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Equivalence of Gibbs and Equilibrium States for Homeomorphisms Satisfying Expansiveness and Specification

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Dedicated to Joel Lebowitz

Abstract. Let M be a compact metrizable space, f: $M \to M$ a homeomorphism satisfying expansiveness and specification, and A: $M \to \mathbb{R}$ a function such that

$$\left|\sum_{k=0}^{n-1} [A(f^k x) - A(f^k y)]\right| \leqq K(\varepsilon) < \infty$$

whenever $n \geqq 1$ and x, y are (ε, n)-close (i.e. $d(f^k x, f^k y) < \varepsilon$ for $k = 0, \ldots, n-1$, some fixed choice of metric d and expansive constant $\varepsilon > 0$). Under these conditions, Bowen has shown that there is a unique *equilibrium state* ρ for A. Assuming that $K(\delta) \to 0$ when $\delta \to 0$, we show that ρ is also the unique *Gibbs state* for A. We further define *quasi-Gibbs states* and show that ρ is the unique f-invariant quasi-Gibbs state for A.

0. Introduction

The concepts of equilibrium state and of Gibbs state come from the statistical mechanics of (spatially) infinite systems. States of thermal equilibrium of such systems can be defined either *globally* by a *variational principle* (this gives *equilibrium states* or locally by specifying certain conditional probabilities (this gives *Gibbs states*). Under fairly general conditions one can prove that equilibrium states coincide with translationally invariant Gibbs states (Dobrushin [4, 5], Lanford and Ruelle [8]). For one-dimensional statistical mechanics (with a natural *mixing* condition and *short range interactions*) Gibbs states are automatically translationally invariant, hence equivalent to equilibrium states; there is in fact one equilibrium state (i.e. these systems have no *phase transitions*).

The invariance under translations for (one-dimensional) statistical mechanics is a special example of invariance under a homeomorphism f of a compact metrizable space M. It is natural to try to extend the theory of equilibrium and Gibbs states to this more general situation. For equilibrium states, this is relatively easy

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and was done first under the assumption of expansiveness and specification (Ruelle [9]), then quite generally (Walters [12]).

The extension of the theory of Gibbs states is more difficult, and was first achieved for *Smale homeomorphisms* of spaces with local product structures (generalizing the Axiom A diffeomorphisms of Smale [11]). For the definition of Gibbs states see Capocaccia [3], Ruelle [10]. The main result of equivalence of Gibbs states and equilibrium states is due to Haydn [7]. This was later extended by Baladi [1] to the *finitely presented* systems of Fried [6].

The present paper establishes the equivalence of equilibrium states and Gibbs states under the assumption of expansiveness and specification. This assumption is satisfied by (mixing) Smale homeomorphisms. Interestingly, expansiveness and specification are topological assumptions, while the definition of Smale homeomorphisms involves a metric.

A preliminary version of the present paper by one of us (N.H.) did not have a proof of the existence of a Gibbs state. Discussion of that problem has led to the present joint work.

Section 1 reviews some necessary definitions and results of topological dynamics. The bulk of the work is in Sect. 2, which contains the main theorem 2.5.

1. Topological Dynamics, Thermodynamic Formalism, and Equilibrium States

Throughout what follows, M is a compact metrizable space, and we choose arbitrarily a metric d compatible with the topology; f is a homeomorphism of M.

1.1. Expansiveness and Specification. We say that f is *expansive* with expansive constant $\varepsilon > 0$ if, for $x, y \in M$,

$$d(f^k x, f^k y) \leqq \varepsilon \quad \text{for } all\ k \in \mathbb{Z}$$

implies $x = y$. By compactness, if $\delta > 0$, one can then choose n such that

$$d(f^k x, f^k y) \leqq \varepsilon \quad \text{for } |k| < n$$

implies $d(x, y) \leqq \delta$.

We say that f satisfies *specification* if for every $\varepsilon > 0$ there is an integer $p = p(\varepsilon) \geqq 0$ such that, given l points $x_1, \ldots, x_l \in M$ and integers $n_1, \ldots, n_l > 0$, $p_1, \ldots, p_l \geqq p$, there exists $z \in M$ such that[1]

$$d(f^{m(j-1)+i} z, f^i x_j) \leqq \varepsilon$$

for $i = 0, \ldots, n_j - 1$ and $j = 1, \ldots, l$, where $m(0) = 0$ and $m(j) = n_1 + p_1 + \ldots . + n_j + p_j$.

1.2. Invariant Measures and Entropy. We denote by $\mathbf{I}$ the set of f-invariant probability measures (or f-invariant *states*) on M. The set $\mathbf{I}$ is convex and compact for the weak (= vague) topology of measures. The entropy $h(\rho)$ of $\rho \in \mathbf{I}$ is defined as

[1] Usually one assumes that z can be taken periodic: $f^{m(\ell)} z = z$, but this will not be needed here

usual. If ε is an expansive constant, and if $\mathscr{A} = (\mathscr{A}_i)$ is a finite Borel partition of M into sets of diameter $\leqq \varepsilon$, then

$$h(\rho) = \lim_{n\to\infty} \frac{1}{n} H\left(\rho, \bigvee_0^{n-1} \mathscr{A}\right) = \inf_n \frac{1}{n} H\left(\rho, \bigvee_0^{n-1} \mathscr{A}\right),$$

where $\bigvee_0^{n-1} \mathscr{A}$ is the partition generated by $\mathscr{A}, f^{-1}\mathscr{A}, \ldots, f^{-n+1}\mathscr{A}$, and

$$H(\rho, \mathscr{A}) = -\sum_i \rho(\mathscr{A}_i) \log \rho(\mathscr{A}_i) \geqq 0$$

(with $0 \log 0 = 0$ as usual). In particular, when f is expansive, h is upper semi-continuous on $\mathbf{I}$.

1.3. Pressure and Variational Principle. We denote by $B_x(\varepsilon)$ the closed ball of radius ε centered at x , and we also let

$$B_x(\varepsilon, n) = \{y \in M : d(f^k y, f^k x) \leqq \varepsilon \text{ for } k = 0, \ldots, n-1\} .$$

We say that x, y are (ε, n)-*close* if $y \in B_x(\varepsilon, n)$. We say that the set E is (ε, n)-*separated* if ($x, y \in E$ and $x \neq y$) implies $y \notin B_x(\varepsilon, n)$. Note that if E is maximal (ε, n)-*separated*, then

$$\bigcup_{x \in E} B_x(\varepsilon, n) = M .$$

We introduce also the set $B_x(\varepsilon, \pm n)$, and $(\varepsilon, \pm n)$-*close* pairs, and $(\varepsilon, \pm n)$-*separated* sets; these are defined as above, but with the interval $[0, \ldots, n-1]$ replaced by $[-n+1, \ldots, n-1]$.

If $A : M \to \mathbb{R}$ is continuous, then $P(A) \in \mathbb{R} \cup \{+\infty\}$ is defined by

$$\{P(A)\} = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \left\{ \sum_{x \in E} \exp \sum_{k=0}^{n-1} A(f^k x) : \ E \text{ is maximal } (\varepsilon, n)\text{-separated} \right\} .$$

(This means that the set of limits when $n \to \infty$ tends to a single point $P(A)$ when $\varepsilon \to 0$. We use the fact that a maximal (ε, n)-separated set is both (ε, n)-separated and (ε, n)-spanning, therefore $\lim_{\varepsilon\to 0} \liminf_{n\to\infty} = \lim_{\varepsilon\to 0} \limsup_{n\to\infty}$ by Walter's results, see [12] or [10].) The function P on $\mathscr{C}_{\mathbb{R}}(M)$ is called the *topological pressure* (or simply pressure); it has various other equivalent definitions. P is either real-valued or identically $+\infty$.

1.4 Theorem (*Variational principle* [12]).

$$P(A) = \sup_{\rho \in \mathbf{I}} (h(\rho) + \rho(A)) .$$

In particular, P is convex and continuous on $\mathscr{C}_{\mathbb{R}}(M)$. If $\rho \in \mathbf{I}$ and $h(\rho) + \rho(A) = P(A)$, then ρ is called an equilibrium state for A. When h is finite and upper semi-continuous (in particular when f is expansive) there is at least one equilibrium state for each A.

1.5 Proposition. *For any $A : M \to \mathbb{R}$ and finite $E \subset M$, we write*

$$Z_n(A, E) = \sum_{x \in E} \exp \sum_{k=0}^{n-1} A(f^k x) .$$

We assume that the homeomorphism f is expansive, with expansive constant 2ε, and that

$$K_A = \sup_n K_A(\varepsilon, n) < \infty ,$$

where

$$K_A(\varepsilon, n) = \sup\left\{\left|\sum_{k=0}^{n-1} (A(f^k x) - A(f^k y))\right| : x \text{ and } y \text{ are } (\varepsilon, n)\text{-close}\right\} < \infty . \quad (1)$$

Note that $K_A < \infty$ if $A = B - B \circ f$ provided B is bounded (not necessarily continuous).

(a) *If $\delta, \delta' \leqq \varepsilon$, and if E, E' are maximal (δ, n)- and (δ', n)-separated sets respectively we have*

$$Z_n(A, E') \leqq Q Z_n(A, E) ,$$

where $Q = Q(\delta, \delta')$ is independent of n. In particular

$$P(A) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(A, E_n) ,$$

where the E_n are any maximal (ε, n)-separated sets.

(b) *If f furthermore satisfies specification and E_m, E_n, E_{m+n} are as above, we have*

$$e^{-a} \leqq \frac{Z_{m+n}(A, E_{m+n})}{Z_m(A, E_m) Z_n(A, E_n)} \leqq e^a ,$$

where a depends on A, ε but not on m, n. In particular

$$\exp(nP(A) - a) \leqq Z_n(A, E_n) \leqq \exp(nP(A) + a) .$$

Note that we have not assumed that A is continuous. The mild extension on the variational principle needed to make sense of $P(A)$ is given in Sect. 1.6 below.

We prove (a). Let N be the cardinal of a maximal $\frac{1}{2}\delta$-separated set. We may choose q such that

$$d(f^k x, f^k y) \leqq \delta \quad \text{for } k = -q, \ldots, n + q - 1$$

implies that x, y are $(\frac{1}{2}\delta', n)$-close, and

$$d(f^k x, f^k y) \leqq 2\delta \quad \text{for} \quad k = -q, \ldots, n + q - 1$$

implies that x, y are (δ, n)-close. Let E^* be such that $f^q E^*$ is a maximal $(\delta, n + 2q)$ separated set. If $x' \in E'$, choose $x^* \in E^*$ such that $f^q x', f^q x^*$ are $(\delta, n + 2q)$-close. Then, x', x^* are $(\frac{1}{2}\delta', n)$-close, hence $x' \mapsto x^*$ is injective. If $x^* \in E^*$, choose $x \in E$ such that x^*, x are (δ, n)-close, the map $x^* \mapsto x$ is at most N^{2q}-to-one. [If $x^*, y^* \mapsto x$ and $f^k x^*$, $f^k y^*$ are $\frac{1}{2}\delta$-close to the same point of a maximal $\frac{1}{2}\delta$-separated set for $k = -q, \ldots, -1$ and $k = n, \ldots, n + q - 1$, then $f^q x^*$, $f^q y^*$ are $(2\delta, n + 2q)$-close, hence x^*, y^* are (δ, n)-close, hence $f^q x^*$, $f^q y^*$ are $(\delta, n + 2q)$-close, hence $x^* = y^*$.] The map $x' \mapsto x$ obtained by composing $x' \mapsto x^* \mapsto x$ is thus at most N^{2q}-to-one and the pairs (x', x^*), (x^*, x) are (ε, n)-close. Therefore

$$Z_n(A, E') \leqq N^{2q} (\exp 2K_A) Z_n(A, E) .$$

i.e., we may take $Q(\delta, \delta') = N^{2q}\exp 2K_A$. We have thus $\limsup_{n\to\infty}\frac{1}{n}\log Z_n(A, E)$ $=\limsup_{n\to\infty}\frac{1}{n}\log Z_n(A, E')$, and similarly for lim inf. In particular, the limit $\varepsilon \to 0$ may be omitted in the definition of the pressure.

The proof of (b) is of the same sort and left to the reader (see [2] Lemma 2).

1.6. An Extension of the Variational Principle. Given $A: M \to \mathbb{R}$ we define $K_A(\varepsilon, n)$ by (1), and do not assume expansiveness. The space

$$\mathscr{W} = \left\{A : \frac{1}{n} K_A(\varepsilon, n) \to 0 \text{ when } \varepsilon \to 0 \text{ and } n \to \infty\right\}$$

is closed with respect to the uniform norm $\|\cdot\|_0$; $\mathscr{W} \supset \mathscr{C}_{\mathbb{R}}(M)$ but $\mathscr{W}$ may contain non-continuous functions. (Note that $K_A(\varepsilon, v)$ is decreasing in ε and subadditive in n, therefore the limits $\varepsilon \to 0$, $n \to \infty$ can be taken simultaneously in the definition of $\mathscr{W}$.)

Given $A \in \mathscr{W}$, $\delta > 0$, and n sufficiently large, there is $B \in \mathscr{C}_{\mathbb{R}}(M)$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n-1} A \circ f^k - B \right\|_0 < \delta \, .$$

Therefore the map$(\rho, A) \mapsto \rho(A)$ extends to $\mathbf{I} \times \mathscr{W} \to \mathbb{R}$ such that $\rho \mapsto \rho(A)$ is (vaguely) continuous on $\mathbf{I}$. The definition of $P(A)$ extends in a natural manner to $A \in \mathscr{W}$ and the variational principle then continues to hold. If h is upper semi-continuous, the set $\mathbf{I}_A$ of equilibrium states for $a \in \mathscr{W}$ is not empty.

2. Gibbs and Quasi-Gibbs States

Throughout this section, the homeomorphism f satisfies expansiveness and specification with the constants ε and p. (We assume that ε is taken so small that 4ε is again an expansive constant.)

For any function $A: M \to \mathbb{R}$ we write as earlier

$$K_A = \sup_n K_A(n, \varepsilon)$$

and define

$$\mathscr{V} = \{A: K_A < \infty\}$$

Note that K is a seminorm on $\mathscr{V}$ and that $\mathscr{V}$ is a Banach space for the norm $\|\cdot\|_0 + K$. It is easily seen that different choices of expansive constant ε give equivalent norms. [*Bowen* [2] was the first to consider the space $\mathscr{V} \cap C(M)$ in this setting.] Write also

$$V_A = \min_{a\in\mathbb{R}} \|A + a\|_0 = \frac{1}{2}\sup_x A(x) - \frac{1}{2}\inf_x A(x) \, ,$$

then the quotient $\mathscr{V}/\mathbb{R}$ by constant functions is a Banach space for the norm $K + V$.

We say that two points $x, y \in M$ are *conjugate* if $\lim_{|k|\to\infty} d(f^k x, f^k y) = 0$. Conjugacy is an equivalence relation, and the points conjugate to a given x are dense in M. We say that x, y are $(\varepsilon, \pm n)$-conjugate if

$$d(f^k x, f^k y) \leqq \varepsilon \quad \text{for } |k| \geqq n .$$

The points x, y are conjugate if and only if they are $(\varepsilon, \pm n)$-conjugate for some n. If N is the cardinal of a maximal $\frac{1}{2}\varepsilon$-separated set, there are at most N^{2n-1} points $(\varepsilon, \pm n)$-conjugate to a given x.

A (continuous) map φ from $U \subset M$ to M is *conjugating* if, for some n and all $z \in U$, the points z, φz are $(\varepsilon, \pm n)$-conjugate. If (U, φ), (U', φ') are conjugating maps, with $x \in U \cap U'$ and $\varphi x = \varphi' x$, then there is a neighborhood V of x such that $\varphi | U \cap U' \cap V = \varphi' | U \cap U' \cap V$, and this restriction is a homeomorphism (Capocaccia [3]). We shall restrict our attention to *conjugating homeomorphisms* (U, φ) with U compact.

In what follows we shall consider sums of the form

$$\sum_{k=a}^{b} [A f^k x - A f^k y],$$

where x, y are conjugate, A is a Borel function $\in \mathscr{V}$, and a, b may be infinite. Our assumptions do not guarantee the convergence of the sum if a or b is infinite (unless we take $\lim_{\delta\to 0} \sup_n K_A(\delta, n) = 0$ as in the abstract).

We therefore *assume* that a value has been attributed to sums with a or b infinite, such that the sum is a Borel function of x, y and

$$\sum_{k=a}^{b} [\cdots] + \sum_{k=b+1}^{c} [\cdots] = \sum_{k=a}^{c} [\cdots] ,$$

$$\sum_{k=a}^{b} [A f^k x - A f^k y] + \sum_{k=a}^{b} [A f^k y - A f^k z] = \sum_{k=a}^{b} [A f^k x - A f^k z] ,$$

$$\sum_{k=a+1}^{b+1} [A f^k x - A f^k y] = \sum_{k=a}^{b} [A f^k (f x) - A f^k (f y)] .$$

If $d(f^k x, f^k y) \leqq \varepsilon$ for all $k \in [a, b]$ we also assume

$$\left| \sum_{k=a}^{b} [A f^k x - A f^k y] \right| \leqq K_A .$$

[We are indebted to Oscar Lanford for pointing out the need of the above assumptions in the study of Gibbs states. For quasi-Gibbs states they are unnecessary, and the assumption that A is Borel is also not needed.]

A probability measure μ on M is called a *Gibbs state* for Borel $A \in \mathscr{V}$ if, for every conjugating homeomorphism (U, φ), the measure $\varphi(\mu | U)$ is absolutely continuous with respect to $\mu | \varphi U$, with Radon–Nikodym derivative

$$\frac{d\varphi\mu}{d\mu} = \exp \sum_{k=-\infty}^{\infty} [A \circ f^k \circ \varphi^{-1} - A \circ f^k] . \tag{2}$$

We say that μ is a *quasi-Gibbs state* if there is a constant C such that, for all (U, φ),

$$\frac{d\varphi\mu}{d\mu} \leqq C \exp \sum_{k=-\infty}^{\infty} [A \circ f^k \circ \varphi^{-1} - A \circ f^k] . \tag{3}$$

2.1 Proposition. *If μ is a quasi-Gibbs state for $A \in \mathscr{V}$, one can choose $c > 0$ such that for all $x \in M$, $n \geqq 0$,*

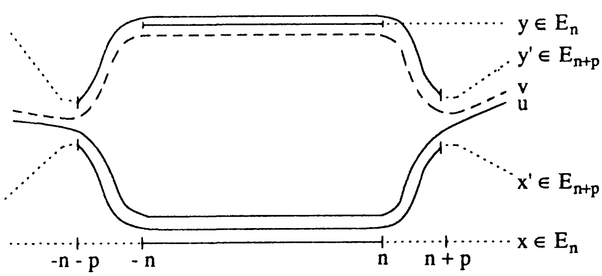
$$\exp\left[\sum_{|k|<n} A(f^k x) - (2n-1)P(A) - c\right] \leqq \mu(B_x(\varepsilon, \pm n))$$

$$\leqq \exp\left[\sum_{|k|<n} A(f^k x) - (2n-1)P(A) + c\right],$$

where $P(A)$ denotes the pressure. The constant c depends on ε and A; it depends on μ only through the constant C of (3).

I. Lower bound. Choose maximal $(\varepsilon, \pm n)$- and $(\varepsilon, \pm(n+p))$-separated sets E_n, E_{n+p} in M. (We may take $x \in E_n \subset E_{n+p}$.) Fix $y \in E_n$.

Consider any $(\varepsilon, \pm(n+p))$-conjugate pair (u, v) with $u \in B_x(\varepsilon, \pm n)$, $v \in B_y(\varepsilon, \pm n)$. The set $B_x(\varepsilon, \pm n)$ (respectively $B_y(\varepsilon, \pm n)$) is covered by at most N^{2p} sets $B_{x'}(\varepsilon, \pm(n+p))$ with $x' \in E_{n+p}$ (respectively $B_{y'}(\varepsilon, \pm(n+p))$ with $y' \in E_{n+p}$).



There are therefore at most N^{4p} conjugating homeomorphisms (U_i, φ_i) such that all pairs (u, v) may be written $(u, \varphi_i u)$. [The set $\{(u, v)$: $u \in B_{x'}(\varepsilon, \pm(n+p)) \cap B_x(\varepsilon, \pm n)$, $v \in B_{y'}(\varepsilon, \pm(n+p)) \cap B_y(\varepsilon, \pm n)$, u and v are $(\varepsilon, \pm(n+p))$ conjugate$\}$ is closed in $M \times M$, hence compact, and (by expansiveness) is the graph of a map. Therefore the domain U_i of this map is compact and the map φ_i is continuous.] We have thus

$$\sum_i \mu(U_i) \leqq N^{4p}\mu(B_x(\varepsilon, \pm n)),$$

$$\mu(B_y(\varepsilon, \pm n)) \leqq \sum_i \mu(\varphi_i U_i),$$

and (3) implies that

$$\mu(\varphi_i U_i) \leqq Ce^{4K+4pV} \cdot \left(\exp \sum_{|k|<n} [A(f^k y) - A(f^k x)]\right) \cdot \mu(U_i)$$

with $K = K_A$, $V = V_A$. Therefore

$$\mu(B_y(\varepsilon, \pm n)) \leqq Ce^{4K+4pV} \cdot \left(\exp \sum_{|k|<n} [A(f^k y) - A(f^k x)]\right) \cdot \sum_i \mu(U_i)$$

$$\leqq Ce^{4K+4pV} N^{4p} \cdot \left(\exp \sum_{|k|<n} [A(f^k y) - A(f^k x)]\right) \cdot \mu(B_x(\varepsilon, \pm n)).$$

Summing over $y \in E_n$, and using Proposition 1.5(b) we obtain

$$1 \leqq e^c \cdot \left(\exp\left[(2n-1)P(A) - \sum_{|k|<n} A(f^k x) \right] \right) \cdot \mu(B_x(\varepsilon, \pm n))$$

with $e^c = Ce^{4K+4pV} N^{4p} e^a$; this is the desired lower bound.

II. Upper Bound. Here we choose maximal $(2\varepsilon, \pm n)$- and $(\varepsilon, \pm(n+p))$-separated sets E_n^*, E_{n+p} (with $x \in E_n^* \subset E_{n+p}$). Fixing $y \in E_n^*$ we consider as before the $(\varepsilon, \pm (n+p))$-conjugate pairs (u, v) with $u \in B_x(\varepsilon, \pm n)$, $v \in B_y(\varepsilon, \pm n)$. There are again at most N^{4p} conjugating homeomorphisms (U_i, φ_i) such that all pairs (u, v) may be written $(u, \varphi_i u)$. We have thus

$$\mu(B_x(\varepsilon, \pm n)) \leqq \sum_i \mu(U_i) ,$$

$$\sum_i \mu(\varphi_i U_i) \leqq N^{4p} \mu(B_y(\varepsilon, \pm n)) ,$$

$$\mu(U_i) \leqq Ce^{4K+4pV} \left(\exp \sum_{|k|<n} [A(f^k x) - A(f^k y)] \right) \cdot \mu(\varphi_i U_i) .$$

Therefore

$$\mu(B_x(\varepsilon, \pm n)) \cdot \exp \sum_{|k|<n} A(f^k y)$$

$$\leqq Ce^{4K+4pV} N^{4p} \left(\exp \sum_{|k|<n} A(f^k x)) \cdot \mu(B_y(\varepsilon, \pm n) \right) .$$

Summing over $y \in E_n^*$, and using Proposition 1.5(b) and the disjointness of the $B_y(\varepsilon, \pm n)$, we obtain

$$\mu(B_x(\varepsilon, \pm n)) \exp[(2n-1)P(A)] \leqq e^c \cdot \exp \sum_{|k|<n} A(f^k x)$$

which is the desired upper bound.

2.2 Corollary. *If μ, ν are quasi-Gibbs states for $A \in \mathscr{V}$, then the measures μ, ν are equivalent, and the Radon–Nikodym derivatives $d\mu/d\nu$, $d\nu/d\mu$ are essentially bounded.*

A measure ν_n on M is defined by

$$\int \nu_n(dx) \Phi(x) = \int \nu(dx) \Phi_n(x) ,$$

where

$$\Phi_n(x) = \frac{1}{\mu(B_x(\varepsilon, \pm n))} \int_{B_x(\varepsilon, \pm n)} \mu(dy) \Phi(y)$$

for all $\Phi \in \mathscr{C}(M)$. Note that $\mu(B_x(\varepsilon, \pm n)) > 0$ by Proposition 2.1, and that $\operatorname{diam} B_x(\varepsilon, \pm n) \to 0$ when $n \to \infty$ (by expansiveness). Therefore when $n \to \infty$, we have $\Phi_n \to \Phi$ (uniformly), and $\nu_n \to \nu$ (vaguely). We have then, if $\Phi \geqq 0$,

$$\begin{aligned}
\int \nu_n(dx)\Phi(x) &= \int \mu(dy)\Phi(y) \int_{B_y(\varepsilon,\, \pm n)} \frac{\nu(dx)}{\mu(B_x(\varepsilon,\, \pm n))} \\
&\leqq \int \mu(dy)\Phi(y) \int_{B_y(\varepsilon,\, \pm n)} \frac{\nu(dx)}{\exp\left[\sum_{|k|<n} A(f^k x) - (2n-1)P(A) - c\right]} \\
&\leqq e^K \int \mu(dy)\Phi(y) \int_{B_y(\varepsilon,\, \pm n)} \frac{\nu(dx)}{\exp\left[\sum_{|k|<n} A(f^k y) - (2n-1)P(A) - c\right]} \\
&\leqq e^{K+2c} \int \mu(dy)\Phi(y) \int_{B_y(\varepsilon,\, \pm n)} \frac{\nu(dx)}{\nu(B_y(\varepsilon,\, \pm n))} \\
&= e^{K+2c} \int \mu(dy)\Phi(y)
\end{aligned}$$

so that $\nu_n \leqq e^{K+2c}\mu$ (for simplicity we have used the same constant c for μ and ν). When $n \to \infty$, ν_n tends vaguely to ν, hence $\nu \leqq e^{K+2c}\mu$. Similarly one shows that $\mu \leqq e^{K+2c}\nu$.

2.3 Remark. Let (U, φ) be an $(\varepsilon, \pm n)$-conjugating homeomorphism, and suppose that $x \in U$, $y = \varphi x$, $U \subset B_x(\varepsilon, \pm n)$, $\varphi U \subset B_y(\varepsilon, \pm n)$. By (3) there is a constant C', independent of n and (U, φ), such that

$$\frac{d\varphi\mu}{d\mu} \leqq C' \exp \sum_{|k|<n} [A(f^k x) - A(f^k y)] .$$

Conversely, if this inequality is satisfied for all n and all (U, φ) as above, then (3) holds and μ is a quasi-Gibbs state.

2.4 Proposition. *For every $A \in \mathscr{V}$ there exists a quasi-Gibbs state μ.*

We assume that 8ε is an expansive constant. For every integer $m > 0$, let E_m be a maximal $(\varepsilon, \pm m)$-separated set, and define

$$\mu_m = [Z_{\pm m}(A, E_m)]^{-1} \sum_{x \in E_m} \left[\exp \sum_{|k|<m} A(f^k x)\right] \delta_x ,$$

where δ_x is the unit mass at x, and $Z_{\pm m}(A, E_m) = Z_{2m-1}(A, f^{-m+1} E_m)$. We shall show that if μ is a limit of the sequence (μ_m), then (3) holds.

We may restrict our attention (cf. the proof of Proposition 2.1) to the case where the graph of (U, φ) is the set of all (x, y) such that

$$\begin{aligned}
&d(f^k x, x_k) \leqq \varepsilon \quad \text{if} \quad |k| < n , \\
&d(f^k y, y_k) \leqq \varepsilon \quad \text{if} \quad |k| < n , \\
&d(f^k x, f^k y) \leqq \varepsilon \quad \text{if} \quad |k| \geqq n ,
\end{aligned}$$

for some choice of n and $(x_k)_{|k|<n}$, $(y_k)_{|k|<n}$. We shall take neighborhoods $\mathcal{N}_\ell$ of U, $\mathcal{N}_\ell^*$ of φU, and construct maps $\varphi_{\ell m}\colon \mathcal{N}_\ell \cap E_m \to \mathcal{N}_\ell^* \cap E_m$ such that

$$\varphi_{\ell m}(\mu_m|\mathcal{N}_\ell) \leqq C^*\left(\exp \sum_{|k|<n} [A(f^k X) - A(f^k Y)]\right)\cdot(\mu_m|\mathcal{N}_\ell^*)$$

for some arbitrary choice of $X \in U$, $Y = \varphi X$.

We let $\mathcal{N}_\ell = \bigcup_{x\in U} B_x(\varepsilon, \pm\ell)$, $\mathcal{N}_\ell^* = \bigcup_{y\in\varphi U} B_y(2\varepsilon, \pm\ell)$ with $n < \ell$, $\ell + p < m$. If $u \in \mathcal{N}_\ell \cap E_m$, choose $x \in U \cap B_u(\varepsilon, \pm\ell)$ and let $y = \varphi x$. We may then by specification choose $w \in B_y(\varepsilon, \pm\ell)$ such that

$$d(f^k w, f^k u) \leqq \varepsilon \quad \text{if } \ell + p \leqq |k| < m\,.$$

Therefore there is $v \in E_m \cap B_y(2\varepsilon, \pm\ell)$ such that

$$d(f^k u, f^k v) \leqq 2\varepsilon \quad \text{if } \ell + p \leqq |k| < m\,.$$

We define $\varphi_{\ell m} u = v$. The possible choices of u satisfy conditions

$$d(f^k u, x_{i(k)}^*) \leqq \varepsilon \quad \text{if } \ell \leqq |k| < \ell + p\,,$$

where the $x_{i(k)}^*$ are taken in a maximal $\frac{1}{2}\varepsilon$-separated subset (x_i^*) of M.

For a given v, we have the following restrictions on u:

$$\begin{aligned} d(f^k u, x_k) &\leqq 2\varepsilon && \text{if } |k| < n\,,\\ d(f^k u, f^k v) &\leqq 4\varepsilon && \text{if } n \leqq |k| < \ell\,,\\ d(f^k u, x_{i(k)}^*) &\leqq \varepsilon && \text{if } \ell \leqq |k| < \ell + p\,,\\ d(f^k u, f^k v) &\leqq 2\varepsilon && \text{if } \ell + p \leqq |k| < m. \end{aligned}$$

If the same conditions are satisfied by $u' \in E_m$ then expansiveness implies $d(f^k u, f^k u') \leqq \varepsilon$ if $|k| \leqq m - q$ for some suitable q. If N is the cardinality of (x_i^*), there are at most N^{2p} choices for the $x_{i(k)}^*$, and the map $\varphi_{\ell m}$ is thus at most N^{2p+2q}-to-one. We have finally

$$\varphi_{\ell m}(\mu_m|\mathcal{N}_\ell) \leqq N^{2p+2q} e^{8K+2pV+4K+4qV+8K} \cdot\left(\exp \sum_{|k|<n} [A(f^k X) - A(f^k Y)]\right)\cdot(\mu_m|\mathcal{N}_\ell^*)$$

with $X \in U$, $Y \in \varphi U$ as announced. If μ is a vague limit of μ_m for $m \to \infty$ we find therefore, taking first $m \to \infty$ then $\ell \to \infty$ and using Remark 2.3, that (3) holds. We have thus shown that μ is a quasi-Gibbs state.

2.5 Theorem. *If (M, f) satisfies expansiveness and specification, and $A \in \mathscr{V}$, A Borel, there is a unique Gibbs state ρ for A, and ρ is also the unique f-invariant quasi-Gibbs state for A, and the unique equilibrium state for A.*

First we shall construct a Gibbs state ρ, and notice that this is an invariant quasi-Gibbs state. We shall then see that an f-invariant quasi-Gibbs state is necessarily an equilibrium state, and then use Bowen's result [2] that there is only one equilibrium state.

We know by Proposition 2.4 that there exists a quasi-Gibbs state μ. For each conjugating homeomorphism (U, φ). Let $F_\varphi : \varphi U \to \mathbb{R}$ be defined by

$$F_\varphi = \frac{d\varphi\mu}{d\mu} \exp - \sum_{k=-\infty}^{\infty} (A \circ f^k \circ \varphi^{-1} - A \circ f^k) .$$

Define $\tilde{\mu}$ to be the smallest real measure such that, for all (U, φ),

$$(\tilde{\mu} - F_\varphi \mu) | \varphi U \geqq 0 .$$

Then, since $C^{-1} \leqq F_\varphi \leqq C$, $\tilde{\mu}$ is a bounded positive measure.

If (U', φ') is a conjugating homeomorphism, then $\varphi'\tilde{\mu}|\varphi'U'$ is the smallest measure such that for all (U, φ) we have

$$\begin{aligned}\varphi'\tilde{\mu}|\varphi'(U' \cap \varphi U) &\geqq (F_\varphi \circ \varphi'^{-1}) \cdot \varphi'\mu|\varphi'(U' \cap \varphi U)\\ &= \left[\frac{d\varphi'\varphi\mu}{d\mu} \cdot \exp - \sum_{k=-\infty}^{\infty} (A \circ f^k \circ (\varphi' \circ \varphi)^{-1} - A \circ f^k \circ \varphi'^{-1})\right] \cdot \mu|\varphi'(U' \cap \varphi U)\\ &= \left[\exp \sum_{k=-\infty}^{\infty} (A \circ f^k \circ \varphi'^{-1} - A \circ f^k)\right] \cdot F_{\varphi'\varphi}\mu|\varphi'(U' \cap \varphi U) .\end{aligned}$$

Therefore $\varphi'\tilde{\mu}|\varphi'U'$ is the smallest of the measures μ^* on $\varphi'U'$ such that, for all (U, φ),

$$\mu^*|\varphi'(U' \cap \varphi U) \geqq \left[\exp \sum_{k=-\infty}^{\infty} (A \circ f^k \circ \varphi'^{-1} - A \circ f^k)\right] \cdot F_{\varphi'\varphi}\mu|\varphi'(U' \cap \varphi U) . \quad (4)$$

By definition of $\tilde{\mu}$, the condition that a measure μ^* on $\varphi'U'$ satisfies (4) is *equivalent to*

$$\mu^*|\varphi'U' \geqq \left[\exp \sum_{k=-\infty}^{\infty} (A \circ f^k \circ \varphi'^{-1} - A \circ f^k)\right] \tilde{\mu}|\varphi'U' \quad (5)$$

(one implication because $\tilde{\mu} - F_\varphi\mu \geqq 0$ on φU, the other because $\tilde{\mu}$ is smallest with this property). Since $\varphi'\tilde{\mu}|\varphi U$ is the smallest μ^* satisfying (5) we have

$$\varphi'\tilde{\mu}|\varphi'U' \geqq \left[\exp - \sum_{k=-\infty}^{\infty} (A \circ f^k \circ \varphi'^{-1} - A \circ f^k)\right] \cdot \tilde{\mu}|\varphi'U' .$$

Therefore the measure $\rho = \tilde{\mu} / \|\tilde{\mu}\|$ is a Gibbs state.[2]

For any Gibbs state ρ, $f\rho$ is also a Gibbs state. If $f\rho \neq \rho$, one obtains mutually singular Gibbs states ρ_1, ρ_2 by normalizing $|f\rho - \rho| \pm (f\rho - \rho)$, but these are also quasi-Gibbs states, contradicting Corollary 2.2. Therefore ρ is an f-invariant quasi-Gibbs state. Note that such a state is necessarily ergodic, otherwise one could decompose ρ into mutually singular quasi-Gibbs states

We now have to show that any f-invariant quasi-Gibbs state ρ is an equilibrium state. For this it suffices (by the variational principle) to prove that

$$h(\rho) + \rho(A) \geqq P(A) .$$

[2] This construction was explained to one of us (D.R.) by D. Sullivan

Choose a a maximal $\frac{1}{2}\varepsilon$-separated set E and a Borel partition $(\mathscr{A}_x)_{x\in E}$ of M such that $\mathscr{A}_x \subset B_x(\frac{1}{2}\varepsilon)$ for $x \in E$. The entropy of ρ is

$$h(\rho) = \lim_{n\to\infty} \frac{1}{2n-1} H(\rho, \mathscr{A}^{(n)}),$$

where $\mathscr{A}^{(n)}$ consists of the sets

$$\mathscr{A}_\xi = \{u \in M : f^k u \in \mathscr{A}_{\xi(k)} \text{ for } |k| < n\}$$

indexed by functions ξ: $\{k\colon |k| < n\} \to E$. If $\mathscr{A}_\xi$ is not empty, say $A_\xi \ni y(\xi)$, then $\mathscr{A}_\xi \subset B_{y(\xi)}(\varepsilon, \pm n)$, hence

$$\rho(\mathscr{A}_\xi) \leqq \exp\left[\sum_{|k|<n} A(f^k y(\xi)) - (2n-1)P(A) + c\right].$$

Therefore

$$\begin{aligned} h(\rho) + \rho(A) \geqq \lim_{n\to\infty} \frac{1}{2n-1} \sum_\xi \rho(\mathscr{A}_\xi) \cdot \\ \left\{\sum_{|k|<n} A(f^k y(\xi)) - K_A - \sum_{|k|<n} A(f^k y(\xi)) \right. \\ \left. + (2n-1)P(A) - c\right\} \\ = P(A). \end{aligned}$$

For the proof that there is a unique equilibrium state, see Bowen [2] Lemma 8; Bowen's proof is based on the fact that we know an ergodic equilibrium state ρ, and we have good estimates for $\rho(B_x(\varepsilon, \pm n))$, it does not use the continuity of A, but only $A \in \mathscr{V}$.

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