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The Relation Between Quantum W Algebras and Lie Algebras

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Abstract: By quantizing the generalized Drinfeld-Sokolov reduction scheme for arbitrary sl_2 embeddings we show that a large set $\mathscr{W}$ of quantum W algebras can be viewed as (BRST) cohomologies of affine Lie algebras. The set $\mathscr{W}$ contains many known W algebras such as W_N and $W_3^{(2)}$. Our formalism yields a completely algorithmic method for calculating the W algebra generators and their operator product expansions, replacing the cumbersome construction of W algebras as commutants of screening operators. By generalizing and quantizing the Miura transformation we show that any W algebra in $\mathscr{W}$ can be embedded into the universal enveloping algebra of a semisimple affine Lie algebra which is, up to shifts in level, isomorphic to a subalgebra of the original affine algebra. Therefore *any* realization of this semisimple affine Lie algebra leads to a realization of the W algebra. In particular, one obtains in this way a general and explicit method for constructing the free field realizations and Fock resolutions for all algebras in $\mathscr{W}$. Some examples are explicitly worked out.

1. Introduction

W algebras were introduced by Zamolodchikov as a new ingredient in the classification program of conformal field (CFT) theories [1] (for a recent review see [2]). As is well known such a classification would correspond to a classification of all possible perturbative groundstates of string theory. However CFT and W algebras have been shown to be related to several other areas of research as well such as integrable systems, 2D critical phenomena and the quantum Hall effect. W symmetries are therefore an interesting new development in theoretical physics and it is the purpose of this paper to provide a step towards understanding their meaning and structure.

The point of view that we shall develop in this paper is that the theory of W algebras is closely related to the theory of Lie algebras and Lie groups. The

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construction of W algebras as Casimir algebras (with as a special case the Sugawara construction), and the coset construction [3, 4] are examples of such a relation, but unfortunately these have some serious drawbacks. We therefore take the Hamiltonian or Drinfeld-Sokolov (DS) [5, 6] reduction perspective which for classical W algebras has been extremely successful.

The DS reduction approach starts with the observation that certain Poisson algebras encountered in the theory of integrable evolution equations can be considered to be classical versions of the W algebras first constructed by Zamolodchikov [7]. Drinfeld and Sokolov had already shown that these Poisson algebras are reductions of Kirillov Poisson structures on the duals of affine Lie algebras thus providing a relation between Lie algebras and W algebras. A first attempt to quantize the classical W algebras found by Drinfeld-Sokolov reduction (nowadays called W_N algebras) was made in [8]. There the Miura transformation was used to realize the generators of the classical W_N algebra in terms of classical free fields. The algebra was then quantized by making the free fields into quantum free fields and normal ordering the expressions fo the W generators. In general this is not a valid quantization procedure however since it is by no means clear that the algebra of quantum W generators will close. In fact it only closes in certain cases [2] (which are of course the cases that were studied in [8]).

Since DS reduction is in essence Hamiltonian reduction in infinitely many dimensions it is possible to apply the techniques of BRST quantization in order to quantize the classical W_N algebras. This was first done in [20] for the special case of the Virasoro algebra and the W_N case was solved by Feigin and Frenkel [16].

Even though W_N algebras have an appealing description as BRST cohomologies of affine Lie algebras the quantum DS method is still rather limited since W_N algebras are by far not the only W algebras. The quantum DS reduction leading to the by now well known $W_3^{(2)}$ algebra [9, 10] was however the first indication that DS reduction can be generalized to include many other W algebras.

In [11] it has been shown that to every sl_2 embedding into the simple Lie algebra underlying the affine algebra there is associated a generalized classical DS reduction of this affine algebra leading to a W algebra. The fact that one considers sl_2 embeddings is closely related to the fact that one wants the reduced algebra to be an extended conformal algebra (i.e. it must contain the Virasoro algebras as a subalgebra and the other generators must be primary fields w.r.t. this Virasoro algebra). Since the numer of inequivalent sl_2 embeddings into sl_n is equal to the number of partitions of the number n the set of W algebras that can be obtained by DS reduction increased drastically. The W_N algebras turn out to be associated to the so-called "principal" sl_2 embeddings. The Polyakov-Bershadsky algebra $W_3^{(2)}$ is associated to the only nonprincipal sl_2 embedding into sl_3.

The reductions considered in [11] are clasical and it is the purpose of this paper to quantize them. The usual formalism developed in [16, 20] constructs the W algebra as the commutant of certain screening operators which is rather difficult to generalize to arbitrary sl_2 embeddings. The main reason for this is that it is difficult to find a complete set of generators of this algebra for arbitrary sl_2 embeddings (one has to make use of character formulas to check if one has obtained all generators. These characters are however not known in advance). Also it makes use of free field realizations of the original affine Lie algebra which means that one obtains, in the end, the W algebra in its free field form. If the W algebra has affine subalgebras this will therefore get obscured by the not very transparent free field form. In this paper we therefore use the formalism that was developed in [13] to quantize finite W algebras. It turns out that this formalism still works, with some modifications, in the infinite dimensional case considered here.

Let us now give an outline of the paper. In Sect. 2 we quantize the generalized DS reductions of [11]. This is done by the same spectral sequence calculation that was used in [13]. We also introduce the quantum Miura transformation for arbitrary sl_2 embeddings and show how to obtain free field realizations for arbitrary W algebras. In Sect. 3 we briefly discuss the conformal properties of the quantum W algebras obtained in Sect. 2. Furthermore a general formula for the central charges of the W algebras in terms of the level of the affine Lie algebra and the defining vector of the sl_2 embedding is given. In the last section we consider some examples in order to illustrate the general procedure. We end the paper with some comments and open problems.

2. Quantization

Let $\{t_a\}$ be a basis of the Lie algebra $\bar{g} \equiv sl_n$. The affine Lie algebra g associated to $\bar{g}$ is the span of $\{J_n^a\}$ and the central element K. The commutation relations are given by

$$[J_n^a, J_m^b] = f_c^{ab} J_{n+m}^c + n g^{ab} K \delta_{n+m,0}\,; \qquad [K, J_n^a] = 0\,, \tag{2.1}$$

where g^{ab} is the inverse of $g_{ab} = \mathrm{Tr}(t_a t_b)$ and $[t_a, t_b] = f_{ab}^c t_c$. As usual we use g_{ab} to raise and lower indices. Let $\mathscr{U}_k g (k \in \mathbf{C})$ be the universal enveloping algebra of g quotiented by the ideal generated by $K - k$.

It was shown in [11] that one can associate to every sl_2 embedding into $\bar{g}$ a Drinfeld-Sokolov reduction of g leading to a classical W algebra. We shall now quantize these algebras. Let $\{t_0, t_+, t_-\}$ be an sl_2 subalgebra of $\bar{g}$, then one can decompose $\bar{g}$ into eigenspaces of the operator ad_{t_0}

$$\bar{g} = \bigoplus_{k \in \frac{1}{2}\mathbf{N}} \bar{g}_k\,, \tag{2.2}$$

where $\bar{g}_k = \{x \in \bar{g} \mid [t_0, x] = kx\}$. This defines a gradation of $\bar{g}$ which is in general half integer. However, it was shown in [12, 13] that in those cases where the grading contains half integers one can replace it by an integer grading which in the end leads to the same Drinfeld-Sokolov reduction. This is done by replacing t_0 by a certain element δ of the Cartan subalgebra which has the property that the grading w.r.t. the operator ad_δ is an integer grading (for some basic facts on sl_2 embeddings and the explicit construction of δ given an sl_2 embedding see the appendix). Without loss of generality we can therefore assume that the gradation (2.2) is integral. The algebra $\bar{g}$ now admits a triangular decomposition into a direct sum of a negative grade piece, a zero grade piece and a positive grade piece denoted by $\bar{g}_-, \bar{g}_0$ and $\bar{g}_+$ respectively.

Under the adjoint action of the sl_2 subalgebra $\{t_0, t_\pm\}$, $\bar{g}$ decomposes into a direct sum of sl_2 multiplets. Let us choose the basis $\{t_a\}$ such that all elements t_a are basis vectors of some sl_2 multiplet. Of course this means that all t_a are homogeneous w.r.t. the gradation. From now on we let latin indices $a, b, \dots$ run over the entire basis of $\bar{g}$, Greed indices $\alpha, \beta, \dots$ over the basis of $\bar{g}_+$ and barred Greek indices $\bar{\alpha}, \bar{\beta}, \dots$ over the basis of $\bar{g}_0 \oplus \bar{g}_-$ (i.e. $\lambda^\alpha t_\alpha + \lambda^{\bar{\alpha}} t_{\bar{\alpha}} = \lambda^a t_a$).

We now come to the constraints. Since the sl_2 subalgebra $\{t_0, t_\pm\}$ is a triplet under its own adjoint action there must be some α_+ such that $t_+ = At_{\alpha_+}$. Define the character χ of g_+ (where g_+ is the affinization of $\bar{g}_+$) by putting $\chi(J^\alpha_n) = A\delta^{\alpha,\alpha_+}\delta_{n+1,0}$. The constraints one imposes are then $J^\alpha_n = \chi(J^\alpha_n)$. These constraints are first class [11] for integral gradings which means that one can use the BRST formalism. Thereto introduce the fermionic ghost variables c^n_α, b^α_n with ghost numbers 1 and -1 respectively and relations $c^n_\alpha b^\beta_m + b^\beta_m c^n_\alpha = \delta_{\alpha\beta}\delta_{n+m,0}$. The algebra generated by these ghost variables is the Clifford algebra $Cl(g_+ \otimes g^*_+)$. As usual one then considers the algebra $\Omega_k = \mathscr{U}_k g \otimes Cl(g_+ \oplus g^*_+)$.

For calculational purposes it is convenient (as is standard practice in conformal field theory) to introduce the following "basic fields" $J^a(z) = \sum_n J^a_n z^{-n-1}$; $c_\alpha(z) = \sum_n c^n_\alpha z^{-n}$; $b^\alpha(z) = \sum_n b^\alpha_n z^{-n-1}$. It is well known that the commutation relations in Ω_k can then be expressed in terms of the operator product expansions (OPE),

$$J^a(z)J^b(w) = \frac{kg^{ab}}{(z-w)^2} + \frac{f^{ab}_c}{z-w}J^c(w) + \dots\,, \tag{2.3}$$

$$c_\alpha(z)b^\beta(w) = \frac{\delta^\beta_\alpha}{z-w}\,. \tag{2.4}$$

Now inductively define the *algebra of fields* $F(\Omega_k)$ as follows: $J^a(z), c_\alpha(z), b^\alpha(z)$ are elements of $F(\Omega_k)$ with "conformal dimensions" $\Delta = 1, 0, 1$ respectively; if $A(z), B(z) \in F(\Omega_k)$, then $\alpha A(z) + \beta B(z) \in F(\Omega_k)$; if $A(z)$ is an element of $F(\Omega_k)$ of conformal dimension Δ, then $\frac{dA}{dz}(z)$ is also an element of $F(\Omega_k)$ and has conformal dimension $\Delta + 1$; if $A(z), B(z)$ are elements of $F(\Omega_k)$ of conformal dimensions Δ_A and Δ_B respectively, then the normal ordered product $(AB)(z) \equiv A_-(z)B(z) \pm B(z)A_+(z)$ (where one has the minus sign if A and B are fermionic) is also an element of $F(\Omega_k)$ and has conformal dimension $\Delta_A + \Delta_B$. Here $A_-(z) = \sum_{n\le-\Delta_A} A_n z^{-n-\Delta_A}$ and $A_+(z) = A(z) - A_-(z)$. We say that $F(\Omega_k)$ is "generated" by the basic fields. Note that $F(\Omega_k) \subset \mathscr{U}_k g[\![z, z^{-1}]\!]$. The algebra $F(\Omega_k)$ is graded by ghost number, i.e. $J^a(z)$, $c_\alpha(z)$ and $b^\alpha(z)$ have degrees 0, 1 and -1 respectively and we have the decomposition

$$F(\Omega_k) = \bigoplus_n F(\Omega_k)^{(n)}\,. \tag{2.5}$$

The algebra of fields $F(\Omega_k)$ is not simply the set of "words" in the fields that can be made using the rules given above, there are also relations. If we denote the operator product expansion of A and B by $A(z)B(w) \sim \sum_r \frac{\{AB\}_r}{(z-w)_r}$, then the relations valid in $F(\Omega_k)$ are [4]

$$\begin{aligned}(AB)(z) - (BA)(z) \equiv [A,B](z) &= \sum_{r>0}(-1)^{r+1}\frac{\partial^r}{r!}\{AB\}_r\,,\\ (A(BC))(z) - (B(AC))(z) &= ([A,B]C)(z)\,,\\ \partial(AB)(z) &= (\partial AB)(z) + (A\partial B)(z)\,.\end{aligned} \tag{2.6}$$

The BRST operator is then [15] $D(.) = [d, .]$, where $d = \oint \frac{dz}{2\pi i}\, d(z)$ and

$$d(z) = (J^\alpha(z) - \chi(J^\alpha(z)))c_\alpha(z) - \tfrac{1}{2} f_\gamma^{\alpha\beta}(b^\gamma(c_\alpha c_\beta))(z)\,. \tag{2.7}$$

D is of degree 1 (i.e. $D(F(\Omega_k)^{(l)}) \subset F(\Omega_k)^{(l+1)}$) and $D^2 = 0$ which means that $F(\Omega_k)$ is a complex. One is then interested in calculating the cohomology (or Hecke algebra) of this complex because the zeroth cohomology is nothing but the quantization of the classical W algebra [15, 16]. This problem has been solved for the so-called "finite W algebras" in [13].

The first step is to split the BRST current into two pieces [16]:

$$d_0(z) = -\chi(J^\alpha(z))c_\alpha(z)\,, \tag{2.8}$$
$$d_1(z) = J^\alpha(z)c_\alpha(z) - \tfrac{1}{2} f_\gamma^{\alpha\beta}(b^\gamma(c_\alpha c_\beta))(z)\,, \tag{2.9}$$

and to make $F(\Omega_k)$ into a double complex $F(\Omega_k) = \bigoplus_{rs} F(\Omega_k)^{(r,s)}$ by assigning the following (bi)grades to its generators:

$$\begin{aligned}
&\deg(J^a(z)) = (-k, k) && \text{if} \quad t_a \in \bar{g}_k\,,\\
&\deg(c_\alpha(z)) = (k, 1-k) && \text{if} \quad t_\alpha \in \bar{g}_k\,,\\
&\deg(b^\alpha(z)) = (-k, k-1) && \text{if} \quad t_\alpha \in \bar{g}_k\,.
\end{aligned} \tag{2.10}$$

The operators $D_0 : F(\Omega_k)^{(r,s)} \to F(\Omega_k)^{(r+1,s)}$ and $D_1 : F(\Omega_k)^{(r,s)} \to F(\Omega_k)^{(r,s+1)}$ associated in the obvious way to d_0 and d_1 satisfy $D_0^2 = D_1^2 = D_0D_1 + D_1D_0 = 0$ verifying that we have obtained a double complex.

Let us now calculate the action of the operators D_0 and D_1 on the generators of $F(\Omega_k)$. For this it is convenient to introduce $\hat{J}^a(z) = J^a(z) + f_\gamma^{\alpha\beta}(b^\gamma c_\beta)(z)$. One then finds by explicit calculation

$$\begin{aligned}
&D_0(\hat{J}^a(z)) = -f_\gamma^{\alpha\beta}\chi(J^\gamma(z))c_\beta(z)\,,\\
&D_0(c_\alpha(z)) = 0\,,\\
&D_0(b^\alpha(z)) = -\chi(J^\alpha(z))\,,\\
&D_1(\hat{J}^a(z)) = f_{\bar{\beta}}^{\alpha a}\hat{J}^{\bar{\beta}}(z)c_\alpha(z) + kg^{a\alpha}\partial c_\alpha(z) - f_\beta^{\alpha e} f_e^{\beta a}\partial c_\alpha(z)\,,\\
&D_1(c_\alpha(z)) = -\tfrac{1}{2} f_\alpha^{\beta\gamma}(c_\beta c_\gamma)(z)\,,\\
&D_1(b^\alpha(z)) = \hat{J}^\alpha(z)\,.
\end{aligned}$$

From these formulas it immediately follows that $D(\hat{J}^\alpha(z)) = 0$ and $D(b^\alpha(z)) = \hat{J}^\alpha(z) - \chi(J^\alpha(z))$. This means that the subspace $F^\alpha(\Omega_k)$ of $F(\Omega_k)$ generated by $J^\alpha(z)$ and $b^\alpha(z)$ is actually a subcomplex. The cohomology of this complex can easily be calculated and one finds $H^*(F^\alpha(\Omega_k); D) = \mathbf{C}$. Note also that due to the Poincaré-Birkhoff-Witt theorem for field algebras (which follows immediately from the relations (2.6)) the normal ordening map

$$(\ldots) : F_{\mathrm{red}}(\Omega_k) \otimes \bigotimes_\alpha F^\alpha(\Omega_k) \to F(\Omega_k) \tag{2.11}$$

defined by $A_1(z)\otimes \ldots \otimes A_l(z) \mapsto (A_1 \ldots A_l)(z)$ (where we always use the convention $(ABC)(z) = (A(BC))(z)$) is an isomorphism of vectorspaces. Due to this and the fact that the BRST operator acts as a derivation [1] on $F(\Omega_k)$ we have

$$H^*(F(\Omega_k); D) \cong H^*(F_{\text{red}}(\Omega_k); D) \otimes \bigotimes_\alpha H^*(F^\alpha(\Omega_k); D) \\ \cong H^*(F_{\text{red}}(\Omega_k); D)\,, \tag{2.12}$$

where in the first step we used a Kunneth like theorem given in [13].

In order to calculate $H^*(F_{\text{red}}(\Omega_k); D)$ one uses the fact that $F_{\text{red}}(\Omega_k)$ is actually a double complex which makes calculation of the cohomology possible via a spectral sequence argument [13, 16, 17]. The first term E_1 of the spectral sequence is the D_0 cohomology of $F_{\text{red}}(\Omega_k)$. Note that we can write $D_0(\hat{J}^{\bar\alpha}(z)) = -\,\mathrm{Tr}([t_+, t^{\bar\alpha}]t^\beta c_\beta(z))$. Therefore $D_1(\hat{J}^{\bar\alpha}(z)) = 0$ iff $t_{\bar\alpha} \in \bar{g}_{lw}$, where $\bar{g}_{lw}$ is the set of elements of $\bar{g}$ that are annihilated by ad_{t_-} (the lowest weight vectors of the sl_2 multiplets) and where we used the fact [13] that $t_{\bar\alpha} \in \mathrm{Ker}(\mathrm{ad}_{t_-})$ iff $t^{\bar\alpha} \in \mathrm{Ker}(\mathrm{ad}_{t_+})$. It can also easily be seen that for all β there exists a linear combination $a(\beta)_{\bar\alpha}\hat{J}^{\bar\alpha}(z)$ such that $D_0(a(\beta)_{\bar\alpha}\hat{J}^{\bar\alpha}(z)) = c_\beta(z)$. From this it follows [13] that purely on the level of vectorspaces we have

$$H^n(F_{\text{red}}(\Omega_k); D_0) \cong F_{lw}(\Omega_k)\delta_{k,0}\,, \tag{2.13}$$

where $F_{lw}(\Omega_k)$ is the subspace of $F(\Omega_k)$ generated by the fields $\{J^{\bar\alpha}(z)\}_{t_{\bar\alpha}\in\bar{g}_{lw}}$. Since the only cohomology that is nonzero is of degree 0 the spectral sequence abuts at the first term, i.e. $E_\infty = E_1$, and we find the end result

$$H^n(F_{\text{red}}(\Omega_k); D) \cong F_{lw}(\Omega_k)\delta_{n,0}\,. \tag{2.14}$$

Having calculated the BRST cohomology at the level of vector spaces one now can construct the cohomology (or W algebra) generators and their OPEs via a procedure called the tic-tac-toe construction [18]. Consider a generator $\hat{J}^{\bar\alpha}(z)$ of degree $(p, -p)$ of the field algebra $F_{lw}(\Omega_k)$ (i.e. $t_{\bar\alpha} \in \bar{g}_{lw}$) then the generator of cohomology associated to this element is given by

$$W^{\bar\alpha}(z) = \sum_{l=0}^{p} (-1)^l W_l^{\bar\alpha}(z)\,, \tag{2.15}$$

where $W_0^{\bar\alpha}(z) \equiv \hat{J}^{\bar\alpha}(z)$ and $W_l^{\bar\alpha}(z)$ can be determined inductively by

$$D_1(W_l^{\bar\alpha}(z)) = D_0(W_{l+1}^{\bar\alpha}(z))\,. \tag{2.16}$$

It is easy to check, using the fact that $D_0(J^{\bar\alpha}(z)) = 0$ for $t_{\bar\alpha} \in \bar{g}_{lw}$ that indeed $D(W^{\bar\alpha}(z)) = 0$.

The formalism presented above provides us with a completely algorithmic procedure of calculating the W algebra associated to a certain sl_2 embedding: First determine the space $\bar{g}_{lw}$. Then take a current $\hat{J}^{\bar\alpha}(z)$ with $t_{\bar\alpha} \in \bar{g}_{lw}$ and inductively calculate the fields $W_l^{\bar\alpha}(z)$ using relations (2.16). The field (2.15) is then the corresponding W generator and the relations in the W algebra are then just the OPEs between the fields $\{W^{\bar\alpha}(z)\}_{t_{\bar\alpha}\in\bar{g}_{lw}}$ calculated using the OPEs in $F(\Omega_k)$.

[1] This follows from the fact that $D(X(w)) = \{d(z)X(w)\}_1$, and from the following general identities for operator product expansions: $\{A\partial B\}_1 = \partial\{AB\}_1$ and $\{A(BC)\}_1 = (-1)^{AB}(B\{AC\}_1) + (\{AB\}_1 C)$

In principle this algebra closes only modulo D-exact terms. But since we computed the D cohomology on a reduced complex generated by $\hat{J}^{\bar{\alpha}}(z)$ and $c_{\alpha}(z)$, and this reduced complex is zero at negative ghost number, there simply aren't any D exact terms at ghost number zero. Thus the algebra generated by $\{W^{\bar{\alpha}}(z)\}_{t_{\bar{\alpha}}\in\bar{g}_{lw}}$ closes in itself.

As was shown in [13] for finite W algebras, the operator product algebra generated by the fields $W^{\bar{\alpha}}(z)$ is isomorphic to the operator product algebra generated by their (bi)grade $(0,0)$ components $W_p^{\bar{\alpha}}(z)$ (the proof in the infinite dimensional case is completely analogous and will not be repated here). The fields $W_p^{\bar{\alpha}}(z)$ are of course elements of the field algebra generated by the currents $\{\hat{J}^{\bar{\alpha}}(z)\}_{t_{\bar{\alpha}}\in\bar{g}_0}$. The relations (i.e. the OPEs) satisfied by these currents are almost identical to the relations satisfied by the unhatted currents,

$$\hat{J}^{\bar{\alpha}}(z)\hat{J}^{\bar{\beta}}(w)=\frac{kg^{\bar{\alpha}\bar{\beta}}+k^{\bar{\alpha}\bar{\beta}}}{(z-w)^2}+\frac{f_{\bar{\gamma}}^{\bar{\alpha}\bar{\beta}}\hat{J}^{\bar{\gamma}}(w)}{z-w}+\dots, \tag{2.17}$$

where $k^{\bar{\alpha}\bar{\beta}}=f_{\gamma}^{\bar{\alpha}\lambda}f_{\lambda}^{\bar{\beta}\gamma}$. Now, it is easy to see that $\bar{g}_0$ is just a direct sum of sl_{p_j} and $u(1)$ algebras, i.e. forgetting for a moment about the $u(1)$ algebras one can write

$$\bar{g}_0\sim\bigoplus_j sl_{p_j}\,. \tag{2.18}$$

Within the sl_{p_j} component of $\bar{g}_0$ we have the identity

$$k^{\bar{\alpha}\bar{\beta}}=g^{\bar{\alpha}\bar{\beta}}(h-h_j)\,, \tag{2.19}$$

where h is the dual coxeter number of $\bar{g}$ and h_j is the dual coxeter number of sl_{p_j}. We therefore find that the field algebra generated by the currents $\{\hat{J}^{\bar{\alpha}}(z)\}_{t_{\bar{\alpha}}\in\bar{g}_0}$, denoted from now on by $\hat{F}_0$, is nothing but the field algebra associated to a semisimple affine Lie algebra the components of which are affine sl_{p_j} and $u(1)$ Lie algebras. This semisimple affine Lie algebra is not simply g_0 (whose field algebra is generated by the unhatted currents) however because in g_0 all components have the same level while in $\hat{F}_0$ the level varies from component to component as follows from Eq. (2.19). This is just a result of the ghost contributions $k^{\bar{\alpha}\bar{\beta}}$ in the OPEs of the hatted currents.

From the above we find that the map

$$W^{\bar{\alpha}}(z)\mapsto(-1)^p W_p^{\bar{\alpha}}(z) \tag{2.20}$$

is an embedding of the W algebra into $\hat{F}_0$. This provides a quantization and generalization to arbitrary sl_2 embeddings of the well known Miura map. In [8] the standard Miura map for W_N algebras was naively quantrised by simly normal ordering the classical expressions. This is known to work only for certain algebras [2]. Our construction gives a rigorous derivation of the quantum Miura transformations for arbitrary Kac-Moody algebras and sl_2 embeddings (the generalized Miura transformations for a certain special class of sl_2 embeddings were also recently given in [23]).

As a result of the generlaized quantum Miura transformation *any* representation or realization of $\hat{F}_0$ gives rise to a representation or realization of the W algebra. In particular one obtains a free field realization of the W algebra by choosing the Wakimoto free field realization of $\hat{F}_0$. Given our formalism it is therefore straightforward to construct free field realizations for any W algebra that can be obtained by Drinfeld-Sokolov reduction.

2 1. *The Stress Energy Tensor*

It is possible to give a general expression for the stress-energy tensor of a W algebra related to an arbitrary sl_2 embedding. For this purpose we write t_0 as $t_0 = s^a t_a$, where the s^a is only nonzero if t_a lies in the Cartan subalgebra. Furthermore, let δ_α be the eigenvalue of ad_{t_0} acting on t_α, thus $[t_0, t_\alpha] = \delta_\alpha t_\alpha$. From this it is easy to see that $\delta_\alpha = s_a f_\alpha^{\alpha a}$. Then the stress-energy tensor is

$$T = \frac{1}{2(k+h)} (g_{a_0 b_0} (\hat{J}^{a_0} \hat{J}^{b_0}) + 2 g_{b\alpha} \hat{J}^b \chi(J^\alpha) - 2(k+h) s_a \partial \hat{J}^a + g_{b\alpha} f_e^{b\alpha} \partial \hat{J}^e) , \quad (2.12)$$

where the indices a_0, b_0 run only over $\bar{g}_0$, and h is again the dual Coxeter number. By adding a D-exact term $D(R)$ to (2.21), where

$$R = \frac{1}{k+h} g_{b\alpha} (J^b J^\alpha) + \frac{1}{2(k+h)} g_{e\alpha} f_\gamma^{e\beta} (b^\alpha (b^\gamma c_\beta)) , \quad (2.22)$$

we can rewrite it as

$$T = \frac{1}{2(k+h)} g_{ab} (J^a J^b) - s_a \partial J^a + (\delta_\alpha - 1) b^\alpha \partial c_\alpha + \delta_\alpha \partial b^\alpha c_\alpha , \quad (2\ 23)$$

which has the familiar form of improved Sugawara stress-energy tensor plus the stress energy tensors of a set of free $b-c$ systems. The other generators of the W algebra cannot in general be written as the sum of a current piece plus a ghost piece. Actually, (2.23) is precisely what one would expect to get from a constrained WZNW model. Notice that δ_α is the degree of t_α with respect to t_0, whereas α in (2.23) runs over $\bar{g}^+$ which was defined with respect to a new, different, integral grading of the Lie algebra.

In terms of the level k and the Cartan elemet of the sl_2 embedding t_0 (called the "defining vector" since it determines the whole sl_2 subalgebra up to inner automorphisms) the central charge of the W algebra is given by

$$c(k; t_0) = \dim(g_0) - \tfrac{1}{2} \dim(\bar{g}_{\frac{1}{2}}) - 12 \left| \frac{\varrho}{\sqrt{k+h}} - t_0 \sqrt{k+h} \right|^2 , \quad (2.24)$$

where $\bar{g}_{\frac{1}{2}}$ is defined by (2.2), and ϱ is half the sum of the positive roots, $\varrho = \frac{1}{2} \sum_{a \in \Delta^+} f_a^{b_0 a} t_{b_0}$.

3. Examples

In this section we consider the three simplest cases of quantum Drinfeld-Sokolov reduction, namely the Virasoro algebra, the Zamolodchikov W_3 algebra and the so-called Polyakov-Bershadsky algebra $W_3^{(2)}$. For notational convenience we shall omit the explicit z of the fields where possible.

3 1. The Virasoro Algebra

The Virasoro algebra is the simplest W algebra and it is well known to arise from the affine sl_2 KM algebra by quantum Drinfeld-Sokolov reduction [20]. It is the W algebra associated to the only nontrivial embedding of sl_2 into itself, namely the identity map. We consider this example here to contrast our methods to the ones used by Bershadsky and Ooguri.

Choose the following basis of sl_2:

$$J^a t_a = \begin{pmatrix} J^2/2 & J^3 \\ J^1 & -J^2/2 \end{pmatrix}, \tag{3.1}$$

where $t_0 = -t_2$, $t_+ = t_1$ and $t_- = t_3$. The positive grade piece of the Lie algebra $\bar{g} = sl_2$ is generated by t_1, and the constraint to be imposed is $J^1 = 1$. The BRST current $d(z)$ is given simply by

$$d(z) = (J^1 c_1) - c_1 . \tag{3.2}$$

The 'hatted" currents are $\hat{J}^1 = J^1$, $\hat{J}^2 = J^2 + 2(b^1 c_1)$ and $\hat{J}^3 = J^3$. The cations of D_0 and D_1 are given by

$$\begin{aligned} D_0(\hat{J}^2) &= -2c_1 , \quad & D_1(\hat{J}^3) &= (\hat{J}^2 c_2) + (k+2)\partial c_1 , \\ D_0(b^1) &= -1 , & D_1(b^1) &= \hat{J}^1 , \end{aligned} \tag{3.3}$$

On the other fields D_0 and D_1 vanish. From (3.3) it is immediately clear that $H(F_{\text{red}}(\Omega_k); D_0)$ is generated by $W_0^3 \equiv \hat{J}^3$, in accordance with the general arguments in Sect. 2. To find the generator of the D-cohomology, we apply the tic-tac-toe construction. We are looking for an element $W_1^3(z) \in F_{\text{red}}(\Omega_k)$ such that $D_0(W_1^3(z)) = D_1(W_0^3)$. The strategy is to write down the most general form of $W_1^3(z)$, and then to fix the coefficients. In general, $W_l^{\bar{\alpha}}$ must satisfy the following two requirements:
1. if $W_l^{\bar{\alpha}}$ has bidegree $(-k, k)$, then $w_{l+1}^{\bar{\alpha}}$ must have bidegree $(-k-1, k+1)$,
2. if we define inductively the *weight* h of a monomial in the $J^{\bar{\alpha}}$ by $h(J^{\bar{\alpha}}) = 1-k$ if $t_{\bar{\alpha}} \in g_k$, $h((AB)) = h(A) + h(B)$ and $h(\partial A) = h(A) + 1$ [2], then $h(W_l^{\bar{\alpha}}) = h(W_{l+1}^{\bar{\alpha}})$.

These two conditions guarantee that the most general form of $W_l^{\bar{\alpha}}$ will contain only a finite number of parameters, so that in a sense the tic-tac-toe construction is a finite algorithm. In the case at hand, the most general form of W_1^3 is $a_1(\hat{J}^2\hat{J}^2) + a_2\partial\hat{J}^2$, and the D_0 of this equals $-4a_1(\hat{J}^2 c_1) - (4a_1 + 2a_2)\partial c_1$. Thus, $a_1 = -1/4$ and $a_2 = -(k+1)/2$. Since $D_1(W_1^3) = 0$, the tic-tac-toeing stops here and the generator of the D cohomology reads

$$W^3 = W_0^3 - W_1^3 = \hat{J}^3 + \tfrac{1}{4}(\hat{J}^2\hat{J}^2) + \frac{(k+1)}{2}\partial\hat{J}^2 . \tag{3.4}$$

As one can easily verify, $T = W^3/(k+2)$ generates a Virasoro algebra with central charge

$$c(k) = 13 - 6(k+2) - \frac{6}{k+2}, \tag{3.5}$$

a result first found by Bershadsky and Ooguri [20].

---

[2] The weight h is similar to the conformal weight, but not always equal to it. It is independent of the way in which the $\hat{J}$ are ordered

Let's now consider the quantum Miura transformation. In the case at hand $\bar{g}_0$ is the Cartan subalgebra spanned by t_2 and $\hat{F}_0$ is an affine $u(1)$ field algebra at level $k+2$ generated by $\hat{J}^2$. Indeed defining the field $\partial\phi \equiv \nu \hat{J}^2$, where $\nu = \sqrt{2(k+2)}$, it is easy to check that

$$\partial\phi(z)\,\partial\phi(w) = \frac{1}{(z-w)^2} + \dots . \tag{3.6}$$

In terms of the field ϕ the (bi)grade $(0,0)$ piece of T is given by

$$T^{(0,0)} = \tfrac{1}{2}\,(\partial\phi\partial\phi) + \alpha_0 \partial^2\phi\,, \tag{3.7}$$

where $\alpha_0 = \frac{1}{2}\,\nu - \dfrac{1}{\nu}$. This is the usual expression for the Virasoro algebra in terms of a free field ϕ.

3.2. *The Zamolodchikov W_3 Algebra*

Having illustrated the construction in some detail for the Virasoro algebra, we will now briefly discuss two other examples. We start with the Zamolodchikov W_3 algebra [1]. This algebra is associated to the so-called "principal" sl_2 [11]. In terms of the following basis of sl_3:

$$J^a t_a = \begin{pmatrix} \dfrac{J^4}{6} + \dfrac{J^5}{2} & \frac{1}{2}\,(J^6 + J^7) & J^8 \\ \frac{1}{2}\,(J^2 + J^3) & -\dfrac{J^4}{3} & \frac{1}{2}\,(-J^6 + J^7) \\ J^1 & \frac{1}{2}\,(J^2 - J^3) & \dfrac{J^4}{6} - \dfrac{J^5}{2} \end{pmatrix}, \tag{3.8}$$

the sl_2 subalgebra is given in this case by $t_+ = 4t_2$, $t_0 = -2t_5$ and $t_- = 2t_7$. The constraints are therefore $J^1 = J^3 = 0$ and $J^2 = 1$ according to the general prescription. The BRST current associated to these (first class) constraints reads

$$d(z) = J^1 c_1 + (J^2 - 1)c_2 + J^3 c_3 + 2(b^1(c_2 c_3))\,. \tag{3.9}$$

The cohomology $H(F_{\rm red}(\Omega_k); D_0)$ is generated by $\hat{J}^7$ and $\hat{J}^8$ since t_7 and t_8 span $\bar{g}_{lw}$. The tic-tac-toe construction gives as generators of $H(F(\Omega_k); D)$ the fields $W^7 = W_0^7 - W_1^7$ and $W^8 = W_0^8 - W_1^8 + W_2^8$, where

$$\begin{aligned}
W_0^7 &= \hat{J}^7\,,\\
W_1^7 &= -\tfrac{1}{6}\,(\hat{J}^4\hat{J}^4) - \tfrac{1}{2}\,(\hat{J}^4\hat{J}^4) - 2(k+2)\,\partial\hat{J}^5\,,\\
W_0^8 &= \hat{J}^8\,,\\
W_1^8 &= -(\hat{J}^5\hat{J}^6) + \tfrac{1}{3}\,(\hat{J}^4\hat{J}^7) - (k+2)\,\partial\hat{J}^6\,,\\
W_2^8 &= -\tfrac{1}{27}\,(\hat{J}^4(\hat{J}^4\hat{J}^4)) + \tfrac{1}{3}\,(\hat{J}^4(\hat{J}^5\hat{J}^5))\\
&\quad + \frac{(k+2)}{3}\,(\hat{J}^4\partial\hat{J}^4) + (k+2)\,(\hat{J}^5\partial\hat{J}^4) + \frac{2(k+2)^2}{3}\,\partial^2\hat{J}^4\,.
\end{aligned} \tag{3.10}$$

With some work, for instance by using the program for computing OPE's by Thielemans [22], one finds that

$$\begin{aligned} T &= \frac{1}{2(k+3)} W^7 , \\ W &= \left(\frac{3}{(5c+22)(k+3)^3} \right)^{\frac{1}{2}} W^8 , \end{aligned} \tag{3.11}$$

generate the W_3 algebra with central charge

$$c(k) = 50 - 24(k+3) - \frac{24}{k+3} . \tag{3.12}$$

For any principal embedding the grade zero subalgebra $\bar{g}_0$ is just the Cartan subalgebra. In the case at hand $\hat{F}_0$ is therefore a direct sum of two affine $u(1)$ field algebras, both of level $k+3$, generated by $\hat{J}^4$ and $\hat{J}^5$. Defining $\partial\phi_1 \equiv \nu_1 \hat{J}^4$and $\partial\phi_2 \equiv \nu_2 \hat{J}^5$, where $\nu_1 = \sqrt{6(k+3)}$ and $\nu_2 = \sqrt{2(k+3)}$ it is easy to check that

$$\partial\phi_i(z)\partial\phi_j(w) = \frac{\delta_{ij}}{(z-w)^2} + \dots . \tag{3.13}$$

According to the general prescription the Miura transformation reads in this case $W^7 \mapsto -W_1^7$ and $W^8 \mapsto W_2^8$, and the fields

$$\begin{aligned} T^{(0,0)} &= -\nu_2^{-2} W_1^7 , \\ W^{(0,0)} &= \left(\frac{3}{(5c+22)(k+3)^3} \right)^{\frac{1}{2}} W_2^8 , \end{aligned}$$

also generate a W_3 algebra with central charge (3.12). Note that according to (3.10) $T^{(0,0)}$ and $W^{(0,0)}$ only depend on ϕ_1 and ϕ_2. This is the well known free field realization of W_3.

3.3. The $W_3^{(2)}$ Algebra

The two examples discussed above are both related to principal sl_2 embeddings. In order to illustrate that our methods work for arbitrary embeddings we now consider the example of the $W_3^{(2)}$ algebra which is associated to the (only) nonprincipal sl_2 embedding into sl_3.

To describe the $W_3^{(2)}$ algebra, we pick a slightly different basis of sl_3, namely

$$J^a t_a = \begin{pmatrix} \frac{J^4}{6} - \frac{J^5}{2} & J^6 & J^8 \\ J^2 & -\frac{J^4}{3} & J^7 \\ J^1 & J^3 & \frac{J^4}{6} + \frac{J^5}{2} \end{pmatrix} . \tag{3.14}$$

The sl_2 embedding reads $t_+ = t_1$, $t_0 = t_5$ and $t_- = t_8$. The gradation of $\bar{g}$ with respect to ad_{t_0} is half-integer which means that there will be class constraints [11] corresponding to the fields with grade $-1/2$. If one wants to use the BRST formalism all constraints should be first class. One way to get around this problem is to introduce auxiliary fields [10]. This is not necessary however as was shown in [12, 13] since it is always possible to replace the half integer grading and the constraints associated to it by an integer grading and a set of first class constraints that nevertheless lead to the same Drinfeld-Sokolov reduction. As mentioned earlier we have to replace the grading by t_0 by a grading w.r.t. an element δ.[3] In this specific case $\delta = \frac{1}{3}\,\mathrm{diag}(1,1,-2)$. With respect to ad_δ, t_1 and t_3 have degree 1 and span $\bar{g}_+$. The BRST current is

$$d(z) = ((J^1 - 1)c_1) + J^3 c_3 \,. \tag{3.15}$$

Notice that there is no need for auxiliary fields, since the constraints $J^1 = 1$ and $J^3 = 0$ are first class. The cohomology $H(F_{\mathrm{red}}(\Omega_k); D_0)$ is generated by $\{\hat{J}^4, \hat{J}^6, \hat{J}^7, \hat{J}^8\}$. Again using the tic-tac-toe construction one finds that $H(F(\Omega_k); D)$ is generated by $W^4 = W_0^4$; $W^6 = W_0^6$; $W^7 = W_0^7 - W_1^7$ and $W^8 = W_0^8 - W_1^8$, where

$$\begin{aligned}
W_0^4 &= \hat{J}^4 \,, \\
W_0^6 &= \hat{J}^6 \,, \\
W_0^7 &= \hat{J}^7 \,, \\
W_1^7 &= \tfrac{1}{2}(\hat{J}^2\hat{J}^5) + \tfrac{1}{2}(\hat{J}^2\hat{J}^4) - (k+1)\partial\hat{J}^2 \,, \\
W_0^8 &= \hat{J}^8 \,, \\
W_1^8 &= -\tfrac{1}{4}(\hat{J}^5\hat{J}^5) - (\hat{J}^2\hat{J}^6) + \frac{(k+1)}{2}\partial\hat{J}^5 \,.
\end{aligned} \tag{3.16}$$

The OPE's of the hatted currents involving shifts in level are in this case

$$\begin{aligned}
\hat{J}^4(z)\hat{J}^4(w) &\sim \frac{6\left(k+\frac{3}{2}\right)}{(z-w)^2} + \dots \,, \\
\hat{J}^5(z)\hat{J}^5(w) &\sim \frac{2\left(k+\frac{5}{2}\right)}{(z-w)^2} + \dots \,, \\
\hat{J}^4(z)\hat{J}^5(w) &\sim \frac{3}{(z-w)^2} + \dots \,, \\
\hat{J}^2(z)\hat{J}^6(w) &\sim \frac{(k+1)}{(z-w)^2} + \frac{\frac{1}{2}(\hat{J}^4 - \hat{J}^5)}{(z-w)} + \dots \,.
\end{aligned} \tag{3.17}$$

If we now define the following generators:

$$\begin{aligned}
H &= -W^4/3 \,, \\
G^+ &= W^6 \,, \\
G^- &= W^7 \,, \\
T &= \frac{1}{k+3}\left(W^8 + \tfrac{1}{12}(W^4W^4)\right),
\end{aligned} \tag{3.18}$$

[3] Essentially what one does is split the set of second class constraints in two halves. The constraints in the first half, corresponding to positive grades w r t δ, are still imposed but have now become first class. The other half can then be obtained as gauge fixing conditions of the gauge invariance generated by the first half

we find that these generate the $W_3^{(2)}$ algebra [10], with

$$c(k) = 25 - 6(k+3) - \frac{24}{k+3}\,, \tag{3.19}$$

a formula that was found in [10] by a counting argument.

In the case at hand the subalgebra $\bar{g}_0$ is spanned by the elements t_4, t_5, t_6 and t_2. Obviously $\bar{g}_0 \cong sl_2 \oplus u(1)$. Therefore $\hat{F}_0$ is the direct sum of an affine sl_2 and an affine $u(1)$ field algebra, and using Eq. (2.19) the levels of these can be calculated to be $k+1$ and $k+3$ respectively. Indeed if we introduce the currents

$$\begin{aligned}
\partial\phi &= \tfrac{1}{4}\,(\hat{J}^4 + 3\hat{J}^5)\,,\\
J^0 &= \tfrac{1}{4}\,(\hat{J}^5 - \hat{J}^4)\,,\\
J^- &= \tfrac{1}{2}\,\hat{J}^2\,,\\
J^+ &= 2\hat{J}^6\,,
\end{aligned}$$

then these satisfy the following OPE's:

$$\begin{aligned}
J^0(z)\,J^\pm(w) &= \frac{\pm J^\pm(w)}{z-w} + \dots\,,\\
J^+(z)\,J^-(w) &= \frac{(k+1)}{(z-w)^2} + \frac{2J^0(w)}{z-w} + \dots\,,\\
J^0(z)\,J^0(w) &= \frac{\tfrac{1}{2}\,(k+1)}{(z-w)^2} + \dots\,,\\
\partial\phi(z)\,\partial\phi(w) &= \frac{\tfrac{3}{2}\,(k+3)}{(z-w)^2} + \dots\,,
\end{aligned}$$

and all other OPEs are regular. As stated before the shifts in the levels that one can see in these OPEs are a result of the ghost contributions.

The quantum Miura transformation in this case reads: $W^4 \mapsto W_0^4$, $W^6 \mapsto W_0^6$, $W^7 \mapsto -W_1^7$, $W^8 \mapsto -W_1^8$. This means that in terms of the currents $J^\pm$, J^0 and ϕ the grade $(0,0)$ components of the W generators read (let's for notational convenience denote the $(0,0)$ components of H, G^+, G^- and T again simply by the same letters since they generate an isomorphic algebra anyway)

$$\begin{aligned}
H &= J^0 - \tfrac{1}{3}\,\partial\phi\,,\\
G^+ &= \tfrac{1}{2}\,J^+\,,\\
G^- &= (J^-J^0) + (J^0J^-) + 2(k+3)\,\partial J^- - 2\partial\phi J^-\,,\\
T &= \frac{1}{2(k+3)}\,(2(J^0J^0) + (J^-J^+) + (J^+J^-)\\
&\quad + (k+3)\,\partial J^0 + \tfrac{2}{3}\,(\partial\phi\partial\phi) + Q_0\partial^2\phi)\,,
\end{aligned}$$

where $Q_0 = -(k+1)$. We recognise in the expression for T the improved sl_2 Sugawara stress energy tensor and the free boson in a background charge. Note that these formulas provide an embedding of $W_3^{(2)}$ into $\hat{F}_0$. In [11] a realization of this type was called a "hybrid field realization" since it represents the W algebra partly in terms of KM currents and partly in terms of free fields.

It is now easy to obtain a realization of $W_3^{(2)}$ completely in terms of free fields by inserting for the sl_2 KM currents $J^{\pm}$, J^0 their Wakimoto free field form

$$\begin{aligned} J^- &= \beta\,, \\ J^+ &= -(\gamma^2\beta) - k\partial\gamma - \sqrt{2k+6}\gamma i\partial\varphi\,, \\ J^0 &= -2(\gamma\beta) - \sqrt{2k+6}i\partial\varphi\,, \end{aligned} \tag{3.20}$$

where as usual β, γ and φ are bosonic fields with the following OPEs:

$$\begin{aligned} \gamma(z)\beta(w) &= \frac{1}{z-w} + \dots\,, \\ i\partial\varphi(z)i\partial\varphi(w) &= \frac{1}{(z-w)^2} + \dots\,. \end{aligned} \tag{3.21}$$

This example gives a nice taste of the general case. By the Miura transformation one can write down for any W algebra a hybrid field realization, i.e. a realization partly in terms of free fields and partly in Kac-Moody currents. When required one can then insert for the KM currents the Wakimoto free field realization giving you a realization of the W algebra completely in terms of free fields.

4. Discussion

In this paper we have quantized all generalized Drinfeld-Sokolov reductions. This was done using a formalism that differs from the formalism first used by Bershadsky and Ooguri. The formalism of Bershadsky-Ooguri makes use of the Fock space resolutions of affine Lie algebras and W algebras. In the calculation of the BRST cohomology they have to prove that the BRST cohomology and the resolution cohomology commute. This they indeed did for the Virasoro algebra but in the case of the $W_3^{(2)}$ algebra it is an assumption [10]. The W algebras are in the end constructed as the commutant of certain screening operators. Calculating this commutant and finding a complete set of generators of it is in general very difficult. In [10] Bershadsky doesn't prove that the generators that he provides are a complete set nor does he show how he has obtained them. Our method on the other hand does not make any assumptions, is completely algorithmic and works for arbitrary sl_2 embeddings. The difference with the cohomology calculations of Feigin and Frenkel [16] is that the spectral sequence they use is different from the one that we use (in principal one can associate two spectral sequences to any double complex).

An important open problem is to find unitary representations of the W algebras in the set $\mathscr{W}$. It is believed that many questions about the representation theory can be answered using the correspondence between Lie algebras and W algebras exhibited in this paper. For example it should be possible to derive character formulas for the W algebras from the affine characters (see also [24]). This is now under investigation.

5. Appendix

In this appendix we review some basic facts on sl_2 embeddings [25]. The sl_2 embeddings into sl_n are in one to one correspondence with the partitions of n. (Let $(n_1, n_2, \ldots)$ be a partition of n with $n_1 \geq n_2 \geq \ldots$, then define a different partition $(m_1, m_2, \ldots)$ of n by letting m_k be the number of i for which $n_i \geq k$. Furthermore let $s_t = \sum_{i=1}^{t} m_i$. Then the sl_2 embedding associated to the partition $(n_1, n_2, \ldots)$ is given by

$$
\begin{aligned}
t_+ &= \sum_{l\geq 1} \sum_{k=1}^{n_l - 1} E_{l+s_{k-1}, l+s_k} , \\
t_0 &= \sum_{l\geq 1} \sum_{k=1}^{n_l} \left(\frac{n_l + 1}{2} - k \right) E_{l+s_{k-1}, l+s_{k-1}} , \\
t_- &= \sum_{l\geq 1} \sum_{k=1}^{n_l - 1} k(n_l - k) E_{l+s_k, l+s_{k-1}} ,
\end{aligned}
$$

where E_{ij} is as usual the $n \times n$ matrix with zeros everywhere except for the matrix element (i, j) which is equal to one. The element δ which defines the grading on sl_n, that we use to impose the constraints is given by [13]

$$
\delta = \sum_{k\geq 1} \sum_{j=1}^{m_k} \left(\frac{\sum_l l m_l}{\sum_l m_l} - k \right) E_{s_{k-1}+j, s_{k-1}+j} . \tag{5.1}
$$

One can check that in case the grading provided by t_0 is integer then $\delta = t_0$.

The fundamental representation of sl_n decomposes into irreducible sl_2 multiplets. This we denote by $\underline{n} \to \bigoplus_i n_i \underline{i}$, where $\underline{i}$ is the i-dimensional representation of sl_2. We then have the following identities that come in useful when trying to calculate the central charge $c(k; \delta)$ for a certain specific case:

$$
\begin{aligned}
\tfrac{1}{2} \dim(\bar{g}_{\frac{1}{2}}) &= \sum_{i>0,\, k\geq 0} i n_i n_{i+2,k+1} , \\
|\varrho|^2 &= \tfrac{1}{12} (n^3 - n) , \\
|t_0|^2 &= \tfrac{1}{12} \sum_i n_i (i^3 - i) , \\
(t_0 \mid \varrho) &= \tfrac{1}{12} \left(\sum_i n_i^2 (i^3 - i) + \sum_{i<r} i(i+1)(3r - i - 2) n_i n_r \right) .
\end{aligned} \tag{5.2}
$$

This concludes our discussion of sl_2 embeddings.

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