Commun. Math. Phys. 149, 377–414 (1992)

Communications in
Mathematical
Physics

Limit Behavior of Saturated Approximations of Nonlinear Schrödinger Equation

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Received January 22, 1992

Abstract. We consider the solution $u_\varepsilon(t)$ of the saturated nonlinear Schrödinger equation

$$i\,\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,, \tag{1_ε}$$

where $N \geq 2$, $\varepsilon > 0$, $1+4/N < q < (N+2)/(N-2)$, $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, φ is a radially symmetric function in $H^1(\mathbb{R}^N)$. We assume that the solution of the limit equation is not globally defined in time. There is a $T > 0$ such that $\lim\limits_{t\to T} \|u(t)\|_{H^1} = +\infty$, where $u(t)$ is the solution of

$$i\,\partial u/\partial t = -\Delta u - |u|^{4/N}u \quad \text{and} \quad u(0,.) = \varphi(.)\,. \tag{1}$$

For $\varepsilon > 0$ fixed, $u_\varepsilon(t)$ is defined for all time. We are interested in the limit behavior as $\varepsilon \to 0$ of $u_\varepsilon(t)$ for $t \geq T$. In the case where there is no loss of mass in u_ε at infinity in a sense to be made precise, we describe the behavior of u_ε as ε goes to zero and we derive an existence result for a solution of (1) after the blow-up time T in a certain sense. Nonlinear Schrödinger equation with supercritical exponents are also considered.

I. Introduction

In the present paper, we consider the saturated nonlinear Schrödinger equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,, \tag{1_ε}$$

where Δ is the Laplace operator on $\mathbb{R}^N$, $u : [0,T) \times \mathbb{R}^N \to \mathbb{C}$, and $\varphi \in H^1(\mathbb{R}^N)$. We assume that $N \geq 2$, $\varepsilon > 0$ and $1 + 4/N < q < (N+2)/(N-2)$.

We say that $u(.)$ is a solution of an equation of the type

$$i\partial u/\partial t = -\Delta u - f(u) \quad \text{and} \quad u(0,.) = \varphi(.)\,,$$

for $t \in [0,T)$, where $f(x)$ is a nonlinear term, if $\forall t \in [0,T)$,

$$u(t) = S(t)\varphi + i\int_0^t S(t-s)\{f(u(s))\}\,ds\,,$$

where $s(.)$ is the group with the infinitesimal generator $i\Delta$ (the free Schrödinger group) and for each t, $u(t)$ denotes the function $x \to u(t,x)$.

For *a fixed* $\varepsilon > 0$, under these assumptions on N and q, it is well known that equation $(1)_\varepsilon$ has a unique solution $u_\varepsilon(t)$ in H^1 defined globally in time: that is $\forall t \in \mathbb{R}$, $u_\varepsilon(t) \in H^1 = H^1(\mathbb{R}^N)$ (see Ginibre and Velo [5,6], Kato [8]). The problem is to understand for a fixed $t > 0$, the *limit behavior of* $u_\varepsilon(t)$ *as* ε *goes to zero.*

Indeed, for both numerical and theoretical reasons, we want to relate this limit behavior to the limit equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u \quad \text{and} \quad u(0,.) = \varphi(.)\,. \tag{1}$$

It is well known that Eq. (1) has a unique solution $u(t)$ in H^1 and there exists $T > 0$ such that $\forall t \in [0,T)$, $u(t) \in H^1$ and either $T = +\infty$ or $\lim\limits_{t\to T}\|u(t)\|_{H^1} = \lim\limits_{t\to T}\|u(t)\|_{L^{4/N+2}} = +\infty$ (see Ginibre and Velo [5,6], Kato [8]). In [15], it is shown that as $\varepsilon \to 0$, $u_\varepsilon(t)$ converge in H^1 to $u(t)$ uniformly in time in $H^1(\mathbb{R}^N)$ on compact sets of $[0,T)$. Let us consider initial data φ such that $T < +\infty$. Moreover, for a fixed $\varepsilon > 0$, $u_\varepsilon(t)$ is globally defined in time. In order to simplify calculations (numerical computations) as well as for physical reasons, the problem is, for fixed t, to relate the behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0$ to the nonlinear Schrödinger equation (1). For example, if we can define $\lim\limits_{\varepsilon\to 0} u_\varepsilon(t)$ for $t > T$, in what sense does it satisfy the nonlinear Schrödinger equation with the nonlinear term $-|u|^{4/N}u$?

The other way to see this problem is to see it as a problem of physical continuation of blow-up solutions of the nonlinear Schrödinger equation. Equation (1) appears as a model in a lot of different fields: in nonrelativistic quantum mechanics, in superconductivity, in plasmas, in laser beam propagation ($N = 2$)... In particular, for $N = 2$ Eq. (1) can be considered to first approximation as a model of a planar laser beam which is propagating along a single direction t in $\mathbb{R}^3$. In a way, the solution $u(t,x_1,x_2)$ measures the intensity of the laser at a point (t,x_1,x_2) and blow-up of the solution is related to the self-focusing of the laser beam. This model does not quite meet the physicist's requirements when a blowing-up in finite time occurs. In fact, the nonlinear term $-|u|^2u$ is the first term of the expansion of the nonlinearity, and this model is valid when the solution is not too large. Since we have $\lim\limits_{t\to T}\|u(t)\|_{L^4} = +\infty$, the approximation is no longer valid at the blow-up time. For this reason, physicists add a corrective term which gives the saturated model of Eq. $(1)_\varepsilon$ with

$$f(u) = -|u|^2u + \varepsilon|u|^{q-1}u$$

as nonlinear term where $0 < \varepsilon \ll 1$ *and* $q > 3$.

In the present paper, we focus on the problem of understanding the behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0$ for a fixed t, in the special case of initial data radially symmetric and in space dimension N greater or equal to 2. All the results are set in that context. The uniqueness of the Cauchy problem for Eqs. (1), $(1)_\varepsilon$ implies then that for a fixed t the functions $u(t)$, $u_\varepsilon(t)$ are radially symmetric and we do not have to control the space evolution of the singular point in time. Nevertheless in [19], we consider the case of initial data with no special symmetry in dimension one ($N = 1$) and give in that case a complete description of the phenomenon at the blow-up time.

Let us recall some well known results on nonlinear Schrödinger equations. We consider for $1 < p < (N+2)/(N-2)$, the following equation

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,. \tag{1*}$$

For $1 < p < (N+2)/(N-2)$, it is well known that Eq. (1*) has a unique solution $u(t)$ in H^1 and there exists $T > 0$ such that $\forall t \in [0,T)$, $u(t) \in H^1$ and either $T = +\infty$ or $\lim_{t\to T} \|u(t)\|_{H^1} = \lim_{t\to T} \|u(t)\|_{L^{p+1}} = +\infty$ (see Ginibre and Velo [5,6], Kato [8]). We have the following conservation laws in time for $t \in [0,T)$,

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}\,, \tag{2}$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2\,dx \frac{1}{p+1}\int |u(t,x)|^{p+1}\,dx = E(\varphi)\,, \tag{3}$$

$$d/dt \int |x|^2|u(t,x)|^2\,dx = 4\,\mathrm{Im}\int ru(t,x)\bar{u}_r(t,x)\,dx\,,$$

$$d^2/dt^2 \int |x|^2|u(t,x)|^2\,dx = 16E(\varphi) + C(p,N)\int |u(t,x)|^{p+1}\,dx\,, \tag{4}$$

where $C(p,N) < 0$ is $p > 1+4/N$, $C(p,N) = 0$ if $p = 1+4/N$, $C(p,N) > 0$ if $p < 1+4/N$, $r = |x|$ and $u_r = \partial u/\partial r$.

For $1 < p < 1+4/N$, the conservation of mass (2) and energy (3) imply that blowing-up in finite time never occurs (Ginibre and Velo [5]). On the other hand, it is well known that for $p \geq 1+4/N$, there are singular solutions of Eq. (1) for suitable initial data (see Zakharov, Sobolev, and Synach [27], Glassey [7] in the case of initial data with negative energy). That is, there exist solutions $u(t)$ of Eq. (1) such that

$$u(.) \in \mathscr{C}([0,T), H^1) \quad \text{and} \quad \lim_{t\to T} \|u(t)\|_{H^1} = \lim_{t\to T} \|u(t)\|_{L^{p+1}} = +\infty\,.$$

In [13], Merle shows that $u_\varepsilon(t)$ converges in H^1 to $u(t)$ as $\varepsilon \to 0$, uniformly in time t on compact sets of $[0,T)$, where $u_\varepsilon(t)$ is the solution of equation

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,.$$

Therefore, we assume that we are not in the situation where $T = +\infty$: that is $1+4/N \leq p < (N+2)/(N-2)$ and the solution of Eq. (1) with initial data φ blows up in finite time (for example the energy of φ, $E(\varphi)$, is negative). We are interested in particular in the critical case $p = 1+4/N$ and we assume now that

$$p = 1 + 4/N\,.$$

Similar techniques used for Eq. $(1)_\varepsilon$ can be applied for supercritical nonlinear Schrödinger equations. In the last section of the paper, we consider solutions of equation

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,,$$

with $1+4/N < p < q < (N+2)/(N-2)$. For physical reasons, a fixed $\varepsilon > 0$, we want to be $u_\varepsilon(.)$ globally defined in time for all initial data, which yields

$$\varepsilon > 0 \quad \text{and} \quad 1+4/N < q < (N+2)/(N-2)\,.$$

Little is known about the behavior of $u_\varepsilon(t)$ as $\varepsilon \to 0$, for $t = T$ and $t > T$. Even in the case of the nonlinear Schrödinger equation (1) little is known about the behavior of $u(t)$ at the blow-up time.

In particular, from the conservation of the L^2 norm in time of the solution and physical reasons, it is important to understand the behavior in L^2 of $u(t)$ and $u_\varepsilon(t)$ as $\varepsilon \to 0$. We give in this paper a description of different types of behavior in L^2.

The blow-up problem of the solution of Eq. (1) is related in some sense to the following elliptic problem. Let us consider the set of solutions of the equation for $\omega > 0$,

$$-\Delta_u + \omega u - |u|^{4/N}u = 0 \quad \text{in } \mathbb{R}^N. \tag{5$_\omega$}$$

Existence of solutions of such equations have been proved by Berestycki, Lions, Peletier in [1,2] (see also Strauss in [22]).

We first remark that if $w(x)$ is a solution of Eq. $(5)_\omega$, then $u(t,x) = e^{i\omega t}w(x)$ is a solution of Eq. (1). For this special nonlinearity $-|u|^{4/N}u$, we recall that the nonlinear Schrödinger equation (1) has one more time invariant. Equation (1) has a pseudoconformal invariance law, which is the following: if $u(x,t)$ is a solution of Eq. (1), then

$$|t|^{-N/2}e^{i|x|^2/4t}\overline{u}(1/t, x/t)$$

is again a solution of (1). From the conformal invariance for $T > 0$,

$$|t-T|^{-N/2}e^{-i\omega/(t-T)}\overline{w}(x/t-T)$$

is a blow-up solution of Eq. (1).

Moreover for this special exponent, the set of nonzero solutions of Eq. $(5)_\omega$ for $\omega > 0$ has a minimal element in the L^2 sense which is called the ground state. Let us consider the unique radial solution $Q(x)$ of the problem

$$-\Delta u + u - |u|^{4/N}u = 0 \quad \text{and} \quad u > 0 \quad \text{in } \mathbb{R}^N$$

(see Kwong [9] and the other references in [9] for the uniqueness). We have the following property: If w is a solution of $(5)_\omega$ for $\omega > 0$, then

$$\|w\|_{L^2} \geq \|Q\|_{L^2}\,.$$

We remark that $\omega^{N/4}Q(\omega^{1/2}x)$ which is the unique solution of

$$-\Delta u + \omega u - |u|^{4/N}u = 0 \quad \text{and} \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

has the following property: $\|\omega^{N/4}Q(\omega^{1/2}x)\|_{L^2} = \|Q\|_{L^2}$.

The number $\|Q\|_{L^2}$ plays an important role in the blow-up problem of Eq. (1). Weinstein first showed in [25] that, if

$$\|\varphi\|_{L^2} < \|Q\|_{L^2}\,,$$

then the solution $u(t)$ of Eq. (1) is globally defined in time and there is no blow-up.

We remark that this bound is optimal. Indeed, for $\omega > 0$ and $\theta \in \mathbb{R}$, we have that

$$S_{\omega,\theta}(t,x) = \left\{\frac{\omega}{|T-t|}\right\}^{-N/2} \exp\left\{i\theta - \frac{i\omega^2}{T-t}\right\} Q\left(\frac{\omega x}{(T-t)}\right),$$

is a solution of Eq. (1) which blows up at T such that $\|\varphi\|_{L^2} = \|Q\|_{L^2}$. In fact, in [17] Merle shows a uniqueness result for blow-up solutions with minimal mass which gives a characterization of the functions $S_{\omega,\theta}$, for $\omega > 0$ and $\theta \in \mathbb{R}$. That is, if the initial data φ is radially symmetric, $\|\varphi\|_{L^2} = \|Q\|_{L^2}$, and the solution of Eq. (1) blows up at time T, there exists then a $\omega > 0$ and $\theta \in \mathbb{R}$ such that $\varphi = S_{\omega,\theta}(0)$ and $u(t) = S_{\omega,\theta}(t)$.

In the nonradial case, Merle shows in [18], using a compactness lemma, that if the initial data φ is such that $\|\varphi\|_{L^2} = \|Q\|_{L^2}$, and the solution of Eq. (1) blows up at time T, then there exists a $\omega > 0$, $x_1, x_2 \in \mathbb{R}^N$ and $\theta \in \mathbb{R}$ such that

$$\varphi = S_{\omega,\theta,x_1,x_2}(0)\,,$$

and

$$u(t) = S_{\omega,\theta,x_1,x_2}(t)\,,$$

where

$$S_{\omega,\theta,x_1,x_2}(t,x) = [\omega/(T-t)]^{N/2} \exp\left\{i\left[\theta + \frac{\omega^2}{(T-t)} - |x-x_1|^2/[4(T-t)]\right]\right\} Q\left(\frac{(x-x_1)\omega}{T-t} - x_2\right).$$

More generally, Merle and Tsutsumi [20] show that a blow-up solution $u(t)$ of Eq. (1) can not have a strong limit in L^2 at the blow-up time (see also Weinstein [24]). In addition, if the initial data φ has spherical symmetry, then an L^2-concentration phenomenon occurs at the origin at the blow-up time [20,24]. More precisely, for all $R > 0$, we have

$$\liminf_{t\to T} \|u(t)\|_{L^2(B(0,R))} \geq \|Q\|_{L^2}\,,$$

where T is the blow-up time, and $\|u\|_{L^2(B(0,R))}$ represents the L^2 norm of the restriction of u to the ball of radius $R > 0$ centered at the origin. In [16], Merle shows that for any arbitrarily chosen k points, we can find a solution of (1) which blows up simultaneously at these points. In addition, for that solution, there is a concentration phenomenon at the blow-up time in the L^2-space.

These results show us that L^2 plays an important role in the phenomena which arise at the blow-up time in the critical case. In fact a local L^2 theory can be set up for the Cauchy problem of Eq. (1). That is, for initial data in L^2, there exists a unique solution of Eq. (1) in $\mathscr{C}([0,T],L^2)$ such that either $T=+\infty$ or $\int_0^T\int |u(t,x)|^{4/N+2}\,dx\,dt=+\infty$ (see Cazenave and Weissler [3]).

We know that $u_\varepsilon(t)\to u(t)$ as $\varepsilon\to 0$ uniformly on compact sets of $[0,T)$ (regular case, see [15]). In this paper, we are interested in the behavior of $u_\varepsilon(t)$ for $t=T$ and $t>T$ as $\varepsilon\to 0$. In [15], it is shown a stability of the phenomenon of saturation, that is

$$\lim_{\varepsilon\to 0}\|u_\varepsilon(T)\|_{H^1}=+\infty\,,$$

and

$$\lim_{\alpha\to 0}\left\{\lim_{\varepsilon\to 0}\left\{\operatorname*{Inf}_{t\in[T-\alpha,T+\alpha]}\|u_\varepsilon(t)\|_{H^1}\right\}\right\}=+\infty.$$

Different types of questions can now be asked on the behavior of $u_\varepsilon(t)$ as $\varepsilon\to 0$.

Question 1 (Compactness). Does the sequence $u_\varepsilon(t)$ as $\varepsilon\to 0$ have a compact behavior for $t>T$ or equivalently, is there a compact set or finite dimensional manifold K such that $d(u_\varepsilon(t),K)\to 0$ as $\varepsilon\to 0$ for a suitable topology? In addition, can we describe and relate the different elements of K?

Question 2 (Uniqueness). Is there some case where the set K defined before is a singleton, that is, there is a function $u^*(t)$ such that $u^*(t)=\lim\limits_{\varepsilon\to 0}u_\varepsilon(t)$ for $t>T$ in some sense to be made precise?

Question 3 (Regularity). Is the blow-up phenomenon unstable in time; that is, does the regularizing sequence have a regular behavior for $t>T$:

$$\limsup_{\varepsilon\to 0}\|u_\varepsilon(t)\|_{H^1}\le c(t)\,,$$

where $c(t)<+\infty$ for $t>T$?

Question 4 (Stability). Is the object we obtain after going through the limit as ε goes to zero continuous with respect to the initial data and in what sense?

In [17], Merle answers these questions for a different type of approximation for some special initial data and proves a surprising result about chaotic behavior of $u_\varepsilon(t)$ with respect to ε. In particular, there is no hope of finding a unique limit for the sequence $u_\varepsilon(t)$ for $t>T$ in the case considered. In [17], we consider the case where φ is a radially symmetric blow-up solution of Eq. (1) with minimal L^2-mass among blow-up solutions, that is, there are $\omega_->0$ and $\theta_-\in\mathbb{R}$, such that

$$\varphi=S_{\omega_-,\theta_-}(0)\,.$$

The approximation of the solution of the conformal nonlinear Schrödinger equation is taken in a slightly different context. We consider the solution w_ε of the following equation:

$$i\partial u/\partial t=-\Delta u-|u|^{4/N}u\quad\text{and}\quad u(0,.)=\varphi_\varepsilon(.)\,,$$

such that w_ε is globally defined in time ($\|\varphi_\varepsilon\|_{L^2}<\|\varphi\|_{L^2}=\|Q\|_{L^2}$ and $\varphi_\varepsilon\to\varphi$) and φ_ε satisfies a variational condition (see [17]), for example $\varphi_\varepsilon=(1-\varepsilon)\varphi$. Similar techniques may be applied to u_ε, the solution of Eq. $(1)_\varepsilon$.

We first have that for all $\alpha > 0$ and $A > 0$ there is a θ_ε such that

$$S_{\omega_-,\theta_\varepsilon}(.) - w_\varepsilon(.) \to 0 \quad \text{in} \quad \mathscr{C}([T+\alpha, T+A], H^1) \quad \text{as } \varepsilon \to 0\,.$$

In particular, we obtain a compactness result and a regularity result. The main point is the following: we have a loss of information on the phase θ_ε for the sequence $w_\varepsilon(t)$ which yields a nonuniqueness phenomenon for the problem (1). For all θ, there is sequence $\varepsilon_n \to 0$ (depending on θ) such that (see [17])

$$w_{\varepsilon_n}(t) \to S_{\omega_-,\theta_-}(t) \text{ in } H^1 \text{ uniformly on compact sets of } [0,T) \text{ as } n \to +\infty\,,$$

$$w_{\varepsilon_n}(t) \to S_{\omega_-,\theta}(t) \text{ uniformly on compact sets of } (T,+\infty) \text{ as } n \to +\infty\,.$$

In this paper, our target is to answer Question 1 and to give compactness results on the sequence $u_\varepsilon(t)$ in the case of spherical initial data φ. This yields some results on the existence of a weak solution $u(t)$ of Eq. (1) after the blow-up time. In the last section of the paper, we give similar results with nonlinear Schrödinger equations involving a supercritical exponent.

The main result is the following

Theorem 1 ($N \geq 2$). *Let us consider initial data $\varphi \in \Sigma = H^1 \cap \{\varphi; |x|\varphi \in L^2\}$ such that φ is radially symmetric and the solution $u(t)$ of Eq.* (1) *blows up in finite time T. For $\varepsilon > 0$ and $1 + 4/N < q < (N+2)/(N-2)$, let us consider u_ε, the solution of Eq. $(1)_\varepsilon$, where*

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,. \tag{1_ε}$$

For $T_0 > T$, we then have the following alternative (eventually extracting a subsequence).

A) $\int |x|^2 |u_\varepsilon(T_0, x)|^2\, dx \to +\infty \quad$ as $\varepsilon \to 0$

B) There is a constant $C > 0$ such that $\int |x|^2 |u_\varepsilon(T_0,x)|^2\, dx \leq C$.

In this case we have the following properties.

i) *Compactness Outside the Origin in L^2 for $t < T_0$.* There is an application $t \to u^*(t)$ defined for $t < T_0$, such that for all $R > 0$,

$$u^*(t) \in \mathscr{C}([0,T_0), L^2(|x| \geq R))\,,$$

and

$$u_\varepsilon(t) \to u^*(t) \quad \text{in} \quad \mathscr{C}([0,T_0), L^2(|x| \geq R)) \quad \text{as } \varepsilon \to 0\,.$$

ii) *Concentration at the Origin.* For $t < T_0$, there is $m(t) \geq 0$ such that $|u_\varepsilon(t,x)|^2 \to m(t)\delta_{x=0} + |u^*(t,x)|^2$ as $\varepsilon \to 0$ in the distribution sense

– if $m(t) \neq 0$ then $\|u_\varepsilon(t)\|_{H^1} \to +\infty$ as $\varepsilon \to 0$ and $m(t) \geq \|Q\|^2_{L^2}$,

– if $m(t) = 0$ there is a constant $c > 0$ such that for all ε, $\|u_\varepsilon(t)\|_{H^1} < c$, and $u_\varepsilon(t) \to u^*(t)$ in L^2.

iii) *Conservation of Mass.* For all $t < T_0$, $m(t) + \int |u^*(t,x)|^2\, dx = \int |\varphi(x)|^2\, dx$.

The following corollary follows from Theorem 1

Corollary 1.1. *Let us consider initial data φ and solutions $u_\varepsilon(t)$ of Eq. $(1)_\varepsilon$ as in Theorem 1. In addition, assume that there is a constant $c > 0$ such that for $\varepsilon \in (0,1)$ we have*

$$\int_0^{T_0} \varepsilon \|u_\varepsilon(t)\|_{L^{q+1}}^{q+1}\, dt \le c\,.$$

Extracting a subsequence, we are then in case B.

Remark. In practice, to prove that we are in case B, we use Corollary 1.1.

Similar results (Theorem 1 and Corollary 1.1) hold for solutions of the following equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi_\varepsilon(.), \tag{$(1)_\varepsilon$}$$

with φ_ε radially symmetric and $\varphi_\varepsilon \to \varphi$ in Σ as $\varepsilon \to 0$. The same conclusion holds under weaker assumptions: there are R and R' such that for $t < T_0$, there is a c such that for all $\varepsilon \in (0,1)$,

$$\int_0^t \int_{R<|x|<R'} |\nabla u_\varepsilon(s,x)|^2\, dx\, ds < c\,.$$

We show that if there is no loss of mass in infinity in an L^2-space with a weight at infinity for $t = T_0$, then the sequence $\{u_\varepsilon(t)\}_{\varepsilon\in(0,1)}$ is compact in $\mathscr{C}([0,T_0)$, $L^2(|x| \ge R))$ for all $R > 0$. An interesting point will be to show the alternatives A and B in Theorem 1 with alternative in L^2 (with no weight at infinity). That is,

Case A: There is a $\delta_0 > 0$ such that $\liminf\limits_{\varepsilon\to 0} \left\{ \int\limits_{|x|\ge R} |u_\varepsilon(T_0,x)|^2\, dx \right\} \ge \delta_0$.

Case B: For all $R > 0$, $\limsup\limits_{\varepsilon\to 0} \left\{ \int\limits_{|x|\ge R} |u_\varepsilon(T_0,x)|^2\, dx \right\} \to 0$. There is then $u^*(t)$ such that for all $R > 0$, $T_1 < T_0$,

– $u^*(t) \in \mathscr{C}([0,T_0), L^2(|x| \ge R))$,

and

– $u_\varepsilon(t) \to u^*(t)$ in $\mathscr{C}([0,T_0), L^2(|x| \ge R))$ as $\varepsilon \to 0$.

The fact that in case B of Theorem 1, information at time $t = T_0$ (that is there is a constant $c > 0$ such that $\int |x|^2 |u_\varepsilon(T_0,x)|^2\, dx \le c$) gives a result for $t < T_0$ follows from the fact that the Schrödinger equation has a regularizing effect involving integrals in time. In fact (see proof of Theorem 1 in Sects. II and III), we have the following estimates: for all $0 < A$ and $0 \le t < T_0$ there is a constant $c > 0$ such that

$$\forall \varepsilon \in (0,1), \quad \int_0^t \int_{A<|x|} |\nabla u_\varepsilon(s,x)|^2\, dx\, ds < c\,.$$

One main problem left is to know if Theorem 1 is an optimal result or not. The result in [17] shows in the case of initial data $\varphi = S_{\omega,\theta}(0)$, for $\omega > 0$ and $\theta \in \mathbb{R}$ the full sequence $\{u_\varepsilon(t)\}_\varepsilon$ is compact for $t > T$ and satisfies case B in Theorem 1, where $u_\varepsilon(t)$ is the solution of the following equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi_\varepsilon(.)\,,$$

with φ_ε a suitable sequence such that $\varphi_\varepsilon \longrightarrow \varphi$ as $\varepsilon \longrightarrow 0$. Moreover, some formal calculations under some stability assumptions by V.M. Malkin in [12] (also see Fedoruk, Khudik, Malkin [13]) show in fact that for generic initial data we have for all $t > T$ the existence of a constant $c(t) > 0$ such that for all ε, $\int_0^t \varepsilon\|u_\varepsilon(s)\|_{L^{q+1}}^{q+1}\,ds < c(t)$ and from Corollary 1.1, we are in case B. Therefore we expect that case B is the generic case.
The question is to find an example of initial data φ such that we are in case A. We conjecture that this behavior can occur in some cases but is very instable. For the physical case ($N = 2$), we think that the sequence satisfying case A has no physical meaning and the nonlinear Schrödinger equation has to be replaced by a Sakharov type equation: consider initial data φ such that u_ε satisfies case A, we then conjecture that there is no loss of mass at infinity (case B) as $\varepsilon \longrightarrow 0$ for v_ε, where v_ε is the solution of the following equation:

$$\begin{aligned}
&i\partial u/\partial t = -\Delta u + nu + \varepsilon n^2 u,\\
&\delta^{-1}[\partial^2 n/\partial t^2] = \Delta n + \Delta|u|^2,\\
&u(0,.) = \varphi(.) \quad \text{and} \quad n(0,.) = |\varphi(.)|^2, \quad \text{where } \delta > 0\,.
\end{aligned}$$

Many questions are still open on the stability of the singular behavior in time. The result in [17] shows, in some special case of approximations and initial data ($\varphi = S_{\omega,\theta}(0)$ for a $\omega > 0$ and $\theta \in \mathbb{R}$) that the singularity is unstable in time. That is, for $t \neq T$ we have

$$\limsup_{\varepsilon\longrightarrow 0} \int |\nabla u_\varepsilon(t,x)|^2\,dx < +\infty\,.$$

and the singularity appears only for $t = T$.
In [12], formal methods suggest that for generic initial data φ and for $t \geq T$ we have

$$\liminf_{\varepsilon\longrightarrow 0} \int |\nabla u_\varepsilon(t,x)|^2\,dx = +\infty\,.$$

In that case, the singular behavior is stable in time. A mathematical proof of this fact is an open problem. In addition, we do not know in addition if there are other types of behavior for u_ε as $\varepsilon \longrightarrow 0$.

One question left is also to consider nonradial functions as initial data.

Let us give an application of Theorem 1 related to the existence of a weak solution of the nonlinear Schrödinger equation for $t > T$. We can show in case B that u^* is a solution in a certain sense of Eq. (1).

Theorem 2 ($N \geq 2$). *Let us consider initial data $\varphi \in \Sigma = H^1 \cap \{\varphi; |x|\varphi \in L^2\}$ such that φ is a radially symmetric and the solution of Eq.* (1) *$u(t)$ blows up in finite time T. We assume that there are $T_0 > T$ and sequences $\varphi_n \to \varphi$ and $\varepsilon_n \to 0$ as $n \to +\infty$ with a constant $c > 0$ such that*

$$\int |x|^2 |u_n(T_0, x)|^2 \, dx \leq c \,,$$

where u_n is the solution of the equation

$$i\partial u/\partial t = -\Delta u - |u|^{4/N} u + \varepsilon_n |u|^{q-1} u \quad \text{and} \quad u(0,.) = \varphi_n(.) \,.$$

There then exists a function $u^*(t)$ such that for all $R > 0$,

– *$u^*(t) \in \mathscr{C}([0, T_0), L^2(|x| \geq R))$,*

– $\int_0^{T_0} \int_{R<|x|} |\nabla u^*(t,x)|^2 \, dx \, dt < +\infty$,

– *$u^*(t)$ is a solution in the distribution sense of the equation*

$$i\partial u/\partial t = -\Delta u - |u|^{4/N} u \quad \text{and} \quad u(0,.) = \varphi(.) \,,$$

on $[0, T_0) \times \mathbb{R}^N \setminus \{0\}$ for $N \leq 4$, and $[0, T_0) \times \mathbb{R}^N$ for $N > 4$.

Remark. In [17], we show that there is no hope of uniqueness of the continuation after the blow-up time. Indeed, let us consider initial data $\varphi = S_{\omega_-, \theta_-}(0)$, we then have, for all $\theta \in \mathbb{R}$, the existence of a sequence $\varepsilon_n \to 0$ such that

$$\text{for } t > T, \quad u_{\varepsilon_n}(t) \to S_{\omega_-, \theta}(t) \,,$$

where $u_\varepsilon(t)$ is the solution of the following equation:

$$i\partial u/\partial t = -\Delta u - |u|^{4/N} u + \varepsilon |u|^{q-1} u \quad \text{and} \quad u(0,.) = \varphi_\varepsilon(.) \,,$$

with φ_ε a suitable sequence such that $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$.

Remark. The solution $u^*(t)$ in Theorem 2, obtained by a compactness procedure, has a physical meaning as a limit of saturated solutions.

The plan of the paper is the following: Sect. II is devoted to local estimates on $u_\varepsilon(t)$ which yield a local control outside the origin on the term $|\nabla u_\varepsilon(t,x)|^2$ and $\varepsilon |u_\varepsilon(t,x)|^{q+1}$ for $x \neq 0$. Using a compactness procedure, we conclude the proof of Theorems 1 and 2 in Sect. III. In Sect. IV, we briefly give some extensions of the methods used to prove similar results in the supercritical case.

II. Estimates on the Saturated Solutions

We consider $\varphi \in \Sigma = \Sigma(\mathbb{R}^N)$ radially symmetric such that the solution $u(t)$ of Eq. (1) blows up in finite time T. For $\varepsilon \in (0, 1)$, let $u_\varepsilon(t)$ be the solution of the equation

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,, \tag{1_ε}$$

where $4/N + 1 < q < (N+2)/(N-2)$, and consider $T_0 > T$.

We assume in this section that there is a constant $c > 0$ such that

$$\text{for all } \varepsilon \in (0,1), \quad \int |x|^2 |u_\varepsilon(T_0, x)|^2\, dx \le c\,. \tag{2.1}$$

Since we are interested in the limit as $\varepsilon \longrightarrow 0$, we assume that

$$\varepsilon \in (0, 1)\,. \tag{2.2}$$

Assuming (2.1)–(2.2) (that is a uniform control in ε of the decay of the solution $u_\varepsilon(t,x)$ as $|x| \longrightarrow +\infty$), we show that $u_\varepsilon(t,x)$ has a "singular behavior" as $\varepsilon \longrightarrow 0$ only for $x = 0$. We produce in fact some uniform estimates in ε, on $\nabla u_\varepsilon(t,x)$ for $x \ne 0$ which turn out to be the key estimates in the proof of Theorem 1. We prove under assumption (2.1) that for all $R > 0$, there is a constant $c > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_0^{T_0} (T_0 - t) \int_{|x| \ge R} |\nabla u_\varepsilon(t,x)|^2\, dx\, dt \le c\,, \tag{2.3}$$

and

$$\int_0^{T_0} (T_0 - t) \int_{|x| \ge R} \varepsilon |u_\varepsilon(t,x)|^{q+1}\, dx\, dt \le c\,. \tag{2.4}$$

In the proof of Theorem 1, (2.3) gives a compactness property of the sequence $u_\varepsilon(t)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. Starting from (2.3) and using energy estimates, in Sect. III we derived a property stronger than (2.4); that is in Sect. III, using some energy estimates we derived from (2.3)

$$\text{for all } R > 0, \quad \lim_{\varepsilon \to 0} \int_0^{T_0} (T_0 - t) \int_{|x| \ge R} \varepsilon |u_\varepsilon(t,x)|^{q+1}\, dx\, dt = 0\,.$$

To obtain (2.3)–(2.4), we use a local version of a conservation law of Eq. $(1)_\varepsilon$. We consider the variation in time of quantities of the form

$$\int \psi(x) |u_\varepsilon(t,x)|^2\, dx\,,$$

where $\psi \equiv 0$ near the origin and $\psi(x) \approx |x|^2$ as $|x| \longrightarrow +\infty$. This kind of techniques was used in [17] (see also [21]). A control in time of the variations of these types of quantities (from the fact that the term with the "bad" sign $-|u|^{4/N}u$ has subcritical behavior for $x \ne 0$) shows that assumption (2.1) is equivalent to assumptions (2.3)–(2.4) for all $R > 0$.

Part II.1 is devoted to some basic lemmas. In Part II.2, we prove the main result of this section.

II.1. Basic Lemmas

We first remark that from the uniqueness of the Cauchy problem of Eq. $(1)_\varepsilon$, the function $u_\varepsilon(t,x)$ is a radial function of x for fixed ε and t:

$$\forall \varepsilon, \forall t, \forall x, \quad u_\varepsilon(t,x) = u_\varepsilon(t,r), \quad \text{where } r = |x|\,.$$

Let us recall some classical lemmas. $u_\varepsilon(t,x)$ satisfies the following conservation laws.

Lemma 2.1. $\forall \varepsilon \in (0,1)$, $\forall t \in \mathbb{R}$

$$\|u_\varepsilon(t)\|_{L^2} = \|\varphi\|_{L^2}\,, \tag{2.5}$$

$$\begin{aligned} E_\varepsilon(u_\varepsilon(t)) &= \frac{1}{2}\int |\nabla u_\varepsilon(t,x)|^2\,dx - \frac{1}{\frac{4}{N}+2}\int |u_\varepsilon(t,x)|^{4/N+2}\,dx \\ &\quad + \frac{\varepsilon}{q+1}\int |u_\varepsilon(t,x)|^{q+1}\,dx \\ &= E_\varepsilon(\varphi)\,, \end{aligned} \tag{2.6}$$

$$d/dt \int |x|^2 u_\varepsilon(t,x)|^2\,dx = 4\,\mathrm{Im}\int r u_\varepsilon(t,x)\bar{u}_{\varepsilon r}(t,x)\,dx\,,$$

$$d^2/dt^2 \int |x|^2|u_\varepsilon(t,x)|^2\,dx = 16E(\varphi) + \varepsilon C(q,N)\int |u_\varepsilon(t,x)|^{q+1}\,dx\,, \tag{2.7}$$

where $C(q,N) > 0$, $r = |x|$ and $u_r = \partial u/\partial r$.

Let us give the following inequalities

Lemma 2.2. *Let $w(x)$ be a radial function in H^1. We have the following estimates for all $R > 0$:*
i) *There is a constant $c > 0$ which depends only on $R > 0$ such that*

$$\|w\|^2_{L^\infty(|x|\geq R)} \leq c\|w\|_{L^2(|x|\geq R)}\|\nabla w\|_{L^2(|x|\geq R)}\,.$$

ii) *For all $\varepsilon > 0$, there is a constant depending only on the L^2 norm of $w(x)$ and R such that*

$$\begin{aligned} &\int\limits_{|x|\geq R} \min\{(|x|-R)^2, 1\}\,|w(x)|^{4/N+2}\,dx \\ &\quad \leq \varepsilon_0 \int\limits_{|x|\geq R} \min\{(|x|-R)^2, 1\}\,|\nabla w(x)|^2\,dx + c\,. \end{aligned}$$

Proof. i) is a classical result.

ii) From i), we have $\forall a \in (R, R+1)$,

$$\int\limits_{|x|\geq a} |w(x)|^{4/N+2}\,dx$$
$$\leq \{\|w\|^2_{L^\infty(|x|\geq R)}\}^{2/N} \int\limits_{|x|\geq a} |w(x)|^2\,dx$$
$$\leq c\left\{\int\limits_{|x|\geq a} |\nabla w(x)|^2\,dx\right\}^{1/N} \leq \varepsilon_0 \int\limits_{|x|\geq a} |\nabla w(x)|^2\,dx + c/(\varepsilon_0)^{N/(N-1)},$$

where the constant c depends on $\int |w(x)|^2\,dx$.

Therefore, $\forall a \in (R, R+1)$,

$$2(a-R) \int\limits_{|x|\geq a} |w(x)|^{\frac{4}{N}+2}\,dx \leq 2(a-R)\varepsilon_0 \int\limits_{|x|\geq a} |\nabla w(x)|^2\,dx + c,$$

where the constant c depends only on $\int |w(x)|^2\,dx$. Integrating this inequality with respect to $a \in (R, R+1)$, we obtain ii) of Lemma 2.2 (for a more detailed proof see [17]).

We may now announce main result of the section.

II.2. Estimates on $u_\varepsilon(t,x)$ for $x \neq 0$

Proposition 2.3. *Let $R > 0$ fixed. The following properties are equivalent.*

i) *There is a constant $c > 0$ such that for all $\varepsilon \in (0,1)$, $\int |x|^2 |u_\varepsilon(T_0, x)|^2\,dx \leq c$.*

ii) *There is a constant $c > 0$ such that for all $\varepsilon \in (0,1)$,*

$$\int_0^{T_0} (T_0 - t) \int\limits_{|x|\geq R} |\nabla u_\varepsilon(t,x)|^2\,dx\,dt \leq c,$$

and

$$\int_0^{T_0} (T_0 - t) \int\limits_{|x|\geq R} \varepsilon |u_\varepsilon(t,x)|^{q+1}\,dx\,dt \leq c.$$

The proof of this result is based on the control of the variation in time of quantities of the form

$$\int \psi(x)|u_\varepsilon(t,x)|^2\,dx,$$

where ψ is such that $\psi \equiv 0$ near the origin, $\psi(x) \approx |x|^2$ as $|x| \to +\infty$, and ψ has convexity and regularity properties. In this case for a fixed $R > 0$, the quantities

$$\int_0^{T_0} (T_0 - t) \int_{|x| \geq R} |\nabla u_\varepsilon(t,x)|^2 \, dx \, dt \leq c,$$

and

$$\int_0^{T_0} (T_0 - t) \int_{|x| \geq R} \varepsilon |u_\varepsilon(t,x)|^{q+1} \, dx \, dt \leq c,$$

are the terms which control $\int \psi(x)|u_\varepsilon(T_0, x)|^2 \, dx$.

We first estimate $\int \psi(x)|u_\varepsilon(t,x)|^2 \, dx$ where ψ is suitably chosen, then conclude the proof of Proposition 2.3.

Lemma 2.4. *Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be such that ψ is radially symmetric and there is a constant $c > 0$ such that*

$$\forall x, \quad |\psi(x)| \leq c(1 + |x|^2), \quad |\nabla \psi(x)| \leq c(1 + |x|), \quad |\Delta \psi(x)| + |\Delta^2 \psi(x)| \leq c.$$

We have $\forall t$,

i) $d/dt \int \psi(x)|u_\varepsilon(t,x)|^2 \, dx = 2 \operatorname{Im} \int \nabla \psi(x) u_\varepsilon(t,x) \bar{u}_{\varepsilon r}(t,x) \, dx$,

ii) $d^2/dt^2 \int \psi(x)|u_\varepsilon(t,x)|^2 \, dx = 2\big\{ -2/(N+2) \int \Delta \psi(x) |u_\varepsilon(t,x)|^{4/N+2} \, dx$
$+ \varepsilon(q-1)/(q+1) \int \Delta \psi(x)|u_\varepsilon(t,x)|^{q+1} \, dx + 2 \int (\partial^2 \psi / \partial r^2) |\nabla u_\varepsilon(t,x)|^2 \, dx$
$- 1/2 \int \Delta^2 \psi(x) |u_\varepsilon(t,x)|^2 \, dx \big\}$.

Proof (see [17] for precise calculations). We carry out the calculations with a regular solution. Using standard approximation arguments, we extend these equalities to a solution in the integral sense of Eq. $(1)_\varepsilon$. By direct calculation,

$$\begin{aligned} d/dt \int \psi(x)|u_\varepsilon(t,x)|^2 \, dx &= 2 \operatorname{Re} \int \psi \bar{u}_\varepsilon \partial u_\varepsilon / \partial t \\ &= 2 \operatorname{Re} \int \psi \bar{u}_\varepsilon [i \Delta u_\varepsilon + i |u_\varepsilon|^{4/N} u_\varepsilon - \varepsilon i |u_\varepsilon|^{q-1} u_\varepsilon] \\ &= -2 \operatorname{Im} \int \psi \bar{u}_\varepsilon \Delta u_\varepsilon = 2 \operatorname{Im} \int \nabla \psi \bar{u}_\varepsilon \nabla u_\varepsilon, \end{aligned}$$

and part i) of the lemma is established

We have from i),

$$\begin{aligned} d^2/dt^2 \int \psi(x)|u_\varepsilon(t,x)|^2 \, dx &= 2 \left\{ \operatorname{Im} \int \nabla \psi \bar{u}_\varepsilon \nabla \partial u_\varepsilon / \partial t + \operatorname{Im} \int \nabla \psi \partial \bar{u}_\varepsilon / \partial t \nabla u_\varepsilon \right\} \\ &= 2 \left\{ - \operatorname{Im} \int \Delta \psi \bar{u}_\varepsilon \partial u_\varepsilon / \partial t + 2 \operatorname{Im} \int \nabla \psi \partial \bar{u}_\varepsilon / \partial t \nabla u_\varepsilon \right\}. \end{aligned}$$

On the one hand,

$$
\begin{aligned}
-\operatorname{Im}\int \Delta\psi \bar{u}_\varepsilon \partial u_\varepsilon/\partial t &= -\operatorname{Re}\int \Delta\psi \bar{u}_\varepsilon[\Delta u_\varepsilon + |u_\varepsilon|^{4/N}u_\varepsilon - \varepsilon|u_\varepsilon|^{q-1}u_\varepsilon] \\
&= -\int \Delta\psi|u_\varepsilon|^{4/N+2} + \varepsilon\int \Delta\psi|u_\varepsilon|^{q+1} \\
&\quad + \int \Delta\psi|\nabla u_\varepsilon|^2 - 1/2\int \Delta^2\psi|u_\varepsilon|^2. \qquad (2.8)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
2\operatorname{Im}\int \nabla\psi\partial\bar{u}_\varepsilon/\partial t\nabla u_\varepsilon &= -\operatorname{Im}\int \nabla\psi\partial u_\varepsilon/\partial t\nabla\bar{u}_\varepsilon \\
&= -2\operatorname{Re}\int \nabla\psi\Delta u_\varepsilon\bar{u}_\varepsilon - 2\operatorname{Re}\int \nabla\psi|u_\varepsilon|^{4/N}u_\varepsilon\nabla\bar{u}_\varepsilon \\
&\quad + 2\varepsilon\operatorname{Re}\int \nabla\psi|u_\varepsilon|^{q-1}u_\varepsilon\nabla\bar{u}_\varepsilon \\
&= N/(2+N)\int \Delta\psi|u_\varepsilon|^{4/N+2} - 2\varepsilon/(q+1)\int \Delta\psi|u_\varepsilon|^{q+1} \\
&\quad - \operatorname{Re}\int \partial\psi/\partial r\, \partial/\partial r(r^{N-1}\partial u_\varepsilon/\partial r)\partial\bar{u}_\varepsilon/\partial r\, r^{N-1}\,dr \\
&= N/(2+N)\int \Delta\psi|u_\varepsilon|^{4/N+2} - 2\varepsilon/(q+1)\int \Delta\psi|u_\varepsilon|^{q+1} \\
&\quad + \int \left[\partial^2\psi/\partial r^2 - (N-1)(\partial\psi/\partial r)/r\right]|\nabla u_\varepsilon|^2. \qquad (2.9)
\end{aligned}
$$

Part ii) of the lemma follows from (2.8) and (2.9)

Lemma 2.5. *Let $a > 0$. There is a function $\psi_a(x)$ satisfying the following properties: there are constants $c_1 > 0.$, $c_2 > 0$ and $r_a > 0$ such that*

i) $\psi_a \in \mathscr{C}^4$ *and* $\psi_a(x) = \psi_a(|x|)$.
ii) $\psi_a \equiv 0$ *for* $|x| \le a$.
iii) $\forall r > 0,\ 0 \le \psi_a(r) \le c_1 r^2$ *and* $\forall r \ge r_a,\ \psi_a(r) \ge c_2 r^2$.
iv) $\forall x,\ |\nabla\psi_a(x)| \le c_1|x|$.
v) $\|\partial^2\psi_a/\partial r^2\|_{L^\infty} + \|\Delta\psi_a\|_{L^\infty} + \|\Delta^2\psi_a\|_{L^\infty} \le c_1$.
vi) $\forall r,\ \partial^2\psi_a/\partial r^2(r) \ge c_2\Delta\psi_a(r)$.
vii) $\forall r \ge a,\ c_2\min\{(r-a)^2, 1\} \le \partial^2\psi_a/\partial r^2(r) \le c_1\min\{(r-a)^2, 1\}$.

Proof (see [17] for a detailed proof). In [17], it is shown that for A large, $\psi_A(x)$ satisfies the conclusion of the lemma for $a = A$, where $\psi_A(x)$ is defined as the solution of

- $\psi_A \equiv 0$ for $|x| \le A$,
- $\Delta\psi_A(x) = (|x| - A)^2$, for $|x| \in (A, A+1)$,
- $\Delta\psi_A(x) = (|x| - A)^2[1 - (|x| - A - 1)^3]$, for $|x| \in (A+1, A+2)$,
- $\Delta\psi_A(x) = 1$, for $|x| \in (A+2, +\infty)$.

For $a > 0$, we check by direct calculation that $\psi_a(x) = \psi_A(xA/a)$ satisfies properties i)–vii) and Lemma 2.5 is proved.

For $a > 0$, we have the following estimate for $\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx$.

Lemma 2.6. *Let $a > 0$. There are constants $c_1 > 0$ and $c_2 > 0$ such that $\forall \varepsilon \in (0, 1)$,*
i) $\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx$

$$\leq c_1 \left\{ 1 + \int_0^{T_0} (T_0 - t) \int_{|x|\geq a} |\nabla u_\varepsilon(t, x)|^2 + \varepsilon |u_\varepsilon(t, x)|^{q+1}\, dx\, dt \right\},$$

ii) $\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx$

$$\geq -c_1 + c_2 \left\{ \int_0^{T_0} (T_0 - t) \int_{|x|\geq a} \min[(|x| - a)^2, 1]\, [|\nabla u_\varepsilon(t, x)|^2 + \varepsilon |u_\varepsilon(t, x)|^{q+1}]\, dx\, dt \right\}.$$

Proof.

$$\begin{aligned}
&\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx \\
&\quad = \int \psi_a(x)|\varphi(x)|^2\, dx + 2T_0 \operatorname{Im} \int \nabla \psi_a \overline{\varphi} \nabla \varphi\, dx \\
&\qquad + \int_0^{T_0} 2(T_0 - t) \Big\{ -2/(N+2) \int \Delta \psi_a |u_\varepsilon|^{4/N+2} \\
&\qquad + \varepsilon (q-1)/(q+1) \int \Delta \psi_a |u_\varepsilon|^{q+1} \\
&\qquad + 2 \int (\partial^2 \psi_a / \partial r^2) |\nabla u_\varepsilon|^2 - 1/2 \int \Delta^2 \psi_a |u_\varepsilon|^2 \Big\}.
\end{aligned}$$

From Lemma 2.5, identity (2.3), and the fact that $\varphi \in H^1$, we have for constants $c_1 > 0$ and $c_2 > 0$, $\forall \varepsilon \in (0, 1)$,

$$\begin{aligned}
&\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx \\
&\quad \geq -c_2 + c_1 \int_0^{T_0} (T_0 - t) \Big\{ -c_2 \int_{|x|\geq a} \min[(|x| - a)^2, 1] |u_\varepsilon(t, x)|^{4/N+2}\, dx \\
&\qquad + \int_{|x|\geq a} \min[(|x| - a)^2, 1] |\nabla u_\varepsilon(t, x)|^2\, dx \\
&\qquad + \int_{|x|\geq a} \varepsilon \min[(|x| - a)^2, 1] |u_\varepsilon(t, x)|^{q+1}\, dx \Big\}\, dt.
\end{aligned}$$

Applying Lemma 2.2 ii) with $\varepsilon_0 = (2c_1)^{-1}$, we obtain

$$\int \psi_a(x)|u_\varepsilon(T_0, x)|^2\, dx$$

$$\geq -c_2 + c_1\left\{ \int_0^{T_0} (T_0 - t) \int_{|x|\geq a} \min[(|x| - a)^2, 1][|\nabla u_\varepsilon(t, x)|^2 + \varepsilon|u_\varepsilon(t, x)|^{q+1}]\, dx\, dt \right\}.$$

This concludes the proof of the lemma.

We are now able to prove Proposition 2.3.

Proof of Proposition 2.3. We fix $R > 0$.

i) $\Rightarrow$ ii). We know that for some constant $c > 0$, for all $\varepsilon \in (0, 1)$,

$$\int |x|^2 |u_\varepsilon(T_0, x)|^2\, dx \leq c\,.$$

Let $a = R/2$. From Lemmas 2.5–2.6, there is a $c > 0$ such that

$$\text{for all } \varepsilon \in (0, 1), \quad \int \psi_{R/2}(x)|u_\varepsilon(T_0, x)|^2\, dx \leq c\,,$$

and

$$\int_0^{T_0} (T_0 - t) \int_{|x|\geq R/2} \min[(|x| - R/2)^2, 1][|\nabla u_\varepsilon(t, x)|^2 + \varepsilon|u_\varepsilon(t, x)|^{q+1}]\, dx\, dt \leq c\,.$$

Therefore for all $\varepsilon \in (0, 1)$,

$$\int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \{|\nabla u_\varepsilon(t, x)|^2 + \varepsilon|u_\varepsilon(t, x)|^{q+1}\}\, dx\, dt \leq c\,,$$

and ii) is proved.

ii) $\Rightarrow$ i). Assume that for all $\varepsilon \in (0, 1)$,

$$\int_0^{T_0} (T_0 - t) \int_{|x|\geq R} |\nabla u_\varepsilon(t, x)|^2 + \varepsilon|u_\varepsilon(t, x)|^{q+1}\, dx\, dt \leq c\,.$$

Fix $a = R$; Lemma 2.6 gives the existence of a constant $c > 0$ such that

$$\text{for all } \varepsilon \in (0, 1), \quad \int \psi_R(x)|u_\varepsilon(T_0, x)|^2\, dx \leq c\,.$$

From Lemma 2.5, we have for $R_0 > 0$,

$$\text{for all } \varepsilon \in (0, 1), \quad \int_{|x|\geq R_0} |x|^2|u_\varepsilon(T_0, x)|^2\, dx \leq c\,.$$

From the conservation of mass of $u_\varepsilon(t)$ (2.3), there is a constant $c > 0$ such that

$$\text{for all } \varepsilon \in (0, 1), \quad \int |x|^2|u_\varepsilon(T_0, x)|^2\, dx \leq c\,.$$

This concludes the proof of Proposition 2.3.

III. Proof of the Main Results

In this section we prove the main results of the paper: Theorem 1 and 2.

III.1 Proof of Theorem 1

Consider initial data $\varphi \in \Sigma = H^1 \cap \{\varphi;\ |x|\varphi \in L^2\}$ such that φ is radially symmetric and the solution $u(t)$ of Eq. (1) blows up in finite time T. Let $T_0 > T$ and consider $u_{\varepsilon n}$ (denoted for convenience u_n) where $\varepsilon_n \to 0$ as $n \to +\infty$, the solution of Eq. $(1)_{\varepsilon n}$,

$$i\,\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon_n |u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi(.)\,,$$

Extracting a subsequence ε_n (such that $\varepsilon_n \to 0$ as $n \to +\infty$) we can assume that $\int |x|^2 |u_n(T_0,x)|^2\,dx$ has a limit as $n \to +\infty$ (eventually infinite). If $\int |x|^2|u_n(T_0,x)|^2\,dx \to +\infty$ as $n \to +\infty$, we are in case A of Theorem 1. We next assume that there is a constant $c > 0$ such that

$$\forall n, \quad \int |x|^2 |u_n(T_0,x)|^2\,dx \leqq c \tag{3.1}$$

and

$$\forall n, \ \ \varepsilon_n \in (0,1)\,. \tag{3.2}$$

Under these assumptions, Proposition 2.3 yields, for all $R > 0$, a constant $c > 0$ such that

$$\forall n, \quad \int_0^{T_0} (T_0 - t) \int_{|x| \geqq R} |\nabla u_n(t,x)|^2\,dx\,dt \leqq c\,, \tag{3.3}$$

$$\forall n, \quad \int_0^{T_0} (T_0 - t) \int_{|x| \geqq R} \varepsilon_n |u_n(t,x)|^{q+1}\,dx\,dt \leqq c\,. \tag{3.4}$$

We want to prove, eventually extracting a subsequence, that there is a map $u^*(t)$ such that $\forall R > 0$,
- $u^*(t) \in \mathscr{C}([0,T_0),\ L^2(R^N \setminus B_R(0)))$ such that
- $u_n(t) \to u^*(t)$ in $\mathscr{C}([0,T_0),\ L^2(R^N \setminus B_R(0)))$.

We then prove various properties of $u^*(t)$.

We proceed in three steps.

In step one, we prove using various estimates and Property (3.3) a stronger version of Property (3.4):

$$\lim_{n \to +\infty} \int_0^{T_0} (T_0 - t) \int_{|x| \geqq R} \varepsilon_n |u_n(t,x)|^{q+1}\,dx\,dt = 0\,. \tag{3.5}$$

In step two, Properties (3.3) and (3.5) yield that for all $R > 0$, the sequence $u_n(t)$ is compact in $L^2(R^N \setminus B_R(0))$. The proof is an application of an abstract compactness lemma and some properties of Eq. $(1)_{\varepsilon_n}$. This step, together with Sect. II, is the crucial part of the proof of Theorem 1.

In step three, using different time-invariants of $u_n(t)$ (mass, energy and momentum, (2.5)–(2.7)), we conclude the proof of Theorem 1.

Step 1. Let us show that property (3.4) can be derived from (3.3). We claim that (3.3) and the conservation laws imply for all $R > 0$,

$$\lim_{n\to+\infty} \int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt = 0\,.$$

Property (3.5) says in a sense that the effect of the perturbation term $\varepsilon_n |u_n(t,x)|^{q-1} u_n(t,x)$ for $x \neq 0$ is negligible as $n \to +\infty$.

Proposition 3.1. *We have* $\forall R > 0$,

$$\lim_{n\to+\infty} \int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt = 0\,.$$

Proof. Using Sobolev imbeddings (Lemma 2.2i)), we have $\forall \varepsilon$, $\forall t$,

$$\begin{aligned}
\int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx &\leq \varepsilon_n \{|u_n(t)|_{L^\infty(|x|\geq R)}\}^{q-1} \int_{|x|\geq R} |u_n(t,x)|^2\, dx \\
&\leq \varepsilon_n \{|u_n(t)|_{L^\infty(|x|\geq R)}\}^{q-1} \int |\varphi(x)|^2\, dx \\
&\leq c\varepsilon_n \left\{ \int_{|x|\geq R} |\nabla u_n(t,x)|^2\, dx \right\}^{(q-1)/4} . \qquad (3.6)
\end{aligned}$$

Consider two cases corresponding to different values of q.

Case 1. $q \leq 5$ (We remark that the assumption $q < (N+2)/(N-2)$ implies $q \leq 5$ for $N \geq 3$.)

From (3.6) we have, for a constant $c > 0$, $\forall n$, $\forall t$,

$$\int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx \leq c\varepsilon_n \left\{ \int_{|x|\geq R} |\nabla u_n(t,x)|^2\, dx + 1 \right\} .$$

Therefore, from property (3.3) we have

$$\begin{aligned}
\int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt &\leq c\varepsilon_n \left\{ \int_0^{T_0} (T_0 - t) \int_{|x|\geq R} |\nabla u_n(t,x)|^2\, dx\, dt + 1 \right\} \\
&\leq c\varepsilon_n\,,
\end{aligned}$$

and

$$\int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt \to 0 \quad \text{as } n \to +\infty\,,$$

which concludes the proof in the case $q \leq 5$.

Case 2. $q > 5$ (The assumption $q < (N+2)/(N-2)$ implies $N = 2$.)
In this case Proposition 3.1 follows from a global estimate of $\int |\nabla u_n(t,x)|^2\,dx$ with respect to ε.

Lemma 3.2. *Assume that $N = 2$ and $q > 5$. There exists a constant $c > 0$ such that*

$$\forall \varepsilon,\ \forall t,\ \varepsilon_n \left\{ n \int |\nabla u_n(t_n, x)|^2\,dx \right\}^{(q-3)/2} \leq c\,.$$

Proof. Let us argue by contradiction. Assume that there is a subsequence of ε_n (also denoted ε_n) such that for some $t_n \in \mathbb{R}$,

$$\varepsilon_n \left\{ \int_{|x| \geq R} |\nabla u_n(t,x)|^2\,dx \right\}^{(q-3)/2} \to +\infty \quad \text{as } n \to +\infty\,.$$

A contradiction follows using scaling and energy arguments. Consider

$$v_n(x) = \lambda_n^{-1} u_n(t_n, x/\lambda_n), \quad \text{where } \lambda_n = \left\{ \int |\nabla u_n(t_n, x)|^2\,dx \right\}^{1/2}.$$

We have that

$$\forall n, \quad \int |\nabla v_n(x)|^2\,dx = 1$$

and

$$\forall n, \quad \int |v_n(x)|^2\,dx = \int |\varphi(x)|^2\,dx\,.$$

Therefore, from the facts that v_n is a radial function and $q < +\infty$, classical compactness lemmas yield (eventually extracting a subsequence v_n) the existence of $v \in H^1$ such that

$$v_n \rightharpoonup v \text{ in } H^1, \quad v_n \to v \text{ in } L^4, \quad v_n \to v \text{ in } L^{q+1} \text{ as } n \text{ goes to infinity.}$$

On the one hand, the conservation of energy and Sobolev imbeddings imply that for a constant $c > 0$ for all n,

$$\begin{aligned} &\varepsilon_n \lambda_n^{-2} \int |u_n(t_n, x)|^{q+1}\,dx \\ &= (q+1)\lambda_n^{-2} \left\{ E_\varepsilon(\varphi) - 1/2 \int |\nabla u_n(t_n, x)|^{q+1}\,dx + 1/4 \int |u_n(t_n, x)|^4\,dx \right\} \\ &\leq c\lambda_n^{-2}\{1 + \lambda_n\}^2 \leq c\,. \end{aligned}$$

Moreover $1/4 \int |v_n(x)|^4\,dx \geq 1/2 \int |\nabla v_n(x)|^2\,dx + \lambda_n^{-2} E_\varepsilon(\varphi) \to 1/2$ as $n \to +\infty$ and $\lim_{\to +\infty} \int |v_n(x)|^4\,dx = \int |v_n(x)|^4\,dx \geq 2$. In particular $v \neq 0$.

On the other hand, by direct computation,

$$\varepsilon_n \lambda_n^{-2} \int |u_n(t_n, x)|^{q+1}\, dx = \varepsilon_n \lambda_n^{(q-1-2)} \int |v_n(x)|^{q+1}\, dx$$
$$\approx \varepsilon_n \lambda_n^{(q-3)} \int |v(x)|^{q+1}\, dx \quad \text{as } n \to +\infty\,.$$

Thus there is a constant $c > 0$ such that $\forall n$,

$$\varepsilon_n \lambda_n^{(q-3)} = \varepsilon_n \left\{ \int |\nabla u_n(t_n, x)|^2\, dx \right\}^{(q-3)/2} \le c\,,$$

which is a contradiction and the lemma is proved.

To conclude the proof of Proposition 3.1, we remark that $\forall \varepsilon$, $\forall t$,

$$\begin{aligned}
&\int\limits_{|x|\ge R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx \\
&\le c\varepsilon_n \left\{ \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx \right\}^{(q-1)/4} \\
&\le c\varepsilon_n \left\{ \int |\nabla u_n(t,x)|^2\, dx \right\}^{(q-5)/4} \left\{ \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx \right\} \\
&\le c\varepsilon_n^{1-(q-5)/[2(q-3)]} \left\{ \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx \right\} \\
&\le c\varepsilon_n^{(q-1)/[2(q-3)]} \left\{ \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx \right\} \\
&\le c\varepsilon_n^{1/2} \left\{ \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx \right\}.
\end{aligned}$$

Therefore, from property (3.3) we have

$$\begin{aligned}
&\int_0^{T_0} (T_0 - t) \int\limits_{|x|\ge R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt \\
&\le c\varepsilon_n^{1/2} \left\{ \int_0^{T_0} (T_0 - t) \int\limits_{|x|\ge R} |\nabla u_n(t,x)|^2\, dx\, dt \right\} \\
&\le c\varepsilon_n^{1/2} \to 0 \quad \text{as } n \to +\infty\,.
\end{aligned}$$

This concludes the proof of Proposition 3.1.

Step 2. Equation $(1)_{\varepsilon_n}$, (3.3) and Proposition 3.1 yield the sequence $u_n(t)$ has compact behavior outside the origin in L^2. In a certain sense, we show that the singular behavior of $u_n(t)$ as $n \to +\infty$ can appear only for $x = 0$.

Proposition 3.2. *Eventually extracting a subsequence $u_n(t)$, there is a function $u^*(t)$ such that for all $R > 0$,*
– $u^*(t) \in \mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$,
– $u_n(t) \to u^*(t)$ *in* $\mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$ *as* $n \to +\infty$.

We deduce Proposition 3.2 from the following result.

Proposition 3.3. *Let $t_0 < T_0$ and $R_0 > 0$. Eventually extracting a subsequence $u_n(t)$, there is a function $u^*(t)$ such that for all $R > 0$,*
– $u^*(t) \in \mathscr{C}([0, t_0], L^2(\mathbb{R}^N \setminus B_{R_0}(0)))$,
– $u_n(t) \to u^*(t)$ *in* $\mathscr{C}([0, t_0], L^2(\mathbb{R}^N \setminus B_{R_0}(0)))$ *as* $n \to +\infty$.

Let us assume that Proposition 3.3 is proved and show Proposition 3.2. The proof follows from a diagonal subsequence argument. Applying Proposition 3.3 with $R_0 = 1/k$ and $t_0 = T_0 - 1/m$, for $k \in \mathbb{N}^*$ and $m \in \mathbb{N}^*$, there is a function $u^*_{m,k}(t)$ such that, up to subsequence,
– $u^*_{m,k}(t) \in \mathscr{C}([0, T_0 - 1/m], L^2(\mathbb{R}^N \setminus B_{1/k}(0)))$,
– $u_n(t) \to u^*_{m,k}(t)$ in $\mathscr{C}([0, T_0 - 1/m], L^2(\mathbb{R}^N \setminus B_{1/k}(0)))$ as $n \to +\infty$.
Let us now fix m and allow k to go to infinity; by a diagonal subsequence argument, there is a function $u^*_m(t)$ such that, up to subsequence for all $R > 0$,
– $u^*_m(t) \in \mathscr{C}([0, T_0 - 1/m], L^2(\mathbb{R}^N \setminus B_R(0)))$,
– $u_n(t) \to u^*_m(t)$ in $\mathscr{C}([0, T_0 - 1/m], L^2(\mathbb{R}^N \setminus B_R(0)))$ as $n \to +\infty$.
Letting m go to infinity and again using a diagonal subsequence argument, we obtain for a subsequence of $u_n(t)$ also denoted $u_n(t)$ the existence of a function $u^*(t)$ such that for all $R > 0$,
– $u^*(t) \in \mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$,
– $u_n(t) \to u^*(t)$ in $\mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$ as $n \to +\infty$.
Let us now prove Proposition 3.3.

Proof of Proposition 3.3. Fix $t_0 < T_0$ and $R_0 > 0$. We claim it as a consequence of an abstract compactness lemma and some properties of the free Schrödinger group.

We have the following abstract compactness lemma.

Lemma 3.4 (see [23] for example). *Consider a sequence $w_n(r)$ such that for a $\alpha > 1$,*
– *$w_n(r)$ is uniformly bounded in $L^2(0, T, H^1_0((0,1)))$,*
– *$\partial w_n(r)/\partial t$ is uniformly bounded in $L^\alpha(0, T, H^{-1}_0((0,1)))$.*

We then have that $w_n(r)$ is compact in $L^2(0, T, L^2((0,1)))$. Extracting a subsequence w_n, there is a $w \in L^2(0, T, H^1_0((0,1)))$ such that
– *$w_n \rightharpoonup w$ in $L^2(0, T, H^1_0((0,1)))$ as $n \to +\infty$,*
– *$w_n \to w$ in $L^2(0, T, L^2((0,1)))$ as $n \to +\infty$.*

Proof. See [23] p. 271.

As an application of this lemma, we have

Corollary 3.5. There is a map $u^*(t,x)$ such that
i) for $R \geq R_0$,
- $u_n(t) \rightharpoonup u^*(t)$ in $L^2([0,t_0],\ H^1(R \geq |x| \geq R_0/2))$ as $n \to +\infty$,
- $u_n(t) \to u^*(t)$ in $L^2([0,t_0],\ L^2(R \geq |x| \geq R_0/2))$ as $n \to +\infty$,

ii) for all $0 < t < t_0$, $u^*(t,x)$ is a radial function of x,

iii) $\int_0^{t_0} \int_{|x|<R_0/2} |\nabla u^*(s,x)|^2 \, dx \, dt < +\infty$.

Proof. We claim it as a consequence of Lemma 3.4 and properties (3.3)–(3.5). Fix $R \geq R_0$. We consider a $\mathscr{C}^1$ function $\varrho_R(x)$ such that there is a constant $c > 0$ (independent of R) such that

$$\varrho_R \equiv 0, \quad \text{for } |x| \leq R_0/4 \quad \text{and} \quad |x| \geq R+1\,, \tag{3.8}$$

$$\varrho_R \equiv 1, \quad \text{for } R \geq |x| \geq R_0/2\,, \tag{3.9}$$

$$\forall x, \ |\nabla \varrho_R(x)| + |\varrho_R(x)| \leq c\,. \tag{3.10}$$

Let be $w_n(r) = \varrho_R(r) u_n(r)$. By direct calculation, for a constant $c > 0$;

$$\|w_n\|^2_{H^1_0(R_0/2,R+1)} \leq c\|\nabla u_n\|^2_{L^2(|x|\geq R_0/2)} + c\|u_n\|^2_{L^2(|x|\geq R_0/2)} \tag{3.11}$$

Moreover, for all $\psi \in H^1(R_0/2, R+1)$,

$$\begin{aligned}
&|(\partial w_n(r)/\partial t, \psi)| \\
&\quad = |(\varrho_R(r)\partial u_n(r)/\partial t, \psi)| \\
&\quad \leq \left| \int -\Delta u_n(t,x)\psi(x)|x|^{-(N-1)}\varrho_R(x) \right. \\
&\qquad - \int |u_n(t,x)|^{4/N} u_n(t,x)\psi(x)|x|^{-(N-1)}\varrho_R(x) \\
&\qquad \left. + \varepsilon_n \int |u_n(t,x)|^{q-1} u_n(t,x)\psi(x)|x|^{-(N-1)}\varrho_R(x) \right| \\
&\quad \leq c\|\nabla u_n\|_{L^2(|x|\geq R_0/4)}[\|\nabla\psi\|_{L^2} + \|\psi\|_{L^2}] \\
&\qquad + c\|u_n\|^{4/N+1}_{L^{4/N+2}(|x|\geq R_0/4)}\|\psi\|_{L^{4/N+2}} \\
&\qquad + c\varepsilon\|u_n\|^q_{L^{q+1}(|x|\geq R_0/4)}\|\psi\|_{L^{q+1}} \\
&\quad \leq c\|\nabla\psi\|_{L^2}[1 + \|\nabla u_n\|_{L^2(|x|\geq R_0/4)} \\
&\qquad + \|u_n\|^{4/N+1}_{L^{4/N+2}(|x|\geq R_0/4)} + \varepsilon_n\|u_n\|^q_{L^{q+1}(|x|\geq R_0/4)}]\,,
\end{aligned}$$

and

$$\|\partial w_n(r)/\partial t\|_{H^{-1}(R_0/4,R+1)} \le [1 + \|\nabla u_n\|_{L^2(|x|\ge R_0/4)} + \|u_n\|^{4/N+1}_{L^{4/N+2}(|x|\ge R_0/4)} + \varepsilon_n \|u_n\|^q_{L^{q+1}(|x|\ge R_0/4)}] \,. \tag{3.12}$$

From (3.3), (3.5), (3.11)–(3.12) and Lemma 3.4 ($\alpha = (q+1)/q$), we have the existence of a map $w_R^*(t)$ such that extracting a sequence w_n

- $w_n(t) \rightharpoonup w_R^*(t)$ in $L^2([0,t_0], H^1(R+1 \ge |x| \ge R_0/4))$ as $n \to +\infty$,
- $w_n(t) \to w_R^*(t)$ in $L^2([0,t_0], L^2(R+1 \ge |x| \ge R_0/4))$ as $n \to +\infty$,

or the existence of a map $u_R^*(t)$ such that extracting a sequence u_n,

- $u_n(t) \rightharpoonup u_R^*(t)$ in $L^2([0,t_0], H^1(R \ge |x| \ge R_0/2))$ as $n \to +\infty$,
- $u_n(t) \to u_R^*(t)$ in $L^2([0,t_0], L^2(R \ge |x| \ge R_0/2))$ as $n \to +\infty$.

By an argument of diagonal subsequence, there is a map $u^*(t)$ such that extracting a sequence u_n, for all $R > 0$,

- $u_n(t) \rightharpoonup u^*(t)$ in $L^2([0,t_0], H^1(R \ge |x| \ge R_0/2))$ as $n \to +\infty$,
- $u_n(t) \to u^*(t)$ in $L^2([0,t_0], L^2(R \ge |x| \ge R_0/2))$ as $n \to +\infty$.

We then remark that for all $R > 0$,

$$\int_0^{t_0} \int_{R\ge|x|\ge R_0/2} |\nabla u^*(t,x)|^2\, dx\, dt \le \liminf_{n\to+\infty} \int_0^{t_0} \int_{R\ge|x|\ge R_0/2} |\nabla u_n(t,x)|^2\, dx\, dt \le c_{R_0} \,.$$

This conclude the proof of the lemma.

Let us show that in fact

Lemma 3.6.
i) *We have that for all* $R > R_0$,

- $u^*(t) \in \mathscr{C}([0,t_0],\, L^2(R \ge |x| \ge R_0))$,
- $u_n(t) \to u^*(t)$ *in* $\mathscr{C}([0,t_0],\, L^2(R \ge |x| \ge R_0))$ *as* $n \to +\infty$.

ii) *We have that*

- $u^*(t) \in \mathscr{C}([0,t_0],\, L^2(\mathbb{R}^N \setminus B_{R_0}(0)))$,
- $u_n(t) \to u^*(t)$ *in* $\mathscr{C}([0,t_0],\, L^2(\mathbb{R}^N \setminus B_{R_0}(0)))$ *as* $n \to +\infty$.

Proof.
i) Fix $R > R_0$. We show that the fact that u_n is solution of Eq. $(1)_{\varepsilon_n}$ and that $u_n(t) \to u^*(t)$ in $L^2([0,t_0],\, L^2(R+1 \ge |x| \ge R_0/2))$ as $n \to +\infty$, yield

- $u^*(t) \in \mathscr{C}([0,t_0],\, L^2(R \ge |x| \ge R_0))$,
- $u_n(t) \to u^*(t)$ in $\mathscr{C}([0,t_0],\, L^2(R \ge |x| \ge R_0))$ as $n \to +\infty$.

We consider a $\mathscr{C}^1$ function $\varrho(x)$ such that

$$\varrho \equiv 0, \quad \text{for } |x| \le R_0/2 \text{ and } |x| \ge R+1\,, \tag{3.13}$$

$$\varrho \equiv 1, \quad \text{for } R \ge |x| \ge R_0\,. \tag{3.14}$$

Let us show that the sequence u_n satisfies the Cauchy criterion in $\mathscr{C}([0,t_0],\, L^2(R \ge |x| \ge R_0))$.

By direct calculation, we have for all $t \in [0,t_0]$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$,

$$\begin{aligned}
&d/dt \int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx \\
&= \operatorname{Im}\Bigg\{ \int -\Delta[u_m(t,x)-u_n(t,x)][\bar u_m(t,x)-\bar u_n(t,x)]\varrho(x) \\
&\quad - \int \varrho(x)[|u_m(t,x)|^{4/N}u_m(t,x)-|u_n(t,x)|^{4/N}u_n(t,x)][\bar u_m(t,x)-\bar u_n(t,x)] \\
&\quad + \varepsilon_n \int \varrho(x)[|u_n(t+\delta,x)|^{q-1}u_n(t+\delta,x) \\
&\quad - |u_n(t,x)|^{q-1}u_n(t,x)][\bar u_m(t,x)-\bar u_n(t,x)]\Bigg\}\,,
\end{aligned}$$

and there is a constant $c > 0$ such that

$$\begin{aligned}
&d/dt \int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx \\
&\le \operatorname{Im} \int \nabla[u_m(t,x)-u_n(t,x)][\bar u_m(t,x)-\bar u_n(t,x)]\nabla\varrho(x) \\
&\quad + c\int \varrho(x)[|u_m(t,x)|^{4/N}+|u_n(t,x)^{4/N}]|u_m(t,x)-u_n(t,x)|^2\,dx \\
&\quad + c\varepsilon_n \int \varrho(x)[|u_m(t,x)|^{q+1}+|u_n(t,x)|^{q+1}]\,dx\,, \\
&d/dt \int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx \\
&\le \int \nabla[u_m(t,x)-u_n(t,x)][\bar u_m(t,x)-\bar u_n(t,x)]\nabla\varrho(x) \\
&\quad + c[|u_m(t)|^{4/N}_{L^\infty(|x|>R_0/2)}+|u_n(t)|^{4/N}_{L^\infty(|x|>R_0/2)}] \\
&\quad \times \int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx \\
&\quad + c\varepsilon_n \int_{|x|>R_0/2} [|u_m(t,x)|^{q+1}+|u_n(t,x)|^{q+1}]\,dx\,.
\end{aligned}$$

From Lemma 2.2, interpolation estimates and properties of ϱ, we have for all ε^* fixed

$$
\begin{aligned}
&d/dt \int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx \\
&\le (\varepsilon^*)^{-1} \int |u_m(t,x) - u_n(t,x)|^2 (\nabla\varrho(x))^2 \\
&+ c\left[1 + \left[\int_{|x|>R_0/2} |\nabla u_m(t,x)|^2\,dx\right]^{1/2}\right. \\
&\left. + \left[\int_{|x|>R_0/2} |\nabla u_n(t,x)|^2\,dx\right]^{1/2}\right] \int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx \\
&+ c\varepsilon_n \int_{|x|>R_0/2} |u_m(t,x)|^{q+1} + |u_n(t,x)|^{q+1}\,dx \\
&+ \varepsilon^* \int_{|x|>R_0/2} |\nabla u_m(t,x)|^2 + |\nabla u_n(t,x)|^2\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&d/dt \int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx \\
&\le + c[1 + (\varepsilon^*)^{-1}]\left[\int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx\right. \\
&\left. + \left[\int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx\right]^2\right] \\
&+ c\varepsilon_n \int_{|x|>R_0/2} |u_m(t,x)|^{q+1} + |u_n(t,x)|^{q+1}\,dx \\
&+ 2\varepsilon^* \int_{|x|>R_0/2} |\nabla u_m(t,x)|^2 + |\nabla u_n(t,x)|^2\,dx\,.
\end{aligned}
$$

In fact this formula is proved for regular solutions. We extend to integral solutions by a classical limit procedure.
Since for a $c > 0$,

$$
\forall n, m, \quad \int \varrho(x)|u_m(t,x) - u_n(t,x)|^2\,dx \le c \int |\varphi(x)|^2\,dx \le c\,, \tag{3.15}
$$

we have by integration

$$\begin{aligned}
&\operatorname*{Sup}_{t\in[0,t_0]}\left\{\int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx\right\}\\
&\quad\le c\varepsilon^*-1\left[\int_0^{t_0}\int_{R+1\ge|x|\ge R_0/2}|u_m(t,x)-u_n(t,x)|^2\,dx\,dt\right]\\
&\quad+c\int_0^{t_0}\left[\varepsilon_n\int_{|x|>R_0/2}|u_m(t,x)|^{q+1}+|u_n(t,x)|^{q+1}\,dx\right.\\
&\quad\left.+2\varepsilon^*\int_{|x|>R_0/2}|\nabla u_m(t,x)|^2+|\nabla u_n(t,x)|^2\,dx\right]dt\,.
\end{aligned}$$

Corollary 3.5, (3.3), (3.5) yield for a $c>0$;

$$\operatorname*{Sup}_{t\in[0,t_0]}\left\{\int \varrho(x)|u_m(t,x)-u_n(t,x)|^2\,dx\right\}\le c\varepsilon^*+\varepsilon_{\varepsilon^*}(n)+\varepsilon_{\varepsilon^*}(m)\,,$$

for all ε^*, where $\varepsilon_{\varepsilon^*}(k)\to 0$ as $k\to+\infty$.

Therefore $\operatorname*{Sup}_{t\in[0,t_0]}\left\{\int_{R\ge|x|\ge R_0}|u_m(t,x)-u_n(t,x)|^2\,dx\right\}\to 0$ as $n,m\to+\infty$, and the sequence u_n is a Cauchy sequence in $\mathscr{C}([0,t_0],\,L^2(R\ge|x|\ge R_0))$. There is then a map $u^{**}(t)$ in $\mathscr{C}([0,t_0],\,L^2(R\ge|x|\ge R_0))$ such that

$$u_n(t)\to u^{**}(t)\quad\text{in}\ \mathscr{C}([0,t_0],\,L^2(R\ge|x|\ge R_0))\ \text{ as }\ n\to+\infty\,.$$

The uniqueness of the limit in $L^2([0,t_0],\,L^2(R\ge|x|\ge R_0))$ yields $u^{**}=u^*$ and the proof of part i) is concluded.

ii) follows directly from part i). Indeed, from (2.7)–(3.1) the quantity

$$\int_0^{T_0}(T_0-t)\int\varepsilon_n|u_n(t,x)|^{q+1}\,dx\,dt$$

is uniformly bounded and there is a constant $c>0$ such that

$$\forall n,\ \forall t\in[0,t_0],\quad \int|x|^2|u_n(t,x)|^2\,dx\le c\,.$$

In particular, for a $c>0$, $\forall R$, $\forall n$, $\forall t\in[0,t_0]$, $\int_{|x|\ge R}|u_n(t,x)|^2\,dx\le c/R^2$, and by a limit procedure $\forall R$, $\forall t\in[0,t_0]$, $\int_{|x|\ge R}|u^*(t,x)|^2\,dx\le c/R^2$.

We conclude the proof of ii) by standard arguments.

This conclude the proof of Proposition 3.3 and 3.2.
We assume now that up to a subsequence there is $u^*(t)$ such that for all $R > 0$,
– $u^* \in \mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$,
– $u_n(t) \to u^*(t)$ in $\mathscr{C}([0, T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$ as $n \to +\infty$

Step 3. Assuming the convergence of the sequence for $u_n(t, x)$ for $x \neq 0$, we consider the behavior of $u_n(t_0, x)$ *for a fixed* t_0 and for $x = 0$. Using the conservation laws, we relate the singular behavior in H^1 for $u_n(t_0, x)$ to a concentration phenomenon in L^2 at the origin of $u_n(t_0, x)$ (see [20, 24, 26] for this type of property of the solution $u(t)$ of Eq. (1)). Fix $t_0 \in [0, T_0)$. We first have the following lemma as direct consequence of step 2.

Lemma 3.7. *We have the following properties:*
i) $u^*(t_0) \in L^2$ *and* $\int |u^*(t_0, x)|^2\, dx \leq \int |\varphi(x)|^2\, dx$.
ii) *There is a constant* $m(t_0) = \int |\varphi(x)|^2\, dx - \int |u^*(t_0, x)|^2\, dx \geq 0$ *such that*

$$|u_n(t_0, x)|^2\, \delta \to |u^*(t_0, x)|^2 + m(t_0)\delta_{x=0} \quad \textit{as } n \to +\infty \textit{ in the distribution sense}.$$

Proof.
i) Fix $R > 0$. From the fact that $u_n(t_0) \to u^*(t_0)$ in $L^2(|x| \geq R)$ as $n \to +\infty$, and the fact that for all n, $\int |u_n(t_0, x)|^2\, dx \leq \int |\varphi(x)|^2\, dx$, we have that $\int_{|x|\geq R} |u^*(t_0, x)|^2\, dx \leq \int |\varphi(x)|^2\, dx$. Letting $R \to 0$, the dominated convergence theorem yields $u^*(t_0) \in L^2(\mathbb{R}^N)$ and $\int |u^*(t_0, x)|^2\, dx \leq \int |\varphi(x)|^2\, dx$.
ii) Since $u^*(t_0) \in L^2$, we derive that there is a constant $m(t_0) \geq 0$ such that $|u_n(t_0, x)|^2 \to |u^*(t_0, x)|^2 + m(t_0)\delta_{x=0}$ as $n \to +\infty$ in the distribution sense.
Let us fix $R > 0$. We have in particular

$$\int_{|x|\leq R} |u_n(t_0, x)|^2\, dx \quad \to \quad \int_{|x|\leq R} |u^*(t_0, x)|^2\, dx + m(t_0).$$

From the property (3.1), we have the existence of a constant $c > 0$ such that for all n, we have

$$\int_{|x|\geq R} |u_n(t_0, x)|^2\, dx \leq c/R^2,$$

and the conservation of mass (2.3) gives

$$\left| \int_{|x|\leq R} |u^*(t_0, x)|^2\, dx + m(t_0) - \int |\varphi(x)|^2\, dx \right| \leq c/R^2.$$

Now letting R go to infinity, we conclude the proof of the lemma.

We claim now that the singular behavior of $u_n(t_0)$ is characterized by the value of $m(t_0)$.
– If $m(t_0) = 0$ then $u_n(t_0)$ has a regular behavior: there is a constant $c > 0$ such that $\int |\nabla u_n(t_0, x)|^2\, dx \leq c$,
– If $m(t_0) \neq 0$ then $u_n(t_0)$ has a singular behavior: $\int |\nabla u_n(t_0, x)|^2\, dx \to +\infty$ as $n \to +\infty$.

Proposition 3.8.
i) *If* $m(t_0) \neq 0$ *then* $\int |\nabla u_n(t_0, x)|^2\, dx \to +\infty$ *as* $n \to +\infty$.
In addition, we have $m(t_0) \geq \int |Q(x)|^2\, dx$.
ii) *If* $m(t_0) = 0$ *there is then a constant* $c > 0$ *such that* $\int |\nabla u_n(t_0, x)|^2\, dx \leq c$.

Proof. Proposition 3.8 is a direct consequence of Lemma 2.4 and the conservation laws which the function $u_n(t)$ satisfies.

i) Assume that $m(t_0) \neq 0$. We first have that $\int |\nabla u_n(t_0, x)|^2\, dx \to +\infty$ as $n \to +\infty$. By way of contradiction, assume that for a subsequence also denoted $u_n(t_0)$ there is a constant $c > 0$ such that $\forall n$, $\int |\nabla u_n(t_0, x)|^2\, dx \leq c$. By Sobolev and interpolation estimates, we have that

$$\text{for all} \quad p \in (2, 2N/(N-2)), \ \forall n, \quad \int |u_n(t_0, x)|^p\, dx \leq c.$$

and there are $\alpha > 0$ and $c > 0$ such that $\forall n$, $\int |u_n(t_0, x)|^2\, dx \leq cR^\alpha$. Letting n go to infinity, we obtain from Lemma 3.7 that

$$\forall R > 0, \ m(t_0) \leq cR^\alpha, \quad \text{and} \quad m(t_0) = 0 \ \text{which is a contradiction.}$$

Therefore

$$\int |\nabla u_n(t_0, x)|^2\, dx \to +\infty \quad \text{as } n \to +\infty.$$

We now claim that

$$m(t_0) \geq \int |Q(x)|^2\, dx.$$

Let us argue again by contradiction: assume that $0 < m(t_0) < \int |Q(x)|^2\, dx$. There are then a $\delta_0 > 0$ and $R_0 > 0$ such that for n large

$$\int_{|x| \leq R_0} |u_n(t_0, x)|^2\, dx \leq \int |Q(x)|^2\, dx - \delta_0. \tag{3.15}$$

We obtain a contradiction using scaling and energy properties of $u_n(t_0)$. As for the nonlinear Schrödinger equation with a critical power, we consider

$$v_n = \lambda_n^{-N/2} u_n\Big(t_0, \frac{x}{\lambda_n}\Big) \quad \text{with } \lambda_n = \Big(\int |\nabla u_n(t_0, x)|^2\, dx\Big)^{1/2}.$$

By direct calculations, we have

$$\int |\nabla v_n(x)|^2\, dx = 1 \tag{3.16}$$

$$\int |v_n(x)|^2\, dx = \int |\varphi(x)|^2\, dx \tag{3.17}$$

$$\begin{aligned} E(v_n) &= 1/2 \int |\nabla v_n(x)|^2\, dx - 1/(4/N+2) \int |v_n(x)|^{4/N+2}\, dx \\ &= \lambda_n^{-2}\left\{ 1/2 \int |\nabla u_n(t_0,x)|^2\, dx - 1/(4/N+2) \int |u_n(t_0,x)|^{4/N+2}\, dx \right\} \\ &\le \lambda_n^{-2}\left\{ E_{\varepsilon_n}(\varphi) - \varepsilon_n/(q+1) \int |u_n(t_0,x)|^{q+1}\, dx \right\} \\ &\le \lambda_n^{-2} E_{\varepsilon_n}(\varphi) \to 0 \quad \text{as } n \to +\infty\,. \end{aligned} \tag{3.18}$$

As a consequence of (3.16)–(3.18) we have

$$\begin{aligned} \liminf_{n\to+\infty} \int |v_n(x)|^{4/N+2}\, dx &\ge \liminf_{n\to+\infty} (4/N+2)\left[1/2 \int |\nabla v_n(x)|^2\, dx - E(v_n) \right] \\ &\ge 2/N+2\,. \end{aligned} \tag{3.19}$$

Moreover, from property (3.15) and the fact that $\lambda_n \to +\infty$, for all $R > 0$,

$$\begin{aligned} \limsup_{n\to+\infty} \int_{|x|\le R} |v_n(x)|^2\, dx &\le \limsup_{n\to+\infty} \int_{|x|\le R/\lambda_n} |u_n(t_0,x)|^2\, dx \\ &\le \limsup_{n\to+\infty} \int_{|x|\le R_0} |u_n(t_0,x)|^2\, dx \le \int |Q(x)|^2\, dx - \delta_0\,. \end{aligned} \tag{3.20}$$

Using now a classical compactness procedure (since v_n is a radial function; see for example [26]), we can assume that there is a $v \in H^1$ such that

$$v_n \rightharpoonup v \text{ in } H^1\,, \quad \text{and} \quad v_n \to v \text{ in } L^{4/N+2}.$$

From (3.16)–(3.20), we have

$$\int |\nabla v(x)|^2\, dx \le 1,\ E(v) \le 0,$$

$$\int |v(x)|^2\, dx \le \int |Q(x)|^2\, dx - \delta_0,$$

$$\int |v(x)|^{4/N+2}\, dx \ge 2/N+2\,.$$

In particular, the function $v \neq 0$ (3.19) is such that $E(v) \leq 0$ and $\int |v_n(x)|^2\,dx < \int |Q(x)|^2\,dx$, which contradicts the definition of Q (see for example Weinstein [25]) and so $m(t_0) \geq \int |Q(x)|^2\,dx$.
ii) Assume that $m(t_0) = 0$. We claim that there is a constant $c > 0$ such that for all n,

$$\int |\nabla u_n(t_0, x)|^2\,dx \leq c\,.$$

Assume by way of contradiction that for a subsequence also denoted $u_n(t_0)$, we have

$$\int |\nabla u_n(t_0, x)|^2\,dx \to +\infty \quad \text{as } n \to +\infty\,.$$

In part i), we derived that $m(t_0) \geq \int |Q(x)|^2\,dx > 0$, which is a contradiction. This concludes the proof of Proposition 3.8.

Theorem 1 follows from Proposition 3.1, 3.2, 3.8 and Lemma 3.7.

Corollary 1.1 follows directly from the conservation laws and Theorem 1. Assume that there is a constant $c > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_0^{T_0} (T_0 - t) \int \varepsilon |u_\varepsilon(t,x)|^{q+1}\,dx\,dt \leq c\,.$$

We have from (2.7), for all $\varepsilon \in (0,1)$,

$$\begin{aligned}\int |x|^2 |u_\varepsilon(T_0, x)|^2\,dx &\leq \int |x|^2 |\varphi(x)|^2\,dx + 2T_0 \left| \operatorname{Im} \int x\varphi(x)\nabla\overline{\varphi}(x)\,dx \right| \\ &\quad + c\int_0^{T_0} (T_0 - t) \int \varepsilon |u_\varepsilon(t,x)|^{q+1}\,dx\,dt \leq c\,.\end{aligned}$$

Therefore, we are in case B) of Theorem 1 and the proof of the lemma is concluded.

III.2 Proof of Theorem 2

Let us consider initial date $\varphi \in \Sigma = H^1 \cap \{\varphi;\, |x|\varphi \in L^2\}$ such that φ is radially symmetric and the solution of Eq. (1) $u(t)$ blows up in finite time T. Let us assume that there are $T_0 > T$ and sequences $\varphi_n \to \varphi$ and $\varepsilon_n \to 0$ as $n \to +\infty$ with a constant $c > 0$ such that

$$\int |x|\,|u_n(T_0, x)|^2\,dx \leq c\,, \tag{3.21}$$

where u_n is the solution of the equation

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon_n |u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi_n(.)\,. \tag*{$(1)_n$}$$

We remark that Theorem 1 is still true if we consider solutions $u_n(t)$ of the equation

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u + \varepsilon_n |u|^{q-1}u \quad \text{and} \quad u(0,.) = \varphi_n(.)\,.$$

and the proof is the same.
From assumption (3.21), we derive that we are in case B of Theorem 1. Extracting a subsequence also denoted $u_n(t)$, we can assume there is a function $u^*(t)$ such that for all $R > 0$,
- $u^*(t)$ in $\mathscr{C}([0,T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$,
- $u_n(t) \to u^*(t)$ in $\mathscr{C}([0,T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$ as $n \to +\infty$.

In addition, from Sect. III.1, we have for all $R > 0$ the existence of a constant $c > 0$ such that

$$\forall n, \int_0^{T_0} (T_0 - t) \int_{|x|\geq R} |\nabla u_n(t,x)|^2\, dx\, dt \leq c\,, \tag{3.22}$$

and

$$\int_0^{T_0} (T_0 - t) \int_{|x|\geq R} \varepsilon_n |u_n(t,x)|^{q+1}\, dx\, dt \to 0 \quad \text{as } n \to +\infty\,. \tag{3.23}$$

From the fact that $u_n(t) \rightharpoonup u^*(t)$ in $L^2([0,T_0), H^1(\mathbb{R}^N \setminus B_R(0))$ as $n \to +\infty$ (Step 2), we have

$$\forall n, \int_0^{T_0} (T_0 - t) \int_{|x|\geq R} |\nabla u^*(t,x)|^2\, dx\, dt \leq c\,. \tag{3.24}$$

We now claim that $u^*(t)$ is a solution in the distribution sense on $[0,T_0)\times\mathbb{R}^N\setminus\{0\}$, of equation

$$i\partial u/\partial t = -\Delta u - |u|^{4/N}u \quad \text{and} \quad u(0,.) = \varphi(.)\,.$$

Proposition 3.6. *For all functions $\psi \in \mathscr{C}^\infty$ with compact support in $[0,T_0)\times\mathbb{R}^N\setminus\{0\}$, and for all $t \in [0,T_0)$, we have*

$$\begin{aligned}
&\int_0^t \int - i\partial\psi(s,x)/\partial t\, u^*(s,x)\, dx\, ds \\
&\quad = -i \int \psi(t,x)u^*(t,x)\, dx + i\int \psi(0,x)\varphi(x)\, dx \\
&\quad + \int_0^t \int -\Delta\psi(s,x)u^*(s,x) - \psi(s,x)|u^*(s,x)|^{4/N}u^*(s,x)\, dx\, ds\,.
\end{aligned}$$

We claim that the proposition is a consequence of the facts that the functions $u_n(t)$ are solutions in the distribution sense on $[0,T_0)\times\mathbb{R}^N\setminus\{0\}$, of Eq. $(1)_n$ and $u_n(t) \to u^*(t)$ in $\mathscr{C}([0,T_0), L^2(\mathbb{R}^N \setminus B_R(0)))$ for all $R > 0$ as $n \to +\infty$, (3.22)–(3.23). The case $N > 4$ follows from the fact the term $|u|^{4/N}u$ can be controlled at the origin with the L^2-norm.

Proof of Proposition 3.9. Consider $\psi \in \mathscr{C}^\infty$ with compact support in $[0, T_0) \times \mathbb{R}^N \setminus \{0\}$, and $t \in [0, T_0)$, we want to prove

$$
\begin{aligned}
&-\int_0^t \int -i\partial\psi(s,x)/\partial t\, u^*(s,x)\, dx\, ds \\
&= -i \int \psi(t,x) u^*(t,x)\, dx + i \int \psi(0,x)\varphi(x)\, dx \\
&\quad + \int_0^t \int -\Delta\psi(s,x) u^*(s,x) - \psi(s,x)|u^*(s,x)|^{4/N} u^*(s,x)\, dx\, ds\,.
\end{aligned}
$$

The proof in the case $N > 4$ is similar and follows from the fact that for $\alpha > 0$ and $C > 0$, we have from the conservation of mass $\forall n$,

$$
\int_{|x|\le R} |u^*(s,x)|^{4/N+1} + |u_n(s,x)|^{4/N+1}\, dx \le cR^\alpha \to 0 \quad \text{as } R \to 0\,.
$$

Since ψ has compact support in $[0, T_0) \times \mathbb{R}^N \setminus \{0\}$, there are $t_0 \in [0, T_0)$ and $R_1 > R_0 > 0$ such that Supp(ψ) is contained in $[0, t_0] \times \{R_0 \le |x| \le R_1\}$. We have to prove that for all $t \in [0, t_0]$,

$$
\begin{aligned}
&-\int_0^t \int_{R_0\le|x|\le R_1} i\partial\psi(s,x)/\partial t\, u^*(s,x)\, dx\, ds \\
&= -i \int_{R_0\le|x|\le R_1} \psi(t,x) u^*(t,x)\, dx + i \int_{R_0\le|x|\le R_1} \psi(0,x)\varphi(x)\, dx \\
&+ \int_0^t \int_{R_0\le|x|\le R_1} -\Delta\psi(s,x) u^*(s,x) - \psi(s,x)|u^*(s,x)|^{4/N} u^*(s,x)\, dx\, ds\,.
\end{aligned} \tag{3.25}
$$

Since $u_n(t)$ is a classical solution of Eq. $(1)_n$, we have

$$
\begin{aligned}
&-\int_0^t \int_{R_0\le|x|\le R_1} i\partial\psi(s,x)/\partial t\, u_n(s,x)\, dx\, ds \\
&= -i \int_{R_0\le|x|\le R_1} \psi(t,x) u_n(t,x)\, dx + i \int_{R_0\le|x|\le R_1} \psi(0,x)\varphi(x)\, dx \\
&\quad + \int_0^t \int_{R_0\le|x|\le R_1} -\Delta\psi(s,x) u_n(s,x) - \psi(s,x)|u_n(s,x)|^{4/N} u_n(s,x) \\
&\quad + \varepsilon_n \psi(s,x)|u_n(s,x)|^{q-1} u_n(s,x)\, dx\, ds\,.
\end{aligned} \tag{3.26}
$$

We claim that letting n going to infinity in (3.26) we obtain (3.25).
As a direct consequence of the fact that

$$u_n(t) \to u^*(t) \ \mathscr{C}([0,T_0],\, L^2(R^N \setminus B_{R_0}(0))) \quad \text{as } n \to +\infty\,,$$

we have as $n \to +\infty$

$$\begin{aligned} &\int_0^t \int_{R_0 \le |x| \le R_1} \partial\psi(s,x)/\partial t\, u_n(s,x)\,dx\,ds \\ &\quad \to \int_0^t \int_{R_0 \le |x| \le R_1} \partial\psi(s,x)/\partial t\, u^*(s,x)\,dx\,ds\,, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\int_0^t \int_{R_0 \le |x| \le R_1} -\Delta\psi(s,x) u_n(s,x)\,dx\,ds \\ &\quad \to \int_0^t \int_{R_0 \le |x| \le R_1} -\Delta\psi(s,x) u^*(s,x)\,dx\,ds\,, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\int_{R_0 \le |x| \le R_1} \psi(t,x) u_n(t,x)\,dx \\ &\quad \to \int_{R_0 \le |x| \le R_1} \psi(t,x) u^*(t,x)\,dx\,, \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\int_{R_0 \le |x| \le R_1} \psi(0,x)\varphi(x)\,dx \\ &\quad \to \int_{R_0 \le |x| \le R_1} \psi(0,x)\varphi(x)\,dx\,. \end{aligned} \tag{3.30}$$

In addition from (3.23)

$$\begin{aligned} &\left| \int_0^t \int_{R_0 \le |x| \le R_1} \varepsilon_n \psi(s,x) |u_n(s,x)|^{q-1}\,dx\,ds \right| \\ &\le c \int_0^{t_0} \int_{R_0 \le |x| \le R_1} \varepsilon_n |(u_n(s,x)|^{q+1} + 1)\,dx\,ds \\ &\le c \int_0^{T_0} (T_0 - t) \int_{|x| \ge R} \varepsilon_n |(u_n(s,x)|^{q+1} + 1)\,dx\,dt \to 0 \quad \text{as } n \to +\infty\,. \end{aligned} \tag{3.31}$$

Let us prove that

$$\int_0^t \int_{R_0\le|x|\le R_1} \psi(s,x)|u_n(s,x)|^{4/N}u_n(s,x) - \psi(s,x)|u^*(s,x)|^{4/N}u^*(s,x)\,dx\,ds \to 0 \quad \text{as } n\to+\infty\,. \tag{3.31}$$

By direct calculation

$$\begin{aligned}
&\left|\int_0^t \int_{R_0\le|x|\le R_1} \psi(s,x)|u_n(s,x)|^{4/N}u_n(s,x) - \psi(s,x)|u^*(s,x)|^{4/N}u^*(s,x)\,dx\,ds\right| \\
&\le c\int_0^{t_0}\int_{R_0\le|x|\le R_1} \{|u_n(s,x)|^{4/N} + |u^*(s,x)|^{4/N}\}(|u_n(s,x)-u^*(s,x)|)\,dx\,ds \\
&\le c\int_0^{t_0}\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)|^{8/N} + |u^*(s,x)|^{8/N}\,dx\right\}^{1/2} \\
&\quad\times\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)-u^*(s,x)|^2\,dx\right\}^{1/2} ds\,.
\end{aligned}$$

We have, using Lemma 2.2 and various interpolation estimates,

$$\begin{aligned}
&\int_0^{t_0}\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)|^{8/N} + |u^*(s,x)|^{8/N}\,dx\right\}^{1/2} ds \\
&\le c\left(1+\int_0^{t_0}\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)|^4 + |u^*(s,x)|^4\right\}^{1/2} ds\right) \\
&\le c\left(1+\int_0^{t_0}\{|u_n(s,x)|^2_{L^\infty(R_0\le|x|\le R_1)} + |u^*(s,x)|^2_{L^\infty(R_0\le|x|\le R_1)}\}^{1/2}\,ds\right) \\
&\le c\left(1+\int_0^{t_0}\{|u_n(s,x)|_{L^\infty(R_0\le|x|)} + |u^*(s,x)|_{L^\infty(R_0\le|x|)}\}\,ds\right) \\
&\le c\left(1+\int_0^{t_0}\left\{\int_{R_0\le|x|} |\nabla u_n(s,x)|^2 + |\nabla u^*(s,x)|^2\,dx\right\}ds\right).
\end{aligned}$$

Therefore from Properties (3.22) and (3.25), we obtain that there is a constant $c>0$ such that for all n, $\int_0^{t_0}\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)|^{8/N} + |u^*(s,x)|^{8/N}\,dx\right\}^{1/2} ds \le c.$

We have

$$\left|\int_0^t \int_{R_0\le|x|\le R_1} \psi(s,x)|u_n(s,x)|^{4/N}u_n(s,x) - \psi(s,x)|u^*(s,x)|^{4/N}u^*(s,x)\,dx\,ds\right|$$

$$\le c \operatorname*{Sup}_{0\le s\le t_0}\left\{\int_{R_0\le|x|\le R_1} |u_n(s,x)-u^*(s,x)|^2\,dx\right\}^{1/2} \to 0 \quad \text{as } n\to+\infty,$$

since $u_n(t)\to u^*(t)$ in $L^2(\mathbb{R}^N\setminus B_{R_0}(0))$ as $n\to+\infty$. Thus (3.31) is proved.

From (3.27)–(3.32), letting n going to infinity in (3.26) we obtain (3.25). This concludes the proofs of Proposition 3.9 and Theorem 2.

IV. The Supercritical Case

In this section, we consider the saturated nonlinear Schrödinger equation with supercritical exponent

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u - \varepsilon|u|^{q-1}u \quad \text{and} \quad u(0,.)=\varphi(.)\,, \qquad (1^*)_\varepsilon$$

where $N\ge 2$, $\varepsilon>0$ and $1+4/N<p<q<(N+2)/(N-2)$, and φ radially symmetric in $H^1(\mathbb{R}^N)$.
For $\varepsilon>0$, we have an unique solution $v_\varepsilon(t)$ of Eq. $(1^*)_\varepsilon$ which is globally defined in time. We assume that the solution $v(t)$ of Eq. $(1^*)_\varepsilon$ blows up at $t=T$, where

$$i\partial u/\partial t = -\Delta u - |u|^{p-1}u \quad \text{and} \quad u(0,.)=\varphi(.)\,, \qquad (1^*)$$

We are again interested in the behavior as $\varepsilon\to 0$ of $v_\varepsilon(t)$ for $t\ge T$.
The techniques used previously in the critical case can be applied and we obtain the following theorem:

Theorem 3. ($N\ge 2$). *For $T_0>T$, we then have the following alternative (eventually extracting a subsequence)*

A) $\int |x|^2|v_\varepsilon(T_0,x)|^2\,dx \to +\infty$ *as* $\varepsilon\to 0$.
B) *There is a constant $C>0$ such that* $\int |x|^2|v_\varepsilon(T_0,x)|^2\,dx\le C$.

In this case we have the following properties.

i) Compactness outside the Origin in L^2
There is a map $v^*(t)$ defined for $t<T_0$, such that for all $R>0$,

$$u^*(t)\in\mathscr{C}([0,T_0),\,L^2(|x|\ge R))\,,$$

and

$$u_\varepsilon(t)\to u^*(t) \text{ in } \mathscr{C}([0,T_0),\,L^2(|x|\ge R)) \quad \text{as } \varepsilon\to 0\,.$$

ii) For $0 < t < T_0$, there is $m(t) \geq 0$ such that

$$|v_\varepsilon(t,x)|^2 \to m(t)\delta_{x=0} + \int |v^*(t,x)|^2\,dx \quad \text{as } \varepsilon \to 0\,,$$

and

$$m(t) + \int |v^*(t,x)|^2\,dx = \int |\varphi(x)|^2\,dx\,.$$

This result is less interesting than the one's for the critical case. Indeed, we conjecture that for all $t < T_0$ we have $m(t) = 0$. That is

$$v(t) \to v^*(t) \text{ in } \mathscr{C}([0,T_0),\, L^2)$$

(see also Merle [14] for this type of results and conjectures).

References

1. Berestycki, H., Lions, P.L., Peletier, L.A.: An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$. Ind. Uni. Math. J. **30**, 141–157 (1981)
2. Berestycki, H., Lions, P.L.: Existence d'ondes solitaires dans des problèmes nonlinéaires du type Klein-Gordon. C.R. Paris **287**, 503–506 (1978); **288**, 395–398 (1979)
3. Cazenave, T., Weissler, F.: Some remarks on the nonlinear Schrödinger equation in the critical case. In: Nonlinear semigroups, Partial differential equations, and attractors. Gill, T.L., Zachary (eds.) Lect. Notes in Math. **347**, pp. 18–29, Berlin-Heidelberg-New York: Springer, 1989
4. Craig, W., Kappeler, T., Strauss, W.: Preprint
5. Ginibre, J., Velo, G.: On a class of nonlinear Schrödinger equations. I. The Cauchy problem. J. Funct. Anal. **32**, pp. 1–32 (1979)
6. Ginibre, J., Velo, G.: The global Cauchy problem for the nonlinear Schrödinger equation revisited. Ann. Inst. Henri Poincaré, Physique Théorique **4**, pp. 309–327 (1985)
7. Glassey, R.T.: On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. J. Math. Phys. **18**, pp. 1794–1797 (1977)
8. Kato, T.: On nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré. Physique Théorique **46**, pp. 113–129 (1987)
9. Kwong, M.K.: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. Arch. Rat. Mech. Ann. **105**, 243–266 (1989)
10. Landmann, M., Papanicolaou, G.C., Sulem, C., Sulem, P.L.: Rate of blow-up for solutions of the nonlinear Schrödinger equation in critical dimension. Phys. Rev. A **38**, 3837–3843 (1988)
11. MacLaughlin, D.W., Papanicolaou, G., Sulem, C., Sulem, P.L.: The focusing singularity of the nonlinear Schrödinger equation. Phys. Rev. A **34**, 1200 (1986)
12. Malkin, V.M.: Personal communication
13. Fedoruk, M.P., Khudik, V.N., Malkin, V.M.: Self similar regimes of supersonic Langmuir collapse. Sov. Phys. JETP **70 (6)**, 1023–1030 (1990)
14. Merle, F.: Limit of the solution of the nonlinear Schrödinger equation at the blow-up time. J. Funct. Anal. **84**, 201–214 (1989)
15. Merle, F.: Sur la dépendance continue de la solution de l'équation de Schrödinger non linéaire près du temps d'explosion. C.R. Paris **16**, 479–482 (1987)
16. Merle, F.: Construction of solutions with exactly k blow-up points for the nonlinear Schrödinger equation with critical nonlinearity. Commun. Math. Phys. **129**, 223–240 (1990)
17. Merle, F.: On uniqueness and continuation properties after blow-up time of self-similar solutions of nonlinear Schrödinger equation with critical exponent and critical mass. To appear in Commun. Pure Appl. Math.

18. Merle, F.: Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equation with critical power. Duke J. (To appear)
19. Merle, F.: Limit of solutions of nonlinear Schrödinger equation at the blow-up time in dimension one (in preparation)
20. Merle, F., Tsutsumi, Y.: L^2-concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity. J.D.E. **84**, 205–214 (1990)
21. Ogawa, T., Tsutsumi, Y.: Blow-up of H^1 solution for the nonlinear Schrödinger equation. Preprint
22. Strauss, W. A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **85**, 149–162 (1977)
23. Temam, R.: Navier Stokes equations, Theory and Numerical Analysis. North Holland, Studies in Math and its Applications
24. Weinstein, M. I.: The nonlinear Schrödinger equation singularity formation stability and dispersion, AMS-SIAM Conference on the connection between infinite dimensional and finite dimensional dynamical systems, July 1987
25. Weinstein, M. I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys. **87**, 567–576 (1983)
26. Weinstein, M. I.: On the structure and formation of singularities in solutions to the nonlinear dispersive evolution equations. Commun. Partial Differential Equations **11**, 545–565 (1986)
27. Zakharov, V. E., Sobolev, V. V., Synach, V. S.: Character of the singularity and stochastic phenomena in self-focusing. Zh. Eksp. Teor. Fiz., Pis'ma Red **14**, 390–393 (1971)

Communicated by T. Spencer