

Deformation Quantization of the Heisenberg Group

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Abstract. A $*$ -product compatible with the comultiplication of the Hopf algebra of the functions on the Heisenberg group is determined by deforming a coboundary Lie-Poisson structure defined by a classical r -matrix satisfying the modified Yang-Baxter equation. The corresponding quantum group is studied and its R -matrix is explicitly calculated.

1. Introduction

The quantization of a dynamical system on a symplectic manifold was introduced in [1, 2] by deforming the pointwise multiplication of the commutative algebra of the classical observables into a one parameter family $*_{\hbar}$ of associative but not necessarily commutative products. The parameter \hbar is physically interpreted as the Planck constant and the deformation is required to satisfy the classical limit conditions

$$\phi *_{\hbar} \psi \xrightarrow{\hbar \rightarrow 0} \phi\psi, \quad (\phi *_{\hbar} \psi - \psi *_{\hbar} \phi)/\hbar \xrightarrow{\hbar \rightarrow 0} \{\phi, \psi\}$$

for any pair of observables ϕ, ψ .

Since its first appearance, the method has found a constantly increasing number of applications and special attention has been devoted to systems with symmetry. For instance the geometric quantization, or coadjoint orbit method, yielding the irreducible representations of nilpotent Lie groups has been reproduced in this approach [3]; the deformation of quotient manifolds of the Heisenberg group by appropriate lattice subgroups has been investigated in [4] in connection with results on quantum tori; a framework for quantizing the linear Poisson structures has been proposed in [5]. Almost all deformations are expressed in terms of formal power series in \hbar with coefficients in the algebra of the observables and a very tiny number of $*_{\hbar}$ -products is explicitly known, the most relevant of which is obtained by the Weyl quantization on \mathbf{R}^{2n} . In [4, 5] the convergence of the power series is discussed and an answer is provided in terms of Fourier transforms.

A natural question that arises in this context concerns the relationship and the applicability of these results to quantum groups, being themselves deformations of the algebras of the representative functions of Lie groups. A general prescription for handling the problem and relating quantum groups to $*_{\hbar}$ -products is presented in [6]. Here the fundamental objects are assumed to be a classical r -matrix and the corresponding coboundary Poisson-Lie structure determined by r . This implies that the resulting $*_{\hbar}$ -product will be compatible with the comultiplication of the Hopf algebra of the functions on the group, differently from [4, 5], where the Poisson brackets to be quantized are required to be left-invariant for the appropriate action of the Lie group, so that the deformed multiplication inherits the same property.

A distinction in the procedure of quantization must be made according to whether the r -matrix satisfies the CYBE or the MYBE (respectively: classical and modified Yang-Baxter equation). While a theory [7] and some explicit results exist for the former case [8], no deformed product, to our knowledge, has been found in the second: applications to quantum groups have only been made with the explicit use of a representation and reproduce, for instance, the expression of the quantum R -matrix of $SL_q(2)$ in the fundamental representation.

In this paper we shall consider the deformation quantization of the one dimensional Heisenberg group $H(1)$. After some remarks on the MYBE for $H(1)$ we determine the coboundary Poisson-Lie structure and the brackets to be quantized. We then look for a deformation of the pointwise multiplication of functions on $H(1)$ and we give an explicit form for the $*_{\hbar}$ -product whose restriction to symplectic leaves is obviously equivalent to Weyl quantization, but has different invariance properties. We finally calculate the R -matrix for the quantized structure thus found, obtaining the same expression as the one found in [9, 10] for the Heisenberg quantum group $H_q(1)$. A straightforward generalization to the Heisenberg group in n dimensions is finally given and the quantum group $H_q(n)$ is determined.

2. Coboundary Poisson-Lie Structure

We denote by a, a^\dagger, h the three generators of the Heisenberg Lie algebra $\mathcal{H}(1)$ that satisfy the commutation relations

$$[a, a^\dagger] = h, \quad [h, \cdot] = 0. \tag{2.1}$$

Let

$$r = \lambda a \wedge a^\dagger + \mu a \wedge h + \nu a^\dagger \wedge h \in \wedge^2 \mathcal{H}(1), \tag{2.2}$$

and denote by $\mathcal{U} = U(\mathcal{H}(1))$ the enveloping algebra of $\mathcal{H}(1)$. Let us also introduce the notation r_{ij} , $i = 1, 2, 3$; $i < j$, where $r_{12} = r \otimes \mathbf{1} \in \oplus^3 \mathcal{U}$ and similarly for the other values of indices.

(2.3) Lemma. *r satisfies the MYBE. If $\lambda = 0$, r satisfies the CYBE.*

Proof. Defining

$$B = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

a direct calculation shows that $\text{Ad}_\kappa B = 0$ for any $\kappa \in \mathcal{H}(1)$, where Ad denotes the diagonal adjoint action of $\mathcal{H}(1)$ on $\otimes^3 \mathcal{U}$. Also $B = 0$ for $\lambda = 0$. \square

Therefore, according to the usual definition, any element of $\wedge^2 \mathcal{H}(1)$ is a classical r -matrix.

Let us define on $H(1)$ the coordinates (β, δ, α) such that for $g \in H(1)$ and for $(z, y, x) = (\beta(g), \delta(g), \alpha(g))$ we have

$$g = e^{zh} e^{ya^\dagger} e^{xa} \equiv (z, y, x).$$

The composition law

$$(z', y', x') \cdot (z, y, x) = (z' + z + x'y, y' + y, x' + x)$$

induces then the comultiplication

$$\begin{aligned} \Delta\delta &= 1 \otimes \delta + \delta \otimes 1, & \Delta\alpha &= 1 \otimes \alpha + \alpha \otimes 1, \\ \Delta\beta &= 1 \otimes \beta + \beta \otimes 1 + \alpha \otimes \delta. \end{aligned}$$

(2.4) Proposition. *The coboundary Lie-Poisson structure associated to an element r as in (2.2) has the following brackets:*

$$\{\delta, \beta\} = \lambda\delta, \quad \{\delta, \alpha\} = 0, \quad \{\alpha, \beta\} = \lambda\alpha.$$

Proof. If ad denotes the adjoint representation of $H(1)$ on $\mathcal{H}(1)$, for $g = (z, y, x)$ we have

$$\text{ad}_g a = a - yh, \quad \text{ad}_g a^\dagger = a^\dagger + xh, \quad \text{ad}_g h = h.$$

Since the right invariant fields on $H(1)$ are

$$X_h = \partial_\beta, \quad X_{a^\dagger} = \partial_\delta, \quad X_a = \delta\partial_\beta + \partial_\alpha, \tag{2.5}$$

the brackets $\{, \} = \eta_{uv} X_u X_v$, ($u, v = a, a^\dagger, h$), determined by the coboundary

$$\eta = \text{ad } r - r = \lambda(a \wedge \alpha h + a^\dagger \wedge \delta h): H(1) \rightarrow \wedge^2 \mathcal{H}(1), \tag{2.6}$$

are of the stated form. \square

(2.7) Remark. It can be observed that the Poisson brackets are not vanishing if and only if λ is not vanishing and in this case they differ by a multiplicative factor, so that the coboundary Lie-Poisson structure is essentially unique. In what follows, without loss of generality, we shall thus fix $\lambda = 1/2$, $\mu = \nu = 0$, so that $r = \frac{1}{2} a \wedge a^\dagger$.

3. The $*_{\hbar}$ -Product

Let \mathcal{U}_{\hbar} be the algebra obtained from \mathcal{U} by extending the field of coefficients to the ring of formal power series in \hbar . Let then π_L and π_R respectively be the representation and the anti-representation of the Lie algebra $\mathcal{H}(1)$ by left and right invariant vector fields on $H(1)$ and use the same notation for the extension of the representation that maps \mathcal{U}_{\hbar} into the left and right invariant differential operators on $H(1)$. Consider an element $F \in \otimes^2 \mathcal{U}_{\hbar}$ of the form

$$F = \mathbf{1} \otimes \mathbf{1} + \frac{\hbar}{2} r + \dots, \tag{3.1}$$

for which

$$(\mathbf{1} \otimes \varepsilon)F = (\varepsilon \otimes \mathbf{1})F = \mathbf{1} \otimes \mathbf{1}.$$

Where $\mathbf{1}$ is the unity, ε the counit of \mathcal{U}_{\hbar} and r is the element specified in (2.7). Moreover F is invertible in $\otimes^2 \mathcal{U}_{\hbar}$: letting $\pi_L^{\otimes} = \pi_L \otimes \pi_L$, $\pi_R^{\otimes} = \pi_R \otimes \pi_R$, we can define

$$\tilde{F} = \pi_L^{\otimes} F, \quad \tilde{F}' = \pi_R^{\otimes} (F^{-1})$$

and a composition law [6]

$$*_\hbar = m \circ \tilde{F} \circ \tilde{F}' \tag{3.2}$$

on the differentiable functions on $H(1)$, m denoting the multiplication of functions. According to the standard use, the law (3.2) will be called $*_\hbar$ -product. If $\phi, \psi \in C^\infty(H(1))$, then the properties

- i) $\phi *_\hbar \mathbf{1} = \mathbf{1} *_\hbar \phi = \phi,$
- ii) $\{\phi, \psi\} = \lim_{\hbar \rightarrow 0} \frac{1}{\hbar} (\phi *_\hbar \psi - \psi *_\hbar \phi),$
- iii) $\Delta(\phi *_\hbar \psi) = \Delta(\phi) *_\hbar \Delta(\psi),$

are easily verified. To deal with the associativity of the $*_\hbar$ -product we recall the following result:

(3.3) Proposition [6]. *Assume that an element F as given in (3.1) satisfies the equation*

$$(\Delta \otimes \mathbf{1})F(F \otimes \mathbf{1}) = \chi(\mathbf{1} \otimes \Delta)F(\mathbf{1} \otimes F), \tag{3.4}$$

where $\chi \in \otimes^3 \mathcal{U}$ is invariant, i.e. $\text{Ad}_\kappa \chi = 0$ for any $\kappa \in \mathcal{H}(1)$. Then the product $*_\hbar$ as in (3.2) is associative. \square

Therefore we shall prove the associativity by solving Eq. (3.4). Let $\hat{\mathcal{U}}_\hbar = \mathcal{U}_\hbar \otimes_Z Q(Z)$ be the localization of \mathcal{U}_\hbar with respect to its center Z , $Q(Z)$ being the field of fractions of Z . Let then $\mathbf{x} = \hbar \otimes \mathbf{1}$, $\mathbf{y} = \mathbf{1} \otimes \hbar$, $\mathbf{1}^\otimes = \mathbf{1} \otimes \mathbf{1}$ and define the elements $\theta_1, \theta_2 \in \otimes^2 \hat{\mathcal{U}}_\hbar$ by the relations

$$\tan \theta_1 = \sqrt{\frac{\mathbf{y}}{\mathbf{x}}}, \quad \tan \theta_2 = \sqrt{\frac{\mathbf{1}^\otimes - e^{-\hbar \mathbf{y}}}{e^{\hbar \mathbf{x}} - \mathbf{1}^\otimes}}.$$

(3.5) Proposition. *If $\theta = \theta_1 - \theta_2$ and $\varrho = 2\theta/\sqrt{\mathbf{x}\mathbf{y}}$, then $\varrho \in \otimes^2 \mathcal{U}_\hbar$ and the element*

$$F = e^{\varrho r}$$

solves (3.4).

Proof. For the first statement it is sufficient to observe that $\tan \theta_2 = \sqrt{\mathbf{y}/\mathbf{x}}(1 - \hbar(\mathbf{x} + \mathbf{y})\Theta(\hbar, \mathbf{x}, \mathbf{y}))$, where Θ is a power series in $\hbar, \mathbf{x}, \mathbf{y}$. We then obtain $\theta = \theta_1 - \theta_2 = \arctan\{\sqrt{\mathbf{x}\mathbf{y}}\hbar\Theta/(1 - \mathbf{y}\hbar\Theta)\}$: thus $\varrho = 2\theta/\sqrt{\mathbf{x}\mathbf{y}} \in \otimes^2 \mathcal{U}_\hbar$. The first terms of the expansion of ϱ in powers of \hbar are

$$\varrho = \frac{\hbar}{2} - \frac{\hbar^2(\mathbf{x} - \mathbf{y})}{48} + O(\hbar^3).$$

To prove the second statement, introduce the notation $F_{12,3} = (\Delta \otimes \mathbf{1})F$, $F_{12} = F \otimes \mathbf{1}$, $F_{23} = \mathbf{1} \otimes F$, $F_{1,23} = (\mathbf{1} \otimes \Delta)F$. Equation (3.4) becomes

$$\chi = F_{12,3}F_{12}F_{23}^{-1}F_{1,23}^{-1},$$

and the invariance of χ reads

$$F_{23}^{-1}F_{1,23}^{-1}(\mathbf{1} \otimes \Delta)\Delta\kappa F_{1,23}F_{23} = F_{12}^{-1}F_{12,3}^{-1}(\mathbf{1} \otimes \Delta)\Delta\kappa F_{12,3}F_{12}, \tag{3.6}$$

with $\kappa = a, a^\dagger$. Defining f, g by

$$f = \cos \theta + \sqrt{\frac{\mathbf{y}}{\mathbf{x}}} \sin \theta, \quad g = \cos \theta - \sqrt{\frac{\mathbf{x}}{\mathbf{y}}} \sin \theta,$$

and using for f and g the same notation as for F , Eq. (3.6) is equivalent to

$$\begin{aligned} & (f_{1,23} - f_{12}f_{12,3})\kappa \otimes \mathbf{1} \otimes \mathbf{1} + (g_{12}f_{12,3} - f_{23}g_{1,23})\mathbf{1} \otimes \kappa \otimes \mathbf{1} \\ & + (g_{23}g_{1,23} - g_{12,3})\mathbf{1} \otimes \mathbf{1} \otimes \kappa = 0. \end{aligned} \tag{3.7}$$

With our definition of θ we can write

$$\begin{aligned} f &= e^{\hbar y/4} \sqrt{\frac{\mathbf{x} + \mathbf{y}}{\sinh(\hbar(\mathbf{x} + \mathbf{y})/2)}} \sqrt{\frac{\sinh(\hbar \mathbf{x}/2)}{\mathbf{x}}}, \\ g &= e^{-\hbar x/4} \sqrt{\frac{\mathbf{x} + \mathbf{y}}{\sinh(\hbar(\mathbf{x} + \mathbf{y})/2)}} \sqrt{\frac{\sinh(\hbar \mathbf{y}/2)}{\mathbf{y}}}. \end{aligned}$$

from which Eq. (3.7) is verified. \square

(3.8) Corollary. *The explicit expression for the $*_{\hbar}$ -product is as follows:*

$$\phi *_{\hbar} \psi = m \circ \exp\{\pi_R^{\otimes}(\varrho\eta)\} (\phi \otimes \psi),$$

where ϱ is given in (3.5) and η in (2.6).

Proof. Indeed

$$\begin{aligned} \tilde{F} \circ \tilde{F}' &= \exp\{\pi_L^{\otimes}(\varrho r)\} \exp\{-\pi_R^{\otimes}(\varrho r)\} \\ &= \exp\{\pi_R^{\otimes}(\varrho(\text{ad } r - r))\} = \exp\{\pi_R^{\otimes}(\varrho\eta)\}. \quad \square \end{aligned}$$

(3.9) Remark. According to (2.4), the relation $\alpha/\delta = \text{cost}$ defines the symplectic leaves of $H(1)$, on which the local Darboux coordinates are $p = \log \alpha$ and $q = 2\beta$. The expression given in (3.8) can be compared with the $*_{\mathcal{W}}$ -product obtained by the Weyl quantization that can be defined on any symplectic leaf: as expected, the latter shows to be not compatible with comultiplication (e.g. compare $\beta *_{\mathcal{W}} \alpha\beta^2$). This fact was already noticed in [6] in connection with the quantization of $SL(2)$.

4. R-Matrix and the Quantum Group $H_q(1)$

In this last section we shall determine the R -matrix from the given quantization and we shall compare the result with what was found in [9, 10]. By a direct calculation the following lemma is easily proved.

(4.1) Lemma. *Let*

$$n = \frac{a^\dagger a}{\hbar}, \quad \xi = h \otimes n + n \otimes h.$$

The element $t \in \otimes^2 \hat{\mathcal{W}}_{\hbar}$ given by

$$t = \frac{1}{2} (a \otimes a^\dagger + a^\dagger \otimes a - \xi)$$

is invariant and satisfies the equation

$$B = [t_{23}, t_{13}],$$

where B is given in (2.3). \square

Denote by σ the flip isomorphism of $\otimes^2 \hat{\mathcal{W}}_{\hbar}$, $\sigma(a \otimes b) = (b \otimes a)$. The main result of this section is formulated as follows.

(4.2) Proposition. *The element of $\otimes^2 \hat{\mathcal{U}}_{\hbar}$*

$$R = (\sigma \circ F)^{-1} e^{\hbar t} F, \tag{4.3}$$

with F as in (3.5) satisfies the QYBE (quantum Yang-Baxter equation) and coincides with the quantum R -matrix of $U_q(\mathcal{H}(1))$ [9].

The proof of this proposition is a consequence of the following two lemmas that stem from the observation that the fundamental objects in terms of which F and t are built are $a \otimes a^\dagger$, $a^\dagger \otimes a$ and ξ .

(4.4) Lemma. *The elements of $\otimes^2 \hat{\mathcal{U}}_{\hbar}$ given by*

$$j_+ = \frac{a \otimes a^\dagger}{\sqrt{\mathbf{xy}}}, \quad j_- = \frac{a^\dagger \otimes a}{\sqrt{\mathbf{xy}}}, \quad j_3 = \frac{\hbar \wedge a^\dagger a}{2\mathbf{xy}},$$

$$k = \frac{\xi}{\gamma} - j_3,$$

with $\gamma = \mathbf{x} - \mathbf{y}$, generate an $u(2)$ Lie algebra.

Proof. Indeed it is immediate to verify that j_\pm, j_3 satisfy the $su(2)$ commutation relations, while k is the central generator of $u(1)$. \square

(4.5) Lemma. *The elements*

$$A = \sqrt{\frac{2 \sinh(\hbar h/2)}{\hbar h}} a, \quad A^\dagger = \sqrt{\frac{2 \sinh(\hbar h/2)}{\hbar h}} a^\dagger, \quad H = h, \tag{4.6}$$

satisfy the commutation relations of the generators of $U_q(\mathcal{H}(1))$ [9], namely

$$[A, A^\dagger] = \frac{2}{w} \sinh(\hbar H/2), \quad [H, \cdot] = 0. \quad \square$$

Proof of (4.2). In terms of the $u(2)$ generators, (4.3) reads

$$R = e^{(\sigma \circ \theta)(j_+ - j_-)} e^{\hbar/2(\sqrt{\mathbf{xy}(j_+ + j_-)} - \gamma k - \gamma j_3)} e^{\theta(j_+ - j_-)}.$$

Using the composition law of the group $U(2)$, the last expression is transformed into

$$R = e^{-\hbar(\gamma k + \gamma j_3)/2} \exp\{2e^{\hbar\gamma/4} \sqrt{\sinh(\hbar\mathbf{x}/2) \sinh(\hbar\mathbf{y}/2)} j_+\}. \tag{4.7}$$

From (4.4) and (4.6), letting

$$N = \frac{wA^\dagger A}{2 \sinh(\hbar H/2)}, \quad \Omega = H \otimes N + N \otimes H, \tag{4.8}$$

Eq. (4.7) with $\hbar = w$ becomes

$$R = e^{-w\Omega/2} \exp\{we^{wH/4} A \otimes e^{-wH/4} A^\dagger\}. \tag{4.9}$$

This expression coincides with the R matrix of $U_q(\mathcal{H}(1))$ found in [9], where it is explicitly shown that R satisfies the QYBE. \square

(4.10) Remarks. (i) We observe that F as given in (3.5) deforms the comultiplication Δ of \mathcal{U} to the comultiplication Δ_q of $U_q(\mathcal{H}(1))$, namely

$$\Delta_q(\kappa) = F^{-1} \Delta(\kappa) F, \quad \kappa \in \mathcal{H}(1),$$

as expected.

(ii) A final remark concerns a straightforward generalization to the deformation of the Heisenberg group $H(n)$ in n degrees of freedom, with algebra $\mathcal{H}(n)$ generated by h, a_i^\dagger, a_i ($i = 1, \dots, n$), relations

$$[a_i, a_j^\dagger] = \delta_{ij} h, \quad [h, \cdot] = 0$$

and corresponding group coordinates β, δ_i, a_i . The classical r matrix

$$r = \sum_{i=1}^n \lambda_i r_i, \quad r_i = \frac{1}{2} a_i \wedge a_i^\dagger \in \wedge^2 \mathcal{H}(n); \quad \lambda_i \in \mathbf{R},$$

gives $H(n)$ a Lie-Poisson structure with coboundary

$$\eta = \sum_{i=1}^n \lambda_i (a_i \wedge \alpha_i h + a_i^\dagger \wedge \delta_i h).$$

Let ϱ_i be defined as in (3.5) with the substitution of \hbar by $\lambda_i \hbar$. Then

$$F = \prod_{i=1}^n e^{\varrho_i r_i}$$

defines a $*_{\hbar}$ -product on $H(n)$. The corresponding quantum group $H_q(n) = U_q(\mathcal{H}(n))$ is generated by H, A_i, A_i^\dagger defined for each $i = 1, \dots, n$ as in (4.6) with $w_i = \lambda_i \hbar$ replacing w . The R -matrix of $H_q(n)$ is $R = \prod_i R_i$, where the R_i are given by (4.8) and (4.9) with the above substitutions.

References

1. Vey, J.: *Comment. Math. Helv.* **50**, 421 (1975)
2. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: *Ann. Phys.* **110**, 61 and 111 (1978)
3. Arnal, D.: *Pacific J. Math.* **114**, 285 (1984)
4. Rieffel, M.A.: *Commun. Math. Phys.* **122**, 531 (1989)
5. Rieffel, M.A.: *American J. Math.* **112**, 657 (1990)
6. Takhtajan, L.A.: *Lectures on quantum groups*. In: *Introduction to quantum group and integrable massive models of quantum field theory*. Nankai Lectures in Mathematical Physics. Singapore: World Scientific, 1990, p. 69
7. Drinfeld, V.G.: *Sov. Math. Dokl.* **28**, 667 (1983)
8. Ohn, Ch.: *Lett. Math. Phys.* **25**, 85 (1992)
9. Celeghini, E., Giachetti, R., Sorace, E., Tarlini, M.: *J. Math. Phys.* **32**, 1155 (1991)
10. Celeghini, E., Giachetti, R., Sorace, E., Tarlini, M.: *Contractions of quantum groups*. In: *Lecture Notes in Mathematics 1510*. Berlin, Heidelberg, New York: Springer 1992, p. 221

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