Commun. Math. Phys. 150, 23–44 (1992)

Communications in
Mathematical
Physics

Local Rules for Quasiperiodic Tilings of Quadratic 2-Planes in R^4

Le Tu Quoc Thang [1], **Sergey Piunikhin** [1], **and Vladimir Sadov** [2,⋆]

[1] Moscow State University, Moscow, Russia
[2] ITP Landau, Kosygina str. 2, 117–334 Moscow, Russia

Received September 15, 1991

Abstract. We prove that quasiperiodic tilings of the plane, appearing in the strip projection method always admit local rules, when the linear embedding of $\mathbb{R}^2$ in $\mathbb{R}^4$ has quadratic coefficients. These local rules are constructed and studied. The connection between Novikov quasicrystallographic groups and the quasiperiodic tilings of Euclidean space is explained. All the point groups in Novikov's sense, compatible with these local rules, are enlisted. The two-dimensional quasicrystals with infinite-fold rotational symmetry are constructed and studied.

Introduction

Quasicrystals (QC) are the quasiperiodic tilings of the Euclid space $\mathbb{R}^k$ by a finite (up to translations) number of polyhedra. For the history and reviews we refer to [2–7]. By now, several approaches have been suggested.

One of them, initiated by S. P. Novikov in 1986, is based upon the following definition of the quasicrystallographic group:

Definition. We shall call a finite-generated abelian subgroup $T \subset \mathbb{R}^k$, which generates $\mathbb{R}^k$ as a linear space the quasilattice in $\mathbb{R}^k$.

Definition. A subgroup G of the group E_k of all isometries of k-dimensional Euclid space is called a *k-dimensional quasicrystallographic group*, iff its intersection with the subgroup $\mathbb{R}^k \subset E_k$ of all translations is some quasilattice $T \subset \mathbb{R}^k$.

Definition. The above defined quasilattice $T \subset \mathbb{R}^k$ is called *the subgroup of translations of the quasicrystallographic group G*, and the factor-group $R = G/T$ is called *the point group*, or *the group of orthogonal parts of the quasicrystallographic group G*.

⋆ Address after September 1, 1992: Dept. of Physics, Harvard University, Cambridge, MA 02138, USA

The quasicrystallographic group is called crystallographic in usual sense, if its subgroup of translations T has rank k (or, equivalently, T is a lattice in $\mathbb{R}^k$).

In the paper [1] it was proved that if the point group $R = G/T$ is finite, then G is isomorphic to some n-dimensional crystallographic group. An example of two-dimensional quasicrystallographic groups, containing rotations of infinite order, was there constructed. The first example of such a group was constructed by A. Veselov.

Another aspect of (quasi)crystallography is the problem of what kind of the QC order might appear in nature as a property of real materials. It seems sound to require the "physical" QC to admit restoration by means of only information of its local structure [i.e., on the finite number of admissible configurations of (possibly decorated) tiles]. The well-known example of such Local(matching) Rules are the de Bruijn' arrowed rhombi for the Penrose tilings. According to [2], quasiperiodic is any tiling of the plane by these rhombi, providing the obvious matching condition is met (that the common edges of neighboring rhombi are to have definite arrows on them).

The term "Local Rules" (LR) was suggested by Katz [8] and Levitov [9] for the matching prescriptions, enforcing quasiperiodicity.

We fix standard Euclidean coordinates in $\mathbb{R}^4$ and $\mathbb{R}^2$. Let $\Theta : \mathbb{R}^2 \to \mathbb{R}^4$ be a linear embedding. For a generic Θ, there is a quasiperiodic tiling on $\Theta(\mathbb{R}^2)$ that appears from the so-called strip projection procedure [5]. Levitov has proved that if the embedding is quadratic (that is, all coefficients of Θ are in $\mathbb{Z}_{[\sqrt{D}]}$) and is "non-degenerate" then the tiling admit "weak" LR, and he proved that if the embedding is not quadratic and "in a general position" then "strong" LR do not exist. He conjectured if the embedding is not quadratic then there are no local rules. In this paper we prove that if the embedding is quadratic, then the tilings admit LR, even in degenerate cases. There is a crucial difference between our and Levitov's concepts: we refine the structure of the quasiperiodic tilings by coloring their prototiles, so some prototiles are regarded as different, but in Levitov's sense are the same. This is the reason why Burkov's result [11] does not contradict ours.

Levitov [9] proved that a quasicrystal, having LR, can admit (finite-fold) rotational symmetry only of order 10, 12. We prove here that the quasilattice (the support of Fourier coefficients) of the quasicrystal, having local rules, can also admit infinite-fold rotational symmetry with the basic rotation on the angle φ, where

$$\cos\varphi = \frac{m \pm \sqrt{m^2 - 4k}}{4}\,; \qquad m, l \in \mathbb{Z}; \qquad -1 < \frac{m \pm \sqrt{m^2 - 4k}}{4} < 1\,.$$

We wish to stress that this is the property of the quasilattice. Any particular quasicrystal of this type (for instance the amplitudes of Fourier coefficients) doesn't admit infinite-fold symmetry.

The group of orthogonal parts here submerges naturally in $SO(3, 1, \mathbb{Z})$ instead of $SO(4, \mathbb{Z})$ in the case of 5-, 8-, 10- or 12-fold symmetry.

To show the connection between these two approaches to QC, we enlist here all the types of rotational symmetry, compatible with quadratic embeddings. The quasiperiodic tilings with these types of rotational symmetry are studied in more detail.

The paper is organized as follows:

In Sect. 1 the definitions and notations are introduced.

In Sect. 2 we recall the cut method and the strip projection method and explain how to define matching rules.

In Sect. 3 we study 2-planes in $\mathbb{R}^4$, they are important in our construction.

In Sect. 4 the local rules are constructed.

In Sect. 5 the connection of the quasicrystallographic groups in Novikov's sense and the quasiperiodic tilings of k-dimensional space is explained.

In Sect. 6 all types of rotational symmetry, compatible with the quadratic embeddings are classified.

In Sect. 7 some interesting examples are given.

1. Basic Definitions and Notations

1.1. Suppose $P_1, P_2, \ldots, P_k$ are convex polygons in $\mathbb{R}^2$. A polygon P is called congruent to P' (we write $P \equiv P'$) if $P' = P + \alpha$, $\alpha \in \mathbb{R}^2$. A tiling of $\mathbb{R}^2$ with prototiles $P_1, \ldots, P_k$ is a partition of $\mathbb{R}^2$ by polygons, congruent to $P_1, \ldots, P_k$, such that intersection of two polygons having non-empty intersection is either a common edge or a common vertex. This means that the plane $\mathbb{R}^2$ has a polyhedral structure. Polygon P_i is colored by i, some of P_i may be congruent but they are distinguished by colors. The set of colors is $\{1, 2, \ldots, k\}$.

Color of a vertex of a tiling is the collection $(i_1, i_2, \ldots, i_m)$, defined up to cyclic permutations, of colors of the polygons surrounding this vertex in the counterclockwise direction (we suppose that orientation on $\mathbb{R}^2$ is fixed). Let A be a finite set of such collections. A tiling is called satisfying A-rules if color of any its vertex is in the set A. A-rules are called quasiperiodic local rules (QPLR) of a quasiperiodic tiling if

i) this tiling satisfies these A-rules,

ii) every tiling satisfying A-rules is quasiperiodic.

Note that local rules defined by Levitov [9] are more general, but it is enough for us to use this special type of local rules.

Two tilings T with prototiles $P_1, P_2, \ldots, P_k$ and T' with prototiles $Q_1, Q_2, \ldots, Q_k$ are topologically equivalent if there exists a piecewise-linear homeomorphism $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$, that transfers the first polyhedral structure to the second. This means that the orders of tiles in the two tilings are the same, only shapes of prototiles are different. If the first tiling admits QPLR, then the second obviously admits the same QPLR.

1.2. In $\mathbb{R}^4$ we fix the standard basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and the standard Euclidean scalar product. A point in $\mathbb{R}^4$ is also regarded as a vector, so that we can define $x + y$, where x, y are points in $\mathbb{R}^4$. If X, Y are subsets of $\mathbb{R}^4$ then we denote $X + Y$ the set $\{x + y, x \in X, y \in Y\}$, $-X$ the set $\{-x, x \in X\}$. Let $\mathbb{Z}^4$ be the integral lattice, γ be the unit hypercube:

$$\gamma = \left\{ \sum_{i=1}^{4} x_i \varepsilon_i \,, \quad x_i \in [0, 1], i = 1, 2, 3, 4 \right\}.$$

For $I = (i_1, i_2)$, $1 \le i_1 < i_2 \le 4$ let

$$\gamma_I = \{x_{i1}\varepsilon_{i1} + x_{i2}\varepsilon_{i2}\}\,, \qquad x_{i1}, x_{i2} \in [0, 1]\,.$$

γ_I is a 2-facet of the unit cube. Let $I^c = \{1, 2, 3, 4\}/I$.

We shall always have things to do with two 2-dimensional planes (or briefly 2-planes) E and E' going through 0 in $\mathbb{R}^4$, such that $E \cap E' = \{0\}$. Denote π the projector along E' on E and π' the projector along E on E'. Put $e_i = \pi(\varepsilon_i)$, $e'_i = \pi'(\varepsilon_i)$, $i = 1, \ldots, 4$. Let $E^\perp$ be the 2-plane perpendicular to E and pr be the projector along $E^\perp$on E.

If $v \in \mathbb{R}^4$, $v \notin E$, $v \notin E'$ then let $F(v)$ be the 2-plane generated by $\pi(v)$ and $\pi'(v)$. The plane $F(v)$ goes through v and intersects E and E' by lines.

A 2-plane E going through 0 is called irrational if there is no integral point belonging to E except 0. We always suppose that E and E' are irrational. We shall call a prism any set of the type $X + Y$, where $X \subset E$, $Y \subset E'$. For example $F(v)$ is a prism (2-dimensional):

$$F(v) = \pi(F(v)) + \pi'(F(v)).$$

More generally $\forall \xi \in \mathbb{R}^4$, $F(v) + \xi$ is a prism.

2. The Cut Method, Strip Projection Method and Matching Rules

2.1. Let us briefly recall these methods used to construct quasiperiodic tiling. The reader is referred to [5, 8] for full expositions on these subjects.

Let E be an irrational 2-plane. We construct a strip in $\mathbb{R}^4$ by shifting the cell γ along an affine 2-plane parallel to E:

$$S_\alpha = E + \gamma + \alpha, \qquad \alpha \in \mathbb{R}^4.$$

It is proved in [5] that for translation α such that the boundary of the strip does not contain any point of $\mathbb{Z}^4$ (i.e. α is generic), the strip S_α contains exactly an unique 2-dimensional continuous surface Π_α built up of 2-dimensional facets of the lattice $\mathbb{Z}^4$ lying in S_α. This surface Π_α goes through all the vertices of the lattice $\mathbb{Z}^4$ falling inside S_α. Π_α has an obvious polyhedral structure. The tiling of E is the projection $\mathrm{pr}\,(\Pi_\alpha)$ along $E^\perp$ on E of the surface Π_α with its polyhedral structure. The prototiles are the projections of a 2-dimensional facet of the lattice $\mathbb{Z}^4$. Note that there are no overlaps: the restriction of pr on Π_α is bijective. Vertices of the tiling are projections of all vertices of $\mathbb{Z}^4$, falling inside S_α.

If instead of projection along $E^\perp$ we take projection along E': $\pi(\Pi_\alpha)$, overlaps may happen. In this case the local rules, similar to those constructed in this paper, can also be constructed. But if there are no overlaps (for example when E' is near to $E^\perp$) we get a new quasiperiodic tiling of E, which is topologically equivalent to the old one: only shapes of prototiles are changed.

Let's now consider another construction of these tilings, known as the cut method [5]. Put

$$C_I = \pi(\gamma_I) - \pi'(\gamma_{I^c}), \qquad C_{I,\xi} = C_I + \xi,$$

where $I = (i_1, i_2)$, $1 \le i_1 < i_2 \le 4$, and $A - B$ denotes the set $\{x - y, x \in A, y \in B\}$.

$$I, I^c \in M = \{(1,2), (1,3), (1,4), (2,3), (2,3), (2,4)\}.$$

Denote $P_I = \pi(\gamma_I)$, $P'_I = \pi'(\gamma_{I^c})$.

Each C_I is a prism. If a 2-plane $E + \alpha$ ($\alpha \in E'$) intersects with a prism $C_{I,\xi}$ then the intersection is congruent to P_I. Consider the collection $\{C_{I,\xi}, I \in M, \xi \in \mathbb{Z}^4\}$. If $E = E^\perp$ then this collection covers the whole $\mathbb{R}^4$ without overlaps and holes, i.e. $\left(\bigcup_{I,\xi} C_{I,\xi}\right)$ is a partition of $\mathbb{R}^4$. This partition is called an "oblique periodic tiling" of $\mathbb{R}^4$ in [5] because it is invariant under translations of $\mathbb{Z}^4$. The union of the six prism C_I, $I \in M$, is a fundamental domain of the group $\mathbb{Z}^4$, one can regard it as a rearrangement of the unit cell γ. Every 2-plane $E + \alpha$, where $\alpha \in E'$ is generic, inherits a unique quasiperiodic tiling from the oblique periodic tiling. This tiling is exactly the tiling obtained by projecting the surface Π with its polyhedral structure on $E+\alpha$.

When $E' \neq E^{\perp}$ in the union $\bigcup_{I,\xi} C_{I,\xi}$ there may be overlaps. For every $\alpha \in E'$ we cover the plane $E+\alpha$ by its intersection with $\{C_{I,\xi}, I \in M, \xi \in \mathbb{Z}^4\}$.

Theorem A. *Suppose α is generic. The cover of $E+\alpha$ by intersections with the collection $\{C_{I,\xi}, I \in M, \xi \in \mathbb{Z}^4\}$ is exactly the cover $\pi(\Pi_\alpha)$ on $E+\alpha$: if an intersection $C_{I,\xi} \cap (E+\alpha)$ is not empty then this intersection is the projection of some 2-facet of Π_α, and inversely every projection of some 2-facet of Π_α is an intersection $(E+\alpha) \cap C_{I,\xi}$ for some (I,ξ).*

The proof is actually contained in [5]. Note that the existence of Π_α has been proved in [5]; after this we need only repeat the arguments in [5, Chap. V].

Note. This cover may have overlaps. This is an actual tiling iff the collection $\{C_{I,\xi}, I \in M, \xi \in \mathbb{Z}^4\}$ covers $\mathbb{R}^4$ without overlaps.

The set $-P'_I + \pi'(\xi)$ is called the existence domain of the tile $P_I + \pi(\xi)$.

Let us denote ∂C the boundary of C_I; it is a cellular complex of dimension 3. We decompose ∂C_I into two parts:

$$\partial C_I = (\partial P_I - P_{I'}) \cup (P_I - (\partial P_{I'}) .$$

Denote the first by $\partial' C_I$, the second by $\partial_{\|} C_I$: Put

$$B = \left(\bigcup_{I \in M} \partial_{\|} C_I \right) + Z^4 , \qquad B' = \left(\bigcup_{I \in M} \partial' C_I \right) + Z^4 .$$

The set of plane-cuts, which do not intersect B, is generic. B is called the forbidden set. Each generic plane-cut $E+\alpha$ defines a quasiperiodic tiling: the intersection of $E+\alpha$ with B' is the set of boundaries of tiles in this tiling.

Remark. If α is not generic then the plane $E+\alpha$ defines not one but several tilings. We shall call them the quasiperiodic tilings defined by this non-generic cut.

2.2. From now till Sect. 5 we always suppose that overlaps don't happen.

We construct matching rules as follows. We divide each prism C_I into smaller prisms by dividing the existence domains into a number of convex polygons: $P'_I = \bigcup_j P'_j$, then $C_I = \bigcup_j (P_I - P'_j) = \bigcup_j (P_j - P'_j)$, where $P_j = P_I$. Let $C_j = P_j - P'_j$.

Instead of six prisms $C_I, I \in M$, we have the refined collection $\{C_i\}$, $i = 1, \ldots, k$. Of course $k \geq 6$. One can divide the boundary of C_i into two parts as before:

$$\partial C_i = \partial_{\|} C_i + \partial' C_i , \qquad \partial' C_i = -P_{i'} + \partial P_i , \qquad \partial_{\|} C_i = P_i - \partial P_i ,$$

Put

$$\tilde{B} = \bigcup_i \partial_{\|} C_i + Z^4 , \qquad \tilde{B}' = \bigcup_i \partial' C_i + Z^4 .$$

A cut $E+\alpha$ is called generic if $(E+\alpha) \cap \tilde{B} = \emptyset$. We suppose $\alpha \in E'$, then α is generic if $\alpha \notin \pi'(\tilde{B})$. We color the prototile P_i by color i. Note that each P_i is congruent to one of six projections $P_I = \pi(\gamma_I)$. Note that $\tilde{B}' = B'$.

We shall call a section a graph of some function $\varrho: E \to E'$. We denote the graph by Ω and the point $\varrho(x)+x$ by $\Omega(x)$. If a section Ω does not intersect the forbidden set $\tilde{B}$ then Ω defines a unique tiling of E by the projection $\pi(\Omega \cap \tilde{B}')$ of its intersection with $\tilde{B}'$ on E. This tiling may not be quasiperiodic, but if Ω is a generic 2-plane parallel to E then the tiling is quasiperiodic. Let W be the strip consisting of all prisms $C_i + \mathbb{Z}^4$ having non-empty intersection with Ω. The tiling is uniquely defined by the strip W. A section $\Omega \subset \mathbb{R}_4 \backslash \tilde{B}$ is called reduced to a plane section $\Omega' = E + \alpha$ if for every $x \in E$ the segment $[\Omega(x), \Omega'(x)]$ does not intersect $\tilde{B}$ or intersects at $\Omega'(x)$. The plane section Ω' is not necessarily generic. Obviously if Ω is reduced to a plane section Ω' then the tiling defined by Ω is the same as that of Ω'.

We shall construct matching rules such that any tiling satisfying these rules is a tiling defined by some section $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$. These matching rules are easy to construct. Then we shall choose partitions of the prisms C_I such that every section $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$ is reduced to a plane section. This is the second part, more difficult.

The first problem is solvable for any partition of the prisms C_I. We fix a generic cut $E + \alpha$ and consider its quasiperiodic tiling with prototiles $P_1, \ldots, P_k$. Denote A the set of all colors of its vertices. It is easy to see that A is the same for all generic plane cuts $E + \beta$: all plane-cut tilings satisfy A-rules.

Theorem 1. *Every tiling satisfying A-rules is a tiling defined by some section $\Omega \subset R^4 \backslash \tilde{B}$.*

Proof. Let T be a tiling of $\mathbb{R}^2 = E$, satisfying A-rules. Fix a tile P of this tiling; we may suppose that $P = P_1$, and $\pi(C_1) = P$. We recall that $\mathbb{R}^4$ is covered by the collection $C_{i,\xi}$, $i = 1, \ldots, k$, $\xi \in \mathbb{Z}^4$. If $(i, \xi) \neq (j, \eta)$ then the projection $(C_{1,\xi})$ with its color is not equal to $\pi(C_{1,\eta})$ because both E and E' are irrational. This means that there is at most one prism from the collection $(C_{1,\xi})$ that projects on a fixed colored tile of T. We call this prism, if it exists, the lifting prism of this tile. The tile P, for example, has a lifting prism. Now we prove that every tile of T has a lifting prism. If Q_1 and Q_2 are two neighbour tiles in T and Q_1 has lifting prism, then by definition of A-rules, Q_2 also has lifting prism. So all tiles of T have lifting prisms. If v is a vertex of the tiling T, $Q_1, \ldots, Q_1$ are tiles surrounding this vertex, then their lifting prisms have a common point $\tilde{v}$ (precisely their intersection is a 2-dimensional convex set). This follows from the definition of A-rules and the uniqueness of the lifting prisms. We have also $\pi(\tilde{v}) = v$. It is easy to see that an appropriate simplicial 2-dimensional complex, generated by all lifting vertices $\{\tilde{v}\}$ is a continuous section and is contained in the union of all the lifting prisms. Theorem 1 is proved. □

2.3. Now we find partitions of the existence domains that solve the second problem. Recall that we have fixed 2-planes E, E'. So we can define 2-planes $F_i = F(\varepsilon_i)$, $i = 1, 2, 3, 4$. It will be shown below that if E is quadratic and irrational and E' is algebraically conjugated to E, then these four 2-planes are always integral. We also have the oblique periodic tiling of $\mathbb{R}^4$ with prototiles $C_I = P_I - P'_I$, $I \in M = \{(1,2), (1,3), (1,4), (2,3), (2,3), (2,4)\}$.

Let $F_5, \ldots, F_n$ be some integral 2-planes going through 0 and intersecting E and E' by lines. Because $\dim(F_i \cap E) = \dim(F_i \cap E') = 1$, we have: $F_i = \pi(F_i) + \pi'(F_i)$, $i = 1, \ldots, n$, i.e. all F_i are 2-dimensional prisms, and so are all $F_i + \xi$, $\forall \xi \in \mathbb{R}^4$.

We also have: $\dim(\tilde{K}_i) = 3$, where $\tilde{K}_i = F_i + E$.

Let $d = \max\limits_{1 \le i \le 4}$ (lengths of e_i) and Z_d be the 2-dimensional ball in E with center at 0 and radius d (Z_d is the closed ball). Put $K_i = F_i + Z_d$.

Of course $\tilde{K}_i \supset K_i$, and $\tilde{K}_i$ is the unique 3-plane containing K_i. Let G_i be subgroups of $\mathbb{Q}^4$ such that $\mathbb{Z}^4 \subset G_i$. $|G_i : \mathbb{Z}^4| < \infty$, $i = 1, \ldots, n$. We shall call a wall any set of the type $K_i + \xi$, $i = 1, \ldots, n$, $\xi \in G_i$.

Proposition 1. *The set of all walls* $\bigcup_{i=1}^{n}(K_i + G_i)$ *is discrete in* $\mathbb{R}^4$*: every compact intersects with only a finite number of walls.*

Proof. At first note that the set $F_i + G_i$ is discrete in $\mathbb{R}^4$. This follows from rationality of F_i $(i = 1, \ldots, n)$. If X is compact in $\mathbb{R}^4$, then $X + Z_d$ is also a compact, and $X + Z_d$ intersects with only a finite of number of 2-planes from $(F_i + G_i)$. This means that X intersects only a finite number of walls. $\square$

Let $\mathscr{F}_i = F_i + G_i$, $\mathfrak{F}_i = K_i + G_i$. The set $\mathfrak{F}_i, \mathscr{F}_i$ depend on the 2-plane F_i and group G_i.

We construct partitions of prototiles of the oblique periodic tiling as follows: Fix $I \in M$, the prism C_I intersects with only a finite number of walls $W_I^1, W_I^2, \ldots, W_I^s$. Each wall W_I^i is contained in a unique 3-plane $\tilde{W}_I^i$ $(i = 1, \ldots, s)$. These 3-planes $\tilde{W}_I^i$ $(i = 1, \ldots, s)$ divide the prism C_I into smaller pieces: $C_I = \bigcup_j C_I^j$. It's easy to see that these pieces are prisms (because all 3-planes $\tilde{W}_I^i$ contain a 2-plane parallel to E), and they have the same projection on E as C_I has: $\pi(C_I) = \pi(C_I^j)$. This is the required partition of C_I; $I \in M$. And this partition defines matching rules.

We see that these matching rules depend on the choice of $F_5, F_6, \ldots, F_n$ and $G_1, G_2, \ldots, G_n$. Denote the forbidden set by $\tilde{B}$.

The following lemma follows immediately from the definitions:

Lemma 1. *If* X *is contained in* $F_i + \mathbb{Z}^4$*, then* $X + Z_d$ *is contained in* $\tilde{B}$.

Proof. $X + Z_d$ is contained in the union of all walls, they all belong to $\tilde{B}$. $\square$

We shall prove that if $F_1, F_2, \ldots, F_n, G_1, G_2, \ldots, G_n$ satisfy some requirements, then every section $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$ is reduced to plane sections.

3. 2-Planes in $\mathbb{R}^4$

Each 2-plane E, going through 0 in $\mathbb{R}^4$ can be defined by a pair of linear equations:

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0\,,$$
$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 = 0\,.$$

Here $\alpha_i, \beta_i \in \mathbb{R}$, x_i are coordinates in $\mathbb{R}^4$ with basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$.

Put $E_{ij} = \det\begin{pmatrix}\alpha_i & \alpha_j \\ \beta_i & \beta_j\end{pmatrix}$, then $E_{ij} = -E_{ji}$, $E_{ij} \neq 0$ for some (i, j), and

$$E_{12}E_{34} + E_{14}E_{23} - E_{13}E_{24} = 0\,. \tag{$*$}$$

Conversely, every collection of six numbers E_{ij}, $1 \le i < j \le 4$, satisfying $(*)$ defines a plane E in $\mathbb{R}^4$, given by four linear equations $\sum_{j=1}^{4} E_{ij}x_j = 0$ $(i = 1, 2, 3, 4)$. There are only two independent among these four equations, and so they define a plane. This construction is the inverse to construction of six numbers by the equations of the plane. If (E_{ij}) and (F_{ij}) define the same plane then there exists a number $\lambda \in \mathbb{R}\backslash\{0\}$ such that $E_{ij} = \lambda F_{ij}$. In other words, the Grassmannian $G(4, 2)$ (the set of all 2-planes in $\mathbb{R}^4$) is a 4-dimensional hypersurface in $\mathbb{R}P^5$, defined by $(*)$.

The projective coordinates on $G(4,2)$ are useful to check whether two planes are in general position: two distinct planes E with projective coordinates $\{E_{ij}\}$ and F with coordinates $\{F_{ij}\}$ intersect along a line iff

$$E_{12}F_{34} + E_{34}F_{12} + E_{14}F_{23} + E_{23}F_{14} - E_{13}F_{24} - E_{24}F_{13} = 0. \qquad (**)$$

All these facts about 2-planes in $\mathbb{R}^4$ can be found in [12].

A plane $E = \{E_{ij}\}$ is called integral (respectively quadratic) if there is a non-zero number λ such that $\lambda E_{ij} \in \mathbb{Z}$ ($\lambda E_{ij} \in \mathbb{Z}_{\lfloor\sqrt{D}\rfloor}$ for some natural number D, respectively).

It is easy to prove that the plane E is integral (respectively quadratic) iff it is defined over the field $\mathbb{Q}$ (respectively, over the field $\mathbb{Q}_{\lfloor\sqrt{D}\rfloor}$), that is, $\dim_{\mathbb{Q}}(E\cap\mathbb{Q}^4) = 2$ (respectively, $\dim_{\mathbb{Q}_{\lfloor\sqrt{D}\rfloor}}(E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4) = 2$).

If E is quadratic, $E_{ij} = a_{ij} + b_{ij}\sqrt{D}$, a_{ij}, $b_{ij} \in \mathbb{Z}$, then $\bar{E}_{ij} = a_{ij} - b_{ij}\sqrt{D}$ also satisfies $(*)$ and so defines some plane. We shall denote it by $\bar{E}$. If $v \in (\mathbb{Q}\lfloor\sqrt{D}\rfloor)^4$, then we can define $\bar{v}$ in the same way.

We shall call the 2-plane E irrational, if $E \cap \mathbb{Q}^4 = \{0\}$.

Proposition 2. *If E is irrational and quadratic, then $\bar{E}$ is also irrational and $E \cap \bar{E} = \{0\}$.*

Proof. Since both E and $\bar{E}$ are defined over $(\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})$ it's sufficient to prove that $\lfloor(E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4\rfloor \cap \overline{\lfloor E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4\rfloor} = \{0\}$.

Let us suppose the opposite, that $(E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4) \cap \overline{(E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4)}$ is some non-zero $(\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})$-subspace $F \subset (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4$. Then $F = \bar{F}$, because $\overline{(E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4)} = \bar{E} \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4$ and $\overline{(\bar{E} \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4)} = E \cap (\mathbb{Q}_{\lfloor\sqrt{D}\rfloor})^4$. Since $\overline{((\bar{A}))} = A$, $\forall A \in \mathbb{Q}_{\lfloor\sqrt{D}\rfloor}$, then F is decomposed in direct sum $F_+ \oplus F_-$ of invariant and anti-invariant rational subspaces with respect to algebraic conjugation. Since either F_+ or F_- are non-zero subspaces and $F_+ = F \cap \mathbb{Q}^4$; $F_- = F \cap \sqrt{D}\mathbb{Q}^4$; then either $F \cap \mathbb{Q}^4 \neq \{0\}$ or $F \cap \sqrt{D}\mathbb{Q}^4 \neq \{0\}$. So, $E \cap \bar{E} \cap \mathbb{Q}^4 \neq \{0\}$, which contradicts the fact that E is irrational. Proposition 2 is proved. $\square$

We shall always suppose that E is quadratic and irrational. From now, let $E' = \bar{E}$. Until Sect. 5 we shall suppose that overlaps do not happen.

Proposition 3. *If $v \in \mathbb{Q}^4 \subset \mathbb{R}^4$, then the 2-plane $F(v)$, generated by v and $\pi(v)$, is integral.*

Proof. If $v = \varepsilon_1 = (1,0,0,0)$, then v lies on the plane F, given by equations

$$\begin{cases} a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \\ b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = 0 \end{cases}.$$

Since $F_{ij} = \det\begin{pmatrix} a_{1i} & a_{1j} \\ b_{1i} & b_{1j} \end{pmatrix}$, when $i,j = 2,3,4$ and $F_{1i} = 0$, then

$$\varepsilon^{ijkl}(a_{ij} \pm b_{ij}\sqrt{D})F_{kl} = \det\begin{pmatrix} a_{12} \pm b_{12}\sqrt{D} & a_{13} \pm b_{13}\sqrt{D} & a_{14} \pm b_{14}\sqrt{D} \\ a_{12} & a_{13} & a_{14} \\ b_{12} & b_{13} & b_{14} \end{pmatrix} = 0,$$

which means, using $(**)$, that F intersects with E and with E' along lines. This fact implies that F contains with each vector its projections on E and E'. So, $\pi(v) \in F$ and for $v = \varepsilon_1$ we have $F(v) = F$ and the proposition is proved.

Now if $v \in \mathbb{Q}^4$ is arbitrary, there exists $\varphi \in GL(4,\mathbb{Q})$ such that $\varphi(v) = \varepsilon_1$. Since φ is rational, $\overline{\varphi(E)} = \varphi(\bar{E})$ and this case is reduced to the case $v = \varepsilon_1$. $\square$

A collection $H_1, \ldots, H_k$ of 2-planes is called independent if their coordinates $(H_1)_{ij}, \ldots, (H_k)_{ij}$ (regarded as lines in $\mathbb{R}^6$) are linear independent.

Proposition 4. *There exist four rational vectors v_1, v_2, v_3, v_4 such that $F(v_1)$, $F(v_2)$, $F(v_3)$, $F(v_4)$ are linear independent.*

Proof. Each quadratic irrational 2-plane E in $\mathbb{R}^4$ is generated by two vectors ξ and η, where $\xi = \xi_1 + \sqrt{D}\xi_2$; $\eta = \eta_1 + \sqrt{D}\eta_2$; $\{\xi_1;\xi_2;\eta_1;\eta_2\} \subset \mathbb{Z}^4$. Let us prove that the vectors $\{\xi_1;\xi_2;\eta_1;\eta_2\}$ are linear independent. Suppose the opposite that $m_1\xi_1 + m_2\xi_2 + n_1\eta_1 + n_2\eta_2 = 0$ for some $\{m_1;m_2;n_1;n_2\} \subset \mathbb{Z}$. Then the nonzero vector $w \in E$; $w = (m_2 + \sqrt{D}m_1)\xi + (n_2 + \sqrt{D}n_1)\eta$ is equal to

$$\begin{aligned}
&(m_2 + \sqrt{D}m_1)\xi_1 + (\sqrt{D}m_2 + Dm_1))\xi_2 + (n_2 + \sqrt{D}n_1)\eta_2 + (\sqrt{D}n_2 + Dn_1))\eta_2 \\
&\quad = \sqrt{D}\,(m_1\xi_1 + m_2\xi_2 + n_1\eta_1 + n_2\eta_2) + (m_2\xi_1 + Dm_1\xi_2 + n_2\eta_1 + Dn_1\eta_2) \\
&\quad = (m_2\xi_1 + Dm_1\xi_2 + n_2\eta_1 + Dn_1\eta_2) \in \mathbb{Z}^4 ,
\end{aligned}$$

which contradicts irrationality of E. So, the vectors $\{\xi_1;\xi_2;\eta_1;\eta_2\}$ are linear independent and we can take them as a new basis in $\mathbb{Q}^4$. Let y_1, y_2, y_3, y_4 are the coordinates of vectors of $\mathbb{R}^4$ in this basis and let

$$\begin{aligned}
H_1 &= \{y_1 = y_2 = 0\}; & H_2 &= \{y_3 = y_4 = 0\}; \\
H_3 &= \{y_1 + y_3 = y_2 + y_4 = 0\}; & H_4 &= \{y_1 + Dy_4 = y_2 - y_3 = 0\}.
\end{aligned}$$

One can check by direct calculation that $H_i \cap H_j = \{0\}$; H_i intersects E and $\bar{E}$ along lines; $\{H_i\}$ are independent. If $v_i \in H_i$; $v_i \in \mathbb{Q}^4$, then $F(v_\imath) = H_i$. $\square$

Let $H_1; H_2; H_3$ be integral 2-planes, lying in a general position and intersecting E and E' by lines.

Proposition 5. *Let $v_1, v_2, v_3 \in \mathbb{Q}^4$.*
i) *If $\pi'(H_1 + v_1) = \pi'(H_1 + v_2)$, then $H_1 + v_1 = H_1 + v_2$;*
ii) *If $\pi'(H_1 + v_1) \cap \pi'(H_2 + v_2) \cap \pi'(H_3 + v_3)$ is nonempty, then $(H_1 + v_1) \cap (H_2 + v_2) \cap (H_3 + v_3)$ is also nonempty.*

Proof. i) Suppose $\pi'(H_1 + v_1) = \pi'(H_1 + v_2)$, $v_1, v_2 \in \mathbb{Q}^4$, then $\pi'(v_2 - v_1) \in \pi'(H_1)$. Because $\ker \pi' = E$ we have $v_2 - v_1 \in H_1 + E$ and so $\overline{v_2 - v_1} \in \bar{H}_1 + \bar{E}$ but $\overline{v_2 - v_1} = v_2 - v_1$, $\bar{H}_1 = H_1$, and we have $v_2 - v_1 \in (H_1 + E) \cap (H_1 + \bar{E})$. $\dim(H_1 + E) = \dim(H_1 + \bar{E}) = 3$. We prove that $\dim(H_1 + E) \cap \dim(H_1 + \bar{E}) < 3$. In fact if $\lfloor \dim(H_1 + E) \cap \dim(H_1 + \bar{E}) \rfloor = 3$, then $H_1 + E = H_1 + \bar{E}$ and $(H_1 + E) \supset \bar{E}$. This means $(H_1 + E) \supset E + \bar{E}$, but $E \cap \bar{E} = \{0\}$, $E + \bar{E} = \mathbb{R}^4$. This is a contradiction.

So $\dim(H_1 + E) \cap \dim(H_1 + \bar{E}) = 2$, i.e. $(H_1 + E) \cap (H_1 + \bar{E}) = H_1$, and $v_2 - v_1 \in H_1$. This means that $v_1 + H_1 = v_2 + H_1$.
ii) $\pi'(H_1 + v_1)$, $\pi'(H_2 + v_2)$, $\pi'(H_3 + v_3)$ are 3 lines of different directions in E', if they have nonempty intersection, then this intersection is a point.

Put $X_3 = (H_1 + v_1) \cap (H_2 + v_2)$, $X_2 = (H_1 + v_1) \cap (H_3 + v_3)$, $X_1 = (H_2 + v_2) \cap (H_3 + v_3)$, then $\pi'(X_1) = \pi'(X_2) = \pi'(X_3)$. We have $X_1 - X_2 \in E$, $X_1 - X_3 \in E$. But $X_1 - X_2$, $X_1 - X_3 \in \mathbb{Q}^4$, so $X_1 - X_2 = X_1 - X_3 = 0$. This means that $(H_1 + v_1) \cap (H_2 + v_2) \cap (H_3 + v_3)$ is a point. $\square$

Let $\Gamma_1; \Gamma_2; \Gamma_3$ be subgroups in $\mathbb{Q}^4$, containing $\mathbb{Z}^4$, in which the group $\mathbb{Z}^4$ has finite index and let $\mathscr{H}_i = H_i + \Gamma_i$, $i = 1, 2, 3$. Each $\mathscr{H}_i$ is a discret system of parallel affine 2-planes in $\mathbb{R}^4$.

We shall say that *the system $\mathscr{H}_1$ is bootstrapped by the systems $\mathscr{H}_2$ and $\mathscr{H}_3$* if $(H_1 + \Gamma_1) \cap (H_2 + \Gamma_2) = (H_1 + \Gamma_1) \cap (H_3 + \Gamma_3)$.

Note. If $\Gamma_1 = \Gamma_2 = \Gamma_3 = \mathbb{Z}^4 = (H_1 \cap \mathbb{Z}^4) \oplus (H_2 \cap \mathbb{Z}^4) = (H_1 \cap \mathbb{Z}^4) \oplus (H_3 \cap \mathbb{Z}^4)$, then the system $\mathscr{H}_1$ is bootstrapped by the systems $\mathscr{H}_2$ and $\mathscr{H}_3$, because $(H_1 + \mathbb{Z}^4) \cap (H_2 + \mathbb{Z}^4) = (H_1 + \mathbb{Z}^4) \cap (H_3 + \mathbb{Z}^4) = \mathbb{Z}^4$ in this case.

Proposition 6. *Let $\Gamma_1 = (H_2 + \mathbb{Z}^4) \cap (H_3 + \mathbb{Z}^4)$; $\Gamma_2 = (H_1 + \mathbb{Z}^4) \cap (H_3 + \mathbb{Z}^4)$; $\Gamma_3 = (H_2 + \mathbb{Z}^4) \cap (H_1 + \mathbb{Z}^4)$. Then the system $\mathscr{H}_1$ is bootstrapped by the systems $\mathscr{H}_2$ and $\mathscr{H}_3$.*

Proof. Because the systems $\mathscr{H}_2$ and $\mathscr{H}_3$ are here symmetric, it is sufficient to prove that $(H_1 + \Gamma_1) \cap (H_2 + \Gamma_2) \subset (H_1 + \Gamma_1) \cap (H_3 + \Gamma_3)$ (the opposite inclusion is proved by changing the indices $_2$ and $_3$ in all places of this proof). So, we have to prove that for each two vectors $v_1 \in \Gamma_1$ and $v_3 \in \Gamma_3$ there exist some vector $v_2 \in \Gamma_2$ such that $(H_1 + v_1) \cap (H_2 + v_2) = (H_1 + v_1) \cap (H_3 + v_3) = X$.

By definitions of Γ_1 and Γ_3 we have $v_1 = (H_2 + a_2) \cap (H_3 + a_3)$; $v_3 = (H_1 + b_1) \cap (H_2 + b_2)$; for some $a_2, a_3, b_1, b_2 \in \mathbb{Z}^4$. We shall prove that one can take $v_2 = (H_1 + b_2 + a_2 - b_1) \cap (H_3 + b_2 + a_2 - a_3)$. To prove it, we shall need the following:

Lemma about Parallelograms. *Let F and G be 2-planes in a general position in $\mathbb{R}^4$, $f_1, f_2, g_1, g_2 \in \mathbb{R}^4$. Then the vectors $x_{ij} = (F + f_i) \cap (G + g_j)$ $(i, j = 1, 2)$ are the vertices of the parallelogram, i.e., $x_{12} - x_{11} = x_{22} - x_{21}$.*

The proof of the lemma is based on the fact that the vector $x_{12} - x_{11}$ has the unique decomposition $f + g$, where $f \in F$, $g \in G$ and $f = x_{12} - x_{11}$, $g = x_{21} - x_{11}$, which implies the statement of the lemma.

Now let us apply the lemma about parallelograms, when $F = H_3$; $G = H_1$; $f_1 = g_1 = v_3$; $f_2 = g_2 = v_1$, and we shall obtain

$$X - v_3 = v_1 - w_1 \tag{1}$$

where $w_1 = (H_1 + b_1) \cap (H_3 + a_3)$. If we take $F = H_2$; $G = H_1$; $f_1 = g_1 = v_3$; $f_2 = g_2 = v_1$, we shall obtain

$$w_3 - v_3 = v_1 - w_2\,, \tag{2}$$

where $w_2 = (H_1 + b_1) \cap (H_2 + a_2)$; $w_3 = (H_1 + v_1) \cap (H_2 + b_2)$. Subtracting (2) from (1), we obtain

$$X - w_3 = w_2 - w_1 \tag{3}$$

Put $w_4 = (H_1 + b_2 + a_2 - b_1) \cap (H_2 + b_2)$ and $z = (H_1 + b_2 + a_2 - b_1) \cap (H_2 + X)$. Then, if we take $F = H_1$; $G = H_2$; $f_1 = g_1 = w_4$; $f_2 = g_2 = z$, we shall obtain

$$z - w_4 = X - w_3\,. \tag{4}$$

Let us prove that

$$v_2 - b_2 = a_2 - w_2 = Y\,. \tag{5}$$

To prove it, let us notice, that

$$v_2 - b_2 = \lfloor b_2 + a_2 + (H_1 - b_1) \cap (H_3 - a_3) \rfloor - b_2 = b_2 + a_2 - v_1 - b_2 = a_2 - v_1$$

and Eq. (5) is proved.

Now using (5) and decomposing the vector Y into the sum Y_1+Y_2, where $Y_1 \in H_1$; $Y_2 \in H_2$; we shall obtain that

$$Y_1 = w_2 - w_1 = v_2 - w_4 \,. \tag{6}$$

Combining (3), (4), and (6), we see that $v_2 - w_4 = z - w_4$, which implies that $v_3 = z$ and $(H_1 + v_1) \cap (H_2 + v_2) = X$, which is to be proved. $\square$

Proposition 7 (Generalized Levitov's Lemma). *Let Ω be the graph of a function $\varrho : E \to E'$, satisfying the modified Lipschitz condition: $\exists c_1; c_2 \in \mathbb{R}$ such that*

$$|\varrho(x) - \varrho(y)| < c_1|x - y| + c_2 \,,$$

H_1, H_2, H_3, H_4 be integral 2-planes, independent, intersecting E, E' by lines. If $\Omega \cap \mathscr{H}_i = \emptyset$, where $\mathscr{H}_i = H_i + \mathbb{Z}^4$, $i = 1, 2, 3, 4$ then the function ϱ is bounded:

$$|\varrho(E)| < \text{const}\,.$$

Proof. (Levitov).

Lemma 2. *A constant c exists such that for any $H \in \cup\mathscr{H}_i$ the function ϱ maps $\pi(H)$ into the c-vicinity of some line h' parallel to $\pi'(H)$ (c-vicinity of h' is the set of points in E' having distance to h' less than c).*

Proof of the Lemma. At first we introduce some definitions. A shadowed line in E' is a pair (l, E'_l) consisting of a line $l \subset E'$ and an half plane E'_l of E' separated by l. As usual we denote a shadowed line (h, E'_l) just as l, understanding that a half plane is bounded by l. The half plane E'_l is opened ($l \notin E'_l$), and is called the shadowed half plane of this shadowed line. A set $X \subset \mathbb{R}^4$ is called good with respect to a shadowed line if it lies in the shadowed half plane of this line. Two parallel shadowed lines have the same direction if the shadowed half plane of one of them contains the other.

Suppose $H \in \mathscr{H}_1$. Note that $\pi(H_1 + \mathbb{Z}^4)$ is dense in E, $\pi'(H_1 + \mathbb{Z}^4)$ is dense in E'. Denote U_δ the δ-vicinity of h in E and $V_\delta = \pi'(\mathscr{H}_1 \cap \pi^{-1}(U_\delta))$. V_δ is a set of lines parallel to h. If δ is small then the distance between two neighbor lines in V_δ is large: distance between two neighbor lines in V is of order $1/\delta$. If $L \in \mathscr{H}_1$ and $\pi(L) \in U_\delta$, then $\varrho(\pi(L))$ doesn't intersect $\pi'(L)$ because in the opposite case the 2-plane $L = \pi(L) + \pi'(L)$ intersects with the set $\Omega(\pi(L)) = \{x + \varrho(x), x \in \pi(L)\}$.

We mark this by shadowing the line $\pi'(L)$ in such a way that $\varrho(\pi(L))$ lies in the shadowed half plane separated by $\pi'(L)$. All the lines in V_δ are shadowed. Because ϱ is continuous not all the lines in $\pi'(H_1 + \mathbb{Z}^4)$ are shadowed in one direction, and there exist two neighbour lines in V_δ such that their shadows are as in Fig. 1:

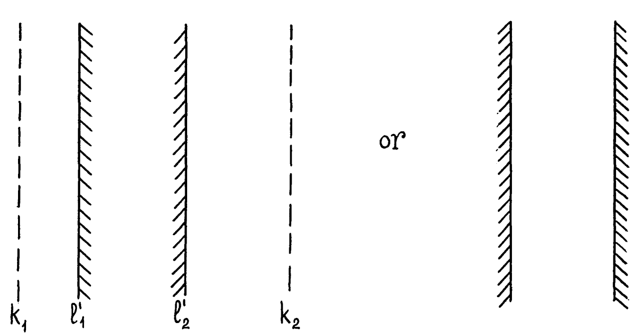


Fig. 1

Denote these lines l'_1, l'_2 and suppose $l'_1 = \pi'(L_1)$, $l'_2 = \pi'(L_2)$. Note that $\pi(L_1)$, $\pi(L_2)$ lie in U_δ. Let k_i be the lines parallel to l_i, lying in the non-shadowed half planes separated by l_i, $i = 1, 2$, in a distance of $c_1 . \delta + c_2$ from l_1. Then by the modified Lipschitz condition $\varrho(l)$ lies between k_1 and k_2.

Proof of Proposition 6. Choose the affine coordinates (y_1, y_2) of E and (z_1, z_2) of E' such that $\pi(H_1)$ is given by $\{y_1 = 0\}$, $\pi(H_2)$ by $\{y_2 = 0\}$, $\pi(H_3)$ by $\{y_1 + y_2 = 0\}$, $\pi'(H_1)$ by $\{z_1 = 0\}$, $\pi'(H_2)$ by $\{z_2 = 0\}$, $\pi(H_3)$ by $\{z_1 + z_2 = 0\}$. Then $\pi(H_4)$ is given by $\{y_1 + ay_2 = 0\}$, $\pi'(H_4)$ by $\{z_1 + bz_2 = 0\}$.

In the affine coordinates (y_1, y_2, z_1, z_2) of $\mathbb{R}^4$ we have

$$H_1 = \{y_1 = z_1 = 0\}\,, \qquad H_2 = \{y_2 = z_2 = 0\}\,,$$
$$H_3 = \{y_1 + y_2 = 0 = z_1 + z_2\}\,, \qquad H_4 = \{y_1 + ay_1 = z_2 + bz_2 = 0\}\,.$$

The projective coordinates $(H_i)_{kl}$ are not difficult to compute, and from the independence of H_1, H_2, H_3, H_4 we have $a \neq 0$, $b \neq 0$, $a \neq b$.

Let for a pair of functions f, g on E the sign $f \equiv g$ means that $|f - g| <$ const. Ω is defined by two functions $z_1(y_1, y_2)$, $z_2(y_2, y_2)$. By applying Lemma 2 to $\mathscr{H}_1$ and $\mathscr{H}_2$ one sees that $z_1(y_1, y_2) \equiv f(y_1)$, $z_2(y_1, y_2) \equiv g(y_2)$. By applying to $\mathscr{H}_3$:

$$f(y_1) + g(y_2) \equiv h(y_1 + y_2) \Rightarrow f \equiv g \equiv h\,.$$

It's easy to see that $f(\alpha x) \equiv \alpha f(x)$ for a fixed $\alpha \in \mathbb{R}$. At last applying Lemma 2 to $\mathscr{H}_4$ we see

$$f(y_1) + bf(y_2) \equiv f(y_1 + ay_2) \Rightarrow bf(y_2) \equiv af(y_2)\,.$$

Because $a \neq b$ we have $f \equiv 0$. This means that the function ϱ is bounded. □

4. Construction of QPLR

Recall that we have defined F_1, F_2, F_3, F_4, $F_i = F(\varepsilon_i)$, $i = 1, 2, 3, 4$. They may be dependent. Choose some new integral 2-plane $F_5, \ldots, F_n$ such that each intersects E and E' by lines and there exist four independent 2-planes in $F_1, F_2, \ldots, F_n$. By Proposition 4 we can always do that. If F_1, F_2, F_3, F_4 are independent we need not add any 2-plane to this collection. Let $H_1 = F_1$, $H_2 = F_2$, $H_3 = F_3$ and define Γ_i as in Prop. 6, $i = 1, 2, 3$. For $i \geq 4$ let $\Gamma_i = \mathbb{Z}^4$. Put $\mathscr{F}_i = F_i + \Gamma_i$. $\mathscr{F}_i$ is a system of parallel 2-planes. These systems satisfy the following conditions:

a) $\mathscr{F}_i$ is discrete in $\mathbb{R}^4$: every compact in $\mathbb{R}^4$ intersects with only a finite number of planes in $\mathscr{F}_i$. F_i intersects E and E' by lines, $F_i \cap F_j = 0$ if $i \neq j$.

b) Each of $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ is bootstrapped by the other two (cr. Proposition 6).

c) If $\varrho: E \to E'$ is a function satisfying the modified Lipschitz condition and its graph doesn't intersect $\bigcup\limits_{i=1}^{n} \mathscr{F}_i$ then $\varrho(E)$ is bounded in E.

d) If $F \in \mathscr{F}_i$, $i \geq 3$ then the set $\pi'(F \cap \mathscr{F}_1 \cap \mathscr{F}_2)$ is dense in the line $\pi'(F)$ (because $\Gamma_1 \supset \mathbb{Z}^4$, $\Gamma_2 \supset \mathbb{Z}^4$).

We construct the partitions of the existence domains as indicated in 2.3. We denote the forbidden set by $\tilde{B}$,

$$\tilde{B} = \left(\bigcup_{i=1}^{n} \partial_{\|} C_i \right) + Z^4\,; \qquad B = \left(\bigcup_{I \in M} \partial_{\|} C_i \right) + \mathbb{Z}^4\,; \qquad B \subset \tilde{B}\,.$$

Of course the set $\mathscr{F}_i$ is contained in $\tilde{B}$. We shall prove that if Ω is a section, $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$, then Ω is reduced to a plane section. This solves the second problem. Fix such a section Ω. Ω is the graph of a function; $\varrho : E \to E'$.

$$\Omega = \{x + \varrho(x), x \in E\}\,, \qquad \Omega \cap \tilde{B} = \emptyset\,.$$

It is easy to see that if Ω is a section, $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$, then Ω satisfies the modified Lipschitz condition and we have:

Corollary. *If $\Omega \subset \mathbb{R}^4 \backslash \tilde{B}$ is a section then $\pi'(\Omega)$ is bounded.* $\square$

Let F be a plane in $\mathscr{F}_i$, $f = \pi(F)$, $f' = \pi'(F)$, $G = \pi^{-1}(f)$. G is a 3-plane, $F \subset G$. Because $\Omega \cap \mathscr{F}_i = \emptyset$ the intersection $(\Omega \cap G)$ lies in one half of G separated by F, and $\pi(\Omega \cap G)$ lies in one half plane of E' separated by $f' = \pi'(F)$. We shadow the line f' in such a way that $\pi(\Omega \cap G)$ lies in the shadowed half. All lines in $\pi'\left(\bigcup_{i=1}^{n} \mathscr{F}_i\right)$ are shadowed. Note that by Proposition 5 no two 2-planes in $\bigcup_{i=1}^{n} \mathscr{F}_i$ have the same projection on E'. The set $\pi'\left(\bigcup_{i=1}^{n} \mathscr{F}_i\right)$ is a dense set E'. A set of shadowed lines is called compatible if there exists a point good with respect to all of them. A set of points is called good with respect to a shadowed line if all of its points are good with respect to this line. If this set is connected then it is good with respect to this line iff one of its points is good.

Lemma 3. *If 3 lines $f_1', f_2', f_3' \in \pi'\left(\bigcup_{i=1}^{n} \mathscr{F}_i\right)$ intersect by a point then they are compatible.*

Proof. Suppose $H_1, H_2, H_3 \in \bigcup_{i=1}^{n} \mathscr{F}_i$ and $\pi'(H_i) = f'$, then by Proposition 5 H_1, H_2, H_3 intersect by a point N. Let E_N' be the 2-plane going through N and parallel to E'. This plane intersects H_1, H_2, H_3 by lines and the point $\Omega(N)$ does not lie on these lines. Its projection $\pi'(\Omega(N)) = \varrho(N)$ is a point good with respect to all f_1', f_2', f_3'. $\square$

Lemma 4. *If $h_1', h_2' \in \pi'(\mathscr{F}_1 \cup \mathscr{F}_2)$, h_1 is parallel to h_2, then there is a point x good with respect to h_1' and h_2'. This means that two lines indicated in Fig. 2 are impossible in $\pi'(\mathscr{F}_1 \cup \mathscr{F}_2)$:*

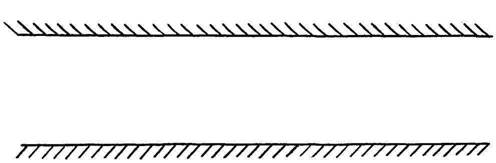


Fig. 2

Proof. Suppose h_1' and h_2' belong to $\pi'(\mathscr{F}_1)$ and are as in Fig. 2. By Proposition 6 there exist lines of $\pi'(\mathscr{F}_2 \cup \mathscr{F}_3)$ located as indicated in Fig. 4: each intersection point is triple, here the two shadowed lines are h_1 and h_2, the non-shadowed are from $\pi'(\mathscr{F}_2 \cup \mathscr{F}_3)$. Now consider possibilities of shadowing lines from $\pi'(\mathscr{F}_2 \cup \mathscr{F}_3)$. According to Lemma 3 there are only 3 variants of shadows of lines going through the point A in Fig. 3:

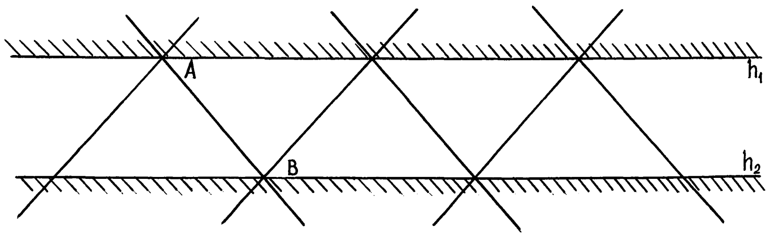


Fig. 3

These variants are introduced in Fig. 4:

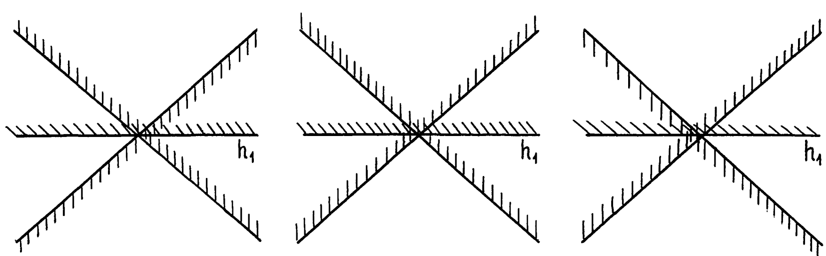


Fig. 4

Let's consider the first variant. At the point B (in Fig. 5) there is a unique shadowing compatible with Lemma 3. In A_1 the shadowing is unique, too. We can continue this process and see that $\varrho(E)$ is not bounded. This contradicts Proposition 7. The other variant of Fig. 4 is analogous.

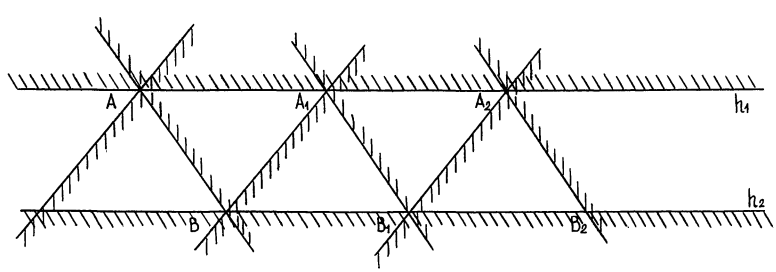


Fig. 5

Lemma is proved. □

Corollary. *There exists a point $N \in E'$ such that for every shadowed line f' in $\pi'(\mathscr{F}_1 \cup \mathscr{F}_2)$ either N lies on f' or N is good with respect to f'.*

Proof. By Lemma 4 every pair of shadowed lines in $\pi'(\mathscr{F}_1)$ are compatible. For $h' \in \pi'(\mathscr{F}_1)$ we denote $E'(h')$ the shadowed half plane of E' separated by h' plus the line h' itself. Then

$$E'(h'_1) \cap E'(h'_2) \neq \emptyset \forall h'_1, h'_2 \in \pi'(\mathscr{F}_1) .$$

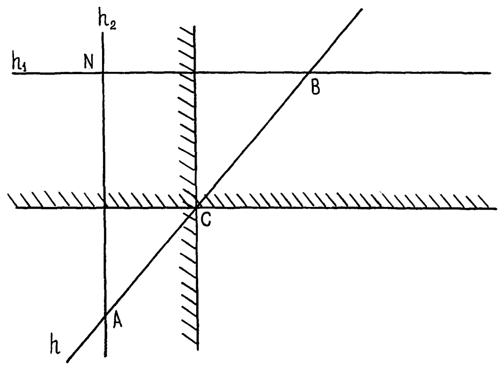
It's easy to see that there are only two possible cases:
a) There exists a line $k_1 \subset \bigcap_{h' \in \pi'(\mathscr{F}_1)} E'(h')$.
b) All the shadowed lines from $\pi'(\mathscr{F}_1)$ have the same direction. But case b) means that $\varrho(E)$ is not bounded, which contradicts Proposition 7. Thus there exists a line k_1 contained in every $E'(h')$ for $h' \subset \pi'(\mathscr{F}_1)$. Analogously there exists a line k_2 contained in $E'(h')$ for $h' \subset \pi'(\mathscr{F}_2)$ and $k_1 \cap k_2$ is the point to find. □

Proposition 8. *For every shadowed line h' in $\pi'\left(\bigcup_{i=1}^{n} \mathscr{F}_i\right)$ the point N of the above corollary either lies on h' or is good with respect to it.*

Proof. If N belongs to h' then we are done. Suppose N does not lie on h' and $h' \notin \pi'(\mathscr{F}_1 \cup \mathscr{F}_2)$. Let h_1, h_2 be lines going through N and parallel respectively to $\pi'(F_1)$ and $\pi'(F_2)$.

The lines h_1 and h_2 intersect h' at point A and B (Fig. 6).

Fig. 6



There is a triple intersection point C in the segment $\lfloor A, B \rfloor$. That is the intersection point of h' and one line from $\pi'(\mathscr{F}_1)$ and one line from $\pi'(\mathscr{F}_2)$. By applying Lemma 2 to these lines we see that N is good with respect to h'. $\square$

By definition the set $\varrho(h)$ is good with respect to h' for every h, h' such that $h = \pi(H)$, $h' = \pi'(H)$, $H \in \bigcup_{i=1}^{n} \mathscr{F}_i$. We prove a stronger result:

Proposition 9. *For $H \in \bigcup_{i=1}^{n} \mathscr{F}_i$, $h = \pi(H)$, $h' = \pi'(H)$, the set $\varrho(U_d(h))$ is good with respect to h', here $U_d(h)$ is the d-vicinity of h (d is defined in 2.3).*

Proof. The set $h' + U_d(h)$ by Lemma 1 belongs to the forbidden set $\tilde{B}$. Moreover $h' + U_d(h)$ is a wall. So $\Omega(U_d(h))$ doesn't intersect with $(h' + U_d(h))$. But $\Omega(U_d(h))$ lies in $(U_d(h) + E')$. Like h' separates E' into two parts, the set $U_d(h) + h'$ separates the set $(U_d(h) + E')$ into two parts, and $\Omega(U_d(h))$ lies in one of them. Projecting this picture to E' by π' we see that $\varrho(U_d(h))$ is good with respect to h'. $\square$

Let x be a point of E, E'_x be the plane going through x and parallel to E'. The intersection of $\tilde{B}$ with E'_x is a partition of E'_x into convex polygons, and $\Omega(x)$ lies in one of them. By projecting the polygon containing $\Omega(x)$ on E' (by π') we obtain a polygon $Q^{(x)}$. Suppose x lies in a tile P of the tiling defined by the section Ω. Note that x may lie on the boundary of P while $\varrho(x)$ always lies in the interior of $Q^{(x)}$. If x and y both lie in the interior of P, then evidently $Q^{(x)} = Q^{(y)}$, and $Q^{(x)} = \pi'(C)$, where C is the prism (or the tile of the refined oblique periodic tiling) containing $\Omega(x)$. If x lies on the boundary of P, for example, if $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$, are all tiles containing x, then it's easy to see that

$$Q^{(x)} = \bigcap_{i=1}^{s} Q^{(y_i)} \tag{***}$$

where y_i $(i = 1, \ldots, n)$ is an interior point of $P^{(i)}$.

Shadow all edges of this polygon $Q^{(x)}$ in the inner direction (Fig. 7).

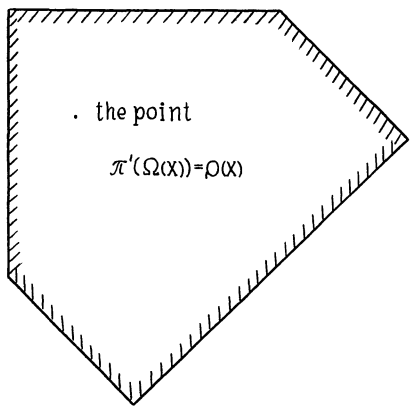


Fig. 7. The polygon $Q^{(x)}$

Proposition 10. *Every edge of the polygon $Q^{(x)}$ is a segment of some line $\pi'(F)$ where $F \in \bigcup_{i=1}^{n} \mathscr{F}_i$, and its shadow coincides with that of this line.*

Proof. Due to $(***)$ it's sufficient to consider the case when x is an interior point of P. We may suppose that the prism C containing $\Omega(x)$ is a piece of the big prism $C_I + \xi$, where $I = \{3, 4\}$ and $\xi \in \mathbb{Z}^4$. We consider only the case $\xi = 0$ because other cases are analogous.

Thus

$$P = \pi(C_I) = P_I = \{x_3e_3 + x_4e_4; x_3, x_4 \in [0, 1]\}.$$
$$C_I = P_I - P'_I. \qquad - P'_I = \{-x_1e_1 - x_2e_2; x_1, x_2 \in \lfloor 0, 1 \rfloor\}.$$

The prism C has the form $C = P - P'$, where $P = P_I$ and $-P'$ is a piece of $-P'_I$. More precisely, the projections of all walls intersecting with C_I divide the parallelogram $-P_I$ into smaller polygons, and $-P'$ is one of them. Because projections of all walls and edges of the parallelogram $-P'_I$ are in $\pi'\left(\bigcup_{i}^{n} \mathscr{F}_i\right)$, we see that all edges of $-P'$ are segments of some line in $\pi'\left(\bigcup_{i}^{n} \mathscr{F}_i\right)$.

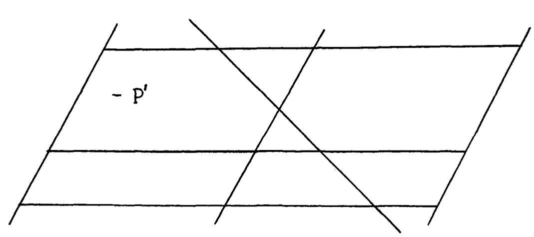


Fig. 8

Fix an edge e of the polygon $-P'$. Now we consider two cases:

1stcase. e is a segment of the line $\pi'(W)$, where W is a wall, $W \cap C_I \neq \emptyset$.

Because W is a wall $W = F + Z_d$, where $F \in \bigcup_i \mathscr{F}_i$, $\pi(W) = \pi(F)$, $e \subset \pi(F) = f$, $W \cap C_I \neq \emptyset \Rightarrow F \cap (Z_d + C_I) \neq \emptyset$.

Let $f = \pi(F)$, we see that $f \cap \{Z_d + P_I\} \neq \emptyset$. $Z_d + P_I$ is the d-vicinity of P_I, so $f \cap \{Z_d + P_I\} \neq \emptyset$ means that $U_d(f) \cap P_I \neq \emptyset$. Let $y \in U_d(f) \cap P_I$. Then $\varrho(y) \in Q^{(x)} = -P'$, and by Proposition 9 $\varrho(y)$ is good with respect to f'. We conclude that shadow of e and $\tilde{f}$ are the same.

2nd case. e is a segment in the edge $\tilde{e}$ of the parallelogram $-P'_I$. There are 4 edges of this parallelogram, two of them go through 0, and two do not. If $\tilde{e}$ goes through 0, for example $\tilde{e} \subset \pi'(F_1)$. It's easy to check that if F_1 intersects with $C_I, \pi'(F_1)$ goes through e, so the proof is just the same as the first case. If $\tilde{e}$ does not go through 0, for example, $\tilde{e}$ lies on $\pi'(F_1) - e'_2$. Let us consider the 2-plane $F = (F_1 - \varepsilon_2)$, and its wall $W = (F_1 - \varepsilon_2 + Z_d)$. We have $\pi'(W)$ $(= \pi'(F))$ contains the edge e. We prove that this wall intersects with C_I. In fact a wall is a prism, and $\pi'(C_I) \cap \pi'(W) \neq \emptyset$; it's sufficient to prove that $\pi'(W) \cap \pi'(C_I) \neq \emptyset$. We have $\pi(W) = \pi(F_1) - e_2 + Z_d$,

$$\pi'(C_I) = \{-x_3 e_3 - x_4 e_4; x_3, x_4 \in [0,1]\}.$$

Because $d > |e_2|$, both sets contain 0, thus $\pi(W) \cap \pi'(C_I) \neq \emptyset$. And this case is also reduced to the previous one. $\square$

Now we are ready to prove the theorem:

Theorem 2. *A tiling satisfying our matching rules is equivalent to a tiling of a plane cut and so is quasiperiodic.*

Proof. By Theorem 1 such a tiling is defined by a section $\Omega \subset R_4 \backslash \tilde{B}$. Let N be the point in Proposition 8, E_N be the plane going through N and parallel to E, $E_N(x)$ be the point-intersection of E_N and E'_x, where x is a point of E. Propositions 8 and 10 show that Ω is reduced to E_N. The theorem is proved. $\square$

5. The Connection Between the Quasiperiodic Tilings and the Quasicrystallographic Groups

It is well-known that if the tiling $\mathfrak{F}$ of the Euclidean space $\mathbb{R}^k$ is quasiperiodic, then it can be obtained by the "cut method." This means that there exists some tiling $\mathfrak{P}$ of $\mathbb{R}^n$, periodic with respect to some lattice $\Lambda^* = \mathbb{Z}^n \subset \mathbb{R}_n$, and there exists some affine immersion $\theta: \mathbb{R}^k \to \mathbb{R}^n$ such that $\mathfrak{F}$ is obtained, as a slice of $\mathfrak{P}$ on $\theta(\mathbb{R}^k)$ Then we can define the projection $p: (\mathbb{R}^n)^* \to (\mathbb{R}^k)^*$, as the operator, adjoint to the linear part of θ, and the lattice $\Lambda \subset (\mathbb{R}^n)^*$, as the lattice, dual to Λ^*.

Definition. The finite-generated abelian subgroup in $\mathbb{R}^k$, which generates it as a linear space, will be called *the quasi-lattice* in $\mathbb{R}^k$.

Definition. The quasicrystallographic group $G \subset E_k$, corresponding to the quasiperiodic tiling $\mathfrak{F}$, is defined as the group of all movements of $(\mathbb{R}^k)^*$, preserving the quasi-lattice $p(\Lambda)$.

Note. The subspace $\lfloor \theta(\mathbb{R}^k) \rfloor^0$ [the annulator of $\theta(\mathbb{R}^k)$ in $(\mathbb{R}^k)^*$] is irrational (i.e., it intersects with Λ only in zero), iff the projection $p: \Lambda \to (\mathbb{R}^k)^*$ has zero kernel.

Everywhere below, unless otherwise specified, we suppose that this situation always takes place. In this situation we shall not distinguish Λ from $p(\Lambda) \subset \mathbb{R}^k$ and $(\mathbb{R}^n)^*$ from $\Lambda \otimes \mathbb{R}$.

If the tiling $\mathfrak{F}$ consists only of parallelotops and is obtained by the strip projection method (cf. 2 of this paper and [6]) (that is, the vertices of the tiling are selected by the following way: The point $x \in \mathbb{R}^k$ is the vertex of the tiling, iff $x = \pi(y)$ for some $y \in \Lambda^* \cap (\theta(\mathbb{R}^k) + I)$, where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is some projection; I is an opened parallelotop, spanned by the set $\{\varepsilon_1, \dots, \varepsilon_n\}$ of some basis vectors in $\Lambda^* \subset \mathbb{R}^n$), then Λ is a quasi-lattice in $(\mathbb{R}^k)^*$, generated by the linear functions $\{h_1, \dots, h_n\}$ on $\mathbb{R}^k$, which are defined as restrictions on $\mathbb{R}^k$ of the linear functions on $\mathbb{R}^n$ from the basis, dual to $\{\varepsilon_1, \dots, \varepsilon_n\}$.

6. Classification of Rotational Symmetry Types of Two-Dimensional Quasicrystals, Obtained by Quadratic Irrational Embedding

If the embedding θ is quadratic and irrational, then the space $\Lambda \otimes \mathbb{Q}$ of rational linear combinations of the quasilattice Λ in $(\mathbb{R}^2)^*$ admits the structure of 2-dimensional linear subspace over $\mathbb{Q}_{\lfloor\sqrt{D}\rfloor}$ in $(\mathbb{R}^2)^*$, and so, the quasilattice T always has rank 4 in this situation.

It was proved in [1] that the angle φ can be an angle of rotation in some two-dimensional quasicrystallographic group G with $\operatorname{rank}(\Lambda) = n$, iff $z = e^{i\varphi}$ is an integral algebraic number of degree n.

So, to classify all possible angles of rotation in two-dimensional quasicrystallographic groups, which are the symmetry groups of some tilings of the plane, obtained by the aid of the strip projection method and admitting above constructed local rules, we have to classify all the integral algebraic numbers of degree 4, lying on the unit circumference. The fact that $\Lambda \otimes \mathbb{Q}$ admits the structure of linear subspace over $\mathbb{Q}_{\lfloor\sqrt{D}\rfloor}$, when Λ is rank-4-subgroup of translations in some 2-dimensional quasicrystallographic group with non-trivial rotational symmetry, is always true. It will follow from the classification of angles.

Theorem 3. *A complex number z, lying on the unit circumference $|z| = 1$, is algebraically integral of degree 4, iff $z + \bar{z}$ is an integral algebraic number of degree 2, or, equivalently,* $z + \bar{z} = \dfrac{m \pm \sqrt{m^2 - 4k}}{2}$, *where* $m, k \in \mathbb{Z}$; $\left|\dfrac{m \pm \sqrt{m^2 - 4k}}{2}\right| < 2$.

Proof. Suppose z is an integral algebraic number of degree 4; $|z| = 1$. Let us denote $z + \bar{z} = A$. Since $\bar{z} = z^{-1}$, then $z^2 - Az + 1 = 0$ which means that A is integral algebraic. Since z is of degree 4, then A is irrational and there exists a quadratic polynomial $(z^2 - Bz + C)$ with some complex coefficients B, C, such, that $(z^2 - Az + 1)(z_2 - Bz + C) \in \mathbb{Z}_{\lfloor z\rfloor}$. This means, that $C \in \mathbb{Z}$; $(A + B) \in \mathbb{Z}$; $(CA + B) \in \mathbb{Z}$; $(AB) \in \mathbb{Z}$. Since because z is of degree 4 and $C \in \mathbb{Q}$, then $B \notin \mathbb{Q}$. Since $C \in \mathbb{Q}$; $(A + B) \in \mathbb{Q}$; $(CA + B) \in \mathbb{Q}$ and A, B are both irrational, then $C = 1$. Let us denote $m = (A + B)$; $k = AB$. Then $z + \bar{z} = A = \dfrac{m \pm \sqrt{m^2 - 4k}}{2}$; $B = \dfrac{m \mp \sqrt{m^2 - 4k}}{2}$;
$\left|\dfrac{m \pm \sqrt{m^2 - 4k}}{2}\right| = |z + \bar{z}| < 2$, and the theorem is proved on the one hand.

On the other hand: let $z \in \mathbb{C}$; $|z| = 1$; $z + \bar{z} = A = \dfrac{m \pm \sqrt{m^2 - 4\bar{k}}}{2}$. Then, $P(z) = z^4 - mz^3 + (2+k)z^2 - mz + 1 = (z^2 - Az + 1)(z^2 - Bz + 1) = 0$, because $(z^2 - Az + 1) = 0$, and so, z is algebraically integral of degree 4. □

Let us denote w_1 and w_2 the roots of the polynomial $P(z)$, other, then z and $\bar{z}$.

Lemma 5. *If either w_1 or w_2 is not real number, then z is a root of unity of degree* 5, 8, 10 *or* 12.

Proof. Conditions of the lemma imply that $w_1 = \bar{w}_2$. Since $z\bar{z}w_1w_2 = P(0) = 1$, then $w_1w_2 = 1$. So, we have $|w_1| = |w_2| = 1$, which implies $B = w_1 + \bar{w}_1$; $|B| < 2$. So, conditions of the lemma are equivalent to $|A| < 2$; $|B| < 2$ and we have to find all $(m, k) \in \mathbb{Z}$ such that $\left|\dfrac{m \pm \sqrt{m^2 - 4\bar{k}}}{2}\right| < 2$; $\left|\dfrac{m \pm \sqrt{m^2 - 4\bar{k}}}{2}\right| < 2$. It is easy to check that there are only four possibilities:

a) $m = k = -1$; $z + \bar{z} = \dfrac{-1 \pm \sqrt{5}}{2}$; $z^5 = 1$;

b) $m = 1$; $k = -1$; $z + \bar{z} = \dfrac{1 \pm \sqrt{5}}{2}$; $z^{10} = 1$;

c) $m = 0$; $k = -2$; $z + \bar{z} = \pm\sqrt{2}$; $z^8 = 1$;

d) $m = 0$; $k = -3$; $z + \bar{z} = \pm\sqrt{3}$; $z^{12} = 1$;

which proves the lemma. □

7. Some Examples

The construction of the quasiperiodic tiling of the plane by the quasicrystallographic group G with the point group R and the group of translations Λ is quite natural.

Since Λ is a commutative normal subgroup in G the action of G on Λ by inner automorphisms is defined. If we factorise this action by the subgroup $\Lambda \subset G$ lying in the kernel of this action, we obtain some representation $\varrho: R = G/\Lambda \to \mathrm{Aut}(\Lambda) = GL(n, \mathbb{Z})$ (here $n = \operatorname{rank} \Lambda$).

Since $G \subset E_2$ and $\Lambda = G \cup \mathbb{R}^2$, then we have the representation $\varphi: R = G/\Lambda \to (E_2/\mathbb{R}^2) = O(2, \mathbb{R})$. The action φ maps each element $z \in G \subset E_2$ onto its Jacobi matrix Z from $O(2, \mathbb{R})$.

According to the definition of representation φ, the quasilattice $\Lambda \subset \mathbb{R}^2$ is $\varphi(R)$-invariant and so, φ is a factor-representation of representation ϱ. If we take the natural inclusion $i: \Lambda \to \mathbb{R}^2$, then the projection $p: \Lambda \otimes \mathbb{R} \to \mathbb{R}^2$; $p\left(\sum\limits_{i=1}^{n} \lambda_i \varepsilon_i\right) = \sum\limits_{i=1}^{n} \lambda_i i(\varepsilon_i)$, $(\lambda_i \in \mathbb{R}, \varepsilon_i \in \Lambda)$; is R-equivariant operator from the space $\Lambda \otimes \mathbb{R}$ of representation ϱ to the space $\mathbb{R}^2$ of representation φ. The kernel of p is an R-invariant subspace in $\Lambda \otimes \mathbb{R}$. We shall denote this subspace $E^{\perp}$.

It follows from Theorem 4, formulated below, that the representation ϱ is always completely reducible. So, we have the R-invariant (and orthogonal with respect to some invariant bilinear form) decomposition $\Lambda \otimes \mathbb{R} = E^{\parallel} \oplus E^{\perp}$, where $E^{\parallel}$ is the 2-dimensional space of representation φ and so we can apply the strip projection method $p: \Lambda \otimes \mathbb{R} \to E^{\parallel}$. If R is a finite group, ϱ preserves some integral symmetric bilinear form and so the projection p is orthogonal. If R is infinite, it is not so. But it appears that ϱ preserves some integral non-positive bilinear form and the spaces $E^{\parallel}$ and $E^{\perp}$ are orthogonal with respect to this form.

In the examples below for $R = \mathbb{Z}_5$ $\mathbb{Z}_8$, $\mathbb{Z}_{12}$ and $\mathbb{Z}$, we shall give the formulas for $E^{\|}$ and $E^{\perp}$, for the action of the group generator and for the planes $\{F_i\}$, which appear in the definition of local rules.

Example 1. $R = \mathbb{Z}_8$. Here $Z(x_1, x_2, x_3, x_4) = (-x_4, x_1, x_2, x_3)$, where Z is the group generator of $\mathbb{Z}_8$.

$$E^{\|} = \begin{cases} x_1 - x_3 + \sqrt{2}x_4 = 0 \\ x_1 - \sqrt{2}x_2 + x_3 = 0 \end{cases}; \quad E^{\perp} = \begin{cases} x_1 - x_3 + \sqrt{2}x_4 = 0 \\ x_1 - \sqrt{2}x_2 + x_3 = 0 \end{cases};$$

$$F(\varepsilon_1) = \begin{cases} x_3 = 0 \\ x_2 + x_4 = 0 \end{cases}; \quad F(\varepsilon_2) = \begin{cases} x_4 = 0 \\ x_1 - x_3 = 0 \end{cases};$$

$$F(\varepsilon_3) = \begin{cases} x_1 = 0 \\ x_2 + x_4 = 0 \end{cases}; \quad F(\varepsilon_4) = \begin{cases} x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}.$$

The planes $F(\varepsilon_1)$; $F(\varepsilon_2)$; $F(\varepsilon_3)$; $F(\varepsilon_4)$ are here dependent, and so, to construct local rules, we have to add the plane

$$F_5 = \begin{cases} x_1 - x_4 = 0 \\ x_2 - x_3 = 0 \end{cases}.$$

To make these local rules $\mathbb{Z}_8$-symmetric, we can add another three 2-planes

$$F_6 = \begin{cases} x_1 + x_2 = 0 \\ x_3 - x_4 = 0 \end{cases}; \quad F_7 = \begin{cases} x_2 + x_3 = 0 \\ x_4 + x_1 = 0 \end{cases}; \quad F_8 = \begin{cases} x_3 + x_4 = 0 \\ x_2 - x_1 = 0 \end{cases}$$

which are obtained from F_5 by $\mathbb{Z}$-symmetry.

The General Example of Rotational Symmetry. Everywhere below we shall fix the angle φ (rational or irrational) and restrict ourselves to consideration of the case when Λ, considered as a $\mathbb{Z}_{\lfloor z \rfloor}$-module, is isomorphic to $\mathbb{Z}_{\lfloor z \rfloor}$. (In the general situation the module Λ is isomorphic to some ideal in $\mathbb{Z}_{\lfloor z \rfloor}$.)

Let us note the operator of rotation on the angle φ, acting on Λ, as Z, and let us fix the basis $\{\varepsilon_1; \varepsilon_2; \varepsilon_3; \varepsilon_4\}$ in Λ such, that $Z(\varepsilon_i) = \varepsilon_{i+1}$ $(i = 1, 2, 3)$.

Lemma 6. *The operator Z, which is given in the basis $\{\varepsilon_1; \varepsilon_2; \varepsilon_3; \varepsilon_4\}$ by the matrix*

$$Z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & m \\ 0 & 1 & 0 & -2-k \\ 0 & 0 & 1 & m \end{pmatrix}$$

has two invariant two-dimensional subspaces E and F in $(\Lambda \otimes \mathbb{Q}_{\lfloor \sqrt{D} \rfloor})$. The restriction of Z on the subspace $E^{\|} = E \otimes \mathbb{R}$ is rotation on the angle φ. The restriction of Z on the subspace $E^{\perp} = F \otimes \mathbb{R}$ is hyperbolic rotation with the eigennumbers w_1 and w_2.

Proof. Let us note that the subspace E in $\Lambda \otimes \mathbb{Q}_{\lfloor \sqrt{D} \rfloor}$, generated by the vectors e_1 and e_2, where $e_1 = \varepsilon_1 + \varepsilon_3 - B\varepsilon_2$; $e_2 = \varepsilon_2 + \varepsilon_4 - B\varepsilon_3$; and the subspace F, generated by the vectors f_1 and f_2, where $f_1 = \varepsilon_1 + \varepsilon_3 - A\varepsilon_2$; $f_2 = \varepsilon_2 + \varepsilon_4 - A\varepsilon_3$; satisfy the conditions of the lemma. □

Note. This example contains all the cases $R = \mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{12}$, and $\mathbb{Z}$. If $m = 0$, $k = \pm 2$, then $R = \mathbb{Z}_8$, and we are in the situation of the previous example. If $m = k = \pm 1$, then $R = \mathbb{Z}_5$; if $m = -k = \pm 1$, then $R = \mathbb{Z}_{10}$ and if $m = 0$, $k = \pm 3$, then $R = \mathbb{Z}_{12}$. In all the other cases $R = \mathbb{Z}$.

The 2-planes $F(\varepsilon_1)$; $F(\varepsilon_2)$; $F(\varepsilon_3)$; $F(\varepsilon_4)$ are independent, if $R \neq \mathbb{Z}_8$. The plane $F(\varepsilon_2)$ is generated by e_1 and f_1; $F(\varepsilon_3)$ is generated by e_2 and f_2; $F(\varepsilon_1) = Z^{-1}(F(\varepsilon_2))$; $F(\varepsilon_4) = Z(F(\varepsilon_3))$.

Theorem 4. *The operator Z preserves the two-parametric family $\{A_{pq}\}$; $p, q \in \mathbb{Z}$ of bilinear symmetric forms on Λ of signature $(1,3)$, if R is infinite, and of signature $(4,0)$, if R is finite, which is given in the basis $\{\varepsilon_1; \varepsilon_2; \varepsilon_3; \varepsilon_4\}$ by the family of matrices*

$$A = \begin{pmatrix} 2p & q & -2p-kp+mq & -mkp+(m^2-k-3)q \\ q & 2p & q & -2p-kp+mq \\ -2p-kp+mq & q & 2p & q \\ -mkp+(m^2-k-3)q & -2p-kp+mq & q & 2p \end{pmatrix}.$$

The subspaces $E^{\|}$ and $E^{\perp}$ are orthogonal with respect to the bilinear form A_{pq} with arbitrary values of p and q.

The proof of this lemma consists of direct checking of the matrix equation $Z^* A_{pq} Z = A_{pq}$ and the formula $A_{pq}(e_i, f_j) = 0$ $(i, j = 1, 2)$. $\square$

Note. It can be proved that the space of all the bilinear forms on $\Lambda \otimes \mathbb{R}$, preserved by the operator Z, is two-dimensional linear space, described by the following way: $E^{\|}$ and $E^{\perp}$ are orthogonal with respect to all the bilinear forms from this space; their restrictions on $E^{\|}$ and on $E^{\perp}$ are proportional to the restrictions of bilinear form $A = A_{11}$ on these subspaces.

Hypothesis. We suppose that variation of the bilinear form in the family $\{A_{pq}\}$, preserving its determinant (there is a one-parametric family of such variations), is connected in some way with the inflation-deflation procedure.

Acknowledgements. The discussions with A. Katz and L. Levitov were very instructive at the initial stage of our work. They stressed in particular the role of 2-dimensional planes in enforcing QPLR. Within the text we use a theorem belonging to L. Levitov (August 1990, private communication). A. Kitaev read the original manuscript and helped to eliminate (so some extent) the ambiguities. We thank him and also P. Kalugin for valuable discussions. We are grateful to Professor S. Novikov for interest in our work and important remarks.

References

1. Piunikhin, S.A.: On quasicrystallographic groups in Novikov's sense. Mat. Zametki **47**, No. 5 (1990), 81–87 (in Russian),
   Some new results on quasicrystallographic groups in Novikov's sense. Mat. Zametki **48**, No. 3 (1990), 100–107 (in Russian)
2. de Bruijn, N.G.: Algebraic theory of Penrose non-periodic tilings. Nederl. Akad. Wetensch. Proc. Ser. **A84**, 39–66 (1981)
3. Kramer, P., Neri, R.: On periodic and non-periodic space tilings of E^m obtained by projection. Acta Cryst. **A40**, 580–587 (1984)
4. Katz, A., Duneau, M.: Quasiperiodic patterns and icosahedral symmetry. J. Phys. France **47**, 181–196 (1986)
5. Katz, A., Duneau, M., Oguey, C.: A geometrical approach to quasiperiodic tilings. Commun. Math. Phys. **118**, 99–118 (1988)
6. Gähler, F., Rhyner, J.: Equivalence of the generalized grid and projection methods for the construction of quasiperiodic tilings. J. Phys. A: Math. Gen. **19**, 267–277 (1986)

7. Whittaker, E.J.W., Whittaker, R.M.: Some generalized Penrose patterns from projections of n-dimensional lattices. Acta Cryst. **A44**, 105–112 (1988)
8. Katz, A.: Theory of matching rules for 3-dimensional Penrose tilings. Commun. Math. Phys. **118**, 263–288 (1988)
9. Levitov, L.S.: Local rules for Quasicrystals. Commun. Math. Phys. **119**, 627–666 (1988)
10. Danzer, L.: Three-dimensional analog of the planar Penrose tilings and quasicrystals. Discrete mathematics. Amsterdam: North-Holland **76**, 1–7 (1989)
11. Burkov, S.E.: Absence of weak local rules for the planar quasicrystalline tiling with 8-fold symmetry. Commun. Math. Phys. **119**, 667–675 (1988)
12. Griffiths, Ph., Harris, J.: Principles of algebraic geometry. New York: Wiley 1978, p. 813

Communicated by Ya. G. Sinai