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Conformal Quantum Field Theory and Half-Sided Modular Inclusions of von-Neumann-Algebras

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Abstract: Let $\mathcal{N}$, $\mathcal{M}$ be von-Neumann-Algebras on a Hilbert space $\mathcal{H}$, Ω a common cyclic and separating vector. Assume Ω to be cyclic and separating also for $\mathcal{N} \cap \mathcal{M}$. Denote by $J_{\mathcal{M}}$, $J_{\mathcal{N}}$ the modular conjugations to $(\mathcal{M}, \Omega)$, $\Delta_{\mathcal{M}}$ and $\Delta_{\mathcal{N}}$ the associated modular operators. If

$$\Delta_{\mathcal{M}}^{-it}(\mathcal{N} \cap \mathcal{M})\Delta_{\mathcal{M}}^{it} \subset (\mathcal{N} \cap \mathcal{M}) \quad \text{for all } t \geqq 0 ,$$

$$\Delta_{\mathcal{N}}^{it}(\mathcal{N} \cap \mathcal{M})\Delta_{\mathcal{N}}^{-it} \subset (\mathcal{N} \cap \mathcal{M}) \quad \text{for all } t \geqq 0 ,$$

and

$$J_{\mathcal{M}} \mathcal{N} J_{\mathcal{M}} = \mathcal{N} ,$$

these data define in a canonical way a conformal quantum field theory on a circle. Conversely, the chiral part of a conformal quantum field theory in two dimensions always yields such data in a natural way.

1. Introduction

It is well known that conformal quantum field theory in two dimensions factor into two chiral conformal theories on the lightrays, see [5]. In the framework of Algebraic Quantum Field Theory, see [6], they are described by a net $\mathcal{A}(I)$ of von-Neumann algebras, indexed by the set $\mathcal{J}$ of proper intervals $I \subset S^1$, with

1. $\mathcal{A}(I) \subset \mathcal{A}(J)$ if $I \subset J$ (isotony)
2. $\mathcal{A}(I) \subset \mathcal{A}(J)'$ if $I \cap J = \emptyset$ (locality),

acting on a Hilbert space $\mathcal{H}$. On $\mathcal{H}$ there is given a strongly continuous unitary positive energy representation U of $Sl(2, \mathbf{R})/\mathbf{Z}_2$ with a unique invariant vacuum vector Ω. The net transforms covariantly under this representation.

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Let $\mathcal{M} = \mathcal{M}(\cap)$ be the algebra of the upper half circle and $\mathcal{N} = \mathcal{M}(\subset)$ be of the left half circle. Assuming the net to be generated by Wightman Fields, which generically seems to be case, [7], we may apply the results of Bisognano and Wichman, [1], to $\mathcal{M}$ and Ω. They showed that the modular group $\sigma^t_{\mathcal{M}}$ associated with $(\mathcal{M}, \Omega)$ acts geometrically as a Lorentz boost. Especially one gets

$$\text{a) } \sigma^{-t}_{\mathcal{M}}(\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N} \cap \mathcal{M} \quad \text{for all } t \geqq 0 ,$$

moreover by the Reeh–Schlieder property

$$\text{b) } \Omega \text{ is a standard vector for } \mathcal{N} \cap \mathcal{M} .$$

By the $Sl(2, R)/Z_2$-covariance this is also true for $\mathcal{M}$ exchanged by $\mathcal{N}$ and $t \leqq 0$. The work of Borchers, [2], shows that $J_{\mathcal{N}}$ acts on the net like a reflection. Especially one gets

$$\text{c) } J_{\mathcal{N}} \mathcal{M} J_{\mathcal{N}} = \mathcal{M} .$$

In this paper we conversely show that any pair of von-Neumann-algebras $\mathcal{N}$ and $\mathcal{M}$ with a common cyclic and separating vector Ω obeying the above relations in a canonical way gives rise to a conformal field theory on the circle in the sense described above. The crucial observation is that half-sided modular inclusions carry a rich symmetry. For the reader's convenience we recall some results obtained in [9, 10].

2. Half-Sided Modular Inclusions and Symmetries

Assume $\tilde{\mathcal{M}} \subset \mathcal{M}$ to be von-Neumann-Algebras acting on a Hilbert space $\mathcal{H}$, and $\Omega \in \mathcal{H}$ a common cyclic and separating vector. Let $\Delta^{-it}_{\mathcal{M}} \tilde{\mathcal{M}} \Delta^{it}_{\tilde{\mathcal{M}}} \subset \tilde{\mathcal{M}}$ for all $t \geqq 0$. We call such an inclusion $(\tilde{\mathcal{M}} \subset \mathcal{M}, \Omega)$ −half-sided modular, see [9, 10]. If one changes $t \geqq 0$ to $t \leqq 0$, we call it $+$ half-sided modular, abbreviated by $\mp$-hsm. Denote by $\Delta_{\mathcal{M}}$, $\Delta_{\tilde{\mathcal{M}}}$ the modular operators associated with $(\mathcal{M}, \Omega)$ and $(\tilde{\mathcal{M}}, \Omega)$, respectively. For such a situation we proved in [9] the following

Theorem 1. *Let $(\tilde{\mathcal{M}} \subset \mathcal{M}, \Omega)$ be a $\mp$-half-sided modular inclusion, $\Delta_{\mathcal{M}}$, $\Delta_{\tilde{\mathcal{M}}}$ the modular operators associated with $(\mathcal{M}, \Omega)$ and $(\tilde{\mathcal{M}}, \Omega)$, respectively. Assume*

$$\Delta^{it}_{\mathcal{M}} \tilde{\mathcal{M}} \Delta^{-it}_{\mathcal{M}} \subset \tilde{\mathcal{M}} \quad \textit{for all } \mp t \geqq 0 .$$

Then

$$\text{a) } \frac{1}{2\pi}(\ln(\Delta_{\tilde{\mathcal{M}}}) - \ln(\Delta_{\mathcal{M}})) \geqq 0$$

is essentially selfadjoint. Denote $U(a)$, $a \in R$, the unitary group on $\mathcal{H}$ with generator $\frac{1}{2\pi}(\ln(\Delta_{\tilde{\mathcal{M}}}) - \ln(\Delta_{\mathcal{M}}))^-$. Then

b) $\Delta^{it}_{\mathcal{M}} U(a) \Delta^{-it}_{\mathcal{M}} = \Delta^{it}_{\tilde{\mathcal{M}}} U(a) \Delta^{-it}_{\tilde{\mathcal{M}}} = U(e^{\mp 2\pi t} a)$ *for all $t, a \in R$,*

c) $J_{\mathcal{M}} U(a) J_{\mathcal{M}} = J_{\tilde{\mathcal{M}}} U(a) J_{\tilde{\mathcal{M}}} = U(-a)$ *for all $a \in R$,*

d) $\Delta^{it}_{\tilde{\mathcal{M}}} \mathcal{M} \Delta^{-it}_{\tilde{\mathcal{M}}} \subset \mathcal{M}$ *for all $\mp t \geqq 0$,*

e) $\tilde{\mathcal{M}} = U(\pm 1) \mathcal{M} U(\mp 1)$.

Proof. See Theorem 3, Cor. 6 and Cor. 7 of [9]. □

This theorem shows already that in the case of half-sided modular inclusions of von-Neumann-algebras their modular operators yield a representation of the two dimensional subgroup of $Sl(2, R)/Z_2$ generated by the translations and dilatations.

Theorem 2. *Let $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ be von-Neumann-algebras on a Hilbert space $\mathcal{H}$, $\Omega \in \mathcal{H}$ a common cyclic and separating vector. Denote by $\Delta_{\mathcal{M}}, \Delta_{\mathcal{M}_1}, \Delta_{\mathcal{M}_2}$ the modular operators to $(\mathcal{M}, \Omega)(\mathcal{M}_1, \Omega), (\mathcal{M}_2, \Omega)$. Assume*

1. $(\mathcal{M}_1 \subset \mathcal{M}, \Omega)$ – hsm,
2. $(\mathcal{M}_2 \subset \mathcal{M}, \Omega)$ + hsm,
3. $(\mathcal{M}_2 \subset \mathcal{M}_1', \Omega)$ – hsm,

where the prime indicates the commutant. Then

$$\Delta_{\mathcal{M}}^{it}, \Delta_{\mathcal{M}_1}^{ir}, \Delta_{\mathcal{M}_2}^{is}, \ t, r, s \in \mathbb{R}$$

generate a representation of the universal covering group $\tilde{Sl(2, R)}$. Denote by $\mathcal{V}$ this representation. For the image of the rotation by π in the first sheet of $\tilde{Sl(2, R)}$, denoted by $\mathrm{rot}(\pi, 1) \in \tilde{Sl(2, R)}$, one computes

$$\mathcal{V}(\mathrm{rot}(\pi, 1)) = J_{\mathcal{M}}\left(\Delta_{\mathcal{M}_1}^{\frac{-i\ln 2}{2\pi}} J_{\mathcal{M}_2} \Delta_{\mathcal{M}_1}^{\frac{i\ln 2}{2\pi}}\right). \tag{1}$$

Proof. See [10, Lemma 3 and Lemma 4 ff].

In the next section we will apply these results in order to formulate a von-Neumann algebraic characterization of conformal quantum field theories on a circle.

3. Half-Sided Modular Inclusions and Conformal Field Theory

Let $\mathcal{N}$, $\mathcal{M}$, $\mathcal{N} \cap \mathcal{M}$ be von-Neumann algebras on a Hilbert space, Ω a common cyclic and separating vector. Assume

1. $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ is – half-sided-modular,
2. $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$ is + half-sided-modular,
3. $J_{\mathcal{N}} \mathcal{M} J_{\mathcal{N}} = \mathcal{M}$,

where the modular objects are indexed by the related von-Neumann algebra as before.[1] Such a situation naturally occurs in Bisognano–Wichmann nets of conformal quantum field theories on a circle, as was mentioned in the introduction. $\mathcal{M}$ denotes in this case the observable algebra of the upper half circle, $\mathcal{N}$ of the left

[1] The importance of the third relation was already noticed by B. Schroer in [8], where the reader can find some preliminary ideas on the representation of conformal field theories from pairs of von-Neumann algebras

half circle. We will show that conversely such a pair of von-Neumann algebras yield in a canonical way a conformal quantum field theory on the circle.[2]

Theorem 3. *Let $(\mathcal{N}, \mathcal{M}, \Omega)$ be as above. Then these data define in a canonical way a local net of von-Neumann algebras on S^1, which transforms co-variantly under $Sl(2, R)/Z_2$. The representation is of positive energy,* i.e. *we get a conformal quantum field model on S^1. This net fulfills Haag Duality and the Reeh–Schlieder property.*

Proof. We will prove this theorem in several steps. Firstly we will show that

$$\Delta_{\mathcal{M}}^{it}, \Delta_{\mathcal{N} \cap \mathcal{M}}^{ir}, \Delta_{\mathcal{N}' \cap \mathcal{M}}^{is}, t, r, s \in \mathbb{R}$$

generate a representation of the group $Sl(2, R)/Z_2$. This will be done by applying Theorem 2 to $\mathcal{M}_1 = \mathcal{N} \cap \mathcal{M}$, $\mathcal{M}_2 = \mathcal{N}' \cap \mathcal{M}$. Secondly we will define a $Sl(2, R)/Z_2$-covariant net on S^1 by using the modular representation of the Moebius group together with defining $\mathcal{M}$ to be the algebra associated to the upper half circle. In the last step we will show that this net is isotonic and local.

In order to apply Theorem 2 we have to prove

a) $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ is $-$ hsm,

b) $((\mathcal{N}' \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ is $+$ hsm,

c) $((\mathcal{N}' \cap \mathcal{M}) \subset (\mathcal{N}' \vee \mathcal{M}'), \Omega)$ is $-$ hsm.

a) is just one of the assumptions, (1).
b) Notice that by assumption 3 we get $[J_{\mathcal{N}}, \Delta_{\mathcal{M}}] = 0$. Applying $\mathrm{Ad}(J_{\mathcal{N}})$ to the $-$ hsm inclusion a), one immediately obtains b).
c) is the most difficult one. Applying Theorem 1 to the $+$ hsm inclusion $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$, one gets a one parameter group

$$\tilde{U}(a) := \exp\left(\frac{1}{2\pi} a (\ln \Delta_{\mathcal{N} \cap \mathcal{M}} - \ln \Delta_{\mathcal{N}})\right) \tag{2}$$

with

$$\mathcal{N} \cap \mathcal{M} = \tilde{U}(-1) \mathcal{N} \tilde{U}(1) \tag{3}$$

and

$$\Delta_{(\mathcal{N}' \vee \mathcal{M}')}^{it} = \tilde{U}(-1) \Delta_{\mathcal{N}}^{-it} \tilde{U}(1) = \tilde{U}(-1 + e^{-2\pi t}) \Delta_{\mathcal{N}}^{-it} . \tag{4}$$

Therefore we get

$$\begin{aligned} \mathrm{Ad}(\Delta_{\mathcal{N}' \vee \mathcal{M}'}^{it})(\mathcal{N}' \cap \mathcal{M}) &= \mathrm{Ad}(\tilde{U}(-1 + e^{-2\pi t}) \Delta_{\mathcal{N}}^{-it})(\mathcal{N}' \cap \mathcal{M}) \\ &= \mathrm{Ad}(\tilde{U}(-1 + e^{-2\pi t}) \Delta_{\mathcal{N}}^{-it} J_{\mathcal{N}})(\mathcal{N} \cap \mathcal{M}) \end{aligned} \tag{5}$$

by assumption 3, and using Theorem 1c),

$$= \mathrm{Ad}(J_{\mathcal{N}} \tilde{U}(1 - e^{-2\pi t}) \Delta_{\mathcal{N}}^{-it})(\mathcal{N} \cap \mathcal{M}) . \tag{6}$$

[2] In an earlier version of this paper the author proposed different conditions on the relative position of two algebras $\mathcal{N}$, $\mathcal{M}$, in order to characterize uniquely a conformal net on the circle. D. Buchholz pointed out to us a serious error in the previous construction of the net. The author thanks D. Buchholz warmly for his valuable comments

By assumption 2 $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$ is $+$ hsm, and Theorem 1b), e) shows

$$\mathrm{Ad}(\tilde{U}(a))(\mathcal{N} \cap \mathcal{M}) \subset (\mathcal{N} \cap \mathcal{M}) \quad \text{for } a \leqq 0 . \tag{7}$$

Putting everything together we get

$$\mathrm{Ad}(\Delta^{it}_{(\mathcal{N}' \vee \mathcal{M}')})(\mathcal{N}' \cap \mathcal{M}) \subset \mathrm{Ad}(J_{\mathcal{N}})(\mathcal{N} \cap \mathcal{M}) = \mathcal{N}' \cap \mathcal{M} \tag{8}$$

for $t \leqq 0$. Therefore we can apply Theorem 2.

In order to reduce the symmetry to the Moebius group $Sl(2, R)/Z_2$, we make use of (4) and calculate

$$\begin{aligned} \Delta_{\mathcal{N} \cap \mathcal{M}}^{\frac{-i\ln 2}{2\pi}}(\mathcal{N}' \cap \mathcal{M})\Delta_{\mathcal{N} \cap \mathcal{M}}^{\frac{i\ln 2}{2\pi}} &= \mathrm{Ad}(\Delta_{\mathcal{N}}^{\frac{-i\ln 2}{2\pi}}\tilde{U}(-1))(\mathcal{N}' \cap \mathcal{M}) \\ &= \mathrm{Ad}(\Delta_{\mathcal{N}}^{\frac{-i\ln 2}{2\pi}}\tilde{U}(-1)J_{\mathcal{N}})(\mathcal{N} \cap \mathcal{M}) \\ &= \mathrm{Ad}(\Delta_{\mathcal{N}}^{\frac{-i\ln 2}{2\pi}}J_{\mathcal{N}}\tilde{U}(1))(\mathcal{N} \cap \mathcal{M}) . \end{aligned} \tag{9}$$

By Theorem 1e) we get

$$= \mathrm{Ad}(\Delta_{\mathcal{N}}^{\frac{-i\ln 2}{2\pi}}J_{\mathcal{N}})(\mathcal{N}) = \mathcal{N}' . \tag{10}$$

Therefore

$$\Delta_{\mathcal{N} \cap \mathcal{M}}^{\frac{-i\ln 2}{2\pi}} J_{(\mathcal{N}' \cap \mathcal{M})} \Delta_{\mathcal{N} \cap \mathcal{M}}^{\frac{i\ln 2}{2\pi}} = J_{\mathcal{N}} . \tag{11}$$

By assumption 3 we have $[J_{\mathcal{N}}, J_{\mathcal{M}}] = 0$, i.e. by Theorem 2 the rotation by 2π lies in the kernel of the representation of $\widetilde{Sl(2, R)}$. The symmetry reduces to the Moebius group $Sl(2, R)/Z_2$.

Let us denote the representation of the Moebius group $Sl(2, R)/Z_2$ by $\mathcal{V}$. The above calculation especially shows

$$\mathcal{V}(\mathrm{rotation}(\pi)) = J_{\mathcal{N}} J_{\mathcal{M}} , \tag{12}$$

where *rotation* is the 1-parameter subgroup of rigid rotations in $Sl(2, R)/Z_2$. Next we want to define a net of algebras indexed by the proper intervals of S^1.

Let $(a, b) \subset S^1$ be a proper interval. There exists an element $g_{(a,b)} \in Sl(2, R)/Z_2$ which maps the upper half circle $(1, -1)$ onto (a, b),

$$g_{(a,b)}((1, -1)) = (a, b) . \tag{13}$$

Let us define

$$\mathcal{M}(a, b) := \mathcal{V}(g_{(a,b)})\mathcal{M}\mathcal{V}(g_{(a,b)})^* . \tag{14}$$

This definition does not depend on the special choice of $g_{(a,b)} \in Sl(2, R)/Z_2$ with the above property. To see this notice that elements of the Möbius group which map the upper half circle onto itself are dilatations. Starting from $g(\pm 1) = \pm 1$ this is a one line calculation. Then it is easily seen that two elements $g_1, g_2 \in Sl(2, R)/Z_2$ mapping the upper half circle onto (a, b) can only differ in the dilatation factor. But the dilatations are represented by the modular group of $\mathcal{M}$,

$$\mathcal{V}(\mathrm{dilatation}(\lambda)) = \Delta_{\mathcal{M}}^{\frac{-i\ln\lambda}{2\pi}} , \tag{15}$$

which proves the well definedness of $\mathcal{M}(a, b)$. By the very construction the net $\mathcal{M}(a, b)$ transforms covariantly w.r.t. to the representation $\mathcal{V}$ of $Sl(2, R)/Z_2$.

Let us make a simple but crucial observation. By the above calculation we know $\mathcal{V}(\mathrm{rotation}(\pi)) = J_{\mathcal{N}}J_{\mathcal{M}}$. Moreover it is easily seen that

$$g_{(b,a)} := g_{(a,b)} \cdot \mathrm{rotation}(\pi) \tag{16}$$

maps the upper half circle onto the complement $S^1\backslash[a, b]$. This yields

$$\begin{aligned}\mathcal{M}(b, a) &= \mathcal{V}(g_{(b,a)})\mathcal{M}\mathcal{V}(g_{(b,a)})^* \\ &= \mathcal{V}(g_{(a,b)})\mathcal{V}(\text{rotation}(\pi))\mathcal{M}\mathcal{V}(\text{rotation}(\pi))^*\mathcal{V}(g_{(a,b)})^* \\ &= \mathcal{V}(g_{(a,b)})J_{\mathcal{M}}J_{\mathcal{N}}\mathcal{M}J_{\mathcal{N}}J_{\mathcal{M}}\mathcal{V}(g_{(a,b)})^* \\ &= \mathcal{V}(g_{(a,b)})\mathcal{M}'\mathcal{V}(g_{(a,b)})^* \\ &= (\mathcal{V}(g_{(a,b)})\mathcal{M}\mathcal{V}(g_{(a,b)})^*)' = (\mathcal{M}(a, b))' . \end{aligned} \tag{17}$$

Let us prove isotony of the net. For this one notices that the translations are represented by

$$\mathcal{V}(\text{translation}(a)) := \exp\left(\frac{ia}{2\pi}(\ln \Delta_{\mathcal{N}\cap\mathcal{M}} - \ln \Delta_{\mathcal{M}})\right), \tag{18}$$

and Theorem 1 implies

$$\mathrm{Ad}(\mathcal{V}(\text{translation}(a)))(\mathcal{M}) \subset \mathcal{M} \quad \text{for } a \geqq 0 . \tag{19}$$

Therefore

$$\mathcal{M}(a, -1) \subset \mathcal{M}(b, -1) \quad \text{for} \quad (a, -1) \subset (b, -1) , \tag{20}$$

and using the rotations this proves

$$\mathcal{M}(a, c) \subset \mathcal{M}(b, c) \quad \text{for} \quad (a, c) \subset (b, c) . \tag{21}$$

Passing to commutants and making use of relation (17) completes the proof of isotony,

$$\mathcal{M}(c, a) \subset \mathcal{M}(c, b) \quad \text{for} \quad (c, a) \subset (c, b) . \tag{22}$$

What is left to be proven is locality. But now this is nearly trivial. Let (a, b), $(c, d) \subset S^1$ be proper intervals with empty intersection. Then $(a, b) \subset (d, c)$, and by isotony and relation (17),

$$\mathcal{M}(a, b) \subset \mathcal{M}(d, c) = (\mathcal{M}(c, d))' . \tag{23}$$

The rest of the theorem follows easily. □

Let me finish with two remarks.

Remark 1. Consider $(\mathcal{N}, \mathcal{M}, \Omega)$ as in Theorem 3. Then in particular $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ is a half-sided modular standard inclusion, i.e. Ω is also cyclic and separating for $(\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}$. We can apply Theorem 2 to $\mathcal{M}_1 = \mathcal{N} \cap \mathcal{M}$, $\mathcal{M}_2 = (\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}$, $\mathcal{M} = \mathcal{M}$, as it was shown in [10]. Again we get a representation of $Sl(2, R)/Z_2$ by the modular groups $\Delta_{(\mathcal{N}\cap\mathcal{M})}$, $\Delta_{(\mathcal{N}\cap\mathcal{M})'\cap\mathcal{M}}$, $\Delta_{\mathcal{M}}$, see [10]. Define

$$U := J_{\mathcal{N}'\cap\mathcal{M}} J_{(\mathcal{N}\cap\mathcal{M})'\cap\mathcal{M}} . \tag{24}$$

Using the various commutation relations one easily sees that

$$[\mathcal{V}(\text{translation}(a)), U] = 0 \quad \text{for all } a \in R . \tag{25}$$

Furthermore one gets

$$U = 1 \Leftrightarrow (\mathcal{N}' \cap \mathcal{M}) = (\mathcal{N}' \vee \mathcal{M}') \cap \mathcal{M} , \tag{26}$$

i.e. strong additivity of the net, see [3]. Let me prove the second part.

Denote

$$U(a) = \exp(ia(\ln \Delta_{\mathcal{N}' \cap \mathcal{M}'} - \ln \Delta_{\mathcal{M}})) ,$$
$$\tilde{U}(a) = \exp(ia(\ln \Delta_{(\mathcal{N}' \vee \mathcal{M}') \cap \mathcal{M}} - \ln \Delta_{\mathcal{M}})) . \tag{27}$$

Then we get from Theorem 1 and the assumptions,

$$\begin{aligned} \Delta^{it}_{(\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}} &= \tilde{U}(-1) \Delta^{it}_{\mathcal{M}} \tilde{U}(1) \\ &= \Delta^{i\frac{\ln 2}{2\pi}}_{\mathcal{M}} \tilde{U}(-2) \Delta^{it}_{\mathcal{M}} \tilde{U}(2) \Delta^{-i\frac{\ln 2}{2\pi}}_{\mathcal{M}} . \end{aligned} \tag{28}$$

But from $U = 1$ we conclude

$$\begin{aligned} \tilde{U}(2) &= J_{(\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}} J_{\mathcal{M}} = J_{\mathcal{N}' \cap \mathcal{M}} J_{\mathcal{M}} \\ &= U(2) , \end{aligned} \tag{29}$$

and therefore

$$\Delta^{it}_{(\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}} = \Delta^{it}_{\mathcal{N}' \cap \mathcal{M}} . \tag{30}$$

Now $(\mathcal{N}' \cap \mathcal{M}) \subset (\mathcal{N} \cap \mathcal{M})' \cap \mathcal{M}$, and the above equality proves equality of the algebras.

The converse is trivial.

Remark 2. We did not use any factor property in this work. In the case $\mathcal{M}$ is a factor we proved in [9] that $\mathcal{M}$ has to be of type III_1. It was also shown in [9] that in this case the associated conformal field theory has a unique translation invariant vector, i.e. a unique vacuum vector.

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