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A Family of Metrics on the Moduli Space of $\mathbf{CP}^2$ Instantons

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Abstract. A family of Riemannian metrics on the moduli space of irreducible self-dual connections of instanton number $k = 1$ over $\mathbf{CP}^2$ is considered. We find explicit formulas for these metrics and deduce conclusions concerning the geometry of the instanton space.

1. Introduction

Let $\mathcal{N}^+$ be the space of gauge equivalence classes of irreducible self-dual connections on a principal $SU(2)$-bundle P over a Riemannian 4-manifold M. Define a Riemannian metric g^s on $\mathcal{N}^+$ for $s \geq 0$ by

$$(g^s)_{[Z]}(u_1, u_2) = ((1 + s\Delta_Z)u_1 \, (1 + s\Delta_Z)u_2)\,,$$

where $[Z] \in \mathcal{N}^+$ and $(\,,)$ denotes the L^2-product. Then g^0 is the usual L^2-metric, whereas g^s is induced by a strong Riemannian metric on the orbit space of all irreducible connections on P for $s > 0$.

Results concerning the L^2-metric g^0 when M is the standard 4-sphere S^4 and the instanton number $k(P)$ is 1 were obtained by several authors (see [5, 8, 10]). In particular, it was shown that

(i) $(\mathcal{N}^+, g^0)$ is incomplete and has finite diameter and volume.
(ii) The completion of $(\mathcal{N}^+, g^0)$ differs from $\mathcal{N}^+$ by a set diffeomorphic to S^4.

Groisser and Parker generalized these results and established some other general properties of g^0 under certain topological assumptions on M and P (cf. [9]).

In [2] we examined the family $\{g^s\}_{s \geq 0}$ in the S^4 example. We showed that $(\mathcal{N}^+, g^s)$ is complete and has infinite diameter and volume for $s > 0$.

In the present paper we will be concerned with the case that M is $\mathbf{CP}^2$ and $k(P) = 1$. Then the moduli space $\mathcal{N}$ of self-dual connections is topologically a cone

on $\mathbf{CP}^2$, where $\mathcal{N}^+$ is the complement of the vertex of this cone (cf. [4]). In [7] Groisser gave a non-twistorial derivation of the formulas for the $\mathbf{CP}^2$ instantons to express the L^2-metric g^0 explicitly. We use this to obtain formulas for the metrics g^s. Among others we deduce for $s > 0$ that
(i) The completion of $(\mathcal{N}^+, g^s)$ is $\mathcal{N}$.
(ii) The diameter and the volume of $(\mathcal{N}^+, g^s)$ are infinite.

2. Preliminaries and Notations

Fix a principal G-bundle $P \to M$ over a closed, oriented Riemannian 4-manifold M, where G is a compact, connected, semisimple Lie group with Lie algebra $\mathbf{g}$. Denote by $\mathcal{C}^+$ the space of irreducible L_2^2-connections on P. The tangent space to $\mathcal{C}^+$ at a connection Z is the space $L_2^2(\Omega^1(\operatorname{Ad} P))$ of 1-forms on M with values in the bundle $\operatorname{Ad} P = P \times_{\operatorname{Ad}} \mathbf{g}$. Thus, a family $\{g^s\}_{s \geq 0}$ of Riemannian metrics on $\mathcal{C}^+$ is defined by

$$(g^s)_Z(u_1, u_2) = ((1 + s\Delta_Z)u_1, (1 + s\Delta_Z)u_2)$$

for $Z \in \mathcal{C}^+$ and $u_1, u_2 \in L_2^2(\Omega^1(\operatorname{Ad} P))$, where $(\,,)$ denotes the usual L^2-product and $\Delta_Z = d_Z^* d_Z + d_Z d_Z^*$ the Laplacian associated with Z (cf. [1]). Recall that g^0 is the (weak) L^2-metric, whereas g^s is a strong Riemannian metric for $s > 0$.

Now the group $\mathcal{G}$ of gauge transformations of P which lie in L_3^2 acts on $\mathcal{C}^+$ such that $\mathcal{M}^+ = \mathcal{C}^+/\mathcal{G}$ is a Hilbert manifold (cf. [6]). Identifying the tangent space to $\mathcal{M}^+$ at the equivalence class [Z] of a connection Z with the kernel of the operator

$$d_Z^* : L_2^2(\Omega^1(\operatorname{Ad} P)) \to L_1^2(\Omega^0(\operatorname{Ad} P)),$$

the restrictions of $(g^s)_Z$ to $\ker d_Z^*$ yield Riemannian metrics on $\mathcal{M}^+$ which we will also denote by g^s.

Another metric tensor on $\mathcal{M}^+$ which was suggested to be considered is described by

$$\hat{g}_Z(u_1, u_2) = (d_Z u_1, d_Z u_2) \quad \text{for} \quad u_1, u_2 \in \ker d_Z^*$$

(cf. [11]). We remark that the notation metric is not quite correct here since in general $\hat{g}$ may be degenerate.

With respect to each of these metrics a connected group K of isometries on M acts by

$$([Z], k) \in \mathcal{M}^+ \times K \mapsto [\tilde{k}^* Z] \in \mathcal{M}^+,$$

where the automorphism $\tilde{k}$ of P projects down to k, isometrically on $\mathcal{M}^+$.

Let $\mathcal{N}$ be the space of gauge equivalence classes of self-dual connections on P, and let $\mathcal{N}^+ \subset \mathcal{N}$ denote the subspace of classes of irreducible connections. Suppose that the Riemannian metric on M is such that $\mathcal{N}^+$ is a (finite dimensional) submanifold of $\mathcal{M}^+$. We will identify the tangent space to $\mathcal{N}^+$ at a point [Z] with the kernel of the Laplacian

$$\Delta_Z^- = d_Z d_Z^* + 2 d_Z^* p_- d_Z : L_2^2(\Omega^1(\operatorname{Ad} P)) \to L^2(\Omega^1(\operatorname{Ad} P)),$$

where p_- is the orthogonal projection onto the space of anti-self-dual 2-forms. Then the metrics g^s and $\hat{g}$ restricted to $\mathcal{N}^+$ are given by restrictions of $(g^s)_Z$ and $\hat{g}_Z$ to $\ker \Delta_Z^-$ for irreducible self-dual connections Z. Note that, using

$$\Delta_Z^- u = \Delta_Z u + *[F^Z, u]$$

for every 1-form u and self-dual connection Z with curvature form F^Z (cf. [2]), we obtain

$$(g^s)_Z(u_1, u_2) = (u_1 - s * [F^Z, u_1],\ u_2 - s * [F^Z, u_2])$$

and

$$\hat{g}_Z(u_1, u_2) = (u_1, \Delta_Z u_2) = -(u_1, *[F^Z, u_2])$$

for $u_1, u_2 \in \ker \Delta_Z^-$.

3. The Calculation of the Riemannian Metrics

In this section we describe the metrics g^s and $\hat{g}$ on the moduli space $\mathcal{N}^+$ for the case that M is the complex projective space $\mathbf{CP}^2$ with the Fubini-Study metric g_0 and P the principal $SU(2)$-bundle with instanton number 1. For this reason we fix local coordinates $z_1 = T_1/T_0$, $z_2 = T_2/T_0$ on $U_0 = \{[T_0 : T_1 : T_2] \in \mathbf{CP}^2 | T_0 \neq 0\}$. In these coordinates the metric on $\mathbf{CP}^2$ becomes

$$g_0 = (2D^2)^{-1}(D\delta_{jk} - \overline{z_j} z_k) dz_j d\overline{z_k}$$

with

$$D = 1 + |z_1|^2 + |z_2|^2 .$$

On the Lie algebra $\mathbf{su}(2)$ of $SU(2)$ we consider the Ad-invariant inner product determined by

$$\langle A, B\rangle = -\operatorname{Tr}(AB) .$$

For the sake of convenience we will identify a matrix $\begin{pmatrix} a_1 & a_2 \\ -\overline{a_2} & -a_1 \end{pmatrix}$ in $\mathbf{su}(2)$ with the vector (a_1, a_2). Then the inner product on $\mathbf{su}(2)$ becomes

$$\langle (a_1, a_2), (b_1, b_2)\rangle = -2\operatorname{Re}(a_1 b_1 - a_2 \overline{b_2}) .$$

Let Q be the Hopf bundle, i.e. the $U(1)$-principal bundle $S^5 \subset \mathbf{C}^3$ with $U(1)$-action

$$((T_0, T_1, T_2), \lambda) \in S^5 \times U(1) \mapsto (T_0\lambda, T_1\lambda, T_2\lambda) \in S^5$$

and projection

$$(T_0, T_1, T_2) \in S^5 \mapsto [T_0 : T_1 : T_2] \in \mathbf{CP}^2 .$$

Then the bundle P under consideration is the associated $SU(2)$-bundle $Q \times_\varrho SU(2)$ by means of the representation

$$\varrho : U(1) \to SU(2) \qquad \varrho(\lambda) = \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda . \end{pmatrix} .$$

In the sequel we will identify forms on P with their local expressions relative to the local section

$$s : U_0 \to P , \qquad s(z_1, z_2) = \left[\frac{1}{\sqrt{D}}(1, z_1, z_2), 1\right] .$$

We now recall the parametrization of the space $\mathcal{N}$ constructed in [7]. Denote by Z^0 the reducible self-dual connection on P induced by the connection

$$Z = \tfrac{1}{2}(\overline{T}_j dT_j - T_j d\overline{T}_j)$$

on Q. Let the 1-form $\eta \in \Omega^1(\operatorname{Ad} P)$ be determined by

$$\eta = \frac{1}{D}(0, \phi),$$

where

$$\phi = z_1 dz_2 - z_2 dz_1 .$$

Note that η lies in the formal tangent space $\ker \Delta^-_{Z^0}$ to $\mathcal{N}$ at $[Z^0]$. For $t \in [0,1)$ let f_t be the automorphism of P induced by

$$(T_0, T_1, T_2) \in Q \mapsto \frac{(\sqrt{1-t^2}T_0, T_1, T_2)}{\|(\sqrt{1-t^2}T_0, T_1, T_2)\|} \in Q$$

and set

$$Z^t = f_t^*(Z^0 + t\eta).$$

Applying $s \circ h_t = f_t \circ s$ for

$$h_t : U_0 \to U_0, \qquad h_t(z_1, z_2) = \frac{1}{\sqrt{1-t^2}}(z_1, z_2),$$

one verifies that

$$Z^t = \frac{1}{D-t^2}\left(\frac{1}{2}(\bar{\partial} D - \partial D), t\phi\right)$$

and

$$F^t = 2\frac{1-t^2}{(D-t^2)^2}(-iD^2 w, t dz_1 \wedge dz_2),$$

where ω is the Kähler form on $\mathbf{CP}^2$ and F^t denotes the curvature form of Z^t. Now consider the $SU(3)$-action on $\mathcal{N}$ corresponding to the usual action of $SU(3)$ on $\mathbf{CP}^2$. Then it holds

Proposition 3.1. (i) *$\mathcal{N}$ is the disjoint union of the orbits $[Z^t] \cdot SU(3)$ with $t \in [0,1)$.*
(ii) *$\mathcal{N}$ differs from $\mathcal{N}^+$ by the orbit $[Z^0] \cdot SU(3) = \{[Z^0]\}$.*
(iii) *Each orbit $[Z^t] \cdot SU(3)$ with $t \in (0,1)$ is the homogeneous space $\mathbf{CP}^2 = SU(3)/S(U(1) \times U(2))$.* $\square$

In particular, Proposition 3.1 yields a foliation $\mathcal{N}^+ = (0,1) \times \mathbf{CP}^2$. Further, setting

$$X^t = \frac{d}{dt}[Z^t] \quad \text{and} \quad Y^t = \frac{d}{ds}[Z^t] \cdot \exp(sY_{\mu_1,\mu_2}),$$

where

$$Y_{\mu_1,\mu_2} = \begin{pmatrix} 0 & \mu_1 & \mu_2 \\ -\overline{\mu_1} & 0 & 0 \\ -\overline{\mu_2} & 0 & 0 \end{pmatrix} \in \mathbf{su}(3) \quad \text{for any} \quad (\mu_1, \mu_2) \in S^3 \subset \mathbf{C}^2,$$

we have

$$X^t = \frac{1}{(D-t^2)^2}\,(t(\bar{\partial}D - \partial D),\,(D+t^2)\phi)$$

and

$$Y^t = \hat{Y}^t - d_{Z^t}W^t$$

with

$$\hat{Y}^t = \frac{t}{(D-t^2)^2}\,(ti\mathrm{Im}\{-2\,\mathrm{Re}(B_1)\partial D + (D-t^2)dB_1\}\,,$$
$$2t^2\,\mathrm{Re}(B_1)\phi - (D-t^2)dB_2)\,,$$
$$W^t = \frac{t^2}{(3-t^2)(D-t^2)}\,(-(1+t^2)\,i\mathrm{Im}(B_1), 2tB_2)$$

and

$$B_1 = \mu_1 z_1 + \mu_2 z_2\,, \qquad B_2 = -\overline{\mu_2} z_1 + \overline{\mu_1} z_2$$

(cf. [7]). Using these facts, we are able to compute the metrics on $\mathscr{N}^+$.

Proposition 3.2. *In terms of the parametrization* $\mathscr{N}^+ = (0,1)\times \mathbf{CP}^2$ *the Riemannian metric* g^s, $s \geq 0$, *is given by*

$$g^s = f^s(t)dt^2 + h^s(t)g_0$$

with

$$f^s(t) = g^s(X^t, X^t) \quad \textit{and} \quad h^s(t) = g^s(Y^t, Y^t)\,.$$

Proof. We regard $(\mathbf{CP}^2, g_0)$ as the Riemannian symmetric space $SU(3)/S(U(1)\times U(2))$ together with the inner product

$$\langle Y_1, Y_2\rangle = -\tfrac{1}{2}\,\mathrm{Tr}(Y_1 Y_2)$$

on the Lie algebra $\mathbf{su}(3)$. Since the vector Y_{μ_1,μ_2} lies in the orthogonal complement to $\mathbf{s}(\mathbf{u}(1)\times\mathbf{u}(2))$ in $\mathbf{su}(3)$ and has unit length, the metric g^s restricted to the orbit $[Z^t]\cdot SU(3) = \mathbf{CP}^2$ is $g^s(Y^t, Y^t)\cdot g_0$. On the other hand, observing that $\hat{Y}^t$ and W^t are odd with respect to (z_1, z_2), whereas the forms Z^t, F^t, and X^t are even, we get

$$g^s(X^t, Y^t) = 0\,. \quad \square$$

Proposition 3.3. *It holds*

$$f^s(t) = 4\pi^2\{f_1(t) + sf_2(t) + s^2 f_3(t)\}$$

and

$$h^s(t) = 4\pi^2\{h_1(t) + sh_2(t) + s^2 h_3(t)\}\,,$$

where

$$f_1(t) = 2\left\{\frac{4-3t^2}{t^4(1-t^2)} + \frac{4-t^2}{t^6}\log(1-t^2)\right\},$$
$$f_2(t) = \frac{8}{15}\cdot\frac{5+t^4}{(1-t^2)^2},$$
$$f_3(t) = \frac{8}{105}\cdot\frac{70-70t^2+91t^4-56t^6+13t^8}{(1-t^2)^3},$$

and

$$h_1(t) = -\left\{ \frac{6 - 9t^2 + t^4}{t^2(3 - t^2)} + \frac{6(1 - t^2)^2}{t^4(3 - t^2)} \log(1 - t^2) \right\},$$
$$h_2(t) = \frac{12}{5} \cdot \frac{t^2(10 - 5t^2 + 5t^4 - 3t^6 + t^8)}{(1 - t^2)(3 - t^2)^2},$$
$$h_3(t) = \frac{4}{105} \cdot \frac{t^2(1260 - 1260t^2 + 2128t^4 - 2037t^6 + 1230t^8 - 425t^{10} + 64t^{12})}{(1 - t^2)^2(3 - t^2)^2}.$$

Proof. Set

$$4\pi^2 f_1(t) = \|X^t\|^2,$$
$$2\pi^2 f_2(t) = -(X^t, *[F^t, X^t]),$$
$$4\pi^2 f_3(t) = \| * [F^t, X^t]\|^2$$

and, analogously,

$$\begin{aligned}
4\pi^2 h_1(t) &= \|Y^t\|^2 = \|\hat{Y}^t\|^2 - 2(\hat{Y}^t, d_{Z^t} W^t) + \|d_{Z^t} W^t\|^2, \\
2\pi^2 h_2(t) &= -(Y^t, *[F^t, Y^t]) = -(\hat{Y}^t, *[F^t, \hat{Y}^t]) + 2(\hat{Y}^t, *[F^t, d_{Z^t} W^t]) \\
&\quad - (d_{Z^t} W^t, *[F^t, d_{Z^t} W^t]), \\
4\pi^2 h_3(t) &= \| * [F^t, Y^t]\|^2 = \| * [F^t, \hat{Y}^t]\|^2 \\
&\quad - 2(*[F^t, \hat{Y}^t], *[F^t, d_{Z^t} W^t]) + \| * [F^t, d_{Z^t} W^t\|^2.
\end{aligned}$$

Here $\| \;\|$ denotes the L^2-norm. The functions f_1 and h_1 were computed in [7]. The expressions for the other functions are obtained in a similar way after checking that

$$\begin{aligned}
*[F^t, X^t] &= -4 \cdot \frac{(1 - t^2)D}{(D - t^2)^4)} (t(D + t^2)(\bar{\partial} D - \partial D),\ D(D + 3t^2)\phi), \\
d_{Z^t} W^t &= \frac{t^2}{(3 - t^2)(D - t^2)^2} \\
&\quad \times (\mathrm{iIm}\{(1 + t^2)[2\,\mathrm{iIm}(B_1)\partial D - (D - t^2) dB_1] - 4t^2 \overline{B_2}\phi\}, \\
&\quad - 2t\{2B_2 \partial D - (1 + t^2)\,\mathrm{iIm}(B_1)\phi - (D - t^2) dB_2\}), \\
*[F^t, \hat{Y}^t] &= \frac{4t(1 - t^2)D}{(D - t^2)^4} (2t\,\mathrm{iIm}\{2t^2\,\mathrm{Re}(B_1)\partial D + (D - t^2)(\overline{B_2}\phi - dB_1)\}, \\
&\quad - 4t^2\,\mathrm{Re}(B_1) D\phi + t^2(D - t^2)\overline{B_1}\phi + (D + t^2)(D - t^2) dB_2)
\end{aligned}$$

and

$$\begin{aligned}
*[F^t, d_{Z^t} W^t] &= -4 \cdot \frac{t^3(1 - t^2)D}{(3 - t^2)(D - t^2)^4} \\
&\quad \times (4t\,\mathrm{iIm}\{(1 + t^2)\,\mathrm{iIm}(B_1)\partial D - (D + t^2)\overline{B_2}\phi - (D - t^2) dB_1\}, \\
&\quad - 4(D + t^2) B_2 \partial D + (1 + t^2)[4D\,\mathrm{iIm}(B_1) + (D - t^2)\overline{B_1}]\phi \\
&\quad + (D - t^2)(2D + 1 + t^2) dB_2),
\end{aligned}$$

applying

$$[\phi,\psi] = 2(-\,\mathrm{i}\,\mathrm{Im}(\varphi_2 \wedge \overline{\psi_2}),\ \varphi_1 \wedge \psi_2 - \varphi_2 \wedge \psi_1)$$

for $\varphi = (\varphi_1, \varphi_2) \in \Omega^k(\mathrm{Ad}\, P)$ and $\psi = (\psi_1, \psi_2) \in \Omega^l(\mathrm{Ad}\, P)$. □

Clearly, our setting implies

Proposition 3.4. *For the Riemannian metric $\hat{g}$ on $\mathcal{N}^+$ it holds that*

$$\hat{g} = 2\pi^2(f_2(t)dt^2 + h_2(t)g_0)\,. \quad \square$$

4. Conclusions

By a result of Groisser and Parker in a more general setting (cf. [9]) we know that $(\mathcal{N}^+, g^0)$ as a metric space is incomplete, where its completion $\mathcal{N}_0^+$ is the disjoint union of $\mathcal{N}$ and a set diffeomorphic to $\mathbf{CP}^2$. Furthermore, $(\mathcal{N}^+, g^0)$ has finite diameter and volume. Here we prove

Proposition 4.1. *Let $s > 0$. Then*
(i) *The completion $\mathcal{N}_s^+$ of $\mathcal{N}^+$ with respect to g^s is $\mathcal{N}$.*
(ii) *The diameter and the volume of $(\mathcal{N}^+, g^s)$ are infinite.*

Proof. The assertions are immediate consequences of

Lemma 4.2. (i) *Let $l^s(r)$ be the length of the curve*

$$t \in (0, r) \mapsto [Z^t] \in \mathcal{N}^+$$

with respect to g^s for $0 < r \leq 1$ and $s > 0$. Then $l^s(r)$ is finite for $r < 1$ and infinite for $r = 1$.
(ii) *For $s > 0$ it holds*

$$\lim_{t \to 0} h^s(t) = \infty\,.$$

Proof. (i) For $0 < r < 1$ the functions f_1, f_2, and f_3 are bounded on the interval $(0, r)$. Thus, $l^s(r) < \infty$. On the other hand,

$$l^s(1) \geq 2\pi \int_0^1 \sqrt{s f_2(t)}dt = 4\pi\sqrt{\frac{2s}{15}} \int_0^1 \frac{\sqrt{t^2+5}}{1-t^2}\,dt$$

$$\geq 4\pi\sqrt{\frac{2s}{3}} \int_0^1 \frac{dt}{1-t^2} = \infty\,.$$

(ii) This is obvious. □

Remark. Clearly, a result similar to Proposition 4.1 holds for the Riemannian manifold $(\mathcal{N}^+, \hat{g})$. □

Computing Taylor series and using formulas relating the sectional curvature k^s of the warped product $(\mathcal{N}^+, g^s)$ to f^s, h^s and the sectional curvature k_0 of $(\mathbf{CP}^2, g_0)$ (see e.g. [3]), one finds

Proposition 4.3. *Let $s \geq 0$ and $t \to 0$. Then*

$$k^s\left(X, \frac{\partial}{\partial t}\right) = \frac{-1}{4\pi^2} \cdot \frac{3(1+24s+96s^2)}{2(1+4s)^4} + 0(t^2)$$

and

$$k^s(X,Y) = \frac{3(k_0 - 1)}{4\pi^2(1+4s)^2} \cdot t^{-2}$$
$$- \frac{5k_0 - 2 + (56k_0 + 16)s + (160k_0 + 128)s^2}{8\pi^2(1+4s)^4} + 0(t^2),$$

where X and Y are tangent vectors to $\mathbf{CP}^2$ and $k_0 = k_0(X,Y)$. □

Our last statement concerns the asymptotic behaviour of the metrics g^s near the equivalence class $[Z^0]$.

Proposition 4.4. *Fix $s \geq 0$ and let l denote the length parameter of the curve*

$$t \in (0,1) \mapsto [Z^t] \in \mathscr{N}^+$$

with respect to g^s. Then

$$g^s = dl^2 + \left[l^2 + \frac{1}{8\pi^2} \cdot \frac{1+24s+96s^2}{(1+4s)^4} \cdot l^4 + 0(l^6)\right] g_0$$

for $l \to 0$.

Proof. The result is obtained by a straightforward computation. □

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