifolds contained in level sets $\lambda_{\hbar}^{-1}(E)$ of the \hbar dependent scalar hamiltonian function λ_{\hbar} obtained by including the curvature term.

Since the curvature 2-form vanishes on L_{\hbar} , we can formulate a quantization condition for cycles in the usual way (including the Maslov correction), which only depends on the homotopy class of the cycle. L_{\hbar} will tend in the limit $\hbar \rightarrow 0$ to a submanifold L_0 which is lagrangian with respect to the unmodified symplectic structure ω , and the Maslov correction can be computed from the corresponding Maslov indices of L_0 . Thus, in the non-degenerate case it is possible to give a completely geometric description of quasiclassical states using globally defined objects.

The approach just described appears to apply only in the non-degenerate case. Nevertheless, we expect that the purely geometric derivation of the transport equation can serve as a guideline to a formulation of a quantization condition in the general case.

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⁵ We think that “adiabatic constraint” would be a better translation of the original Russian in this instance.

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