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Real Polarization of the Moduli Space of Flat Connections on a Riemann Surface*

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Abstract. We prove that the moduli space of flat $SU(2)$ connections on a Riemann surface has a real polarization, that is, a foliation by Lagrangian subvarieties. This polarization may provide an alternative quantization of the Chern–Simons gauge theory in higher genus, in line with the results of [11] for genus one.

I. Introduction

Let Σ^g be a 2-manifold of genus g. The space $\bar{\mathscr{S}}_g$ of conjugacy classes of representations $\rho:\pi_1(\Sigma^g)\to G$, where G is a compact lie group, is an algebraic variety containing an open set $\mathscr{S}_g$ which is a symplectic manifold. The symplectic form ω is the Chern class of a line bundle $\mathscr{L}\to\mathscr{S}_g$ which extends to a line bundle $\bar{\mathscr{L}}\to\bar{\mathscr{S}}_g$. The line bundle $\bar{\mathscr{L}}\to\bar{\mathscr{S}}_g$ is endowed with a canonical connection and hermitian metric. Furthermore, a choice of a metric on Σ^g endows $\mathscr{S}_g$ with a complex structure making the symplectic form ω Kähler and the line bundle $\mathscr{L}$ holomorphic. Thus, ignoring for a moment the singularities of $\bar{\mathscr{S}}_g$, we have arrived at the natural setting for quantization; namely, we have been given a symplectic manifold $\mathscr{S}_g$, a line bundle $\mathscr{L}\to\mathscr{S}_g$ with connection of curvature ω, and a polarization of the sheaf of local sections of $\mathscr{L}\to\mathscr{S}_g$.

Recent developments have emphasized the importance of this system in relation to the theory of representations of loop groups, conformal field theory, and 3-dimensional topological quantum field theory. For example, the quantization of the above system in $g=1$ can be naturally associated to the Weyl–Kac characters of the integrable representations of the Kac–Moody lie algebra $\hat{G}$ associated to G; while this quantization for general g yields a projectively flat bundle over moduli space associated to the conformal field theory of G current algebra.

The main motivation for our study of this system is however related to Chern–Simons gauge theory and the topological field theory related to it [12].

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This topological field theory is supposed to assign to every 2-manifold Σ^g a vector space $\mathscr{H}(\Sigma^g)$, and to every 3-manifold M a vector $v(M)\in\mathscr{H}(\partial M)$ in the vector space corresponding to its boundary. The construction of the vector space is given by quantizing the system corresponding to $\mathscr{L}\to\mathscr{S}_g$ as described above. This construction can be shown to be well-defined and nearly independent of the polarization. But it sheds no light on the construction of the vectors corresponding to three-manifolds.

One hint is however provided by the following. Consider the subspace $\bar{L}_M\subset\bar{\mathscr{S}}_g$ of representations of $\pi_1(\Sigma^g)\to G$ which extend to representations of $\pi_1(M)\to G$ under the natural inclusion. Then $L_M=\bar{L}_M\cap\mathscr{S}_g$ is an isotropic subspace of $\mathscr{S}_g$; i.e., $\omega|_{L_M}=0$. Furthermore, if M is a handlebody with boundary Σ^g, then L_M is in fact Lagrangian (or, maximal isotropic). If we could produce an element of $\mathscr{H}(\Sigma)$ given a Lagrangian subspace, the above axiom would be verified in the case where the 3-manifold M is produced by such a Heegaard splitting; this would be an important step in constructing a topological field theory.

Lagrangian submanifolds however are not easily associated with the holomorphic sections of a line bundle, which arise when a system of this type is quantized in a Kahler polarization. They do occur naturally in the context of real polarizations, which are foliations of the given symplectic manifold by Lagrangian subspaces. The question we shall attempt to answer in this paper is the following: is there a foliation of $\mathscr{S}_g$ by Lagrangian submanifolds so that L_M occurs as a leaf? If we could answer this in the affirmative, the next step would be to quantize the system in this polarization. We may then expect to assign an element of the quantum Hilbert space to the leaf L_M. We will address this issue in [7].

In this paper we construct such a foliation, where the three manifold with boundary M is the handlebody bounding Σ^g. Since any three manifold can be formed by gluing two handlebodies along their boundaries, the construction in this case might be expected to suffice. Our main result is as follows. Let $G=SU(2)$, and let $\bar{\mathscr{S}}_g,\mathscr{S}_g,\mathscr{L},\omega$ be as above.

Theorem. *There exists a map $\mathscr{F}:\bar{\mathscr{S}}_g\to\mathbb{R}^{3g-3}$ whose fibres $\{\mathscr{F}^{-1}(x)\}_{x\in\mathrm{Im}\,\mathscr{F}}$ are generically Lagrangian; that is, $\omega|_{\mathscr{F}^{-1}(x)\cap\mathscr{S}_g}=0$, and, for $x\in\mathrm{Im}\,\mathscr{F}$ generic, $\dim\mathscr{F}^{-1}(x)=1/2\dim(\mathscr{S}_g)$. The subspace L_M is a leaf of the corresponding foliation.*

Note that we do *not* claim that the leaves $\mathscr{F}^{-1}(x)$ are manifolds. This is in fact false, even in the case $g=1$, where $\bar{\mathscr{S}}_g\cong S^2$ is a smooth manifold. The failure of the leaves to be smooth manifolds is of importance in quantization, as described in a companion paper [11]. However, we note that in genus $g>2$ the generic leaf of the foliation above does not intersect the singularities of the variety $\bar{\mathscr{S}}_g$. Therefore, away from the singularities, we obtain a foliation of a smooth (though noncompact) manifold by Lagrangian subvarieties, and we see that $\bar{\mathscr{S}}_g$ is generically fibred by Lagrangian *tori*. An explicit description of this fibration will appear in [6].

Finally, a comment about some previous work is in order. In [5], Hitchin showed that the cotangent bundle of the moduli space $\mathscr{S}_g$ could be foliated by submanifolds which were Lagrangian with respect to the natural symplectic structure on $T^*\mathscr{S}_g$. In fact this was done in two different ways. The leaves of one of Hitchin's foliations intersects $\mathscr{S}_g$ at points; the other contains $\mathscr{S}_g$ as a leaf. Neither of them therefore produces a foliation of $\mathscr{S}_g$ itself.

A closer relation to our theorem is found in the work of Goldman [3]. The leaves of our foliation are given as intersections of inverse images of some real valued functions on $\mathcal{S}_g$. The proof that these leaves are isotropic boils down to demonstrating that the Poisson brackets of these functions are zero. We do this in Lemma 3.6 below. Goldman's work contains a formula for Poisson brackets of such functions which can be used to give an alternative proof of this result.

II. Definition of the Foliation $\mathscr{F}$

In this section we shall construct a base $B_g \subset \mathbb{R}^{3g-3}$ and a map $\mathscr{F}:\bar{\mathcal{S}}_g \to B_g$ whose inverse image sets $\{\mathscr{F}^{-1}(x):x\in B_g\}$ will be the leaves of our foliation. We recall first the definition of the space $\bar{\mathcal{S}}_g$ in terms of the fundamental group of the surface; we will define the map $\mathscr{F}$ using this description.

Now the space $\bar{\mathcal{S}}_g$ is defined as the space of conjugacy classes of representations $\rho:\pi_1(\Sigma^g)\to SU(2)$. More explicitly, we denote by R_g the space of all representations of $\pi_1(\Sigma)$ in $SU(2)$; the group $SU(2)$ acts on R_g by conjugation, and $\bar{\mathcal{S}}_g$ is then the quotient $R_g/SU(2)$. The space $\bar{\mathcal{S}}_g$ contains a large open cell $\mathcal{S}_g$ corresponding to *irreducible* representations of $\pi_1(\Sigma^g)$ in $SU(2)$, which is a smooth manifold. The space $\mathcal{S}_g$ is in fact a symplectic manifold. We recall here the construction of the symplectic form ω on $\mathcal{S}_g$; the proofs may be found, e.g., in Atiyah–Bott [2].

The symplectic form ω is most conveniently described in terms of an alternative definition of $\bar{\mathcal{S}}_g$, given by gauge fields. Let $T^g = \Sigma^g \times SU(2)$ be the trivial $SU(2)$ bundle over Σ^g, and let $\mathscr{A}$ be the space of (smooth) connections on T^g. We may identify $\mathscr{A}$ with the affine space $\Omega^1(\Sigma^g)\otimes \mathfrak{su}(2)$ of $\mathfrak{su}(2)$-valued one forms on Σ^g. On $\mathscr{A}$ there acts the group $\mathscr{G} = \mathrm{Maps}(\Sigma^g, SU(2))$ of (smooth) gauge transformations, which preserves the submanifold $\mathscr{A}_F \subset \mathscr{A}$ of flat connections.

Any flat connection on T^g defines a monodromy representation of the fundamental group of the surface Σ. For any simple closed oriented curve C in Σ^g, and any two points x and y on C, we write $\mathrm{hol}_{x\to y}{}^C(A)$ for the holonomy of the connection A, along C, from x to y. Choose a set of curves ξ_i, $i=1,\dots,2g$ based at $*$ whose equivalence classes generate $\pi_1(\Sigma^g)$.

Proposition 2.1. *Let $\tilde{r}:\mathscr{A}^F \to R_g$ be given by*

$$\tilde{r}(A) = (\mathrm{hol}^{\xi_1}_{*\to *}(A),\dots,\mathrm{hol}^{\xi_{2g}}_{*\to *}(A),\dots).$$

Then $\tilde{r}:\mathscr{A}_F \to R_g$ descends to a homeomorphism $r:\mathscr{A}_F/\mathscr{G}\to \bar{\mathcal{S}}_g$.

On the other hand, we have a $\mathscr{G}$-invariant two-form on $\mathscr{A}_F$ defined as follows. The tangent space $T\mathscr{A}|_A$ may be identified with the space $\Omega^1(\Sigma^g)\otimes \mathfrak{su}(2)$ of $\mathfrak{su}(2)$-valued one-forms on Σ^g; we may therefore define a two form $\tilde{\omega}$ on $\mathscr{A}$ by

$$\tilde{\omega}_A(\psi,\psi') = \mathrm{tr}\int_{\Sigma^g} \psi \wedge \psi' \tag{2.1}$$

for $\psi,\psi' \in \Omega^1(\Sigma^g)\otimes \mathfrak{su}(2)$.

Now $\mathscr{A}_F$ is defined as a subspace of $\mathscr{A}$. The subspace $T\mathscr{A}_{\mathscr{F}}|_A \subset T\mathscr{A}|_A$ for $A\in\mathscr{A}_F$ can be identified with the space of $\mathfrak{su}(2)$-valued one forms ψ satisfying the condition

$$0 = d_A\psi \equiv d\psi + [A,\psi]. \tag{2.2}$$

In any event, we may define a two-form, also denoted $\tilde{\omega}$, on $\mathscr{A}_F$, by pulling back the two-form from $\mathscr{A}$. We then have the following result.

Proposition 2.2. *The two form $\tilde{\omega}$ on $\mathscr{A}_F$ descends to a two form ω on $\mathscr{S}_g$ which turns $\mathscr{S}_g$ into a symplectic manifold.*

We now turn to describing the map $\mathscr{F}$. To do this we will choose some distinguished curves in Σ^g and study the holonomy about these curves of connections represented by points in $\bar{\mathscr{S}}_g$. This will provide us with some functions on $\bar{\mathscr{S}}_g$.

The curves in question are obtained as follows. Let D denote the three-hold sphere; this is a space homeomorphic to the subset

$$D=\{z\in\mathbb{C}:|z|\leqq 1,|z-\tfrac{1}{2}|\geqq\tfrac{1}{4},|z+\tfrac{1}{2}|\geqq\tfrac{1}{4}\}$$

of the complex plane. The Riemann surface Σ^g may be written as the union of $2g-2$ copies of D, with their boundaries identified pairwise. We thus obtain $3g-3$ simple closed curves in Σ^g by considering the boundary circles of the three-holed spheres. Let us denote these disjoint curves by C_i, and orient them arbitrarily.

In order to view these curves as elements of the fundamental group, we must choose some base points. So choose a base point $x\in\Sigma^g$, and choose also points $x_i\in C_i$ for all $i=1,\dots,3g-3$. Choose an arc α_i connecting x to x_i for each i. Using these basepoints and arcs, the curves $\tilde{C}_i=\alpha_i^{-1}*C_i*\alpha_i$ give rise to equivalence classes $[\tilde{C}_i]\in\pi_1(\Sigma^g)$.

Suppose we are given a representation $\rho\in R_g$; then ρ is a map $\rho:\pi_1(\Sigma^g)\to SU(2)$. We now define a map $\tilde{\mathscr{F}}:R_g\to\mathbb{R}^{3g-3}$ by setting

$$\tilde{\mathscr{F}}(\rho)=\{\operatorname{tr}\rho([\tilde{C}_1]),\dots,\operatorname{tr}\rho([\tilde{C}_{3g-3}])\}. \tag{2.3}$$

We let B_g denote the image of $\tilde{\mathscr{F}}$. Since the trace of an $SU(2)$ matrix is invariant under conjugation, we have the following lemma which defines the map constructing our foliation.

Lemma 2.3. *The map $\tilde{\mathscr{F}}:R_g\to B_g$ descends to a map $\mathscr{F}:\bar{\mathscr{S}}_g\to B_g$.*

In terms of connections, we may describe the map $\mathscr{F}$ as the map which assigns to any flat connection the traces of the holonomies of that connections about the curves $\tilde{C}_i$.

We now state more carefully the main result of our paper. The fibres of the map $\mathscr{F}$ will all be isotropic, meaning that the restriction of ω to $\mathscr{F}^{-1}(x)\cap\mathscr{S}_g$ will vanish for all $x\in B_g$. However the fibres will have maximal dimension only generically. For our purpose we restrict our attention to the fibres $\mathscr{F}^{-1}(x)$ lying above points $x\in\operatorname{Int}B_g$.

Theorem 2.4. *Let $\mathscr{F}:\bar{\mathscr{S}}_g\to B_g$ be the map defined in Lemma 2.3 above. Let $x\in\operatorname{Int}B_g$.*

(i) *The dimension of the space $\mathscr{F}^{-1}(x)\cap\mathscr{S}_g$ is $\frac{1}{2}\dim\mathscr{S}_g=3g-3$.*

(ii) *The spaces $\mathscr{F}^{-1}(x)\cap\mathscr{S}_g$ are isotropic*; i.e.,

$$\omega|_{\mathscr{S}_g\cap\mathscr{F}^{-1}(x)}=0.$$

(iii) *Consider the point $p=(2,\dots,2)\in B_g$. Then the leaf $\mathscr{F}^{-1}(p)\subset\bar{\mathscr{S}}_g$ is the image under r of those flat connections on Σ^g which extend to flat connections on the handlebody H bounding Σ; that is, in the notation of the introduction*

$$\mathscr{F}^{-1}(p)=r(\bar{L}_H).$$

Proof. To verify part (iii) note that any flat connection on Σ extending to a flat connection on H must have trivial holonomy on all the curves C_i; likewise any connection with this property will extend to H as a flat connection. Unwinding the definition of $\mathscr{F}$ shows that the points of $\mathscr{F}^{-1}(p)$ are exactly those with trivial holonomy along all the curves C_i. Note that although $p \notin \operatorname{Int} B_g$, the fibre $\mathscr{F}^{-1}(p)$ is in fact Lagrangian.

As for parts (i) and (ii), we distinguish two cases.

Case I. $g=1$. In this case the spaces $\mathscr{F}^{-1}(x)$ are one dimensional, hence of necessity isotropic. Since $\dim \mathscr{S}_1 = 2$, they are Lagrangian.

Case II. $g \geqq 2$. The dimension of the fibre $\mathscr{F}^{-1}(x)$ is $3g-3$ since the dimension of $\bar{\mathscr{S}}_g$ is $6g-6$ while $\operatorname{Int} B_g$ is a (nonempty) open subset of $\mathbb{R}^{3g-3}$. It remains to prove isotropy; this is the result of Proposition 3.7.

III. Isotropy of the Leaves $\mathscr{F}^{-1}(x)$

In this section we complete the proof of Theorem 2.4 by proving the isotropy of the fibres $\mathscr{F}^{-1}(x)$ of the map $\mathscr{F}$ for genus $g \geqq 2$. Our method will be as follows. We consider the components of the map $\mathscr{F}: \bar{\mathscr{S}}_g \to \mathbb{R}^{3g-3}$, as functions $f_i: \bar{\mathscr{S}}_g \to \mathbb{R}$. We now consider the Hamiltonian vector fields V_{f_i} of these functions, defined by

$$\omega_x(V_{f_i}, w) = df_i|_x(w)$$

for $w \in T\mathscr{S}_g|_x$.

Isotropy of the leaves of the foliation would follow if we could show that $\omega_x(V_{f_i}, V_{f_j}) = 0$ for all $x \in \mathscr{S}_g$. To do this we show in Sect. 3.1 that we may lift the computation of the Poisson brackets $\{f_i, f_j\} = \omega(V_{f_i}, V_{f_j})$ to the computation of the Poisson brackets of functions $\{\tilde{f}_i, \tilde{f}_j\}$ on $\mathscr{A}_F$; we also obtain explicit expressions for the Hamiltonian vector fields of these functions. We use these expressions in Sect. 3.2 to show that the leaves are indeed isotropic.

3.1. The Hamiltonian Vector Fields V_{f_i}. In this section we develop explicit expressions for the Hamiltonian vector fields V_{f_i}. These are most easily obtained by constructing functions $\tilde{f}_i: \mathscr{A}_F \to \mathbb{R}$ which descend to the functions f_i under the identification $\bar{\mathscr{S}}_g = \mathscr{A}_F / \mathscr{G}$. We may define such functions as follows.

Recall we have been given a collection $\tilde{C}_i$ of closed curves on Σ^g. Define $\tilde{f}_i: \mathscr{A}_F \to \mathbb{R}$ by

$$\tilde{f}_i(A) = \operatorname{tr} \operatorname{hol}^{\tilde{C}_i}_{x_i \to x_i}(A). \tag{3.1}$$

Then $\omega(V_{f_i}, V_{f_j}) = \tilde{\omega}(V_{\tilde{f}_i}, V_{\tilde{f}_j})$.

We further simplify the computation of $\omega(V_{f_i}, V_{f_j})$ by converting it to a computation on the affine space $\mathscr{A}$. We do this by choosing an extension of the functions $\tilde{f}_i$ to smooth real valued functions on all of $\mathscr{A}$. Now of course one way to do this is to extend the definition (3.1) of the functions $\tilde{f}_i$ directly to $\mathscr{A}$. This however will yield Hamiltonian vector fields which will be distributional $\mathfrak{su}(2)$-valued one forms. The reason for this is that the functions $\tilde{f}_i$ are given, in terms of the gauge field A, by traces of the holonomy of A about a collection of simple closed curves in Σ^g. They therefore depend on the values of A in a one-dimensional submanifold of Σ^g. This will cause the associated Hamiltonian vector fields to be distributional $\mathfrak{su}(2)$-valued one-forms.

In order to get around this problem, we define in this section different extensions $\bar{f}_i$ of the functions $\tilde{f}_i$, obtained by "smearing" the loops about which we take the holonomy. If A is a flat connection, the trace of the holonomy of A about a curve is invariant, not only under homotopy with fixed base point (which preserves the holonomy itself), but also under free homotopy. We may thus smear the functions $\tilde{f}_i$ by integrating over a collar about the appropriate curve, thus obtaining functions $\bar{f}_i$ which coincide with $\tilde{f}_i$ on the space of flat connections but which will give rise to Hamiltonian vector fields represented by smooth $\mathfrak{su}(2)$-valued one forms.

In addition, we may vary the functions $\bar{f}_i$ by free homotopy of the collar, thus allowing us, in the next section, to choose *disjoint* collars, which will simplify the proof of isotropy. In this section we begin by computing the Hamiltonian vector fields $V_{\bar{f}_i}$ on $\mathscr{A}_F$, and exhibiting them explicitly as smooth, $\mathfrak{su}(2)$-valued one forms.

We begin by recalling the following property of the holonomy of a flat connection on a principal bundle.

Lemma 3.1. *Let C, C' be simple closed oriented curves based at $*, *'$, respectively. Suppose C and C' are freely homotopic. Then if A is a flat connection on Σ^g,*

$$\operatorname{tr}\operatorname{hol}^{C}_{*\to *}(A) = \operatorname{tr}\operatorname{hol}^{C'}_{*'\to *'}(A).$$

Now in terms of the curves $\tilde{C}_i$, the functions $\tilde{f}_i$ are given by the holonomies

$$\operatorname{tr}\operatorname{hol}^{\tilde{C}_i}_{*\to *}(A) = \tilde{f}_i(A), \quad i = 1, \dots, 3g-3.$$

Now recall that the curves C_i were freely homotopic to the based curves $\tilde{C}_i$. Then, by Lemma 3.1,

$$\tilde{f}_i(A) = \operatorname{tr}\operatorname{hol}^{C_i}_{*_i\to *_i}(A) \tag{3.2}$$

where $*_i \in C_i$ is arbitrary.

We now 'smear' the $\tilde{f}_i$. Let $T_i \supset C_i$ be a tubular neighborhood of C_i, chosen with a diffeomorphism $\phi_i : T_i \to [0,1] \times S^1$ such that $\phi_i(\{0\} \times S^1) = C_i$. Let $C_i(y) = \phi_i^{-1}(\{y\} \times S^1)$. Then C_i and $C_i(y)$ are freely homotopic for each y. Let $X_i(y) = \phi_i^{-1}((y,1)) \in C_i(y)$. Then

$$\operatorname{tr}\operatorname{hol}^{C_i(y)}_{X_i(y)\to X_i(y)}(A) = \tilde{f}_i(A). \tag{3.3}$$

Now let $\chi \in C^\infty([0,1])$ satisfy

$$\begin{aligned} &\text{(i)} \quad \chi \geqq 0, \\ &\text{(ii)} \quad \chi(0) = \chi(1) = 0, \\ &\text{(iii)} \quad \int_0^1 \chi(x)\,dx = 1. \end{aligned} \tag{3.4}$$

Then, by (3.3), for $A \in \mathscr{A}_F$

$$\tilde{f}_i(A) = \int_0^1 \chi(y)\,dy \cdot \operatorname{tr}\operatorname{hol}^{C_i(y)}_{X_i(y)\to X_i(y)}(A). \tag{3.5}$$

We will use (3.5) to produce an explicit formula for the Hamiltonian vector fields $V_{\bar{f}_i}$.

Now recall that if (M, ω) is a symplectic manifold, and $f: M \to \mathbb{R}$, the Hamiltonian vector field V_f associated to f is defined by

$$\omega_x(V_f, w) = df_x(w)$$

for any tangent vector $w \in TM_x$. Motivated by this we have the following.

Definition 3.2. *Let $\bar{f}_i: \mathscr{A} \to \mathbb{R}$ be a function with $\bar{f}_i | \mathscr{A}_F = \tilde{f}_i$. The Hamiltonian vector field $V_{\bar{f}_i}$ associated to $\bar{f}_i$ is the distributional $\mathfrak{su}(2)$-valued one form defined by*

$$\tilde{\omega}_A(V_{\bar{f}_i}, w) = (d\bar{f}_i)_A(w)$$

for $w \in T\mathscr{A}|_A$.

In other words, we take $V_{\bar{f}_i}$ to be the distributional $\mathfrak{su}(2)$-valued one form dual to the $\mathfrak{su}(2)$-valued current Ξ defined by

$$\int_{\Xi} \alpha = (d\bar{f}_i)_A(\alpha)$$

for any $\mathfrak{su}(2)$-valued one form α.

Suppose that $A \in \mathscr{A}_F$ and $w \in T\mathscr{A}|_A$; then $\tilde{\omega}_A(V_{\bar{f}_i}, w)$ is independent of the choice of extension $\bar{f}_i$. We show now that $V_{\bar{f}_i}$ may be taken to be smooth, and in Lemma 3.5 that $d_A V_{\bar{f}_i} = 0$. Thus $V_{\bar{f}_i}$ descends to a tangent vector field V_{f_i} on $\mathscr{S}_g$ independent of the extension $\bar{f}_i$.

Let $x \in C_i(y)$. We define $\tilde{U}^i_y(x) \in SU(2) \subset \mathrm{gl}(2)$ by

$$\tilde{U}^i_y(x) = \mathrm{hol}^{C_i(y)}_{x \to x}(A) \tag{3.6}$$

and let $U^i_y(x) \in \mathfrak{su}(2) \subset \mathrm{gl}(2)$ be given by

$$U^i_y(x) = \tilde{U}^i_y(x) - \tfrac{1}{2} \mathrm{tr}\, \tilde{U}^i_y(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $i_y: C_i(y) \to \Sigma^g$ denote the inclusion. We extend $\tilde{f}_i$ to $\mathscr{A}$ by the function $\bar{f}_i$ given by

$$\bar{f}_i(A) = \int_0^1 \chi(y) dy \cdot \mathrm{tr}\, \mathrm{hol}^{C_i(y)}_{X_i(y) \to X_i(y)}(A). \tag{3.7}$$

Lemma 3.3. *Let $w \in T\mathscr{A}|_A$, $A \in \mathscr{A}_F$. Then*

$$\tilde{\omega}(V_{\bar{f}_i}, w) = \int_0^1 \chi(y) dy \oint_{C_i(y)} dx\, \mathrm{tr}\, U^i_y(x)(i_y^* w)(x).$$

Proof. We have

$$\begin{aligned}
\tilde{\omega}_A(V_{\bar{f}_i}, w) &= d\bar{f}_i(w) \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1}(\bar{f}_i(A + \varepsilon w) - \bar{f}_i(A)) \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^1 \chi(y) dy [\mathrm{tr}\, \mathrm{hol}^{C_i(y)}_{X_i(y) \to X_i(y)}(A + \varepsilon w) - \mathrm{tr}\, \mathrm{hol}^{C_i(y)}_{X_i(y) \to X_i(y)}(A)] \\
&= \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^1 \chi(y) dy\, \mathrm{tr} \oint_{C_i(y)} dx\, \mathrm{hol}^{C_i(y)}_{X_i(y) \to x}(A) \circ (i_y^* w)(x) \cdot \mathrm{hol}^{C_i(y)}_{X \to X_i(y)}(A) \\
&= \int_0^1 \chi(y) dy \oint dx\, \mathrm{tr}\, U^i_y(x)(i_y^* w)(x).
\end{aligned}$$

Corollary 3.4. *The distributional vector field $V_{\bar{f}_i}$ may be represented by a smooth one form (also denoted $V_{\bar{f}_i}$) so that* $\operatorname{supp}(V_{\bar{f}_i}) \subset T_i$.

Proof. Let $p_1: [0,1] \times S^1 \to [0,1]$, $p_2: [0,1] \times S^1 \to S^1$ denote the projections. We define

$$\begin{aligned} V_{\bar{f}_i}(s) &= \chi(p_1(s)) \cdot \phi_i^*(p_1^*(dt)) \otimes U^i_{p_1(s)}(s), \quad s \in T_i \\ &= 0 \quad s \notin T_i, \end{aligned}$$

where dt denotes the standard one form on $[0,1]$. It is clear that $V_{\bar{f}_i}$ is smooth and that

$$\tilde{\omega}_A(X_{\bar{f}_i}, w) = d\bar{f}_i(w).$$

3.2. Isotropy of the Leaves. We now complete the proof of Theorem 2.4. We begin by showing that, as expected for Hamiltonian vector fields of gauge invariant functions, the vector fields $V_{\bar{f}_i}$ descend to $\mathscr{S}_g$.

Lemma 3.5. *The vector fields $V_{\bar{f}_i}$ are tangent to $\mathscr{A}_F$; that is, $d_A V_{\bar{f}_i} = 0$.*

Proof. We must show that

$$\operatorname{tr} \int_\Sigma d_A V_{\bar{f}_i} \phi = 0$$

for any $\phi \in C^\infty(\Sigma^g) \otimes \mathfrak{su}(2)$.

But by Lemma 3.3,

$$\begin{aligned} \operatorname{tr} \int_\Sigma d_A V_{\bar{f}_i} \cdot \phi &= -\operatorname{tr} \int_\Sigma V_{\bar{f}_i} \wedge d_A \phi \\ &= -\operatorname{tr} \int_0^1 \chi(y) dy \oint_{C_i(y)} dx\, U^i_y(x) i^*_y(d_A \phi) \\ &= \operatorname{tr} \int_0^1 \chi(y) dy \oint_{C_i(y)} dx (i^*_y d_{i^*_y(A)} U^i_y)(x) \phi(x). \end{aligned}$$

We claim $i^*_y d_{i^*_y(A)} U^i_y = 0$.

To see this, let $x \in C_i(y)$, and write

$$P^i_y(x) = \operatorname{hol}^{C_i(y)}_{X_i(y) \to x}(A).$$

Then $i^*_y A = (P^i_y)^{-1} i^*_y dP^i_y$ while

$$\tilde{U}^i_y(x) = (P^i_y(x))^{-1} \tilde{U}^i_y(0) P^i_y(x).$$

Hence

$$\begin{aligned} i^*_y d\tilde{U}^i_y &= -(P^i_y)^{-1} i^*_y dP^i_y \cdot \tilde{U}^i_y + \tilde{U}^i_y (P^i_y)^{-1} i^*_y dP^i_y \\ &= -[i^*_y A, \tilde{U}^i_y] \end{aligned}$$

so that $i^*_y dU^i_y + [i^*_y A, U^i_y] = 0$ as needed.

Lemma 3.6. The symplectic form $\tilde{\omega}$ vanishes on the span of the vector fields $V_{\bar{f}_i}$; we have $\tilde{\omega}_A(V_{\bar{f}_i}, V_{\bar{f}_j}) = 0$.

Proof. For $i = j$, this is immediate. Suppose $i \neq j$. Since $d_A V_{\bar{f}_i} = 0$, $\tilde{\omega}_A(V_{\bar{f}_i}, V_{\bar{f}_j}) = \tilde{\omega}_A(V_{\bar{f}_i}, V_{\bar{f}'_j})$, for any two extensions $\bar{f}_j, \bar{f}'_j$ of $\tilde{f}_j$. Now since the curves C_i were chosen as boundary curves of three-holed spheres joined together to form Σ^g, we may choose the curves C_i and C_j to be disjoint; likewise the neighbourhoods T_i and T_j may be chosen disjoint. For the corresponding $\bar{f}_i, \bar{f}_j$, we then have $\operatorname{supp} V_{\bar{f}_i} \subset T_i$, $\operatorname{supp} V_{\bar{f}_j} \subset T_j$, so that $\tilde{\omega}_A(V_{\bar{f}_i}, V_{\bar{f}_j}) = \operatorname{tr} \int_\Sigma V_{\bar{f}_i} \wedge V_{\bar{f}_j} = 0$.

Combining Lemmas 3.5 and 3.6, we have the following result.

Proposition 3.7. *The leaves $\mathscr{F}^{-1}(x)$ satisfy*

$$\omega|_{\mathscr{F}^{-1}(x) \cap \mathscr{S}_g} = 0.$$

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