Commun. Math. Phys. 168, 493–533 (1995)

Communications in Mathematical Physics

Generalized Hypergeometric Functions and Rational Curves on Calabi–Yau Complete Intersections in Toric Varieties

Victor V. Batyrev[1], **Duco van Straten**[2]

[1] FB Mathematik, Universität-GHS-Essen, Universitätsstraße 3, D-45141 Essen, Germany, e-mail: victor.Batyrev@aixrs1.hrz.uni-essen.de
[2] FB Mathematik, Universität Kaiserslautern, Erwin-Schrödingerstraße 48, D-67663 Kaiserslautern, Germany, e-mail: straten@mathematik uni-kl.de

Received: 2 August 1993/in revised form: 27 December 1993

Abstract. We formulate general conjectures about the relationship between the A-model connection on the cohomology of a d-dimensional Calabi–Yau complete intersection V of r hypersurfaces $V_1, \ldots, V_r$ in a toric variety $\mathbf{P}_\Sigma$ and the system of differential operators annihilating the special generalized hypergeometric series Φ_0 constructed from the fan Σ. Using this generalized hypergeometric series, we propose conjectural mirrors V' of V and the canonical q-coordinates on the moduli spaces of Calabi–Yau manifolds.

In the second part of the paper we consider some examples of Calabi–Yau 3-folds having Picard number > 1 in products of projective spaces. For conjectural mirrors, using the recurrent relation among coefficients of the restriction of the hypergeometric function Φ_0 on a special line in the moduli space, we determine the Picard–Fuchs equation satisfied by periods of this special one-parameter subfamily. This allows to obtain some sequences of integers which can be conjecturally interpreted in terms of Gromov–Witten invariants. Using standard techniques from enumerative geometry, first terms of these sequence of integers are checked to coincide with numbers of rational curves on Calabi–Yau 3-folds.

1. Introduction

In this paper we consider complex projective smooth algebraic varieties V of dimension d whose canonical bundles $\mathcal{K}_V$ are trivial, i.e. $\mathcal{K}_V \cong \mathcal{O}_V$, and the Hodge numbers $h^{p,0}(V)$ are zero unless $p = 0$, or $p = d$. These varieties are called *d-dimensional Calabi–Yau varieties*, or *Calabi–Yau d-folds*. For each dimension $d \geqq 3$, there are many examples of topologically different Calabi–Yau d-folds which can be constructed from hypersurfaces and complete intersections in weighted projective spaces [5, 6, 7, 24, 23, 26].

Physicists have discovered a fascinating phenomen for Calabi–Yau manifolds, so-called *mirror symmetry* [12, 17, 27, 29]. Using mirror symmetry, Candelas et al. in [9] have computed the coefficients of the q-expansion of the Yukawa coupling for Calabi–Yau hypersurfaces of degree 5 in $\mathbf{P}^4$. The method of Candelas et al. was applied to Calabi–Yau 3-folds in weighted projective spaces [14, 33, 21] and complete intersections in weighted and ordinary projective spaces [22, 28]. The q-expansions for Yukawa couplings have been calculated also for Calabi–Yau hypersurfaces of dimension $d > 3$ in projective spaces [18].

The interest of algebraic geometers in Yukawa couplings is explained by the conjectural relationship between the coefficients of the q-expansion of the Yukawa couplings and the intersection theory on the moduli spaces of rational curves on Calabi–Yau d-folds [18, 19]. For small values of the degree of rational curves, this relationship was verified in some cases by S. Katz [20]. However, the main problem which remains unsolved is to find a general rigorous mathematical explanation of the relation between the coefficient of q-expansions and counting of rational curves (instantons) on Calabi–Yau manifolds.

The purpose of this paper is to show that the calculation of the Yukawa couplings for d-dimensional Calabi–Yau complete intersections in toric varieties bases essentially on the theory of special generalized hypergeometric functions. We remark that these hypergeometric functions satisfy the hypergeometric differential system considered by Gelfand, Kapranov and Zelevinsky in [15]. We propose also a general method for computing the normalized canonical q-coordinates.

The paper is organized as follows:

In Sect. 2, we give a review of the calculation of Candelas et al. in [9] of the coefficients Γ_d of the q-expansion of the normalized Yukawa 3-point function

$$K_q^{(3)} = 5 + \sum_{d \geqq 1} \Gamma_d \frac{q^d}{1 - q^d} .$$

The coefficients $\Gamma_d = n_d d^3$ conjecturaly coincide with the Gromov–Witten invariants (introduced by D. Morrison in [35]) for rational curves on quintic hypersurfaces in $\mathbf{P}^4$. Our review is greatly influenced by the work of D. Morrison [32, 33], but we want to emphasize the fact that the computation of the prediction for the number of rational curves on quintic 3-folds bases essentially on the properties of the special generalized hypergeometric series

$$\Phi_0(z) = \sum_{n \geqq 0} \frac{(5n)!}{(n!)^5} z^n ,$$

which admits a combinatorial definition in terms of curves on $\mathbf{P}^4$.

In Sect. 3, we explain a Hodge-theoretic framework for mirror symmetry and the ideas due to P. Deligne [11] and D. Morrison [34, 35]. The key-point here is the existence of a new type of connection on cohomology of Calabi–Yau manifolds. Following a suggestion of D. Morrison, we call it *A-model connection* (see also [43]). The mirror symmetry identifies the A-model connection on the cohomology of a Calabi–Yau d-fold V with the classical Gauß–Manin connection on cohomology of its mirror manifold V'.

Section 4 contains a review of the standard computational technique based on the recurrent relations satisfied by coefficients of formal solutions of Picard–Fuchs equations. We use this technique later in explicit calculations of q-expansions for Yukawa couplings for some examples of Calabi–Yau complete intersections in toric varieties.

Section 5 is devoted to complete intersections in ordinary projective spaces. Using explicit description of the series $\Phi_0(z)$ for Calabi–Yau complete intersections in projective spaces, we calculate the d-point Yukawa coupling and propose the explicit construction for mirrors of such Calabi–Yau d-folds[3].

In Sect. 6, we give a general definition of special generalized hypergeometric functions and establish the relationships between these functions and combinatorial properties of rational curves on toric varieties containing Calabi–Yau complete intersections. It is easy to see that these generalized hypergeometric functions form a special subclass of the generalized hypergeometric functions with *resonance* parameters considered by Gelfand et al. in [15]. We formulate general conjectures about the differential systems and canonical q-coordinates defined by the generalized hypergeometric series corresponding to Calabi–Yau complete intersections in toric varieties. Using a combinatorial interpretation of Calabi–Yau complete intersections in toric varieties due to Yu. I. Manin [30], we propose an explicit construction of mirrors.

In Sect. 7, we consider in more detail the example of Calabi–Yau hypersurfaces V of degree $(3,3)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. We use this example to illustrate the general computational method we used in Sect. 8, where we calculate the q-expansions of Yukawa couplings for some Calabi–Yau complete intersections in products of projective spaces. For this, we restrict the hypergeometric function $\Phi_0(z)$ to a very special line, such that the resulting function of one parameter satisfies a fourth-order differential equation to which we apply the methods described in Sect. 4. The actual calculations were done on the computer, using a general program written inside *MAPLE*. Applying methods of enumerative geometry, we check that first numbers in the resulting sequences of integers (conjectural Gromov–Witten invariants) coincide with numbers of rational curves of small degree on the corresponding Calabi–Yau 3-folds. So our results can be seen as a confirmation of the conjectures related to mirror symmetry.

2. Quintics in $\mathbf{P}^4$

In this section we give a review of the (conjectural) computation of the Gromov–Witten invariants Γ_d and predictions n_d for numbers of rational curves of degree d on quintics V in $\mathbf{P}^5$ due to P. Candelas, X. de la Ossa, P.S. Green, and L. Parkes [9]. The main ingredients of these computations were considered in papers of D. Morrison [32, 33]. The purpose of this review is to stress that this computation depends only on properties of the special generalized hypergeometric function $\Phi_0(z)$. We begin with the algorithm for computing the coefficients in the q-expansion of the Yukawa coupling and the predictions for number of rational curves.

2.1. The Coefficients in the q-Expansion of the Yukawa Coupling. Consider the series

$$\Phi_0(z) = \sum_{n \geqq 0} \frac{(5n)!}{(n!)^5} z^n .$$

Step 1. If we put $a_n = \frac{(5n)!}{(n!)^5}$, then the numbers a_n satisfy the recurrent relation

$$(n+1)^4 a_{n+1} = 5(5n+1)(5n+2)(5n+3)(5n+4) a_n .$$

[3] Recently L. Borisov proposed a general combinatorial duality which includes as a particular case this our construction [4].

This immediately implies that the series $\Phi_0(z)$ is the solution to the differential equation

$$\mathcal{D}\Phi(z) = 0 ,$$

where

$$\mathcal{D} = \Theta^4 - 5z(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4), \ \Theta = z\frac{\partial}{\partial z} .$$

One can rewrite the differential operator $\mathcal{D}$ in powers of Θ as follows:

$$\mathcal{D} = A_4(z)\Theta^4 + A_3(z)\Theta^3 + \cdots + A_0(z) .$$

We denote by $C_i(z)$ the rational function $A_i(z)/A_4(z)$ $(i = 0,\ldots,3)$.

Step 2. Following [32], define the normalized Yukawa 3-differential as

$$\mathcal{W}_3 = K_z^{(3)} \left(\frac{dz}{z}\right)^{\otimes 3} ,$$

where $K_z^{(3)} = W_3(z)/\Phi_0^2(z)$ is the 3-point coupling function. The function $W_3(z)$ satisfies the differential equation

$$\Theta W_3(z) = -\frac{1}{2}C_3(z)W_3(z)(10)$$

and the normalizing condition $W_3(0) = 5$.

One easily obtains

$$\mathcal{W}_3 = \frac{5}{(1 - 5^5 z)\Phi_0^2(z)} \left(\frac{dz}{z}\right)^{\otimes 3} .$$

Step 3. The equation $\mathcal{D}\Phi = 0$ is a Picard–Fuchs differential equation with maximal unipotent monodromy (in the sense of Morrison [32]) at $z = 0$. Therefore, there exists a unique solution $\Phi_1(z)$ to $\mathcal{D}\Phi = 0$ such that $\Phi_1(z) = (\log z)\Phi_0(z) + \Psi(z)$, where $\Psi(z)$ is regular at $z = 0$ and $\Psi(0) = 0$. We define the new local coordinate $q = q(z)$ near the point $z = 0$ as

$$q(z) = \exp\left(\frac{\Phi_1(z)}{\Phi_0(z)}\right) = z\exp\left(\frac{\Psi(z)}{\Phi_0(z)}\right) .$$

Then, we rewrite the normalized Yukawa 3-differential $\mathcal{W}_3$ in the coordinate q as

$$\mathcal{W}_3 = K_q^{(3)} \left(\frac{dq}{q}\right)^{\otimes 3} .$$

The function $K_q^{(3)}$ is called the Yukawa 3-point coupling. This function has the power expansion

$$K_q^{(3)} = 5 + \sum_{d \geqq 1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d} ,$$

where $\Gamma_d = n_d d^3$ are conjectured to be the Gromov–Witten invariants of rational curves of degree d on a quintic 3-fold in $\mathbf{P}^4$ [20, 35]. The numbers n_d are predictions for numbers of rational curves of degree d on quintic 3-folds.

It is important to remark that in the above algorithm for calculation of the numbers n_d one needs to know only properties of the series $\Phi_0(z)$ and the normalization condition $W_3(0) = \deg V = 5$ for $W_3(z)$, i.e., one does not need to know anything about mirrors of quintics.

2.2. Philosophy of Mirrors and the Series $\Phi_0(z)$. The central role in the computation of Candelas et al. in [9] is played by the orbifold construction of mirrors for quintics in $\mathbf{P}^4$ [17]. In [1], this construction of mirrors was generalized for hypersurfaces in toric Fano varieties with Gorenstein singularities.

In the above algorithm, we have shown that one can forget about mirrors. However, the philosophy of mirrors proves to be very helpful. For quintic 3-folds this philosophy appears as the following twofold interpretation of the series $\Phi_0(z)$.

The first interpretation. We compute the coefficients a_n of the power series $\Phi_0(z)$ using combinatorial properties of curves $C \subset \mathbf{P}^4$ of degree n.

Notice that any such curve C meets a generic quintic V at $5n$ distinct points $p_1, \dots, p_{5n}$. There exists a degeneration of V into a union of 5 hyperplanes $H_1 \cup \dots \cup H_5$. Every such hyperplane H_i intersects C at n points $p_{i_1}, \dots, p_{i_n}$ which can be considered as deformations of a subset of n points from the set $\{p_1, \dots, p_{5n}\}$. It remains to remark that there exists exactly $(5n)!/(n!)^5$ ways to divide $\{p_1, \dots, p_{5n}\}$ into 5 copies of n-element disjoint subsets.

The second interpretation. We find the coefficients a_n from an integral representation of $\Phi_0(z)$.

Let $\mathbf{T} \cong (\mathbf{C}^*)^4$ be the 4-dimensional algebraic torus with coordinate functions X_1, X_2, X_3, X_4. Take the Laurent polynomial

$$f(u, X) = 1 - (u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4 + u_5(X_1X_2X_3X_4)^{-1})$$

in variables X_1, X_2, X_3, X_4, where the coefficients $u_1, \dots, u_5$ are considered as independent parameters. Let $z = u_1u_2u_3u_4u_5$.

Proposition 2.2.1.

$$\Phi_0(u_1 \dots u_5) = \Phi_0(z) = \frac{1}{(2\pi\sqrt{-1})^4} \int_{|X_i|=1} \frac{1}{f(u, X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4} .$$

Proof. One has

$$\frac{1}{f(u, X)} = \sum_{n \geqq 0} (u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4 + u_5(X_1X_2X_3X_4)^{-1})^n$$
$$= \sum_{m \in Z^4} c_m(u) X^m .$$

It is straightforward to see that $c_0(u) = \Phi_0(u_1 \dots u_5)$. Now the statement follows from the Cauchy residue formula. $\square$

The second interpretation of $\Phi_0(z)$ implicitly uses mirrors of quintics, since zero-locus of $f(u, X)$ defines the affine Calabi–Yau 3-fold Z_f in $\mathbf{T}$ whose smooth Calabi–Yau compactification is mirror symmetric with respect to quintic 3-folds (see [1]). Moreover, the holomorphic 3-form $\omega(z)$ on Z_f that extends to a regular form on a smooth compactification of Z_f depends only on z, i.e., only on the product $u_1 \dots u_5$. This 3-form can be written as

$$\omega(z) = \frac{1}{(2\pi\sqrt{-1})^4} \operatorname{Res} \frac{1}{f(u, X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4} .$$

This shows that $\Phi_0(z)$ is exactly the monodromy invariant period of the 3-form $\omega(z)$ near $z = 0$.

Proposition 2.2.2. *The differential* 3*-form* $\omega(z)$ *satisfies the same Picard–Fuchs differential equation* $\mathcal{D}\Phi = 0$ *as the series* $\Phi_0(z)$. *In particular, all periods of* $\omega(z)$ *satisfy the Picard–Fuch differential equation with the operator*

$$\Theta^4 - 5z(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4) .$$

Proof. In order to prove the statement, it is sufficient to check that

$$\left(\mathcal{D}\frac{1}{f(u, X)}\right) \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}$$

is a differential of a rational 3-form on $\mathbf{T}\backslash Z_f$. The latter follows from a standard arguments using reduction by the Jacobian ideal J_f (see [2]). □

2.3. A-model Connection. The Yukawa coupling can be described in terms of a nilpotent connection ∇_A on the cohomology of quintic 3-fold V,

$$\nabla_A : H^*(V, \mathbf{C}) \to H^*(V, \mathbf{C}) \otimes \mathbf{C}\left\langle \frac{dz}{z} \right\rangle .$$

This connection is homogeneous of degree 2, i.e.,

$$\nabla_A : H^i(V, \mathbf{C}) \to H^{i+2}(V, \mathbf{C}) \otimes \mathbf{C}\left\langle \frac{dz}{z} \right\rangle ,$$

and hence ∇_A vanishes on $H^3(V, \mathbf{C})$. For this reason, we consider only the cohomology subring

$$H^{2*}(V, \mathbf{Z}) = \bigoplus_{i=0}^{3} H^{2i}(V, \mathbf{Z}) \subset H^*(V, \mathbf{Z})$$

of even-dimensional classes on a quintic 3-fold $V(\mathrm{rk} H^{2i}(V, \mathbf{Z}) = 1)$. Let η_i be the positive generator of $H^{2i}(V, \mathbf{Z})$. Then, in the basis $\eta_0, \eta_1, \eta_2, \eta_3$, the multiplication by η_1 is the endomorphism of $H^{2*}(V, \mathbf{Z})$ having as matrix

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Following [10] and [35], we define the 1-parameter connection on $H^{2*}(V, \mathbf{C}) \otimes \mathbf{C}[[q]]$ considered as a trivial bundle over $\operatorname{Spec} \mathbf{C}[[q]]$ as follows:

$$\begin{pmatrix} \nabla_A \eta_0 \\ \nabla_A \eta_1 \\ \nabla_A \eta_2 \\ \nabla_A \eta_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{q} & 0 & 0 \\ 0 & 0 & K_q^{(3)} \frac{dq}{q} & 0 \\ 0 & 0 & 0 & \frac{dq}{q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} .$$

The matrix

$$K(q) = \begin{pmatrix} 0 & \frac{dq}{q} & 0 & 0 \\ 0 & 0 & K_q^{(3)} \frac{dq}{q} & 0 \\ 0 & 0 & 0 & \frac{dq}{q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

can be considered as the deformation of the matrix Λ such that

$$\Lambda = \operatorname{Res}|_{q=0} K(q) .$$

The mirror philosophy shows that the matrix (1) can be identified with the matrix of the classical Gauß–Manin connection on the 4-dimensional cohomology space $H^3(\hat{Z}_f, \mathbf{C})$ in a special symplectic basis. We notice that the quotients F^i/F^{i+1} of the Hodge filtration

$$H^3(\hat{Z}_f, \mathbf{C}) = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0$$

are 1-dimensional. There is also the monodromy filtration on the homology $H_3(\hat{Z}_f, \mathbf{Z})$,

$$0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 \subset W_3 = H_3(\hat{Z}_f, \mathbf{Z})$$

such that W_i/W_{i-1} are also 1-dimensional. We choose the symplectic basis $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $H_3(\hat{Z}_f, \mathbf{Z})$ in such a way that $\{\gamma_0, \ldots, \gamma_i\}$ form a $\mathbf{Z}$-basis of W_i. We choose also the basis $\omega_0, \omega_1, \omega_2, \omega_3$ of $H^3(\hat{Z}_f, \mathbf{C})$ such that $\{\omega_0, \ldots, \omega_i\}$ form a $\mathbf{C}$-basis of F^{3-i} and

$$p_{ij} = \int_{\gamma_j} \omega_i = \delta_{ij} \text{ for } i \geqq j .$$

So the period matrix $\Pi = (p_{ij})$ has the form [18, 35]

$$\Pi = \begin{pmatrix} 1 & p_{12} & p_{13} & p_{14} \\ 0 & 1 & p_{23} & p_{34} \\ 0 & 0 & 1 & p_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Notice that all coefficients $p_{ij}(i < j)$ are multivalued functions of z near $z = 0$.

Applying the Griffiths transversality property, we obtain that the Gauß–Manin connection in the z-coordinate has the form

$$\begin{pmatrix} \nabla \omega_0 \\ \nabla \omega_1 \\ \nabla \omega_2 \\ \nabla \omega_3 \end{pmatrix} = \begin{pmatrix} 0 & (\Theta p_{12}) \frac{dz}{z} & 0 & 0 \\ 0 & 0 & (\Theta p_{23}) \frac{dz}{z} & 0 \\ 0 & 0 & 0 & (\Theta p_{34}) \frac{dz}{z} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} ,$$

where $\Theta_{p_{i,i+1}}$ are single valued functions.

Then the Yukawa 3-differential is simply the tensor product

$$\mathscr{W}_3 = K_z^{(3)} \left(\frac{dz}{z}\right)^{\otimes 3} = (\Theta p_{12})\frac{dz}{z} \otimes (\Theta p_{23})\frac{dz}{z} \otimes (\Theta p_{34})\frac{dz}{z} .$$

By Griffiths transversality, one has $\omega_0 \wedge \omega_2 = 0$, i.e. we can assume that $p_{12} = p_{34}$. The differential form ω_0 can be defined as $\omega/\Phi_0(z)$. Moreover, $p_{12} = \Phi_1(z)/\Phi_0(z)$. In the new coordinate q, we have $p_{12} = \log q$. Then the Gauß–Manin connection can be rewritten as

$$\begin{pmatrix} \nabla\omega_0 \\ \nabla\omega_1 \\ \nabla\omega_2 \\ \nabla\omega_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{q} & 0 & 0 \\ 0 & 0 & K_q^{(3)}\frac{dq}{q} & 0 \\ 0 & 0 & 0 & \frac{dq}{q} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} .$$

2.4. The q-Coordinate and the Yukawa Coupling. Since the coordinate q was defined intrinsically as the ratio $\Phi_1(z)/\Phi_0(z)$ of two solutions of the differential equation $\mathscr{D}\Phi = 0$, it is natural to ask about the form of the differential operator $\mathscr{D}$ in the new coordinate q. Denote by Ξ the differential operator $q\frac{\partial}{\partial q}$.

Proposition 2.4.1. *The differential* 3*-form* ω_0 *satisfies the Picard–Fuchs differential equation with the differential operator*

$$\Xi^4 + c_3(q)\Xi^3 + c_2(q)\Xi^2 ,$$

where

$$c_3(q) = -2\frac{\Xi K_q^{(3)}}{K_q^{(3)}}, \quad c_2(q) = \frac{\Xi K_q^{(3)}}{(K_q^{(3)})^2} - \frac{\Xi^2 K_q^{(3)}}{K_q^{(3)}} .$$

Proof. By properties of the nilpotent connection, one has

$$\Xi\omega_0 = \omega_1, \quad \Xi\omega_1 = K_q^{(3)}\omega_2, \quad \Xi\omega_2 = \omega_3, \quad \Xi\omega_3 = 0 .$$

So

$$\begin{aligned} \Xi^4\omega_0 &= \Xi^2 K_q^{(3)}\omega_2 = \Xi((\Xi K_q^{(3)})\omega_2 + K_q^{(3)}\omega_3) \\ &= (\Xi^2 K_q^{(3)})\omega_2 + 2(\Xi K_q^{(3)})\omega_3 . \end{aligned}$$

On the other hand,

$$\omega_2 = \frac{1}{K_q^{(3)}}\Xi^2\omega_0 ,$$

$$\omega_3 = \Xi\left(\frac{1}{K_q^{(3)}}\Xi^2\omega_0\right) = -\frac{\Xi K_q^{(3)}}{(K_q^{(3)})^2}\Xi^2\omega + \frac{1}{K_q^{(3)}}\Xi^3\omega_0 .$$

□

Remark. 2.4.2. The differential equation for ω_0 can be written also as

$$\Xi^2(K_q^{(3)})^{-1}\Xi^2\omega_0 = 0 .$$

In this form this equation first arose in [13].

The differential operator $\mathscr{D}$ which annihilates the function $\Phi_0(z)$ defines the connection in the basis $\omega, \Theta\omega, \Theta^2\omega, \Theta^3\omega$ of $H^3(\hat{Z}_f, \mathbf{C})$:

$$\begin{pmatrix} \nabla\omega \\ \nabla\Theta\omega \\ \nabla\Theta^2\omega \\ \nabla\Theta^3\omega \end{pmatrix} = \begin{pmatrix} 0 & \frac{dz}{z} & 0 & 0 \\ 0 & 0 & \frac{dz}{z} & 0 \\ 0 & 0 & 0 & \frac{dz}{z} \\ -C_0(z)\frac{dz}{z} & -C_1(z)\frac{dz}{z} & -C_2(z)\frac{dz}{z} & -C_3(z)\frac{dz}{z} \end{pmatrix} \begin{pmatrix} \omega \\ \Theta\omega \\ \Theta^2\omega \\ \Theta^3\omega \end{pmatrix} .$$

The basis ω, $\Theta\omega\,\Theta^2\omega$, $\Theta^3\omega$ is also compatible with the Hodge filtration in $H^3(\hat{Z}_f, \mathbf{C})$. Thus there exists a matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{34} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{pmatrix}$$

such that

$$\begin{pmatrix} \omega \\ \Theta\omega \\ \Theta^2\omega \\ \Theta^3\omega \end{pmatrix} = R \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} .$$

It is easy to see that

$$r_{11} = \Phi_0(z), r_{22} = \Phi_0(z)(\Theta p_{12}), r_{33} = \Phi_0(z)(\Theta p_{12})(\Theta p_{23}),$$
$$r_{44} = \Phi_0(z)(\Theta p_{12})(\Theta p_{23})(\Theta p_{34}) .$$

3. Quantum Variations of Hodge Structure on Calabi–Yau Manifolds

3.1. A-Model Connection and Rational Curves. A general approach to the definition of a new connection on cohomology of algebraic and symplectic manifolds V was proposed by Witten [42]. The construction of Witten bases on the interpretation of third partial derivatives

$$\frac{\partial^3}{\partial z_i \partial z_j \partial z_k} P(z)$$

of a function $P(z)$ on the cohomology space $H^*(V, \mathbf{C})$ as structure constants of a commutative associative algebra. The function $P(z)$ is defined via the intersection theory on the moduli spaces of mappings of Riemann surfaces S to V. Using Poincare duality, one obtains the structure coefficients of the connection on $H^*(V, \mathbf{C})$.

We consider a specialization of the general construction to the case when V is a Calabi–Yau 3-fold. We put $n = \dim H^2(V, \mathbf{C}) = \dim H^4(V, \mathbf{C})$. Let η_0 be a generator of $H^0(V, \mathbf{Z}), \{\eta_1, \ldots, \eta_n\}$ a $\mathbf{Z}$-basis of $H^2(V, \mathbf{Z})$, $\{\zeta_1, \ldots, \zeta_n\}$ the dual $\mathbf{Z}$-basis of $H^4(V, \mathbf{Z})(\langle \eta_i, \zeta_j \rangle = \delta_{ij})$, and ζ_0 the dual to η_0 generator of $H^6(V, \mathbf{Z})$. We can always assume that the cohomology classes $\eta_1, \ldots, \eta_n$ are contained in the closed Kähler cone of V.

Definition 3.1.1. *Let $R = \mathbf{C}[[q_1,\dots,q_n]]$ be the ring of formal power series in n independent variables. We denote by $H(V)$ the scalar extension*

$$\left(\bigoplus_{i=0}^{3} H^{2i}(V,\mathbf{C})\right) \bigotimes_{\mathbf{C}} R .$$

We consider a flat nilpotent holomorphic connection

$$\nabla_A : H(V) \to H(V) \otimes \Omega^1_R(\log q)$$

defined by the following formulas [10, 35]

$$\nabla_A \eta_0 = \sum_{i=1}^{n} \eta_i \otimes \frac{dq_i}{q_i} ;$$

$$\nabla_A \eta_k = \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ijk}\zeta_j \otimes \frac{dq_i}{q_i}, \quad k = 1,\dots,n ;$$

$$\nabla_A \zeta_j = \zeta_0 \frac{dq_j}{q_j}, \quad j = 1,\dots,n ;$$

$$\nabla_A \zeta_0 = 0 .$$

The coefficients K_{ijk} are power series in $q_1,\dots,q_n$ defined by rational curves C on V, i.e., morphisms $f : \mathbf{P}^1 \to V$ as follows:

$$K_{ijk} = \langle \eta_i,\eta_j,\eta_k\rangle + \sum_{\substack{C\subset V\\ [C]\neq 0}} n_{[C]}\langle C,\eta_i\rangle\langle C,\eta_j\rangle\langle C,\eta_k\rangle \frac{q^{[C]}}{1-q^{[C]}} ,$$

where $q^{[C]} = q_1^{c_1}\cdots q_n^{c_n}$ $(c_i = \langle C,\eta_i\rangle)$. The integer

$$\Gamma_{[C]}(\eta_i,\eta_j,\eta_k) = n_{[C]}\langle C,\eta_i\rangle\langle C,\eta_j\rangle\langle C,\eta_k\rangle$$

is called the *Gromov–Witten invariant* [20, 35] *of the class* $[C]$. If the classes η_i, η_j and η_k are represented by effective divisors D_i, D_j and D_k on V, then $\Gamma_{[C]}(\eta_i,\eta_j,\eta_k)$ is the number of pseudo-holomorphic immersions $\imath : \mathbf{P}^1 \to V$ such that $[\imath(\mathbf{P}^1)] = [C]$ and $\imath(0)\in D_i$, $\imath(1)\in D_j$, $\imath(\infty)\in D_k$ for sufficiently general almost complex structure on V. One could hope that under favorable circumstance the number $n_{[C]}$ would be equal to the number of rational curves $C\subset V$ in the class $[C]$ is always non-negative.

The connection ∇_A will be called the A-*model connection.* The connection ∇_A defines on $H(V)$ a variation of Hodge structure of type $(1,n,n,1)$. We call this variation *the quantum variation of Hodge structure on V.*

Remark. 3.1.2. The Picard–Fuchs differential system satisfied by η_0 was considered in detail in [10].

One immediately obtains:

Proposition 3.1.3. *Let $\eta = l_1\eta_1 + \cdots + l_n\eta_n \in H^2(V,\mathbf{Z})$ be a class of an ample divisor on V. Define the 1-parameter connection with the new coordinate q by putting $q_1 = q^{l_1},\dots,q_n = q^{l_n}$. Then the connection ∇_A on $H(V)$ induces the connection*

$$\nabla_q : \left(\bigoplus_{i=0}^{3} H^{2i}(V, \mathbf{C}[[q]])\right) \to \left(\bigoplus_{i=0}^{3} H^{2i}(V, \mathbf{C}[[q]])\right) \bigotimes_{\mathbf{C}} \Omega^1_{\mathbf{C}[[q]]}(\log q) .$$

In particular, the residue of the connection operator ∇_q at $q = 0$ is the Lefschetz operator L_η : $H^{2i}(V, \mathbf{C}) \to H^{2i+2}(V, \mathbf{C})$, and

$$\langle (\nabla_q)^3 \eta_0, \eta_0 \rangle = \left(\langle \eta, \eta, \eta \rangle + \sum_{d>0} n_d \frac{d^3 q^d}{1 - q^d} \right) \left(\frac{dq}{q} \right)^{\otimes 3} ,$$

where

$$n_d = \sum_{\langle C, \eta \rangle = d} n_{[C]} .$$

Corollary 3.1.4. *The connection ∇_q defines a differential operator of order* 4 *annihilating η_0.*

3.2. The Gauß–Manin Connection for Mirrors. Let W be a Calabi–Yau 3-fold such that $\dim H^3(W, C) = 2n + 2$. Assume that we are given a variation W_z of complex structure on W near a boundary point p of the n-dimensional moduli space $\mathcal{M}_W$ of complex structures on W in holomorphic coordinates $z_1, \ldots, z_n$ near p such that $p = (0, \ldots, 0)$.

Definition 3.2.1. *The family W_z is said to have the maximal unipotent monodromy at $z = 0$ if the weight filtration*

$$0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 \subset W_3 = H^3(W_z, C)$$

defined by N is orthogonal to the Hodge filtration $\{F^i\}$, i.e.,

$$H^3(W_z, C) = W_i^{\perp} \oplus F^{3-i},\ i = 0, \ldots, 3 .$$

(This is essentially the same definition given in [34, 35].)

Choose a symplectic basis

$$\{\gamma_0, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots \delta_n, \delta_0\}$$

of $H_3(W_z, \boldsymbol{Z})$ in such a way that γ_0 generates $W_0, \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ is a $\mathbf{Z}$-basis of W_1,

$$\{\gamma_0, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots \delta_n\}$$

is a $\mathbf{Z}$-basis of W_2. Then we choose a symplectic basis in $H^3(W_z, \mathbf{C})$:

$$\{\omega_0, \omega_1, \ldots, \omega_n,\ v_1, \ldots, v_n, v_0\} ,$$

such that ω_0 generates F^3, $\{\omega_0, \omega_1, \ldots, \omega_n\}$ is the basis of $F^2, \{\omega_0, \omega_1, \ldots, \omega_n, v_1, \ldots, v_n\}$ is the basis of F^1 such that

$$\langle \omega_i, \gamma_i \rangle = \langle v_i, \delta_i \rangle = 1,\ i = 0, \ldots, n ,$$

$$\langle \omega_i, \gamma_0 \rangle = \langle v_j, \gamma_0 \rangle = \langle v_i, \gamma_j \rangle = \langle v_0, \gamma_j \rangle = \langle v_0, \delta_j \rangle = 0,\ i = 1, \ldots, n,\ j = 0, \ldots, n .$$

The choice of the basis of $H^3(W_z, C)$ defines the splitting into the direct sum

$$H^3(W_z, C) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} ,$$

such that all direct summands acquire *canonical integral structures*. By Griffiths transversality property, the Gauß–Manin connection ∇ sends $H^{3-i,i}$ to $H^{3-i-1,i+1} \otimes \Omega^1(\log z)$.

Two Calabi–Yau 3-folds V and W are called mirror symmetric if the quantum variation of Hodge structure for V is isomorphic to the classical VMHS for W. In this case the q-co-ordinates near p up to constants are defined by the formula [34]

$$q_i = \exp(2\pi\sqrt{-1}) \int_{\gamma_i} \omega_0 .$$

4. Picard–Fuchs Equations

In this section we recall standard facts about Picard–Fuchs differential equations which we use in computations of Yukawa d-point functions and predictions for numbers of rational curves on Calabi–Yau manifolds.

4.1. Recurrent Relations and Differential Equations. Let a_n $(n = 0, 1, 2, \ldots)$ be an infinite sequence of complex numbers. For our purposes, it will be more convenient to define a_n for all integers $n \in \mathbf{Z}$ by putting $a_n = 0$ for $n < 0$. We define the generating function for the sequence $\{a_i\}$ as the formal power series

$$\Phi(z) = \sum_{i \geqq 0}^{\infty} a_i z^i \in \mathbf{C}[[z]] .$$

Consider the following two differential operators acting on C[[z]]:

$$\Theta : f \mapsto z\frac{\partial}{\partial z} f ,$$
$$z : f \mapsto z \cdot f .$$

They satisfy the relation

$$[\Theta, z] = \Theta \circ z - z \circ \Theta = z . \tag{2}$$

These operators generate the algebra $\mathbf{D} = \mathbf{C}[z, \Theta]$ of "logarithmic" differential operators which are polynomials in non-commuting operators Θ and z.

Fix a positive integer d. Assume that there exist $m + 1 (m \geqq 1)$ polynomials,

$$P_0(y), \ldots, P_m(y) \in \mathbf{C}[y]$$

of degree $d + 1$ such that for every $n \in \mathbf{Z}$ the numbers $\{a_i\}$ satisfy the recurrent relation:

$$P_0(n)a_n + P_1(n + 1)a_{n+1} + \cdots + P_m(n + m)a_{n+m} = 0 . \tag{3}$$

(Here we consider y as a new complex variable having no connection to our previous variable z.) Then $\Phi(z)$ is a formal solution of the linear differential equation

$$\mathscr{D}\Phi(z) = 0$$

with the differential operator

$$\mathscr{D} = z^m P_0(\Theta) + z^{m-1} P_1(\Theta) + \cdots + P_m(\Theta) . \tag{4}$$

This differential equation of order $d+1$ can be rewritten in powers of Θ as

$$\mathscr{D} = A_{d+1}(z)\Theta^{d+1} + \cdots + A_1(z)\Theta + A_0(z)\,, \tag{5}$$

where A_i are some polynomials in z. It is easy to check the following:

Proposition 4.1.1. *A power series $\Phi(z)$ is a formal solution to a differential equation $\mathscr{D}\Phi(z) = 0$ of order $d+1$ for some element $\mathscr{D} \in \mathbf{D}$ if and only if the coefficients $\{a_i\}$ satisfy a recurrent relation as in* (3) *for some polynomials $P_0(y), \ldots, P_m(y)$ of degree $d+1$.*

4.2. Picard–Fuchs Operators. Recall that a differential operator $\mathscr{D}$ as in (5) is called *a Picard–Fuchs operator at point $z = 0$* if $A_{d+1}(0) \neq 0$. Solutions of the Picard–Fuchs equations $\mathscr{D}\Phi$ are said to have *maximal unipotent monodromy at $z = 0$* [33] if $A_i(0) = 0$ for $i = 0, \ldots, d$. The above conditions on the operator $\mathscr{D}$ can be reformulated in terms of properties of the polynomial $P_m(y)$ in (3) as follows:

A differential operator $\mathscr{D}$ is a Picard–Fuchs operator if and only if the polynomial $P_m(y)$ has degree $d+1$, i.e., its leading coefficient is nonzero. Moreover, solutions of the equations $\mathscr{D}$ have maximal unipotent monodromy at $z = 0$ if and only if the polynomial $P_m(y)$ equals cy^k for some nonzero constant c.

Picard–Fuchs operators having the maximal unipotent monodromy at $z = 0$ will be objects of our main interest. Therefore, we introduce the following definition:

Definition 4.2.1. *A Picard–Fuchs operator $\mathscr{D}$ with the maximal unipotent monodromy will be called a MU-operator. We will always assume that the corresponding polynomial $P_m(y)$ in* (3) *for any MU-operator $\mathscr{D}$ is y^k, i.e., $c = 1$.*

The fundamental property of MU-operators is the following one:

Theorem 4.2.2. *If $\mathscr{D}$ is MU-operator, then the subspace in* $\mathbf{C}[[z]]$ *of solutions of the linear differential equation*

$$\mathscr{D}\Phi(z) = 0$$

has dimension 1. *Moreover, every solution is defined uniquely by the value $\Phi(0) = a_0$.*

Proof. If we have chosen a value of a_0, all coefficients a_i for $i \geqq 0$ are uniquely defined from the recurrent relation (3). (We recall that we put $a_i = 0$ for $i < 0$.) □

Definition 4.2.3. *Let $\mathscr{D}$ be a MU-operator. Then the power series solution $\Phi_0(z)$ of the equation $\mathscr{D}\Phi(z) = 0$ normalized by the condition $\Phi_0(0) = 1$ will be called the socle-solution.*

4.3. Logarithmic Solutions and the q-Coordinate. Let $\mathscr{D}$ be a MU-operator of order $d+1$. Putting $C_i(z) = A_i(z)/A_{d+1}(z)$ we can define another differential operator

$$\mathscr{P} = \mathscr{P}(\Theta) = \sum_{i=0}^{d+1} C_i(z)\Theta^i\,,$$

which is also a MU-operator of order $d+1$, where $C_i(z)$ are rational functions in z, and $C_{d+1}(z) \equiv 1$. Assume that we have a formal regular solution

$$\Phi(z) = \sum_{i=0}^{\infty} a_n z^n .$$

Consider a formal polynomial extension

$$M_z = \mathbf{C}[[z]][\log z] ,$$

where $\log z$ is considered as a new transcendent variable. We can define the structure of a left $\mathbf{D}$-module on M_z putting by definition $\Theta \log z = 1$. In fact, M_z will be a module over the larger algebra $\mathbf{D}_z$ containing the new operator $Log\, z$ such that

$$z \circ (\Theta \circ Log\, z) = (\Theta \circ Log\, z) \circ z = 1 ,$$
$$\Theta \circ Log\, z - Log\, z \circ \Theta = 1 ,$$

and $Log\, z$ acts on M_z by multiplication on $\log z$.

Proposition 4.3.1. *Let* $\mathscr{P} = \sum_{i=0}^{d+1} C_i(z)\Theta^i$ *be any operator in* $\mathbf{D}$. *Then*

$$[\mathscr{P}, Log\, z] = \sum_{i=1}^{d+1} i\, C_i(z)\Theta^{i-1} = \mathscr{P}'_\Theta ,$$

where $\mathscr{P}'_\Theta$ *is a formal derivative of* $\mathscr{P}$ *with respect to* Θ.

Proof. The statement follows from relation

$$\Theta^i \circ Log\, z - Log\, z \circ \Theta^i = i\,\Theta^{i-1} ,$$

which can be proved by induction. □

Assume that we want to find a element $\Phi_1(z)$ in M_z such that $\mathscr{P}\Phi_1(z) = 0$ and $\Phi_1(z)$ has form

$$\Phi_1(z) = \log z \cdot \Phi_0(z) + \Psi(z) ,$$

where $\Psi(z)$ is an element of $\mathbf{C}[[z]]$, and $\Psi(0) = 0$.

Proposition 4.3.2. *The element* $\Psi(z)$ *satisfies the linear non-homogeneous differential equation*

$$\mathscr{P}'_\Theta \Phi_0(z) + \mathscr{P}\Psi(z) = 0 , \tag{6}$$

or, formally,

$$\Psi(z) = -\mathscr{P}^{-1}\mathscr{P}'_\Theta \Phi_0(z) = \partial_\Theta \log \mathscr{P} \cdot \Phi_0(z) .$$

Proof. Since Φ_0 and Φ_1 are solutions, we obtain

$$\begin{aligned} 0 = \mathscr{P}\Phi_1 &= \mathscr{P}\log z\,\Phi_0 + \mathscr{P}\Psi \\ &= (Log\, z \circ \mathscr{P} + [\mathscr{P}, Log\, z])\Phi_0 + \mathscr{P}\Psi = [\mathscr{P}, Log\, z] \circ \Phi_0(z) + \mathscr{P}\Psi \\ &= \mathscr{P}'_\Theta \Phi_0 + \mathscr{P}\Psi_0 = 0 . \end{aligned}$$

□

Proposition 4.3.3. *If* $\Phi_0(z)$ *is the socle solution, then the function* $\Psi(z)$ *is uniquely defined by Eq.* (6) *and the condition* $\Psi(0) = 0$ *as an element of* $\mathbf{C}[[z]]$.

Proof. Let $\Psi(z) = \sum_{i=-\infty}^{+\infty} b_i z^i$ be an element of $z\mathbf{C}[[z]]$, i.e., $b_i = 0$ for $i \leqq 0$. By 4.3.2, for any $n \in \mathbf{Z}$, the coefficient by z^n in $\mathscr{P}\Psi(z)$ is

$$P_m(n)b_n + P_{m-1}(n-1)b_{n-1} + \cdots + P_0(n-m)b_{n-m} .$$

On the other hand, the coefficient by z^n in $\mathscr{P}'_\Theta \Phi_0(z)$ is

$$P'_m(n)a_n + P'_{m-1}(n-1)a_{n-1} + \cdots + P'_0(n-m)a_{n-m} .$$

Thus, we obtain the recurrent linear non-homogeneous relation

$$P_m(n)b_n + \sum_{i=1}^{m} P_{m-i}(n-i)b_{n-i} + \sum_{i=0}^{m} P'_{m-i}(n-i)a_{n-i} = 0 . \tag{7}$$

Since $P_m(n) = n^{d+1} \neq 0$ for $n \geqq 1$, one can find all coefficients b_i $(i \geqq 1)$ using (7). For instance, we obtain

$$\begin{aligned} b_1 &= -(d+1)a_1 - P'_{m-1}(0)a_0 , \\ 2^{d+1}b_2 &= -(d+1)2^d a_2 - P_{m-1}(1)b_1 - P'_{m-1}(1)a_1 - P'_{m-2}(0)a_0; \ \ldots \text{etc.} \end{aligned}$$

Corollary 4.3.4. *Let $\mathscr{P}$ be a MU-operator, then the quotient Ψ/Φ of the solutions of the linear system*

$$\mathscr{P}\Phi = 0, \quad \mathscr{P}'_\Theta \Phi + \mathscr{P}\Psi, \quad \Psi(0) = 0$$

is a function depending only on $\mathscr{P}$.

We come now to the most important definition:

Definition 4.3.5. *The element*

$$q = \exp\left(\frac{\Phi_1(z)}{\Phi_0(z)}\right) = z \exp\left(\frac{\Psi(z)}{\Phi_0(z)}\right)$$

is called the q-parameter for the MU-operator $\mathscr{P}$.

4.4. Generalized Hypergeometric Functions and 2-Term Recurrent Relations. Since the number $m+1$ of terms in a recurrent relation (3) is at least 2, 2-term recurrences are the simplest ones. Any such relation is defined by two polynomials $P_0(y)$ and $P_1(y)$ of degree $d+1$:

$$P_0(n)a_n = P_1(n+1)a_{n+1} . \tag{8}$$

Without loss of generality we again assume that the leading coefficient of $P_1(y)$ is 1.

Definition 4.4.1. *Denote by*

$$G_{d+1}(\alpha, \beta;\, w) = G_{d+1}\begin{pmatrix} \alpha_1, \ldots, \alpha_{d+1} & \\ & ; w \\ \beta_1, \ldots, \beta_{d+1} & \end{pmatrix}$$

the series

$$\sum_{n \geqq 0} \frac{\prod_{i=1}^{d+1} \Gamma(\alpha_i)}{\prod_{i=1}^{d+1} \Gamma(\beta_i)} \times \left(\frac{\prod_{i=1}^{d+1} \Gamma(\beta_i + n)}{\prod_{i=1}^{d+1} \Gamma(\alpha_i + n)}\right) w^n ,$$

which is the generalized hypergeometric function with parameters $\alpha_1,\dots,\alpha_{d+1}$, $\beta_1,\dots,\beta_{d+1}$. *(This is a slight modification of the well-known generalized hypergeometric function* ${}_{d+1}F_d$ *(see* [31,39]).)

Proposition 4.4.2. *Assume that*

$$P_1(y) = \prod_{i=1}(y+\alpha_i)\,,$$
$$P_0(y) = \lambda\prod_{i=1}(y+\beta_i)\,,$$

then the function $G_{d+1}(\alpha,\beta;\lambda z)$ *is a formal solution of the differential equation*

$$\mathcal{P}\Phi = (P_1(\Theta) - zP_0(\Theta))\Phi = 0\,.$$

Consider now the case when $\mathcal{D}$ is a MU-operator, i.e., $P_1(y) = y^{d+1}$, and the recurrent relation has the form

$$(n+1)^{d+1}a_{n+1} = P_0(n)a_n\,.$$

Then for the power series $\Psi(z) = \sum_{i\geqq 1} b_i z^i$ which is the solution to

$$\mathcal{D}'_\Theta\Phi_0(z) + \mathcal{P}\Psi(z) = 0\,,$$

where

$$\Phi_0(z) = \sum_{i=0}^{\infty} a_i z^i\,,$$

is a regular solution to $\mathcal{P}\Psi = 0$, the coefficients $\{b_i\}$ satisfy the recurrent relation

$$n^{d+1}b_n = P_1(n-1)b_{n-1} + P_1'(n-1)a_{n-1} - (d+1)n^d a_n\,.$$

4.5. d-Point Yukawa Functions. Let $\pi : V_z \to S$ be a 1-parameter family of Calabi–Yau d-folds, where $S = \operatorname{Spec}\mathbf{C}[[z]]$. Let T be the corresponding monodromy transformation acting on $H_d(W_z,\mathbf{C}), T_u$ the unipotent part of $T, N = Log\,T_u$.

Definition 4.5.1. *The family* V_z *is said to have the maximal unipotent monodromy at* $z=0$ *if the weight filtration*

$$0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{d-1} \subset W_d = H^d(V_z,\mathbf{C})$$

defined by N *is orthogonal to the Hodge filtration* $\{F^i\}$, *i.e.,*

$$H^d(V_z,\mathbf{C}) = W_\imath^{\perp} \oplus F^{d-i}\; i = 0,\dots,d\,.$$

(This is similar to definitions given in [18,35].)

Assume that the family V_z has the maximal unipotent monodromy at $z=0$ and $\dim F^\imath/F^{i+1} = 1$ for $i = 0,\dots,d$. Then the Jordan normal form of N has exactly *one* cell of size $(d+1)\times(d+1)$. This means that there exists a d-cycle $\gamma\in H_d(V_z,\mathbf{Z})$ such that $\gamma, N\gamma,\dots,N^d\gamma$ are linearly independent in $H_d(V_z,\mathbf{Z})$, and $N^d\gamma = \gamma_0$ is a monodromy invariant d-cycle. Take a 1-parameter family $\omega(z)$ of holomorphic d-forms on W_z. It is well-known that the periods of $\omega(z)$ over the d-cycles in $H_d(V_z,\mathbf{C})$ satisfy a Picard–Fuchs differential equation of order $d+1$ defined by some differential MU-operator

$$\mathcal{P} = \Theta^{d+1} + C_d(z)\Theta^d + \cdots + C_0(z)\,. \tag{9}$$

Definition 4.5.2. *Define the coupling functions* $W_{k,l}(z)$ $(k, l \geqq 0,\ k, l \in \mathbf{Z})$ *as follows*:

$$W_{k,l} = \int_{V_z} \Theta^k \omega(z) \wedge \Theta^l \omega(z) .$$

(*By definition, we put* $\Theta^0 = 1$ *to be the identity differential operator.*)

Definition 4.5.3 [32]. *The coupling function* $W_{d,0}$ *is called unnormalized d-point Yukawa function.*

Proposition 4.5.4. *The coupling functions* $W_{k,l}(z)$ *satisfy the properties*

(i) $W_{k,l}(z) = (-1)^d W_{l,k}$;
(ii) $W_{k,l}(z) = 0 \quad for \quad k + l < d$;
(iii) $\Theta W_{k,l}(z) = W_{k+1,l}(z) + W_{k,l+1}(z)$;
(iv) $W_{d+k+1,0}(z) + C_d(z)W_{d+k,0}(z) + \cdots + C_0(z)W_{k,0}(z) = 0$.

Proof. The statements follow immediately from the properties of the cup-product and from the Griffiths transversality property.

Theorem 4.5.5. *The d-point Yukawa function* $W_{d,0}(z)$ *satisfies the linear differential equation of order one,*

$$\Theta W_{d,0}(z) + \frac{2}{d+1} C_d(z) W_{d,0} = 0 . \tag{10}$$

Proof. By 4.5.4(ii), we have

$$W_{d-i,i}(z) + W_{d-i-1,i+1}(z) = 0 \text{ for } i = 0, 1, \ldots, d . \tag{11}$$

Therefore, $W_{d,0}(z) = (-1)^i W_{d-i,i}$. On the other hand, by 4.5.4(iii), we have

$$\Theta W_{d-i,i} = W_{d-i+1,i}(z) + W_{d-i,i+1}(z) \quad \text{for} \quad i = 0, 1, \ldots, d . \tag{12}$$

It follows from (11) and (12) that

$$k\, \Theta W_{d,0}(z) = \sum_{i=0}^{k-1} (-1)^i W_{d-i,i}(z) = W_{d+1,0}(z) + (-1)^{k-1} W_{d-k+1,k}(z) . \tag{13}$$

Case I. d is odd. Since

$$W_{\frac{d+1}{2},\frac{d+1}{2}}(z) = 0 \ \ (4.5.4(\mathrm{i})) ,$$

we obtain

$$\Theta W_{\frac{d+1}{2},\frac{d-1}{2}}(z) = W_{\frac{d+3}{2},\frac{d-1}{2}}(z) .$$

Using (11) and (13) for $k = (d+1)/2$, we obtain

$$\frac{(d+1)}{2} \Theta W_{d,0}(z) = W_{d+1,0}(z) .$$

By 4.5.4(ii) and (iv), this implies Eq. (10) for $W_{d,0}(z)$.

Case II. d is even. One has

$$\Theta W_{\frac{d}{2},\frac{d}{2}}(z) = W_{\frac{d+2}{2},\frac{d}{2}}(z) + W_{\frac{d}{2},\frac{d+2}{2}}(z) = 2 W_{\frac{d+2}{2},\frac{d}{2}}(z) .$$

Using (11) and (13) for $k = d/2$, we obtain

$$(d+1)\Theta W_{d,0}(z) = 2W_{d+1,0}(z) .$$

The latter again implies the same linear differential equation for $W_{d,0}(z)$.

Corollary 4.5.6.

$$W_{d,0}(z) = c_0 \exp\left(-\frac{2}{d+1}\int_0^z C_d(v)\frac{dv}{v}\right)$$

for some nonzero constant $c_0 = W_{d,0}(0)$.

Example 4.5.7. Assume that $\mathscr{P} = \Theta^{d+1} - zP_0(\Theta)$ be the MU-operator corresponding to a 2-term recurrent relation $(n+1)^{d+1}a_{n+1} = P_0(n)a_n$, where $P_0(y) = \lambda y^{d+1} + \cdots$. Then the Yukawa d-point function $W_{d,0}(z)$ equals

$$W_{d,0}(z) = \frac{c_0}{1-\lambda z} ,$$

i.e., $W_{d,0}(z)$ is a rational function in z.

4.6. Multidimensional Picard–Fuchs Differential Systems with a Symmetry Group. So far we considered only the case of the 1-parameter family of Calabi–Yau d-folds V_z such that $\dim F^i/F^{i+1} = 1$ for $i = 0,\ldots,d$. It is easy to see that the same methods can be applied to the case $\dim F^i/F^{i+1} \geqq 1$, provided V_z has a large automorphisms group.

Proposition 4.6.1. *Let V_z be a 1-parameter family of Calabi–Yau d-folds with* $\dim F^i/F^{i+1} \geqq 1$. *Assume that there exists an action of a finite group G on V_z such that the G-invariant part $(F^i/F^{i+1})^G$ is 1-dimensional for all $i = 0,\ldots,d$. Then the holomorphic differential d-form $\omega(z)$ again satisfies the Picard–Fuchs differential equation of order $d+1$.*

Proof. The statement immediately follows from the fact that the cohomology classes of $\omega(z), \Theta\omega(z), \ldots, \Theta^d\omega(z)$ form the basis of the G-invariant subspace $H^d(V_z, \mathbf{C})^G \subset H^d(V_z, \mathbf{C})$. □

5. Calabi–Yau Complete Intersections in $\mathbf{P}^N$

5.1. Rational Curves and Generalized Hypergeometric Series. Let V be a complete intersection of r-hypersurfaces $V_1,\ldots,V_r$ of degrees $d_1,\ldots,d_r$ in $\mathbf{P}^{d+r}$. Then V is a Calabi–Yau d-fold if and only if $d+r+1 = d_1+\cdots+d_r$. A rational curve C of degree n in $\mathbf{P}^{d+r}$ has nd_i intersection points with a generic hypersurface V_i. On the other hand, there exists a degeneration of every divisor V_i into the union of d_i hyperplanes. Each of these hyperplanes has n intersection points with C. This motivates the definition of the corresponding generalized hypergeometric series $\Phi_0(z)$ as

$$\sum_{i=0}^{\infty}\frac{(nd_1!)}{(n!)^{d_1}}\cdots\frac{(nd_r!)}{(n!)^{d_r}}z^n . \tag{14}$$

The coefficients

$$a_n = \frac{(nd_1!)\cdots(nd_r!)}{(n!)^{d+r+1}}$$

satisfy the recurrent relation

$$(n+1)^{d+1}a_{n+1} = P(n)a_n ,$$

where $P(y)$ is the polynomial of degree $d+1$:

$$P(n) = \frac{(nd_1+d_1)!}{(nd_1)!}\cdots\frac{(nd_r+d_r)!}{(nd_r)!}(n+1)^{-r} = \lambda n^{d+1} + \cdots .$$

In particular, the leading coefficient of $P(y)$ is $\lambda = \prod_{i=1}^{r}(d_i)^{d_i}$.

Example 5.1.1. Let V be a complete intersection of two cubics in $\mathbf{P}^5$. The corresponding generalized hypergeometric series is

$$\Phi_0(z) = \sum_{n\geqq 0}\frac{(3n!)^2}{(n!)^6}z^n .$$

This series was found in [28] using the explicit construction of mirrors for V by orbifolding the 1-parameter family of special complete intersections of two cubics in $\mathbf{P}^5$:

$$Y_1^3 + Y_2^3 + Y_3^3 = 3\psi Y_4 Y_5 Y_6 ;$$
$$Y_4^3 + Y_5^3 + Y_6^3 = 3\psi Y_1 Y_2 Y_3 ,$$

by an abelian group G of order 81, where $z = (3\psi)^{-6}$.

We will give another interpretation of the construction of mirrors V' for V which immediately implies that $\Phi_0(z)$ is the monodromy invariant period for the regular differential 3-form on V'.

Let $Z_{f_1 f_2}$ be the complete intersection of two hypersurfaces in a 5-dimensional algebraic torus $\mathbf{T} = \mathrm{Spec}[X_1^{\pm 1},\ldots,X_5^{\pm 1}]$ defined by the Laurent polynomials

$$f_1(u,X) = 1 - (u_1X_1 + u_2X_2 + u_3X_3),$$
$$f_2(u,X) = 1 - (u_4X_4 + u_5X_5 + u_6(X_1\cdots X_5)^{-1}) .$$

We define the differential 3-form ω on $Z_{f_1 f_2}$ as the residue of the rational differential 5-form on $\mathbf{T}$:

$$\omega = \frac{1}{(2\pi\sqrt{-1})}\,\mathrm{Res}\,\frac{1}{f_1(u,X)f_2(u,X)}\frac{dX_1}{X_1}\wedge\cdots\wedge\frac{dX_5}{X_5} .$$

Let $z = u_1\cdots u_6$. By the residue theorem, we obtain

$$\Phi_0(z) = \frac{1}{(2\pi\sqrt{-1})}\int_{|X_i|=1}\frac{1}{f_1(u,X)f_2(u,X)}\frac{dX_1}{X_1}\wedge\cdots\wedge\frac{dX_5}{X_5} .$$

In this interpretation, the mirrors for V are smooth Calabi–Yau compactifications of affine 3-folds $Z_{f_1 f_2}$.

The equivalence between the above two constructions of mirrors for V follows by the substitution

$$X_1 = Y_1^3/(Y_3Y_4Y_5),\quad X_2 = Y_2^3/(Y_3Y_4Y_5),\quad X_3 = Y_1^3/(Y_3Y_4Y_5) ,$$
$$X_4 = Y_4^3/(Y_1Y_2Y_3),\quad X_5 = Y_5^3/(Y_1Y_2Y_3) ,$$
$$u_1 = \cdots = u_6 = (3\psi)^{-1} .$$

Proposition 5.1.2. *The normalized Yukawa d-differential for Calabi–Yau complete intersections has the form*

$$\mathscr{W}_d = \frac{d_1 \cdots d_r}{(1-\lambda z)\Phi_0^2(z)} \cdot \left(\frac{dz}{z}\right)^{\otimes d},$$

where $d_1, \ldots, d_i$ *are degrees of hypersurfaces.*

Proof. The statement follows from 4.5.7 and the normalizing condition $d_1 \ldots d_r = W_d(0)$. □

5.2. The Construction of Mirrors. Let V be a d-dimensional Calabi–Yau complete intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$ in $\mathbf{P}^{d+r}$. We propose the explicit construction of candidates for mirrors with respect to V as follows:

Let $E = \{v_1, \ldots, v_{d+r+1}\}$ be a generating set in the $(d+r)$-dimensional lattice $N \cong \mathbf{Z}^{d+r}$ such that there exists the relation

$$v_1 + \cdots + v_{d+r+1} = 0 .$$

We divide E into a disjoint union of r subsets $E_i \subset E$ such that Card $E_i = d_i$. For $i = 1, \ldots, r$, we define the Laurent polynomial $P_i(u,X)$ in variables $X_1, \ldots, X_{d+r}$ as

$$P_i(X) = 1 - \left(\sum_{v_j \in E_i} u_j X^{v_j}\right),$$

where $u_1, \ldots, u_{d+r+1}$ are independent parameters. We denote by V' a Calabi–Yau compactification of d-dimensional affine complete intersections Z in $\mathbf{T} = \operatorname{Spec}[X_1^{\pm 1}, \ldots, X_{d+r}^{\pm 1}]$ defined by the polynomials $P_1(u,X), \ldots, P_r(u,X)$ with sufficiently general coefficients u_i. It is easy to see that up to an isomorphism the affine varieties $Z \subset \mathbf{T}$ depend only on $z = u_1 \cdots u_{d+r+1}$. Thus, we have obtained a 1-parameter family of d-dimensional varieties V'.

Conjecture 5.2.1. *The* 1*-parameter family of d-dimensional varieties* V' *yields the mirror family for* V.

This conjecture is motivated by the combinatorial interpretation proposed in [1] of the well-known construction of mirrors for hypersurfaces of degree $d+2$ in $\mathbf{P}^{d+1}$ (see [17]). On the other hand, the conjecture is supported by the following property:

Proposition 5.2.2. *The hypergeometric series* $\Phi_0(z)$ *in* (14) *is the monodromy invariant period function of the holomorphic d-form* ω *on* V'.

Proof. The statement follows from the equality

$$\sum_{i=0}^{\infty} \frac{(nd_1!)}{(n!)^{d_1}} \cdots \frac{(nd_r!)}{(n!)^{d_r}} z^n = \frac{1}{(2\pi\sqrt{-1})^{d+r}} \int\limits_{|X_j|=1} \frac{1}{P_1(X)\cdots P_r(X)} \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_{d+r}}{X_{d+r}} .$$

□.

6. Complete Intersections in Toric Varieties

6.1. The generalized Hypergeometric Series Φ_0. Let N be a free abelian group of rank $(d+r)$. Consider r finite sets

$$E_i = \{v_{i,1}, \ldots, v_{i,k_i}\}, \ i = 1, \ldots, r$$

consisting of elements $v_{i,j} \in N$. Let E be the union $E_1 \cup \cdots \cup E_r$.

We put $k = \operatorname{Card} E = k_1 + \ldots + k_r$ and assume that E generates the group N. Let $R(E)$ be the subgroup in $\mathbf{Z}^n$ consisting of all integral vectors $\lambda = \{\lambda_{i,j}\}$ such that

$$\sum_{i=1}^{r} \sum_{j=1}^{k_i} \lambda_{i,j} v_{i,j} = 0 .$$

We denote by $R^+(E)$ a submonoid in $R(E)$ consisting of all $\lambda = \{\lambda_{i,j}\} \in R(E)$ such that $\lambda_{i,j} \geqq 0$.

Definition 6.1.1. *Let $u_{i,j}$ be k independent complex variables parametrized by k integral vectors $v_{i,j}$. Define the power series $\Phi_0(u)$ as*

$$\Phi_0(u) = \sum_{\lambda \in R^+(E)} \prod_{i=1}^{r} \left(\sum_{j=1}^{k_i} \lambda_{i,j} \right)! \left(\prod_{j=1}^{k_i} \frac{u_{i,j}^{\lambda_{i,j}}}{(\lambda_{i,j})!} \right) .$$

Let $\lambda^{(1)}, \ldots, \lambda^{(t)}$ be a $\mathbf{Z}$-basis of the lattice $R(E)$ such that every element $\lambda \in R^+(E)$ is a non-negative integral linear combination of $\lambda^{(i)}$. We define new r complex variables $z_1, \ldots, z_s$ as follows:

$$z_s = \prod_{i=1}^{t} \prod_{j=1}^{k_i} u_{i,j}^{\lambda_{i,j}^{(s)}}; \quad s = 1, \ldots, t .$$

Thus, the series $\Phi_0(u)$ can be rewritten as the power series $\Phi_0(z)$ in t variables $z_1, \ldots, z_t$.

Example 6.1.2. Let $E = \{v_1, \ldots, v_{d+1}\}$ be vectors generating d-dimensional lattice N and satisfying the integral relation $v_1 + \cdots + v_{d+1} = 0$, i.e., the group $R(E)$ is generated by the vector $(1, \ldots, 1) \in \mathbf{Z}^{d+1}$. Then the corresponding generalized hypergeometric series is

$$\Phi_0(u) = \sum_{n \geqq 0} \frac{(nd+n)!}{(n!)^{d+1}} (u_1 \ldots u_{d+1})^n = \sum_{n \geqq 0} \frac{(nd+n)!}{(n!)^{d+1}} z^n = \Phi_0(z) ,$$

where $z = u_1 \cdots u_{d+1}$. The integral representation of this series is the monodromy invariant period function for mirrors of hypersurfaces of degree $(d+1)$ in $\mathbf{P}^d$.

Definition 6.1.3. *Let $\mathbf{T}$ be a $(d+r)$-dimensional algebraic torus with the Laurent coordinates $X = (X_1, \ldots, X_{d+r})$. We define r Laurent polynomials $P_{E_1}(X)$, $\ldots, P_{E_r}(X)$ as follows:*

$$P_{E_i}(X) = 1 - \sum_{v_{i,j} \in E_i} u_{i,j} X^{v_{i,j}} .$$

Proposition 6.1.4. *The series $\Phi_0(u)$ admits the following integral representation*

$$\Phi_0(u) = \frac{1}{(2\pi\sqrt{-1})^{d+r}} \int_{|X_j|=1} \frac{1}{P_{E_1}(X)\cdots P_{E_r}(X)} \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_{d+r}}{X_{d+r}} .$$

Proof. The statement follows immediately from the residue formula. □

6.2. Calabi–Yau Complete Intersections. Let $\mathbf{P}_\Sigma$ be a quasi-smooth $(d+r)$-dimensional projective toric variety defined by a $(d+r)$-dimensional simplicial fan Σ. Assume that there exist r line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_r$ such that each $\mathscr{L}_i$ is generated by global sections and the tensor product

$$\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_r$$

is isomorphic to the anticanonical bundle on $\mathscr{K}^{-1}$ on $\mathbf{P}_\Sigma$. If V_i is the set of zeros of a generic global section of $\mathscr{L}_i$, then the complete intersection $V = V_1 \cap \cdots \cap V_r$ is a d-dimensional Calabi–Yau variety having only Gorenstein toroidal singularities which are analytically isomorphic to toric singularities of $\mathbf{P}_\Sigma$.

Now let $E = \{v_1, \ldots, v_k\}$ be the set of all generators of 1-dimensional cones in Σ. Denote by D_j the toric divisor on $\mathbf{P}_\Sigma$ corresponding to v_j. Notice that

$$\mathscr{K}^{-1} = \bigotimes_{j=1}^{k} \mathscr{O}_{\mathbf{P}_\Sigma}(D_j) .$$

Following a suggestion of Yu. I. Manin [30], we assume that one can represent E as a disjoint union

$$E = E_1 \cup \cdots \cup E_r$$

such that the line bundle $\mathscr{L}_i$ is isomorphic to the tensor product

$$\bigotimes_{v_j \in E_i} \mathscr{O}_{\mathbf{P}_\Sigma}(D_j) .$$

The elements of the group $R(E)$ can be identified with the homology classes of 1-dimensional algebraic cycles on $\mathbf{P}_\Sigma$. Moreover, one has the following property

Proposition 6.2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be an arbitrary element of $R(E)$ representing the class of an algebraic 1-cycle C. Then*

$$\lambda_i = \langle D_i, C \rangle, \quad i = 1, \ldots, k .$$

We can always choose a $\mathbf{Z}$-basis $\lambda^{(1)}, \ldots, \lambda^{(t)}$ of $R(E)$ such that every *effective* algebraic 1-cycle on $\mathbf{P}_\Sigma$ is a non-negative linear combination of the elements $\lambda^{(1)}, \ldots, \lambda^{(t)}$. Since the submonoid $R^+(E)$ consists of classes of *nef-curves*, this implies that every element of $R^+(E)$ is also a non-negative linear combination of $\lambda^{(1)}, \ldots, \lambda^{(t)}$. This allows us to rewrite the series $\Phi_0(u)$ in t algebraically independent variables $z_1, \ldots, z_t$ ($t = \operatorname{rk} R(E)$).

Corollary 6.2.2. *The series $\Phi_0(z)$ can be interpreted via the intersection numbers of classes $[C]$ of curves C on $\mathbf{P}_\Sigma$ as follows*:

$$\Phi_0(z) = \sum_{[C] \in R^+(E)} \frac{(\langle V_1, C\rangle)! \cdots (\langle V_r, C\rangle)!}{\langle D_1, C\rangle! \cdots \langle D_k, C\rangle!} z^{[C]} ,$$

where $z^{[C]} = z_1^{c_1} \ldots z_t^{c_t}$, $[C] = c_1\lambda^{(1)} + \cdots + c_t\lambda^{(t)}$.

6.3. General Conjectures. Let V be a d-dimensional Calabi–Yau complete intersection of hypersurfaces $V_1, \ldots, V_r$ in a $(d+r)$-dimensional quasi-smooth toric variety defined by a simplicial fan Σ. Choose a $\mathbf{Z}$-basis $\lambda^{(1)}, \ldots, \lambda^{(t)}$ in $R(E)$ such that the classes of all effective algebraic 1-cycles have non-negative integral coordinates. We assume that the divisors $V_1, \ldots, V_r$ are numerically effective (in particular, they are not assumed to be necessary ample divisors). We assume also that the following conditions are satisfied:

(i) V is smooth;
(ii) the restriction mapping $\mathrm{Pic}\,\mathbf{P}_\Sigma \to \mathrm{Pic}\,V$ is injective.

In this situation, there exist two flat A-model connections: the connection ∇_{AP} on $H^*(\mathbf{P}_\Sigma)$ and the connection ∇_{AV} on $H^*(V, \mathbf{C})$. Let $\tilde{H}^i$ be the image of $H^i(\mathbf{P}_\Sigma, \mathbf{C})$ in $H^i(V, \mathbf{C})$. The connection ∇_{AP} defines the quantum variation on cohomology of toric variety $\mathbf{P}_\Sigma$. It follows from the result in [3] the following.

Proposition 6.3.1. *The complex coordinates $z_1, \ldots, z_t$ on $\tilde{H}^2$ can be identified with flat coordinates with respect to ∇_{AP}.*

Conjecture 6.3.2. *The generalized hypergeometric series $\Phi_0(z)$ as a function of ∇_{AP}-flat z-coordinates on $\tilde{H}^2$ is a solution of the differential system $\mathscr{D}$ defined by the A-model connection ∇_{AV} on $\tilde{H}^2$ which defines the quantum variation of Hodge structures on the subring in $\bigoplus_{i=0}^d H^{2i}(V, \mathbf{C})$ generated by restrictions of the classes in* $\mathrm{Pic}\,\mathbf{P}_\Sigma$ *to V.*

Remark. 6.3.3. One can check in many examples that the differential system $\mathscr{D}$ is already defined by the generalized hypergeometric series $\Phi_0(z)$. Probably there exists a general explanation of this fact.

Conjecture 6.3.4. *The ∇_{AV}-flat coordinates $q_1, \ldots, q_t$ on $\tilde{H}^2$ are defined as*

$$q_i = \exp(\Phi_i(z)/\Phi_0(z)), \quad i = 1, \ldots, t\,,$$

where $\Phi_i(z)$ is a logarithmic solution to the differential system $\mathscr{D}$ having the form

$$\Phi_i(z) = (\log z_i)\Phi_0(z) + \Psi_i(z), \quad \Psi_i(0) = 0$$

for some regular at $z = 0$ power series $\Psi_i(z)$.

Moreover, all coefficients of the expansion of ∇_{AV}-flat coordinates q_i as power series of ∇_{AP}-flat z-coordinates are **integers**.

Remark. 6.3.5. This conjecture establishes a general method for normalizing the logarithmic solutions defining the canonical q-coordinates for the differential system $\mathscr{D}$. There are two motivations for this conjecture. First, the conjecture is true for all already known examples of q-coordinates for Picard–Fuchs equations corresponding to Calabi–Yau complete intersections in products of weighted projective spaces (see examples in the remaining part of the paper). Second, the Lefschetz theorem and the calculation of the quantum cohomology ring of toric varieties [3] imply the relation

$$q_i = z_i + O\left(|z|^2\right), \quad (i = 1, \ldots, t)\,.$$

Conjecture 6.3.6. *Assume that V has dimension 3. Let $K_{i,j,k}(z)$ be structure constants defining the A-model ∇_{AV} connection in the z-coordinates. Then*

$$\Phi_0^2(z)K_{i,j,k}(z)$$

is a rational function in z-coordinates.

Conjecture 6.3.7. *The mirror Calabi–Yau varieties with respect to V are Calabi–Yau compactifications of the complete intersection of the affine hypersurfaces in the $(d+r)$-dimensional algebraic torus $\mathbf{T}$ defined by the equations*

$$P_{E_1}(X) = \cdots = P_{E_r}(X) = 0 .$$

Remark. 6.3.8. Recall that two Calabi–Yau d-folds V and V' are called mirror symmetric if $h^{p,d-p}(V) = h^{d-p,d-p}(V')$ and the superconformal field theories corresponding to V and V' are isomorphic. In [1] a general method for constructing pairs of mirror symmetric Calabi–Yau hypersurfaces in toric varieties was proposed, based on the duality among so-called *reflexive polyhedra* Δ and Δ^* (see also [36, 37]). However, the equality $h^{1,1}(\hat{Z}_f) = h^{d-1,1}(\hat{Z}_g)$ for the pair of Calabi–Yau d-folds $\hat{Z}_f$ $\hat{Z}_g$ corresponding to the polyhedra Δ and Δ^* are not sufficient to prove the mirror duality between $\hat{Z}_f$ and $\hat{Z}_g$ in full strength. One needs to prove more: the isomorphism between the quantum cohomology of $\hat{Z}_f$ and $\hat{Z}_g$. Since the quantum cohomology is defined by the canonical form of the A-model connection ∇_A in q-coordinates, Conjecture 6.3.2 and Proposition 6.1.4 yield more evidence for validity of Conjecture 6.3.7. We give below one example showing that Conjecture 6.3.7 agrees with an orbifold construction of mirrors for complete intersections in the product of projective spaces inspired by superconformal field theories.

Example 6.3.9. Let V be a Calabi–Yau complete intersection of two hypersurfaces of degrees $(3,0)$ and $(1,3)$ in the product $\mathbf{P}^3 \times \mathbf{P}^2$.

It is known that the mirrors for V can be obtained by orbifolding the complete intersection of two special hypersurfaces

$$S_1T_1^3 + S_2T_2^3 + S_3T_3^3 = \phi S_4T_1T_2T_3 ,$$

$$S_1^3 + S_2^3 + S_3^3 + S_4^3 = \psi S_1S_2S_3$$

by the group G of order 27, where $(S_1 : S_2 : S_3 : S_4)$ and $(T_1 : T_2 : T_3)$ are the homogeneous coordinates on $\mathbf{P}^3$ and $\mathbf{P}^2$ respectively.

On the other hand, the 5-dimensional fan Σ defining $\mathbf{P}^3 \times \mathbf{P}^2$ has 7 generators $\{v_1, \ldots, v_7\} = E$ satisfying the relations

$$v_1 + v_2 + v_3 + v_6 = v_4 + v_5 + v_7 = 0 .$$

We choose vectors $v_1, \ldots, v_5$ as the basis of the 5-dimensional lattice N. The complete intersection V is defined by dividing E into two subsets $E_1 = \{v_1, v_2, v_3\}$ and $E_2 = \{v_4, v_5, v_6, v_7\}$. The corresponding polynomials $P_{E_1}(X)$ and $P_{E_2}(X)$ are

$$PE_1 = 1 - (u_1X_1 + u_2X_2 + u_3X_3),$$

$$P_{E_2} = 1 - (u_4X_4 + u_5X_5 + u_6(X_1X_2X_3)^{-1} + u_7(X_4X_5)^{-1}) .$$

We obtain the equivalence between two constructions of mirrors by putting $u_1 = u_2 = u_3 = \phi^{-1}, u_4 = u_5 = u_6 = u_7 = \psi^{-1}$, and

$$X_1 = \frac{S_1 T_1^2}{S_4 T_2 T_3}, \quad X_2 = \frac{S_2 T_2^2}{S_4 T_1 T_3}, \quad X_3 = \frac{S_3 T_3^2}{S_4 T_1 T_2},$$
$$X_4 = \frac{S_1^2}{S_2 S_3}, \quad X_5 = \frac{S_2^2}{S_1 S_3}.$$

6.4. Calabi–Yau 3-Folds with $h^{1,1} = 1$. We consider below examples of generalized hypergeometric series corresponding to smooth Calabi–Yau complete intersections V of r hypersurfaces in a toric variety $\mathbf{P}_\Sigma$ such that $h^{1,1}(V) = 1$. By the Lefschetz theorem, $h^{1,1}(\mathbf{P}_\Sigma)$ must be also 1. So Σ is a $(r+3)$-dimensional fan with $(r+4)$ generators. There exists the unique primitive integral linear relation $\sum \lambda_i v_i = 0$ among the generators $\{v_i\}$ of Σ, i.e., rk $R(E) = 1$, where $E = \{v_i\}$ is the whole set of generators of Σ (Card $E = r + 4$).

In all these examples the MU-operator $\mathscr{P}$ has the form

$$\mathscr{P} = \Theta^4 - \mu z(\Theta + \alpha_1)(\Theta + \alpha_2)(\Theta + \alpha_3)(\Theta + \alpha_4),$$

where the numbers $\alpha_1, \ldots, \alpha_4$ are positive rationals satisfying the relations

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = 1.$$

The Yukawa 3-differential in the z-coordinate has the form

$$\mathscr{W}_3 = \frac{W(0)}{(1 - \mu z)\Phi_0^2(z)} \left(\frac{dz}{z}\right)^{\otimes 3}.$$

Example 6.4.1. Hypersurfaces in weighted projective spaces: In this case we obtain Calabi–Yau hypersurfaces in the following weighted projective spaces $\mathbf{P}(\lambda_1, \ldots, \lambda_5)$

$(\lambda_1, \ldots, \lambda_5)$	$\Phi_0(z)$	$W(0)$	μ	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$
$(1,1,1,1,1)$	$\sum_{n \geqq 0} \frac{(5n)!}{(n!)^5} z^n$	5	5^5	$(1/5, 2/5, 3/5, 4/5)$
$(2,1,1,1,1)$	$\sum_{n \geqq 0} \frac{(6n)!}{(n!)^4 (2n!)} z^n$	3	$2^5 3^6$	$(1/6, 2/6, 4/6, 5/6)$
$(4,1,1,1,1)$	$\sum_{n \geqq 0} \frac{(8n)!}{(n!)^4 (4n!)} z^n$	2	2^{18}	$(1/8, 3/8, 5/8, 7/8)$
$(5,2,1,1,1)$	$\sum_{n \geqq 0} \frac{(10n)!}{(n!)^3 (2n!)(5n!)} z^n$	1	$2^9 5^6$	$(1/10, 3/10, 7/10, 9/10)$

The q-expansion of the Yukawa 3-point function and predictions n_d for the number of rational curves on these hypersurfaces were obtained in [33, 21, 14].

Example 6.4.2. Complete intersections in ordinary projective spaces: Let $V_{d_1,\ldots,d_r}$ denote the complete intersection of hypersurfaces of degrees $d_1,\ldots,d_r$.

	$\Phi_0(z)$	$W(0)$	μ	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$
$V_{3,3} \subset \mathbf{P}^5$	$\sum_{n\geqq 0} \frac{((3n)!)^2}{(n!)^6} z^n$	9	3^6	$(1/3,1/3,2/3,2/3)$
$V_{2,4} \subset \mathbf{P}^5$	$\sum_{n\geqq 0} \frac{(2n)!(4n)!}{(n!)^6} z^n$	8	2^{10}	$(1/4,2/4,2/4,3/4)$
$V_{2,2,3} \subset \mathbf{P}^6$	$\sum_{n\geqq 0} \frac{((2n)!)^2(3n)!}{(n!)^7} z^n$	12	$2^4 3^3$	$(1/3,1/2,1/2,2/3)$
$V_{2,2,2,2} \subset \mathbf{P}^8$	$\sum_{n\geqq 0} \frac{((2n)!)^4}{(n!)^8} z^n$	16	2^8	$(1/4,1/4,1/4,1/4)$

These Calabi–Yau complete intersections in ordinary projective spaces were considered by Libgober and Teitelbaum [28].

Example 6.4.3. Complete intersections in weighted projective spaces:

	$\Phi_0(z)$	$W(0)$	μ	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$
$V_{4,4} \in \mathbf{P}(1,1,1,1,2,2)$	$\sum_{n=0}^{\infty} \frac{(4n!)^2}{(n!)^4(2n!)^2} z^n$	4	2^{12}	$(1/4,1/4,3/4,3/4)$
$V_{6,6} \in \mathbf{P}(1,1,2,2,3,3)$	$\sum_{n=0}^{\infty} \frac{(6n!)^2}{(n!)^2(2n!)^2(3n!)^2} z^n$	1	$2^8 3^6$	$(1/6,1/6,5/6,5/6)$
$V_{3,4} \in \mathbf{P}(1,1,1,1,1,2)$	$\sum_{n=0}^{\infty} \frac{(4n!)(3n!)}{(n!)^5(2n!)} z^n$	6	$2^6 3^3$	$(1/4,1/3,2/3,3/4)$
$V_{2,6} \in \mathbf{P}(1,1,1,1,1,3)$	$\sum_{n=0}^{\infty} \frac{(6n!)(2n!)}{(n!)^5(3n!)} z^n$	4	$2^8 3^3$	$(1/6,1/2,1/2,5/6)$
$V_{4,6} \in \mathbf{P}(1,1,1,2,2,3)$	$\sum_{n=0}^{\infty} \frac{(6n!)(4n!)}{(n!)^3(2n!)^2(3n!)} z^n$	2	$2^{10} 3^3$	$(1/6,1/4,3/4,5/6)$

The coefficients of the Yukawa 3-point function $K_q^{(3)}$ for these five examples of Calabi–Yau 3-folds V having the Hodge number $h^{1,1}(V) = 1$ were obtained by A. Klemm and S. Theisen [22].

7. Calabi–Yau 3-folds on $\mathbf{P}^2 \times \mathbf{P}^2$

7.1. The Generalized Hypergeometric Series Φ_0. Calabi–Yau 3-folds V in $\mathbf{P}^2 \times \mathbf{P}^2$ are hypersurfaces of degree $(3,3)$. The homology classes of rational curves on $\mathbf{P}^2 \times \mathbf{P}^2$ are parametrized by pairs of integers (l_1, l_2). Let γ_1, γ_2 be the homology classes of $(1,0)$-curves and $(0,1)$-curves respectively. Then for any Kähler class η

we put

$$z_i = \exp\left(- \int_{\gamma_i} \eta \right), \quad (i = 1, 2) .$$

The generalized hypergeometric series corresponding to the fan Σ defining $\mathbf{P}^2 \times \mathbf{P}^2$ is

$$\Phi_0(z_1, z_2) = \sum_{l_1, l_2 \geqq 0} \frac{(3l_1 + 3l_2)!}{(l_1!)^3 (l_2!)^3} z_1^{l_1} z_2^{l_2} .$$

There are obvious two recurrent relations for the coefficients a_{l_1,l_2} of the series

$$\Phi_0(z_1, z_2) = \sum_{l_1, l_2 \geqq 0} a_{l_1,l_2} z_1^{l_1} z_2^{l_2} \ :$$

$$(l_1 + 1)^3 a_{l_1+1,l_2} = (3l_1 + 3l_2 + 1)(3l_1 + 3l_2 + 2)(3l_1 + 3l_2 + 3) a_{l_1,l_2} \ ;$$
$$(l_2 + 1)^3 a_{l_1,l_2+1} = (3l_1 + 3l_2 + 1)(3l_1 + 3l_2 + 2)(3l_1 + 3l_2 + 3) a_{l_1,l_2} \ .$$

Let

$$\Theta_1 = z_1 \frac{\partial}{\partial z_1} , \quad \Theta_2 = z_2 \frac{\partial}{\partial z_2} .$$

Then the function $\Phi_0(z_1, z_2)$ satisfies the Picard–Fuchs differential system $\mathscr{D}$:

$$(\Theta_1^3 - z_1(3\Theta_1 + 3\Theta_2 + 1)(3\Theta_1 + 3\Theta_2 + 2)(3\Theta_1 + 3\Theta_2 + 3))\Phi_0 = 0 ,$$
$$(\Theta_1^3 - z_2(3\Theta_1 + 3\Theta_2 + 1)(3\Theta_1 + 3\Theta_2 + 2)(3\Theta_1 + 3\Theta_2 + 3))\Phi_0 = 0 .$$

The differential system $\mathscr{D}$ has the maximal unipotent monodromy at $(z_1, z_2) = (0, 0)$. There are two uniquely determined regular at (0,0) functions $\Psi_1(z_1, z_2)$ and $\Psi_2, (z_1, z_2)$ such that

$$(\log z_1)\Phi_0(z_1, z_2) + \Psi_1(z_1, z_2) ,$$
$$(\log z_2)\Phi_0(z_1, z_2) + \Psi_2(z_1, z_2)$$

are solutions to $\mathscr{D}$, and $\Psi_1(0, 0) = \Psi_2(0, 0) = 0$. If we put

$$\Psi_j(z_1, z_2) = \sum_{\substack{l_1, l_2 \geqq 0 \\ (l_1, l_2) \neq (0,0)}} b^{(j)}_{l_1,l_2} z_1^{l_1} z_2^{l_2} ,$$

then one finds the coefficients $b^{(j)}_{l_1,l_2}$ from the simple recurrent relations based on 4.3.2.

The q-coordinates q_1, q_2 defined by the formulas

$$q_1 = z_1 \exp(\Psi_1/\Phi_0,) \, , q_2 = z_2 \exp(\Psi_2/\Phi_0)$$

are the power series with integral coefficients in z_1, z_2 of the form

$$q_j(z_1, z_2) = z_j \left(1 + \sum_{\substack{l_1, l_2 \geqq 0 \\ (l_1, l_2) \neq (0,0)}} c^{(j)}_{l_1,l_2} z_1^{l_1} z_2^{l_2} \right) , \quad j = 1, 2 .$$

By symmetry, one has $c^{(1)}_{l_1,l_2} = c^{(2)}_{l_2,l_1}$.

7.2. Mirrors and the Discriminant. Let f be the Laurent polynomial

$$f(X,u) = 1 - u_1X_1 - u_2X_2 - u_3(X_1X_2)^{-1} - u_4X_3 - u_5X_4 - u_6(X_3X_4)^{-1} .$$

Let γ_0 be a generator of $H_4((\mathbf{C}^*)^4, \mathbf{Z})$, i.e., the cycle defined by the condition $|X_i| = 1$ for $i = 1, \ldots, 4$.

By the residue theorem, the integral

$$I(u) = \frac{1}{(2\pi\sqrt{-1})^4} \int_{\gamma_0} \frac{1}{f(X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}$$

is the power series

$$I(u) = \sum_{k,m \geqq 0} \frac{(3k+3m)!}{(k!)^3(m!)^3} (u_1u_2u_3)^k (u_4u_5u_6)^m .$$

Thus, putting $z_1 = u_1u_2u_3$; $z_2 = u_4u_5u_6$, we obtain exactly the generalized hypergeometric function $\Phi_0(z_1, z_2)$.

It was proved in [1] and [2] that the function $I(u)$ can be considered as the monodromy invariant period of the holomorphic differential 3-form

$$\omega = \frac{1}{(2\pi\sqrt{-1})^4} \mathrm{Res} \frac{1}{f(X)} \frac{dX_1}{X_2} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}$$

for the family of Calabi–Yau 3-folds $\hat{Z}_f$ which are smooth compactifications of the affine hypersurfaces Z_f in $(\mathbf{C}^*)^4$ defined by Laurent polynomial f. One has $h^{1,1}(\hat{Z}_f) = 83$, $h^{2,1}(\hat{Z}_f) = 2$. The coordinates z_1, z_2 are natural coordinates on the moduli space of Calabi–Yau 3-folds $\hat{Z}_f$.

The mirror construction helps to understand the discriminant of the differential system $\mathscr{D}$ as a polynomial function in z_1, z_2.

By definition [16], the zeros of the discriminant are exactly those values of the coefficients $\{u_i\}$ of $f(X)$ such that the system

$$f(X) = X_1 \frac{\partial}{\partial X_1} f(X) = X_2 \frac{\partial}{\partial X_2}(X) = X_3 \frac{\partial}{\partial X_3} f(X) = X_4 \frac{\partial}{\partial X_4} f(X) = 0$$

has a solution in the toric variety $\mathbf{P}_\Delta$, where Δ is the Newton polyhedron of f. Since $\mathbf{P}_\Delta$ is isomorphic to the subvariety of $\mathbf{P}^6$ defined as

$$\mathbf{P}_\Delta = \{(Y_0 : \ldots : Y_6) \in \mathbf{P}^6 | Y_0^3 = Y_1Y_2Y_3, Y_0^3 = Y_4Y_5Y_6\} ,$$

or equivalently, the system of the homogeneous equations

$$\begin{gathered} u_0Y_0 + \cdots + u_6Y_6 = u_1Y_1 - u_3Y_3 \\ = u_2Y_2 - u_3Y_3 = u_4Y_4 - u_6Y_6 = u_5Y_5 - u_6Y_6 = 0 ; \\ Y_0^3 = Y_1Y_2Y_3 = Y_4Y_5Y_6 \end{gathered}$$

has a non-zero solution.

If we put

$$A = u_3Y_3,\ B = u_6Y_6,\ C = u_0Y_0 ,$$

then the last system can be rewritten as

$$3A + 3B + C = A^3 + z_1 C^3 = B^3 + z_2 C^3 = 0 \,.$$

So the discriminant of the two-parameter family is the resultant of two binary homogeneous cubic equations in A and B:

$$27z_1(A+B)^3 - A^3 = 0, \; 27z_2(A+B)^3 - B^3 = 0 \,.$$

Put $27z_1 = x$, $27z_2 = y$.

Proposition 7.2.1. *The discriminant of the* 2*-parameter family of Calabi–Yau* 3*-folds* $\hat{Z}_f$ *is*

$$\mathrm{Disc}\, f = 1 - (x+y) + 3(x^2 - 7xy + y^2) - (x^3 + 3x^2y + 3xy^2 + y^3) \,.$$

7.3. The Diagonal One-Parameter Subfamily. We consider the diagonal one-parameter subfamily of Kähler structures η on V which are invariant under the natural involution of $H^{1,1}(V)$, i.e., we assume that

$$\int_{\gamma_1} = \int_{\gamma_2} \eta \,.$$

This is equivalent to the substitution $z = z_1 = z_2$.

Remark. 7.3.1. In this case we obtain the one-parameter family of mirrors

$$f_\psi(X) = X_1 + X_2 + (X_1X_2)^{-1} + X_3 + X_4 + (X_3X_4)^{-1} - 3\psi = 0, \psi^3 = (27z)^{-1} \,,$$

which is an analog to mirrors of quintic 3-folds [9].

It is easy to check that the discriminant of $f_\psi(X)$ vanishes exactly when $\psi = \alpha + \beta$, where $\alpha^3 = \beta^3 = 1$, i.e., $\psi^3 \in \{8, -1\}$, or $z \in \{-(3)^{-3}, (2 \cdot 3)^{-3}\}$.

The monodromy invariant period function is

$$F_0(z) = \Phi_0(z,z) = \sum_{n \geqq 0} \left(\sum_{k+m=n} \frac{(3n)!}{(k!)^3(m!)^3} \right) z^n \,.$$

It satisfies an ordinary Picard–Fuchs differential equation

$$\mathscr{D} : \left(\Theta^4 + \sum_{i=0}^{3} C_i(z)\Theta^i \right) F(z) = 0, \; \Theta = z\frac{\partial}{\partial z} \,.$$

We compute the Picard–Fuchs differential equation $\mathscr{E}$ for $F_0(z)$ from the recurrent formula for the coefficients

$$a_n = \sum_{k+m=n} \frac{(3n)!}{(k!)^3(m!)^3} = \frac{(3n)!}{(n!)^3} \left(\sum_{k=0}^{n} \binom{n}{k}^3 \right)$$

in the power expansion

$$F_0(z) = \sum_{n \geqq 0} a_n z^n \,.$$

Proposition 7.3.2 ([40]). *Let*

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^3 .$$

Then the numbers b_n satisfy the recurrent relation

$$(n+1)^2 b_{n+1} = (7n^2+7n+2)b_n + 8n^2 b_{n-1} .$$

Corollary 7.3.3. *The numbers a_n satisfy the recurrent relation*

$$\begin{aligned}(n+1)^4 a_{n+1} =& 3(7n^2+7n+2)(3n+2)(3n+1)a_n \\ &+ 72(3n+2)(3n+1)(3n-1)(3n-2)a_{n-1} .\end{aligned}$$

Corollary 7.3.4. *The monodromy invariant period function $F_0(y)$ is annihilated by the differential operator $\mathscr{P}$:*

$$\begin{gathered}\Theta^4 - 3z(7\Theta^2+7\Theta+2)(3\Theta+1)(3\Theta+2) \\ -72z^2(3\Theta+5)(3\Theta+4)(3\Theta+2)(3\Theta+1) .\end{gathered}$$

The last operator can be rewritten also as

$$\begin{gathered}(1-216z)(1+27z)\Theta^4 - 54z(7+432z)\Theta^3 - 3z(10584z+95)\Theta^2 \\ -48z(351z+2)\Theta - 12z - 2880z^2 .\end{gathered}$$

In particular, one has the coefficient

$$C_3(z) = \frac{-54z(7+432z)}{(1-216z)(1+27z)} .$$

The z-normalized Yukawa coupling $K_z^{(3)}$ is the solution to the differential equation

$$\frac{dK_z^{(3)}}{dz} = \frac{27(7+432z)}{(1-216z)(1+27z)} K_z^{(3)} .$$

Let H be the cohomology class in $H^2(V,\mathbf{Z})$ such that $\langle H,\gamma_1\rangle = \langle H,\gamma_2\rangle = 1$. Since $H^3 = 18$, we obtain the normalization condition

$$K_z^{(3)}(0) = 18 .$$

Applying the general algorithm in 4.3.3, we find the q-expansion of the z-coordinate

$$z(q) = q - 48q^2 - 18q^3 + 7976q^4 - 1697115q^5 + \mathscr{O}(q^6) ,$$

and the q-expansion of the q-normalized Yukawa coupling is

$$\begin{aligned}K_q^{(3)} =& 18 + 378q + 69498q^2 + 7724862q^3 + 1030043898q^4 \\ &+ 132082090128q^5 + \mathscr{O}(q^6) .\end{aligned}$$

We expect that

$$K_q^{(3)} = 18 + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1-q^d} ,$$

where n_d are predictions for numbers rational curves of degree d relative to the ample divisor of type (1,1) on V. In particular, one has $n_1 = 378$.

7.4. Lines on a Generic Calabi–Yau 3-Fold in $\mathbf{P}^2 \times \mathbf{P}^2$. We show how to check the prediction for the number of lines on a generic Calabi–Yau 3-fold in $\mathbf{P}^2 \times \mathbf{P}^2$.

First we formulate one lemma which will be useful in the sequel.

Lemma 7.4.1. *Let M be a complete algebraic variety, $\mathscr{L}_1$ and $\mathscr{L}_2$ two invertible sheaves on M such that the projectivizations $\mathbf{P}(\mathscr{L}_i) = \mathbf{P}(H^0(M, \mathscr{L}_i))\,(i = 1,2)$ are nonempty. Define the morphism*

$$p_\lambda : \mathbf{P}(\mathscr{L}_1) \times \mathbf{P}(\mathscr{L}_2) \to \mathbf{P}(\mathscr{L}_1 \otimes \mathscr{L}_2) = \mathbf{P}(H^0(M, \mathscr{L}_1 \otimes \mathscr{L}_2))$$

by the natural mapping

$$\lambda : H^0(M, \mathscr{L}_1) \otimes H^0(M, \mathscr{L}_2) \to H^0(X, \mathscr{L}_1 \otimes \mathscr{L}_2)\,.$$

Then the pullback $p_\lambda^* \mathcal{O}(1)$ of the ample generator $\mathcal{O}(1)$ of the Picard group of $\mathbf{P}(\mathscr{L}_1 \otimes \mathscr{L}_2)$ is isomorphic to $\mathcal{O}(1,1)$ on $\mathbf{P}(\mathscr{L}_1) \times \mathbf{P}(\mathscr{L}_2)$.

Proof. The statement follows immediately from the fact that λ is bilinear. □

Proposition 7.4.2. *A generic Calabi–Yau hypersurface in $\mathbf{P}^2 \times \mathbf{P}^2$ contains* 378 *lines relative to the $\mathcal{O}(1,1)$-polarization.*

Proof. There are two possibilities for the type of lines: (1,0) and (0,1). By symmetry, it is sufficient to consider only (1,0)-lines whose projections on the second factor in $\mathbf{P}^2 \times \mathbf{P}^2$ are points. Let

$$\pi_2 : V \to \mathbf{P}^2$$

be the projection of V on the second factor. Then for every point $p \in \mathbf{P}^2$ the fiber $\pi_2^{-1}(p)$ is a cubic in $\mathbf{P}^2 \times p$. We want to calculate the number of those fibers $\pi_2^{-1}(p)$ which are unions of a line L and a conic Q in $\mathbf{P}^2 \times p$. The space of the reducible cubics $L \cup Q$ is isomorphic to the image $\Lambda \subset \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3))$ of the morphism

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1)) \times \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2))) = \mathbf{P}^2 \times \mathbf{P}^5 \to \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3))\,.$$

By 7.4.1, Λ has codimension 2 and degree 21.

On the other hand, a generic Calabi–Yau hypersurface V defines a generic Veronese embedding

$$\phi : \mathbf{P}^2 \hookrightarrow \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3)), \quad \phi(p) = \pi_2^{-1}(p)\,.$$

The degree of the image $\phi(\mathbf{P}^2)$ is 9. The number of (1,0)-lines is the intersection number of two subvarieties $\phi(\mathbf{P}^2)$ and Λ in $\mathbf{P}^9$, i.e., $9 \times 21 = 189$. Thus, the total amount of lines is $2 \times 189 = 378$ □.

8. Further Examples

In this section we consider more examples of Calabi–Yau 3-folds V obtained as complete intersections in the product of projective spaces. In all these examples for simplicity we restrict ourselves to one-parameter subfamilies invariant under permutations of factors. The latter allows to apply the Picard–Fuchs operators of order 4 to the calculation of predictions for numbers of rational curves on Calabi–Yau 3-folds with $h^{1,1} > 1$.

8.1. Calabi–Yau 3-folds in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. We consider the diagonal subfamily of Kähler classes on Calabi–Yau hypersurfaces of degree (1,1,1,1) in $(\mathbf{P}^1)^4$. Repeating the same procedure as for hypersurfaces of degree (3,3) in $\mathbf{P}^2 \times \mathbf{P}^2$, we obtain:

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k_1+k_2+k_3+k_4=n} \frac{(2k_1+2k_2+2k_3+2k_4)!}{(k_1!)^2(k_2!)^2(k_3!)^2(k_4!)^2} \right) z^n$
$\mathscr{P}$	$\Theta^4 - 4z(5\Theta^2+5\Theta+2)(2\Theta+1)^2 + 64z^2(2\Theta+3)(2\Theta+1)(2\Theta+2)^2$
$K_z^{(3)}$	$\frac{48}{(1-64z)(1-16z)}$
$K_q^{(3)}$	$48 + 192q + 7872q^2 + 278400q^3 + 9445056q^4 + 315072192q^5 + \mathscr{O}(q^6)$
n_i	$n_1 = 192,\ n_2 = 960,\ n_3 = 10304,\ n_4 = 147456,\ n_5 = 2520576$

Proposition 8.1.1. *The number of lines on a generic Calabi–Yau hypersurface in* $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ *relative to the* $(1,1,1,1)$*-polarization is equal to* 192.

Proof. Let f be the polynomial of degree $(2,2,2,2)$ defining a Calabi–Yau hypersurface V in $(\mathbf{P}^1)^4$. If V contains a $(0,0,0,1)$-curve whose projection on the product of the first three $\mathbf{P}^1$ is a point (p_1, p_2, p_3), then all three coefficients of the binary quadric obtained from f by substitution of (p_1, p_2, p_3) must vanish. Hence, the number of $(0,0,0,1)$ curves on V equals the intersection number of 3 hypersurfaces of degree $(2,2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. This number is 48. By symmetry, the total amount of lines on V is $4 \times 48 = 192$. □.

Proposition 8.1.2. *The number of conics on a generic Calabi–Yau hypersurface in* $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ *with respect to the* $(1,1,1,1)$*-polarization is equal to* 960.

Proof. By symmetry, it is sufficient to compute the number of rational curves of type $(0,0,1,1)$. Let M be the product of first two $\mathbf{P}^1$ in $(\mathbf{P}^1)^4$. Then we obtain the natural embedding

$$\phi : M \hookrightarrow \mathbf{P}^8 = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2)) .$$

On the other hand, the points on M corresponding to projections of $(0,0,1,1)$-curves on V are intersections of $\phi(M)$ with the 6-dimensional subvariety $\Lambda \subset \mathbf{P}^8$ which is the image of the morphism

$$\phi' : \mathbf{P}(\mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1))^2 = \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbf{P}^8 = \mathbf{P}(\mathscr{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2)) .$$

The image $\phi(M)$ has degree 8. On the other hand, ϕ has degree two onto its image. Hence, the subvariety Λ has degree 10. Hence, we obtain $8 \times 20 = 160$ points on

M. There are 6 possibilities for the choice of the type of conics. Thus, the total amount of conics is $6 \times 160 = 960$.

8.2. Complete Intersections of Three Hypersurfaces in $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$. We consider two examples of 3-dimensional complete intersections with trivial canonical class in $(\mathbf{P}^2)^3$.

Calabi–Yau complete intersections of 3 hypersurfaces of degree (1,1,1):

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m+l=n} \frac{((k+m+l)!)^3}{(k!)^3(m!)^3(l!)^3} \right) z^n$
$\mathcal{P}$	$25\Theta^4 - 15z(5+30\Theta+72\Theta^2+84\Theta^3+51\Theta^4)$ $+6z^2(15+155\Theta+541\Theta^2+828\Theta^3+531\Theta^4)$ $-54z^3(1170+3795\Theta+4399\Theta^2+2160\Theta^3+423\Theta^4)$ $+243z^4(402+1586\Theta+2270\Theta^2+1368\Theta^3$ $+279\Theta^4) - 59049z^5(\Theta+1)^4$
$K_z^{(3)}$	$\dfrac{90+162z}{(27z-1)(27z^2+1)}$
$K_q^{(3)}$	$90+108q+2916q^2+57456q^3+834084q^4+137431108q^5+\mathcal{O}(q^6)$
n_i	$n_1 = 108,\ n_2 = 351,\ n_3 = 2124,\ n_4 = 12987,\ n_5 = 109944$

Proposition 8.2.1. *A generic complete intersection of* 3 *hypersurfaces of degree* $(1,1,1)$ *in* $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$ *contains* 108 *lines relative to the* $\mathcal{O}(1,1,1)$*-polarization.*

Proof. Let V be the complete intersection of three generic hypersurfaces V_1, V_2, V_3 in $M_1 \times M_2 \times M_3$, where $M_i \cong \mathbf{P}^2$.

By symmetry, it is sufficient to consider lines having the class $(0,0,1)$ whose projections on $M_1 \times M_2$ are points. There is the morphism

$$\phi : M_1 \times M_2 \to \mathbf{P}^8 = \mathbf{P}(E),$$

where E is the space of all 3×3-matrices. By definition, ϕ maps a point $(p_1, p_2) \in M_1 \times M_2$ to the matrix of coefficients of three linear forms obtained from the substitution of p_1 and p_2 in the equations of V_1, V_2, and V_3. The morphism ϕ is the Segre embedding and its image has degree 6. On the other hand, if a point $(p_1, p_2) \in M_1 \times M_2$ is a projection of a $(0,0,1)$-curve on V, then the image $\phi(p_1, p_2)$ must correspond to a matrix of rank 1 in E. Thus, the number of $(0,0,1)$-curves equals $6 \times 6 = 36$, the intersection number of two Segre subvarieties in $\mathbf{P}^8$. So the number of lines on V is $3 \times 36 = 108$. □

Abelian 3-folds: The complete intersection of three hypersurfaces of degrees $(3,0,0)$, $(0,3,0)$, $(0,0,3)$ are abelian 3-folds constructed by taking products of 3 elliptic cubic curves in $\mathbf{P}^2$. Although abelian varieties are not Calabi–Yau manifolds from the view-point of algebraic geometers, these manifolds also present interest for physicists.

$F_0(z)$	$\sum_{p+q+r=n} \left(\frac{((3p)!)^3((3q)!)^3((3r)!)^3}{(p!)^3(q!)^3(r!)^3} \right) z^n$
$\mathcal{P}$	$\Theta^4 - 3z(6 + 29\Theta + 56\Theta^2 + 54\Theta^3 + 27\Theta^4)$ $+81z^2(27\Theta^2 + 54\Theta + 40)(\Theta + 1)^2$ $-2187z^3(3\Theta + 5)(3\Theta + 4)(\Theta + 2)(\Theta + 1)$
$K_q^{(3)}$	162

Thus, we obtain that all Gromov–Witten invariants for the abelian 3-folds are zero which agrees with the fact that there are no rational curves on abelian varieties.

8.3. Calabi–Yau 3-Folds in $\mathbf{P}^3 \times \mathbf{P}^3$.

Complete intersections of a hypersurface of degree (2,2) **and** 2 **copies of hypersurfaces of degree** (1,1):

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{(2k+2m)!((k+m)!)^2}{(k!)^4(m!)^4} \right) z^n$
$\mathcal{P}$	$\Theta^4 - 4z(3\Theta^2 + \Theta + 1)(2\Theta + 1)^2 - 4z^2(4\Theta + 5)$ $(4\Theta + 6)(4\Theta + 2)(4\Theta + 3)$
$K_z^{(3)}$	$\frac{40}{(1+16z)(1-64z)}$
$K_q^{(3)}$	$40 + 160q + 12640q^2 + 393280q^3$ $+17420640q^4 + 662416160q^5 + O(q^6)$
n_i	$n_1 = 160,\ n_2 = 1560,\ n_3 = 14560,\ n_4 = 272000,\ n_5 = 5299328$

Proposition 8.3.1. *The number of lines on a generic complete intersection of a hypersurface of degree* (2,2) *and* 2 *copies of hypersurfaces of degree* (1,1) *is equal to* 160.

Proof. Let $W = \mathrm{Gr}(2,4) \times \mathbf{P}^3$ be the 7-dimensional variety parametrizing all (1,0)-lines on $\mathbf{P}^3 \times \mathbf{P}^3$. Let $\mathcal{E}$ be the tautological rank-2 locally free sheaf on Gr(2,4). We put $c_1(\mathcal{E}) = c_1$, $c_2(\mathcal{E}) = c_2$, and h be the first Chern class of the ample generator H of $\mathrm{Pic}(\mathbf{P}^3)$. Let $S^2(\mathcal{E})$ be the 2^{nd} symmetric power of $\mathcal{E}$. By standard arguments, we obtain:

Lemma 8.3.2. *The Chern classes* c_1, c_2 *generate the cohomology ring of* Gr(2,4). *The elements* $1, c_1, c_2, c_1^2, c_1c_2, c_1^2c_2$ *form a* $\mathbf{Z}$*-basis of* $H^*(\mathrm{Gr}(2,4), \mathbf{Z})$, *and one has the following*:

$$c_1^4 = c_2^2 = c_1^2,\ c_1^3 = 2c_1c_2\ ,$$

$$c_1(S^2(\mathcal{E})) = 3c_1,\ c_2(S^2(\mathcal{E})) = 2c_1^2 + 4c_2,\ c_3(S^2(\mathcal{E})) = 4c_1c_2\ .$$

Moreover, for any invertible sheaf $\mathscr{L}$ *on* $\mathbf{P}^3$, *one has*

$$c_1(S^2(\mathscr{E})\otimes\mathscr{L}) = 3c_1 + 3c_1(\mathscr{L}),$$
$$c_2(S^2(\mathscr{E})\otimes\mathscr{L}) = 2c_1^2 + 4c_2^2 + 2c_1(\mathscr{L})(3c_1) + 3c_1^2(\mathscr{L}),$$
$$c_3(S^2(\mathscr{E})\otimes\mathscr{L}) = 4c_1c_2 + c_1(\mathscr{L})(2c_1^2 + 4c_2) + c_1^2(\mathscr{L})(3c_1) + c_1^3(\mathscr{L}).$$

Then the number of (1,0)-lines equals the following product in the cohomology ring of W:

$$\begin{aligned} &c_2(\mathscr{E}\otimes\mathscr{O}(H))\cdot c_2(\mathscr{E}\otimes\mathscr{O}(H))\cdot c_3(S^2(\mathscr{E})\otimes\mathscr{O}(2H)) \\ &\quad = (h^2 + c_1h + c_2)^2(8h^3 + 3c_1\cdot 4h^2 + 2(c_1^2 + 2c_2)\cdot 2h + 4c_1c_2) \\ &\quad = (8c_1^2c_2h^3 + 4(c_1^2 + 2c_2)^2h^3 + 24c_1^2c_2h^3 + 8c_2^2h^3) \\ &\quad = (8 + 4\times 10 + 24 + 8)c_1^2c_2h^3 = 80c_1^2c_2h^3\,. \end{aligned}$$

Thus, the number of (1,0)-lines is 80. By symmetry, the total amount of lines is 160. □

Complete intersections of hypersurfaces of degrees (1,1), (1,2) **and** (2,1):

$F_0(z)$	$\sum_{n=0}^{\infty}\left(\sum_{k+m=n}\frac{(2k+m)!(k+2m)!((k+m)!)}{(k!)^4(m!)^4}\right)z^n$
$\mathscr{P}$	$529\Theta^4 - 23z(92 + 621\Theta + 1644\Theta^2 + 2046\Theta^3 + 921\Theta^4)$ $-z^2(221168 + 1033528\Theta + 1772673\Theta^2 + 1328584\Theta^3 + 380851\Theta^4)$ $-2z^3(-27232 + 208932\Theta + 1028791\Theta^2 + 1310172\Theta^3 + 475861\Theta^4)$ $-68z^4(-976 - 1664\Theta + 5139\Theta^2 + 14020\Theta^3 + 8873\Theta^4)$ $+6936z^5(3\Theta + 4)(3\Theta + 2)(\Theta + 1)^2$
$K_z^{(3)}$	$\frac{46 + 68z}{(54z - 1)(z^2 - 11z - 1)}$
$K_q^{(3)}$	$46 + 160q + 9416q^2 + 251530q^3 + 9120968q^4 + 289172660q^5 + O(q^6)$
n_i	$n_1 = 160,\ n_2 = 1157,\ n_3 = 9310,\ n_4 = 142368,\ n_5 = 2313380$

Proposition 8.3.3. *The number of lines on a generic complete intersection of hypersurfaces of degrees* (2,1), (1,2), *and* (1,1) *is equal to* 160.

Proof. We use the same notations as in 8.3.1. The number of (1,0)-lines equals the following product in the cohomology ring of W:

$$\begin{aligned} &c_2(\mathscr{E}\otimes\mathscr{O}(H))\cdot c_2(\mathscr{E}\otimes\mathscr{O}(2H))\cdot c_3(S^2(\mathscr{E})\otimes\mathscr{O}(H)) \\ &\quad = (h^2 + c_1h + c_2)\cdot(4h^2 + 2c_1h + c_2)\cdot(h^3 + 3c_1h^2 + 2(c_1^2 + 2c_2)h + 4c_1c_2) \\ &\quad = (24c_1^2c_2 + 2(5c_2 + 2c_1^2)(2c_2 + c_1^2) + 9c_1^2c_2 + c_2^2)h^3 \\ &\quad = (24 + 2(5 + 10 + 4 + 4) + 9 + 1)c_1^2c_2h^3 = 80c_1^2c_2h^3\,. \end{aligned}$$

Thus, the number of (1,0)-lines is 80. By symmetry, the total amount of lines is 160. □

Hypersurfaces in product of two Del Pezzo surfaces of degree 3:
A Calabi–Yau hypersurface in product of two Del Pezzo surfaces of degree 3 is a complete intersections of (1,1), (3,0) and (0,3)-hypersurfaces in $\mathbf{P}^3 \times \mathbf{P}^3$.

$F_0(z)$	$\sum_{n=0}^{\infty}\left(\sum_{k+m=n}\frac{(3k)!(3m)!(k+m)!}{(k!)^4(m!)^4}\right)z^n$
$\mathcal{P}$	$\Theta^4 - 3z(4+23\Theta+53\Theta^2+60\Theta^3+48\Theta^4)$ $+9z^2(304+1344\Theta+2319\Theta^2+1980\Theta^3+873\Theta^4)$ $-162z^3(800+3348\Theta+5259\Theta^2+3888\Theta^3+1269\Theta^4)$ $+2916z^4(688+2952\Theta+4653\Theta^2+3240\Theta^3+891\Theta^4)$ $-1417176z^5(3\Theta+4)(3\Theta+2)(\Theta+1)^2$
$K_z^{(3)}$	$\frac{54-972z}{(1-54z)(1-27z)^2}$
$K_q^{(3)}$	$54+162q+7290q^2+119232q^3+3045114q^4+79845912q^5+O(q^6)$
n_i	$n_1=162,\ n_2=891,\ n_3=4410,\ n_4=47466,\ n_5=638766$

Proposition 8.3.4. *Let S_1, S_2 be two Del Pezzo surfaces of degree* 3. *Then the number of lines on a generic Calabi–Yau hypersurface V in $S_1 \times S_2$ is* 162.

Proof. If C is a line of type (1,0) on $S_1 \times S_2$, then $\pi_1(C)$ is one of 27 lines on S_1, and $\pi_2(C)$ is a point on S_2. Let $\mathcal{O}_{S_i}(-K)$ denotes the anticanonical bundle over S_i. Then the zero set of a generic global section s of $\pi_1^*\mathcal{O}_{S_1}(-K)\otimes\pi_2^*\mathcal{O}_{S_2}(-K)$ defines a morphism

$$\phi : S_2 \to \mathbf{P}^3 = \mathbf{P}(\mathcal{O}_{S_1}(-K)) .$$

On the other hand, for any line $L \in S_1$, one has the linear embedding

$$\phi' : \mathbf{P}(\mathcal{O}_{S_1}(-K-L)) \cong \mathbf{P}^1 \hookrightarrow \mathbf{P}^3 = \mathbf{P}(\mathcal{O}_{S_1}(-K)) .$$

The intersection number of $\operatorname{Im}\phi$ and $\operatorname{Im}\phi'$ in $\mathbf{P}^3$ equals 3, i.e., one has exactly 3 lines C on a generic V such that $\pi_1(C) = L$ and $\pi_2(C)$ is a point on S_2. Thus, there are $3 \times 27 = 81$ lines of type (1,0) on V. By symmetry, the total amount of lines is 162. □

Proposition 8.3.5. *Let S_1, S_2 be two Del Pezzo surfaces of degree* 3. *Then the number of conics on a generic Calabi–Yau hypersurface V in $S_1 \times S_2$ is* 891.

Proof. If C is a conic of type (1,1) on $S_1 \times S_2$, then $L_1 = \pi_1(C)$ is one of 27 lines on S_1, and $L_2 = \pi_2(C)$ is one of 27 lines on S_2. On the other hand, for any pair of lines $L_1 \in S_1, L_2 \in S_2$, the intersection of the product $L_1 \times L_2 \subset S_1 \times S_2$ with V is a conic of type (1,1). So we obtain $27 \times 27 = 729$ conics of type (1,1) on V. On the other hand, the number of $(2,0)$ and $(0,2)$ conics obviously equals the number of $(1,0)$ and $(0,1)$ lines. Thus, the total number of conics is equal to $729+162 = 891$. □

8.4. Calabi–Yau 3-Folds in $\mathbf{P}^4 \times \mathbf{P}^4$.

Complete intersection of hypersurfaces of degrees $(2,0)$, $(0,2)$ **, and** 3 **copies of hypersurfaces of degree** $(1,1)$:

$F_0(z)$	$\sum_{n=0}^{\infty}\left(\sum_{k+m=n}\frac{((k+m)!)^3(2k)!(2m)!}{(k!)^5(m!)^5}\right)z^n$
$\mathcal{P}$	$25\Theta^4-20z(5+30\Theta+72\Theta^2+84\Theta^3+36\Theta^4)$ $-16z^2(-35-70\Theta+71\Theta^2+268\Theta^3+181\Theta^4)$ $+256z^3(\Theta+1)(165+375\Theta+248\Theta^2+37\Theta^3)$ $+1024z^4(59+232\Theta+331\Theta^2+198\Theta^3+39\Theta^4)+32768z^5(\Theta+1)^4$
$K_z^{(3)}$	$\frac{80+128z}{(1+4z)(1-4z)(1-32z)}$
$K_q^{(3)}$	$80+128q+3776q^2+65792q^3+1299136q^4+23104128q^5+O(q^6)$
n_i	$n_1=128,\ n_2=456,\ n_3=2432,\ n_4=20240,\ n_5=184832$

Proposition 8.4.1. *The number of lines on a generic complete intersection of hypersurfaces of degrees* $(2,0)$, $(0,2)$, *and* 3 *copies of hypersurfaces of degree* $(1,1)$ *is equal to* 128.

Proof. Let $W = \mathrm{Gr}(2,5)\times\mathbf{P}^4$ be the 10-dimensional variety parametrizing all $(1,0)$-lines on $\mathbf{P}^4\times\mathbf{P}^4$. Let $\mathcal{E}$ be the tautological rank-2 locally free sheaf on Gr(2,5). We put $c_1(\mathcal{E})=c_1$, $c_2(\mathcal{E})=c_2$, and h be the first Chern class of the ample generator H of $\mathrm{Pic}(\mathbf{P}^4)$. Let $S^2(\mathcal{E})$ be the 2nd symmetric power of $\mathcal{E}$. Again, by standard arguments, we obtain:

Lemma 8.4.2. *The Chern classes* c_1, c_2 *generate the cohomology ring of* Gr(2,5). *The elements* 1, c_1, c_2, c_1^2, c_1c_2, c_1^3, $c_1^2c_2$, c_1^4, $c_1c_2^2+c_2^3$ *form a* $\mathbf{Z}$*-basis of* $H^*(\mathrm{Gr}(2,5),\boldsymbol{Z})$, *and satisfy the following relations:*

$$c_1^4c_2 = 2c_1^2c_2^2 = 2c_2^3,\ c_1^5 = 5c_1c_2^2,\ c_1^6 = 5c_1^2c_2^2 = 5c_2^3,\ c_1^3c_2 = 2c_1c_2^2\,.$$

Then the number of $(1,0)$-lines equals the following product in the cohomology ring of W:

$$c_1(\mathcal{O}(H))\cdot(c_2(S^2(\mathcal{E})))^3\cdot c_3(S^2(\mathcal{E}))$$

$$=(2h)\cdot(h^2+c_1h+c_2)^3\cdot(4c_1c_2)=64c_1^2c_2^2h^4\,.$$

Thus, the number of $(1,0)$-lines is 64. By symmetry, the total amount of lines is 128. □

Complete intersection of 5 hypersurfaces of degree $(1,1)$:

$F_0(z)$	$\sum_{n=0}^{\infty}\left(\sum_{k+m=n}\frac{((k+m)!)^5}{(k!)^5(m!)^5}\right)z^n$
$\mathscr{P}$	$49\Theta^4-7z(14+91\Theta+234\Theta^2+286\Theta^3+155\Theta^4)$ $-z^2(15736+66094\Theta+102261\Theta^2+68004\Theta^3+16105\Theta^4)$ $+8z^3(476+3759\Theta+9071\Theta^2+8589\Theta^3+2625\Theta^4)$ $-16z^4(184+806\Theta+1439\Theta^2+1266\Theta^3+465\Theta^4)+512z^5(\Theta+1)^4$
$K_z^{(3)}$	$\dfrac{70-40z}{(32z-1)(z^2-11z-1)}$
$K_q^{(3)}$	$K_q(q)=70+100q+5300q^2+79750q^3+1966900q^4$ $+37143850q^5+O(q^6)$
n_l	$n_1=100,\ n_2=650,\ n_3=2950,\ n_4=30650,\ n_5=297150$

Proposition 8.4.3. *A generic complete intersection of generic* 5 *hypersurfaces of degree* $(1,1)$ *in* $\mathbf{P}^4\times\mathbf{P}^4$ *contains* 100 *lines.*

Proof. We give below two different proofs of the statement.

I: We keep the notation from the proof of 8.4.1. Then the number of $(1,0)$-lines equals the following product in the cohomology ring of W:

$$\begin{aligned}(c_2(S^2(\mathscr{E})))^5 &= (h^2+c_1h+c_2)^5\\ &= (c_1h+c_2)65+5(c_1h+c_2)64h^2+10(c_1h+c_2)^3h^4\\ &= 5c_1^4c_2h^4+5\binom{4}{2}c_1^2c_2^2h^4r+10c_2^3h^4\\ &= (5\times 2+5\times 6+10)c_1^4c_2h^4=50c_1^2c_2^2h^4\ .\end{aligned}$$

Thus, the number of $(1,0)$-lines is 50. By symmetry, the total amount of lines is 100.

II: Let $M=M_1\times M_2$, where $M_i\cong\mathbf{P}^4 (i=1,2)$. By symmetry, we consider only lines of type $(0,1)$ whose projections on M_1 are points. The substitution of a point $p\in M_1$ in the equations of the hypersurfaces $H_1,\dots,H_5\subset M$ gives 5 linear forms $f_1,\dots,f_5$ in homogeneous coordinates on M_2. A point $p\in M_1$ is a projection of a $(0,1)$-line on $H_1\cap\ldots\cap H_5$ if the system of linear forms has rank 3. The space of 5 copies of linear forms can be identified with the space L of matrices 5×5. We are interested in the determinantal subvariety D in $\mathbf{P}^{24}$ consisting of matrices of rank $\leqq 3$. The subvariety D has the codimension 4, the ideal of D is generated by all 4×4 minors. Using the free graded resolution of the homogeneous coordinate ring of D as a module over the homogeneous coordinate ring of $\mathbf{P}^{24}$, we can compute the degree of D which is equal to 50 (The Hilbert-Poincaré series equals $(1+4t+10t^2+20t^3+10t^4+4t^5+t^6)/(1-t)^{21})$. On the other hand, the equations of generic hypersurfaces $H_1,\dots,H_5$ define a generic embedding

$$\mathbf{P}^4 \hookrightarrow \mathbf{P}^4$$

of $\mathbf{P}^4$ as a linear subspace. So the number of lines of type $(0,1)$ on a generic complete intersection is 50. Thus, the total number of lines is 100. □

Hypersurfaces in product of two Del Pezzo surfaces of degree 4:

A Calabi–Yau hypersurface in the product of two Del Pezzo surfaces of degree 4 is a complete intersection of 5 hypersurfaces in $\mathbf{P}^4 \times \mathbf{P}^4$: two copies of type $(2,0)$, two copies of type $(0,2)$, and one copy of type $(1,1)$.

$F_0(z)$	$\sum_{n=0}^{\infty}\left(\sum_{k+m=n}\frac{((2k)!)^2((2m)!)^2(k+m)!}{(k!)^5(m!)^5}\right)z^n$
$\mathcal{P}$	$9\Theta^4-12z(6+33\Theta+73\Theta^2+80\Theta^3+64\Theta^4)$ $+128z^2(75+315\Theta+527\Theta^2+440\Theta^3+194\Theta^4)$ $-4096z^3(66+261\Theta+397\Theta^2+288\Theta^3+94\Theta^4)$ $+131072z^4(19+77\Theta+117\Theta^2+80\Theta^3+22\Theta^4)-8388608z^5(\Theta+1)^4$
$K_z^{(3)}$	$\dfrac{96-1024z}{(1-32z)(1-16z)^2}$
$K_q^{(3)}$	$96+128q+3456q^2+38144q^3+572800q^4+9344128q^5+O(q^6)$
n_i	$n_1=128,\ n_2=416,\ n_3=1408,\ n_4=8896,\ n_5=74752$

Proposition 8.4.4. *Let S_1, S_2 be two Del Pezzo surfaces of degree* 4. *Then the number of lines on a generic Calabi–Yau hypersurface V in $S_1 \times S_2$ is* 128.

Proof. If C is a line of type $(1,0)$ on $S_1 \times S_2$, then $\pi_1(C)$ is one of 16 lines on S_1, and $\pi_2(C)$ is a point on S_2. Let $\mathcal{O}_{S_i}(-K)$ denotes the anticanonical bundle over S_i. Then the zero set of a generic global section s of $\pi_1^*\mathcal{O}_{S_1}(-K)\otimes\pi_2^*\mathcal{O}_{S_2}(-K)$ defines a morphism

$$\phi : S_2 \to \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

On the other hand, for any line $L \in S_1$, one has the linear embedding

$$\phi' : \mathbf{P}(\mathcal{O}_{S_1}(-K-L)) \cong \mathbf{P}^2 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

The intersection number of $\operatorname{Im}\phi$ and $\operatorname{Im}\phi'$ in $\mathbf{P}^3$ equals 4, i.e., one has exactly 4 lines C on a generic V such that $\pi_1(C)=L$ and $\pi_2(C)$ is a point on S_2. Thus, there are $4\times 16=64$ lines of type $(1,0)$ on V. By symmetry, the total amount of lines is 128. □

Proposition 8.4.5. *Let S_1, S_2 be two Del Pezzo surfaces of degree* 3. *Then the number of conics on a generic Calabi–Yau hypersurface V in $S_1 \times S_2$ is* 416.

Proof. If C is a conic of type $(1,1)$ on $S_1 \times S_2$, then $L_1=\pi_1(C)$ is one of 16 lines on S_1, and $L_2=\pi_2(C)$ is one of 16 lines on S_2. On the other hand, for any pair of lines $L_1 \in S_1$,, $L_2 \in S_2$, the intersection of the product $L_1 \times L_2 \subset S_1 \times S_2$ with V is a conic of type $(1,1)$. So we obtain $16 \times 16 = 256$ conics of type $(1,1)$ on V.

In order to compute the number of $(2,0)$-conics, we notice that S_1 has exactly 10 conic bundle structures. Moreover, these conic bundle structures can be divided into 5 pairs such that the union of degenerate fibers of each pair is the set of all 16 lines on S_1. A generic global section s of $\pi_1^* \mathscr{O}_{S_1}(-K) \otimes \pi_2^* \mathscr{O}_{S_2}(-K)$ defines the anticanonical embedding

$$\phi : S_2 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathscr{O}_{S_1}(-K)) .$$

On the other hand, the points $p \in S_2$ such that $\phi(p)$ splits into the union of two conics $C_1 \cup C_2$ are exactly intersection points of $\phi(S_2)$ and the image of the embedding

$$\phi' : \mathbf{P}(\mathscr{O}_{S_1}(C_1) \times \mathscr{O}_{S_1}(C_2) \cong \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathscr{O}_{S_1}(-K)) .$$

Since the image of ϕ' has degree 2, we obtain 8 points $p \in S_2$. Each such a point yields 2 conics on $\pi_2^{-1}(p)$. Therefore, for each of 5 pairs of conic bundle structures we have 16 $(2,0)$-conics.

Thus, the total number of conics is equal to $256 + 2 \times 80 = 416$. □

Acknowledgements. This work benefited greatly from conversations with F. Beukers, P. Candelas, B. Greene, D. Morrison, Yu.I. Manin, R. Plesser, R. Schimmrigk, Yu. Tschinkel and A. Todorov. We want to thank J. Stienstra who independently checked our calculations and pointed out some misprints in coefficients.

We are very grateful to D. Morrison whose numerous remarks concerning a preliminary version of the paper helped us to give precise references on his work, especially on the forthcoming papers [18, 35].

We would like to express our thanks for hospitality to the Mathematical Sciences Research Institute where this work was supported in part by the National Science Foundation (DMS-9022140), and the DFG (Forschungsschwerpunkt Komplexe Mannigfaltigkeiten).

References

1. Batyrev, V.V.: Dual Polyhedra and Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties. Nov. 18 (1992) Preprint, Uni-Essen
2. Batyrev, V.V.: Variations of the Mixed Hodge Structure of Affine Hypersurfaces in Algebraic Tori. Duke Math. J. **69**, 349–409 (1993)
3. Batyrev, V.V.: Quantum Cohomology Rings of Toric Manifolds. Preprint MSRI, (1993)
4. Borisov, L.A.: Towards the Mirror Symmetry for Calabi–Yau Complete Intersection in Gorenstein Toric Fano Varieties. Preprint 1993, alg-geom 9310001
5. Candelas, P., Dale, A.M., Lütken, C.A., Schimmrigk, R.: Complete Intersection Calabi–Yau Manifolds I, II. Nucl. Phys. **B 298**, 493–525 (1988), Nucl. Phys. **B 306**, 113–136 (1988)
6. Candelas, P., Lynker, M., Schimmrigk, R.: Calabi–Yau Manifolds in Weighted $\mathbf{P}^4$. Nucl. Phys. **B 341**, 383–402 (1990)
7. Candelas, P., He, A.: On the Number of Complete Intersection of Calabi–Yau Manifolds. Commun. Math. Phys. **135**, 193–199 (1990)
8. Candelas, P., de la Ossa, X.: Nucl. Phys. **342**, 246 (1990)
9. Candelas, P., de la Ossa, X.C., Green, P.S., Parkes, L.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory. Nucl. Phys. **B 359**, 21–74 (1991)
10. Ceresole, A., D'Auria, R., Ferrara, S., Lerche, W., Louis, J.: Picard-Fuchs Equations and Special Geometry. CERN-TH.6441/92, UCLA/92/TEP/8, CALT-8-1776, POLFIS-TH.8/92
11. Deligne, P.: Letter to D. Morrison, 6 November 1991

12. Dixon, L.: In Superstrings. Unified Theories and Cosmology 1987, G. Furlan et al., eds., Singapore: World Scientific, 1988
13. Ferrara, S., Louis, J.: Phys. Lett. **B 273**, 246 (1990)
14. Font, A.: Periods and duality symmetries in Calabi–Yau compactifications. Nucl. Phys. **B 389**, 153 (1993)
15. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Hypergeometric functions and toric manifolds. Funct. Anal. Appl. **28:2**, 94–106 (1989)
16. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants of polynomials in several variables and triangulations of Newton polytopes. Algebra i analiz (Leningrad Math. J.) **2**, 1–62 (1990)
17. Greene, B.R., Plesser, M.R.: Duality in Calabi–Yau moduli space. Nucl. Phys. **B 338**, 15–37 (1990)
18. Greene, B.R., Morrison, D.R., Plesser, M.R.: Mirror manifolds in higher dimension. In preparation
19. Katz, S.: Rational curves on Calabi–Yau 3-folds. In: Essays on Mirror Manifolds (Ed. S.-T. Yau), Hong Kong: Int. Press Co., 1992 pp. 168–180
20. Katz, S.: Rational curves on Calabi–Yau manifolds: Verifying of predictions of Mirror Symmetry. Preprint, alg-geom/9301006 (1993)
21. Klemm, A., Theisen, S.: Consideration of One Modulus Calabi–Yau Compactification: Picard–Fuchs Equation, Kähler Potentials and Mirror Maps. Nucl. Phys. **B 389**, 753 (1993)
22. Klemm, A., Theisen, S.: Mirror Maps and Instanton Sums for Intersections in Weighed Projective Space. LMU-TPW 93–08, Preprint April 1993
23. Klemm, A., Schimmrigk, R.: Landau–Ginzburg String Vacua. CERN-TH-6459
24. Kim, J.K., Park, C.J., Yoon, Y.: Calabi–Yau manifolds from complete intersections in products of weighted projective spaces. Phys. Lett. **B 224**, 108–114 (1989)
25. Kreuzer, M., Schimmrigk, R., Skarke, H.: Abelian Landau–Ginzburg Orbifolds and Mirror Symmetry. Nucl. Hys. **B 372**, 61–86 (1992)
26. Kreuzer, M., Skarke, H.: No Mirror Symmetry in Landau–Ginzburg Spectra!. CERN-TH 6461/92, to appear in Nucl. Phys. B
27. Lerche, W., Vafa, C., Warner, N.: Nucl. Phys. **B 324**, 427 (1989)
28. Libgober, A., Teitelbaum, J.: Lines on Calabi–Yau complete intersections, mirror symmetry, and Picard-Fuchs equations. Duke Math. J., Int. Math. Res Notices **129**, (1993)
29. Lynker, M., Schimmrigk, R.: Phys. Lett. **249**, 237 (1990)
30. Manin, Yu.I.: Private communication
31. Mathai, A.M., Saxens, R.K.: Generalized Hypergeometric Functions with Applications in Statistic and Physical Sciences. Lect. Notes Math **348**, (1973)
32. Morrison, D.: Mirror symmetry and rational curves on quintic threefolds: A guide for mathematicians. J. Am. Math. Soc. **6**, 223–247 (1993)
33. Morrison, D.: Picard–Fuchs equations and mirror maps for hypersurfaces. In: Essay on Mirror Manifolds. Ed. Yau, S.-T., Hong Kong: Int. Press. Co., 1992, pp. 241–264
34. Morrison, D.: Compactifications of moduli spaces inspired by mirror symmetry. Preprint (1993)
35. Morrison, D.: Hodge-theoretic aspects of mirror symmetry. In preparation
36. Roan, S.-S.: The mirror of Calabi–Yau Orbifold. Internat. J. Math. **2**, 439–455 (1991)
37. Roan, S.-S.: Topological Properties of Calabi–Yau Mirror Manifolds. Preprint (1992)
38. Schimmrigk, R.: A new construction of a three-generation Calabi–Yau manifold. Phys. Lett. **B 193**, 175–180 (1987)
39. Slater, L.J.: Generalized Hypergeometric Functions. Cambridge: Cambridge University Press, XIII, 1966
40. Stienstra, J., Beukers, F.: On the Picard–Fuchs Equation and the Formal Brauer Group of Certain Elliptic K3-surfaces. Math. Ann. **271**, 269–304 (1985)
41. Witten, E.: Topological sigma models. Commun. Math. Phys. **118**, 411–449 (1988)
42. Witten, E.: Two-dimensional gravity and intersection theory on moduli space. Surveys in Diff. Geom. **1**, 243–310 (1991)
43. Witten, E.: Mirror Manifolds and Topological Field Theory. In: Essays on Mirror Manifolds, Ed. S.-T. Yau, Hong Kong: Int. Press Co., 1992, pp. 120–180

Communicated by S.-T. Yau