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String Vertices, Overlap Equations, τ Functions and the Hirota Equation

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Abstract. String vertices, V, are shown to satisfy a new type of overlap equation of the form $V\exp\{ip\cdot Q^i(\xi^i)\} = V\exp\{ip\cdot Q^j(\xi^j)\}\left(\frac{d\xi^j}{d\xi^i}\right)^{p^2/2}$ as well as corresponding equations for A_n and B_n cycles. A special case of such an equation, when integrated, is shown to be the Hirota equation for the K–P hierarchy.

1. Introduction

There are a number of different approaches to string theory; the dual model [1], the light-cone string field theory [2], the sum over world sheet surfaces (the Polyakov approach) [3], gauge covariant string field theory [4] and the new oscillator formalism. The latter approach was developed with the objectives of providing an efficient method of calculating string perturbation theory and giving in some sense, a more fundamental definition of string theory. While the extent to which the latter aim has been achieved is not clear, it did succeed in giving perhaps the most efficient method of computing string perturbation theory. The formalism has a number of features in common with the original dual model approach to string theory; in particular it works with multi-string vertices. However, in the new oscillator formalism ghost oscillators were introduced into the vertices [5]. Although four distinct groups worked on different variants of the new oscillator formalism, substantial use was made of a new kind of relations called *overlap equations* which were discovered in Refs. [5–7]. These equations came in two types called unintegrated overlap equations and integrated overlap equations. The integrated overlap equations were often subsequently called conserved charges in the literature.

One of the new oscillator formalisms was called the group theoretic approach [7, 8] since calculations were reduced to essentially an exercise in manipulating conformal transformations. The basis of this approach were the overlap equations and the decoupling of zero norm physical states. The relations between the different new oscillator formalisms were discussed in the final parts of the papers in Ref. [9] where many more references can be found to the Copenhagen approaches [18, 19] as well as the Grassmannian approach [20].

Reviews of the group theoretic approach are given in Ref. [8 and 9] and so here we limit ourselves to a very brief summary. The scattering of N physical string states $|\chi\rangle_i$, $i = 1, \ldots, N$ is of the form

$$W = \int \prod_i{}' dz_i \prod_r dv_r \bar{f}(z_i, v_r) V |\chi\rangle_1 \cdots |\chi\rangle_N,$$

where V is the N string vertex, $z_i, i = 1, \ldots, N$ and v_r are the Koba–Nielsen co-ordinates and moduli respectively and $\bar{f}$ is a function of the moduli and Koba–Nielsen co-ordinates.

The method specifies how to calculate the function $\bar{f}$ and the vertex V. The vertex V is determined by requiring that it satisfy overlap equations which we will specify shortly, while the function $\bar{f}$ is uniquely determined, once we have found the vertex V, by demanding that the zero norm physical states decouple.

The scattering vertex V for N string scattering obeys the unintegrated overlap equation [5–7],

$$VR^i(\xi^i) = VR^j(\xi^j)\left(\frac{d\xi^j}{d\xi^i}\right)^d, \tag{1}$$

where R is a conformal operator of dimension d and ξ^i is a coordinate on the patch which includes the point where the i^{th} string is emitted and has its origin at that point. If z is a coordinate system in common to a number of patches then $\xi^i = (V^i)^{-1} z$ is an analytic function of z vanishing at the point z_i where the string is emitted. The relation between two coordinate patches ξ^i and ξ^j is then $\xi^j = ((V^j)^{-1} V^i)(\xi^i)$. In this paper we have changed our definition of a conformal field from those of previous works by the substitution $R(\xi) \to \xi^d R(\xi)$, that is in this paper $R(\xi)(d\xi)^d$ is invariant under a conformal transformation.

Such a relation (1) is valid for the fundamental conformal operators out of which the other conformal operators may be built. For the bosonic string, the fundamental operator is

$$Q^\mu(\xi) = q - i\alpha_0^\mu \ln \xi + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^\mu \xi^{-n} \tag{2}$$

which has conformal weight zero. One has found, however, that Eq. (1) is in fact valid for all the conformal operators R for which it has so far been tested. For the bosonic string, besides Q^μ itself these are

$$P^\mu(\xi) = i\frac{dQ^\mu(\xi)}{d\xi} \tag{3}$$

and

$$L(\xi) = \frac{1}{2} \lim_{\xi^1 \to \xi} \left\{ P^\mu(\xi) P^\nu(\xi^1) \eta_{\mu\nu} - \frac{D}{(\xi - \xi^1)^2} \right\}. \tag{4}$$

By considering the integral

$$V \oint_{C_i} d\xi^i R^i(\xi^i) \varphi, \tag{5}$$

where C_i is the contour around $\xi^i = 0$ and includes no other points where strings are emitted, and deforming the contour one derives [6–8] using Eq. (1) that

$$V\left\{\sum_{j=1}^{N} \oint d\xi^j \left(\frac{d\xi^j}{d\xi^i}\right)^{d-1} R^j(\xi^j)\varphi\right\} = 0, \tag{6}$$

provided φ has poles only at the points where strings are emitted. These latter equations are called *integrated overlap* equations.

Equation (1) for the fundamental conformal operator provides us with a definition of V. The above also applies to string scattering corresponding to surfaces of genus greater than zero, but in this case one has additional overlap equations, corresponding to A_n and B_n cycles which are discussed below and are required in the derivation of Eq. (6).

In this paper we show that the vertex V obeys a new type of overlap equations for the operator $\exp ip\cdot Q(\xi)$. Such a relation when integrated and applied to a special two string vertex, with restricted loop momentum, is shown to be none other than the Hirota equation, which determines the K–P equation. From this viewpoint the relation between solutions of the K–P equation and Riemann surfaces becomes intuitively clear. The result also implies that the special one string vertex is a solution to the K–P equation, a fact which is consistent with previous empirical observations [15, 12, 13] on the form of certain string vertices. Although we give the explicit form of the string vertices we note that the derivation depends only on the form of the overlap equations and does not require a knowledge of the details of the vertex.

2. Derivation of the Overlap Equation for the Operator $\exp ip\cdot Q(\xi)$

The operator

$$:\exp ip\cdot Q(\xi): = \exp - ip\cdot\left(\sum_{n=1}^{\infty} i\frac{\alpha_{-n}\xi^n}{n}\right) e^{ipq} z^{p\cdot\alpha_0} \exp ip\cdot\left(\sum_{n=1}^{\infty} i\frac{1}{n}\alpha_n \xi^{-n}\right) \tag{7}$$

is well known to be a conformal operator of dimension $\frac{p^2}{2}$ and might therefore be expected to obey Eq. (1). The object of this part of the paper is to prove this overlap equation for vertices, beginning with the relation

$$VQ^{\mu i}(\xi^i) = VQ^{\mu j}(\xi^j). \tag{8}$$

If $p^2 = 0$ there is no nomal ordering required, writing

$$\exp ip\cdot Q(\xi) = \sum_{n=0}^{\infty} \frac{(ip\cdot Q(\xi))^n}{n!},$$

and using Eq. (8) repeatedly we find that [10]

$$V \exp ip\cdot Q^i(\xi^i) = V \exp ip\cdot Q^j(\xi^i). \tag{9}$$

The case $p^2 \neq 0$ is more complicated to show, but we also begin by writing

$$\exp ip\cdot Q(\xi) = \sum_{n=0}^{\infty} :\frac{(ip\cdot Q(\xi))^n}{n!}: \tag{10}$$

and use the relation

$$Q^{\mu}(z)Q^{\nu}(w) = :Q^{\mu}(z)Q^{\nu}(w): - \eta^{\mu\nu}\ln(z-w) \tag{11}$$

to carry out the normal ordering.

It will prove advantageous to streamline our notation, we let $ip\cdot Q^i = \varphi, ip\cdot Q^j = \psi$ and $\xi^i = z, \xi^j = w$. An obvious consequence of Eq. (8) is that

$$V\varphi(z_1)\varphi(z_2)\cdots\varphi(z_n) = V\psi(w_n)\psi(w_{n-1})\cdots\psi(w_1). \tag{12}$$

For the case of two Q's, using Eq. (11), we find that

$$V:\frac{\varphi(z_1)\varphi(z_2)}{2}: = V:\frac{\psi(w_2)\psi(w_1)}{2}: + \frac{P^2}{2}\ln\left(\frac{w_1-w_2}{z_1-z_2}\right). \tag{13}$$

Taking the limit $z_1 \to z_2 = z$ under which $w_1 \to w_2 = w$, the equation becomes

$$V:\frac{\varphi(z)^2}{2}: = V:\frac{\psi(w)^2}{2}: + \frac{p^2}{2}\ln\left[\frac{dw}{dz}\right]. \tag{14}$$

The above equation is an example of a more general result;

$$V:\frac{\varphi(z)^n}{n!}: = V\sum_{l=0}{}' :\frac{\psi(w)^{n-2l}}{(n-2l)!}:\frac{1}{l!}\left(\frac{p^2}{2}\ln\left(\frac{dw}{dz}\right)\right)^l, \tag{15}$$

which we now prove by induction. The sum $\sum'$ as has its upper term such that the power of ψ is non-negative. When proving this relation we must normal order the $n\varphi$'s using the relation

$$\varphi(z_1)\cdots\varphi(z_n) = :\varphi(z_1)\cdots\varphi(z_n): + \sum_{\substack{i,j\\ i<j}}\prod_{\substack{k=1\\ k\neq i,}}^{n} :\varphi(z_k):p^2\ln(z_i-z_j)$$
$$+ \sum_{\substack{i,j\\ i<j}} p^2\ln(z_i-z_j)\cdot \sum_{\substack{k,l\\ k<l\\ k\neq i,j}} p^2\ln(z_k-z_l) \prod_{\substack{s\\ s\neq i,j,k,l}} :\varphi(z_s): + \cdots. \tag{16}$$

We may write this equation in the generic form

$$:\frac{\varphi^n}{n!}: = \sum_{l=0}{}' \frac{1}{(n-2l)!}\frac{1}{2^l}\frac{1}{l!}:\varphi^{n-2l}:(p^2\ln\Delta z)^l, \tag{17}$$

where the z arguments have been suppressed and Δz is some z difference.

Applying Eq. (17) to Eq. (12) we find that

$$V\sum_{l=0}{}' \frac{1}{(n+2-2l)!}\frac{1}{2^l}\frac{1}{l!}:\varphi^{n+2-2l}:(p^2\ln\Delta z)^l$$
$$= V\sum_{l=0}{}' \frac{1}{(n+2-2l)!}\frac{1}{2^l}\frac{1}{l!}:\psi^{n+2-2l}:(p^2\ln(\Delta w))^l. \tag{18}$$

Assuming Eq. (15) to hold for the integer n, we can use it to swop $:\varphi^p:$ for $:\psi^q:$ for $p \leqq n$ in the above equation;

$$V:\frac{\varphi^{n+2}}{(n+2)!} = V\sum_{l=0}{}' \frac{1}{(n+2-2l)!}\frac{1}{2^l}\frac{1}{l!}(p^2\ln(\Delta w))^l$$

$$- V \sum_{l=1}' \frac{1}{l!} \frac{1}{2^l} (p^2 \ln \Delta z)^l \sum_{n=0}' :\psi^{n+2-2l-2n}:$$

$$\cdot \frac{1}{(n+2-2l-2n)!} \frac{1}{n!} \frac{1}{2^n} \left(p^2 \ln\left(\frac{\Delta w}{\Delta z} \right) \right)^n . \tag{19}$$

Carrying out the sum and taking the appropriate limit we find that

$$V : \frac{\varphi^{n+2}}{(n+2)!} : = V \sum_{l=0}' : \frac{\psi^{n+2-2l}}{(n+2-2l)} \frac{1}{l!} \left(\frac{p^2}{2} \ln\left(\frac{dw}{dx} \right) \right)^l . \tag{20}$$

By induction Eq. (15) is true for all n. Summing on n we find the desired result,

$$V : e^{ip \cdot Q^i(\xi^i)} : = V e^{ip \cdot Q^j(\xi^j)} : \left(\frac{d\xi^j}{d\xi^i} \right)^{p^2/2} , \tag{21}$$

which holds for string scattering vertices corresponding to surfaces of any genus. The reader may verify that the use of the generic formulae above for arguments and differences is justified by the limiting procedure; essentially because $\frac{w_j - w_i}{z_j - z_i}$ gives $\frac{dw_i}{dz_i}$ in the limit $z_j \to z_i$ no matter what the initial value of z_j.

While for tree level (i.e. sphere) string scattering Eq. (8) determines the vertex; for loop string scattering a further overlap equation is required.

Let us choose a Schottky representation of the Riemann surface. Let $P_n, n = 1, .., g$, be the $SL(2, C)$ transformations whose isometric circle C_n is mapped under P_n into C'_n. The Riemann surface is the region exterior to the $2g$ circles. We refer the reader to Ref. [11] for discussions of the Schottky representation. The B_n cycles of the Riemann surface are any path connecting an arbitrary point z_0 to the point $P_n z_0$. If z_0 lies on the circle C_n, then $P_n z_0$ lies on the circle C'_n, while if z_0 lies outside all the circles $P_n z_0$ will lie inside C^1_n. The A_n cycle is any path encircling C_n. The required overlap equations for the multiloop vertex [7, 8] are

$$V Q^i(\xi^i) = V Q^i(P^i_n \xi^i) \tag{22}$$

and

$$V P^i(\xi^i) = V P^i(A^i_n \xi^i) \left(\frac{d(A^i_n \xi^i)}{d\xi^i} \right), \tag{23}$$

where P^i_n are the transformation P_n in the ξ^i coordinate system and similarly A^i_n for an A_n cycle. Differentiating Eq. (22) implies that

$$V P^i(\xi^i) = V P^i(P^i_n \xi^i) \frac{d(P^i_n \xi^i)}{d\xi^i} . \tag{24}$$

In the above and what follows we have suppressed the μ index on Q^μ and P^μ.

Using the overlap equations (8), (22) and (23), the multi-spring g loop string scattering vertex which was first found in the third work of Ref. (7) to be

$$V = \langle \mu | U^{\text{loop}} U^{\text{tree}}, \tag{25}$$

where

$$U^{\text{tree}} = \exp - \left\{ \sum_{i \leq i < j \leq N}^{N} \left\{ \sum_{n,m=1}^{\infty} a_n^i (\Gamma (V^i)^{-1} V^j)_{nm} a_m^j \right.\right.$$

$$+ \sum_{n=1}^{\infty} \frac{a_n^i}{\sqrt{n}} (\Gamma (V^i)^{-1} V^j(0))^n \alpha_0^j + \sum_{n=1}^{\infty} \alpha_0^i \frac{a_n^j}{\sqrt{n}} (\Gamma (V^j)^{-1} V^i(0))^n$$

$$+ \frac{1}{2} \left(\ln \left\{ \frac{d}{dz} (\Gamma (V^i)^{-1} V^i(z)) \right\} \right) \Bigg|_{z=0} \alpha_0^i \alpha_0^j, \tag{26}$$

$$U^{\text{loop}} = \exp \left[- \sum_{i,j=1} \left\{ \frac{1}{2} \sum_{n,m=1}^{\infty} a_n^i (\bar{E}^i (V^i)^{-1} V^j)_{nm} a_m^j \right.\right.$$

$$\left.\left. + \sum_{n,m=1}^{\infty} \alpha_0^i \frac{[(V^j)^{-1} V^i)(0)]^n}{\sqrt{n}} \bar{E}_{nm}^j a_m^j \right\} + \sum_{1 \leq i \leq j \leq N} \alpha_0^i \alpha_0^j \ln \chi^i ((V^i)^{-1} V^j(0), 0) \right], \tag{27}$$

where

$$\bar{E}_{nm}^j = \frac{1}{n!m!} \left(\frac{\partial}{\partial \xi^j} \right)^n \left(\frac{\partial}{\partial \zeta^j} \right)^m \ln \chi^j(\xi^j, \zeta^j) |_{\xi^i = \zeta^j = 0} \ln \chi^j(\xi^j, \zeta^j)$$

$$= - \ln \frac{E^j(\xi^j, \zeta^j)}{\xi^j - \zeta^j} - \frac{1}{2} \sum_{n,m=1}^{g} \left(\int_{\zeta^j}^{\xi^j} w_m \right) (\tau^{-1})_m{}^n \left(\int_{\zeta^j}^{\xi^j} w_n \right), \tag{28}$$

where

$$\alpha_n^\mu = \sqrt{n} a_n^\mu, n \geqq 1 \quad \text{and} \quad \langle \mu | = \prod_{i=1}^{N} \int dp_{ii} \langle 0, p_i | \tag{29}$$

and E^j is the prime form in the ξ^j coordinate system; in other words it involves $T_\alpha^j = (V^j)^{-1} T_\alpha^j V^j$ rather than T_α.

A more elegant expression can be found by recognising the appropriate Taylor expansion; and Eq. (26) and (27) can be written as

$$U^{\text{loop}} = \exp \frac{1}{2} \sum_{i,j} \left\{ \oint d\xi \oint d\zeta P^i(\xi) \ln \left\{ E \frac{(V^i(\xi), V^j(\zeta)}{(V^i(\xi) - V^j(\zeta))} \right\} P^j(\zeta) \right.$$

$$\left. - \frac{1}{2} \left(\oint d\xi P^i(\xi) \int_{z_0}^{V^i(\xi)} w_m \right) \left(\frac{1}{2\pi i} \right) (\tau^{-1})_m{}^n \left[\left(\oint d\zeta P^j(\zeta) \int_{z_0}^{V^j(\zeta)} w_n \right) \right] \right\}, \tag{30}$$

$$U^{\text{tree}} = \exp \left\{ \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{2} \oint d\xi \oint d\zeta p^i(\xi) \ln (V^i(\xi) - V^j(\zeta)) P^j(\zeta) \right.$$

$$\left. + \frac{1}{2} \sum_{i=1}^{n} \oint \frac{d\xi}{\xi} \alpha_0^i P^i(\xi) \ln V_i'(\xi) \right\}, \tag{31}$$

where $w_m; m = 1, \ldots, g$ are the g holomorphic differentials on the Riemann surface.

In Schottky representation these are given by

$$w_n(z) = \sum_{T_\alpha}^{(n)} \left\{ \frac{1}{z - T_\alpha(\alpha_n)} - \frac{1}{z - T_\alpha(\beta_n)} \right\} dz,$$

where T_α is any element of the Schottky group, $\sum^{(n)}$ mean a sum over all T_α except those that have P_n or P_n^{-1} as their right-most factor and α_n and β_n are the two fixed points of P_n. These differentials are normalized so that

$$\oint_{A_k} w_n = \delta_{nk}. \tag{33}$$

In this paper, all closed contour integrals are assumed to contain a $(2\pi i)^{-1}$ factor which is not explicitly shown, and, if no closed contour is indicated, it is assumed to be around the origin.

We define the Riemann matrix, as usual, by the integral

$$2\pi i \oint_{B_k} w_n \equiv B_{nk} \equiv 2\pi i \tau_{nk}. \tag{34}$$

It will be advantageous to consider string vertices before the loop momentum $k^n; n = 1, \ldots, g$ are integrated or summed; we denote such vertices by $V(k^n)$. The corresponding overlap equations are slightly different. As before, we adopt Eq. (8), and also Eqs. (23) and (24), but Eq. (22) is replaced by [12],

$$V \oint_{A_n} P^i(\xi^i) d\xi^i = V k^n \quad n = 1, \ldots, g, \tag{35}$$

where k^n is the loop momentum. We can regard this equation as a definition of k^n. Equation (35) can be rewritten as

$$V\{Q^i(\xi^i) - Q^i(A_n^i \xi^i)\} = -2\pi V k^n. \tag{36}$$

Using these overlap equations which completely determine $V(k)$ we can derive the corresponding integrated overlap equation, in the standard way. Given a function φ which is analytic everywhere except at the points where the strings are emitted and satisfies

$$\varphi(P_n z) = \varphi(z) + d_n, \varphi(A_n z) = \varphi(z) + c_n, \tag{37}$$

where c_n and d_n are constants, then we find that the analogue of Eq. (6) is given by

$$\sum_{n=1}^{g} d_n \oint_{A_n} V(k) P^i(\xi^i) d\xi^i - \sum_{n=1}^{g} c_n \oint_{B_n} V(k) P^i(\xi^i) d\xi^i = \sum_{j=1}^{N} V(k) \oint_{C_j} \varphi P^j(\xi^j) d\xi^j. \tag{38}$$

The derivation of this equation follows a discussion in the fourth paper in reference [7]. We have used the relation $P_n^i = (V^i)^{-1} P_n V^i$, and the fact that going counterclockwise around C_n leads under P_n to a clockwise motion around C_n'.

Using Eq. (35), this last equation becomes

$$\sum_{j=1}^{N} V(k) \oint_{C_j} \varphi P^j(\xi^i) d\xi^j = V(k) \sum_{n=1}^{g} k_n d_n - \sum_{n=1}^{g} c_n V(k) \oint_{A_n} P^i(\xi^i) d\xi^i. \tag{39}$$

Let us take the function

$$\varphi(\xi) = \int_{V^i(\xi_0^i)}^{V^i(\xi^i)} w_m = \int_{z_0}^{z} w_m \tag{40}$$

for which

$$d_n = 2\pi i \tau_{mn} \quad \text{and} \quad c_m = 2\pi i \delta_{n,m}. \tag{41}$$

Substituting into Eq. (39) we find that

$$2\pi i \oint_{B_n} V(k)P^i(\xi^i)d\xi^i = V(k)\left\{ + \sum_{m=1}^{g} k^m 2\pi i \tau_{mn} - \sum_{j=1}^{N} \oint d\xi^j P^j(\xi^j) \oint_{V^j(\xi_0^j)}^{V^j(\xi^j)} w_n \right\}$$

$$= iV(k)\{Q^i(P_n^i\xi^i) - Q^i(\xi^i)\}. \tag{42}$$

To determine $V(k)$, we take as our set of functions

$$\phi_n(z) = \frac{1}{n!}\frac{\partial^n}{\partial w^n} \ln E(z,w)|_{w=0}, \tag{43}$$

where $E(z, w)$ is the prime form:-

$$E(z,w) = \frac{1}{2}\frac{(z-w)}{\sqrt{dz\,dw}} \prod_{\substack{\alpha \\ \alpha \neq I}} \frac{(z - T_\alpha(w))\,(w - T_\alpha(z))}{(z - T_\alpha(z))\,(w - T_\alpha(w))}.$$

Although the prime form is inert under any A_n cycle it transforms as

$$E(P_n z, w) = -E(z,w)\exp\left\{ -\pi i \tau_{nn} - \int_w^z w_n \right\}$$

under a B_n cycle and hence for ϕ_n we have

$$c_m = 0; \quad d_m = A_n^{(m)} = \frac{1}{n!}\frac{\partial^n}{\partial w^n}\int_w^z w_m|_{w=0}. \tag{45}$$

The Taylor expansion of ϕ_n around the emission points of the strings can be taken as

$$\phi_n(\xi^i) = \begin{cases} \dfrac{1}{n}\dfrac{1}{(\xi^i)^n} + \sum\limits_{n=0}^{\infty} E_{nm}^{ii}(\xi^i)^m \text{ around } \xi^i = 0 \\[2ex] \sum\limits_{m=0}^{\infty} t_{nm}^{ij}(\xi^j)^m + \sum\limits_{m=0}^{\infty} E_{nm}^{ij}(\xi^j)^m \text{ around } \xi^j = 0 \end{cases}, \tag{46}$$

where

$$\frac{1}{n}\frac{1}{(\xi^i)^n} = \sum_{m=0}^{\infty} t_{nm}^{ij}(\xi^j)^m = \sum_{m=1}^{\infty} \frac{(\Gamma(V^i)^{-1}V^j)_{nm}}{\sqrt{nm}}(\xi^j)^m + \frac{(\Gamma(V^i)^{-1}V^j(0)^n)}{n} \tag{47}$$

around $\xi^j = 0$.

Substituting the ϕ_n of Eq. (39), the multi-string g loop vertex is

$$V(k) = \langle \mu | U^{\text{loop}}(k) U^{\text{tree}}, \tag{48}$$

where U^{tree} is as in Eq. (31) and

$$U^{\text{loop}}(k) = \exp\left\{ -\frac{1}{2}\sum_{n,m=0}^{\infty} \sum_{i,j} \alpha_n^i E_{nm}^{ij} \alpha_m^j \right\}$$

$$\cdot \exp\left\{ \frac{1}{2}\sum_{m,n=1}^{g} k^m 2\pi i \tau_{mn} k^n \right\} \exp\left(-\sum_{n=1}^{\infty}\sum_{m=1}^{g} \alpha_n^i A_n^{i(m)} k^m \right) f(k^n). \tag{49}$$

The $\alpha_0^i \alpha_0^j$ piece is not determined by the above calculation, but is found from Eq. (8) as in Ref. [7]. The dependence on k^m is also not completely determined as one can multiply U^{loop} by an arbitrary function $f(k^m)$, as indicated, and it is still a solution of all the overlap conditions.

Using the appendix of Ref. [13] we can rewrite U^{loop} as

$$U^{\text{loop}}(k) = \exp\left\{\frac{1}{2}\sum_{m,n=1}^{g} k^m 2\pi i\tau_{mn} k^n - \sum_{m=1}^{g}\sum_{i=1}^{N} \oint d\xi P^i(\xi) \int_{z_0}^{V^i(\xi)} w_m k^m \right.$$
$$\left. + \frac{1}{2}\sum_{i,j=1}^{N} \oint d\xi \oint d\zeta P^i(\zeta) \ln\left\{\frac{E(V^i(\xi), V^j(\zeta))}{(V^i(\xi) - V^j(\zeta))}\right\} P^j(\zeta)\right\} f(k^n). \tag{50}$$

We note that provided we take the additional function $f(k) = 1$ in the vertex $V(k)$, then Eq. (41) can be written as

$$V(k)\{-Q^i(\xi^i) + Q^i(P^i_n \xi^i)\} = -i\frac{\partial}{\partial k^n} V(k). \tag{51}$$

When k^n are real numbers, as for the bosonic string, we may integrate the vertex with respect to the loop momentum, where upon the right-hand side of Eq. (51) vanishes and we recover Eq. (22) as we should. One can reverse the above steps and demand Eq. (22) for the integrated vertex. This would then fix the loop momentum dependence of the vertex as it is in Eq. (49) or (50) with $f(k) = 1$.

Since the vertices $V(k)$ obey Eq. (8) they will also obey Eq. (21). For later considerations we will require the analogue of Eq. (21) for A_n and B_n cycles. Equations (36) and (51) can, using Eq. (8), be written in the form

$$V(k) Q^i(\xi^i) = V(k)(Q^j(\zeta^j) + \Delta), \tag{52}$$

where for an A_n cycle, $\zeta^j = A^j_n \xi^j$ and $\Delta = -2\pi k^n$, but for a B_n cycle, $\zeta^j = P^j_n \xi^j$ and $\Delta = i\frac{\overleftarrow{\partial}}{\partial k_n}$.

We note that Δ commutes with Q^i and Q^j. Examining the previous argument and identifying $\varphi = ip\cdot Q^i, z = \xi^i, \psi = ip\cdot(Q^j + \Delta)$ and $w = \zeta^j$ the previous argument leads to the results

$$V(k) e^{ip\cdot Q^i(\xi^i)} = V(k) e^{ip\cdot Q^i(A^i_n \xi^i)} \exp(-2\pi i p\cdot k^n)\left(\frac{d(A^i_n \xi^i)}{d\xi^i}\right)^{p^2/2} \tag{53}$$

and

$$V(k) e^{ip\cdot Q^i(\xi^i)} = V(k) e^{ip\cdot Q^i(P^i_n \xi^i)} \exp\left\{-p\frac{\overleftarrow{\partial}}{\partial k^n}\right\}\left(\frac{d(A^i_n \xi^i)}{d\xi^i}\right)^{p^2/2} \tag{54}$$

after using Eq. (21). The extra factor (i.e. $\exp(-2\pi i p\cdot k^n)$ in Eq. (53) will vanish if

$$p\cdot k^n \in Z. \tag{55}$$

The effect of the additional factor in Eq. (54) is to shift k^m to $k^m - p\delta_{m,n}$. Upon summing over k^m this factor will disappear by reordering the sum if $k^m - p\delta_{n,m}$ is in the same set as k^m, i.e. if $k^m \in S$ then

$$k^m - \delta_{m,n} p \in S. \tag{56}$$

For the usual bosonic string $k^m \in \mathbb{R}$ and clearly (56) holds for any $p \in \mathbb{R}$. Provided Eq. (55) and (56) hold then for the summed or integrated vertex V, $Ve^{ip \cdot Q^i(\xi^i)}$ is inert under all A_n and B_n cycles and in this case we can integrate Eq. (21) to find the corresponding integrated overlap equation,

$$\sum_{j=1}^{N} \oint V e^{ip \cdot Q^j(\xi^j)} d\xi^j \varphi = 0, \tag{57}$$

where φ is any form of degree $\left(\frac{p^2}{2} - 1\right)$ having poles only at string emission points. A case which will be of interest to us is when $p^2 = 2$ for which φ is a function, a particularly important example being $\varphi = 1$.

The vertex with only one leg $V^{(1)}(k)$, for the local coordinate choice $V^i(z) = z$, can be written in the form

$$V^{(1)}(k) = \langle \mu | \exp\left\{\frac{1}{2} \sum_{m,n=1}^{g} k^m 2\pi i \tau_{mn} k^n - \sum_{m=1}^{g} \oint dz P(z) \int_{z_0}^{z} w_m \right. \\ \left. + \frac{1}{2} \oint dz \oint dy P(z) \ln\left\{\frac{E(z,y)}{(z-y)}\right\} P(y) \right\}. \tag{58}$$

The identities of Eqs. (53) and (54) will hold regardless of what choice of k^m we take. Let us in particular assume that the space-time has two Euclidean dimensions (i.e. the μ indices have been suppressed).

Let us also suppose that

$$k^m_\mu = n^{(m)}_\mu + \alpha^{(m)}, n^{(m)}_\mu \in \mathbb{Z}, \alpha^{(m)} \in \mathbb{R}. \tag{59}$$

Now if $p^\mu = (p, -p)$, then $p.k = p(n^{(m)}_1 - n^{(m)}_2)$ and Eq. (55) will hold if $p \in \mathbb{Z}$. Further, if we sum the vertex over all such k^μ then $k^m_\mu - \delta^{m,p} p_\mu$ is of the same form as k^m_μ and Eq. (56) holds. If we now demand that $p^2 = 2$ then the only possible value for p^μ is up to a sign, $p^\mu = (1, -1)$.

As a consequence, taking the function $\varphi = 1$, we may conclude that

$$\oint d\xi V\{e^{iQ^1(\xi)} - e^{iQ^2(\xi)}\} = 0, \tag{60}$$

where

$$V = \sum_{k^m \in S \oplus S} V(k). \tag{61}$$

3. Relation to the Hirota Equation

The special example of an integrated overlap of Eq. (60) is none other than the Hirota Bilinear equation [14]. We now explain this connection.

Consider a one string vertex, denoted $\bar{V}(k)$, for a string propagating in one dimensional space whose loop momentum are as in Eq. (59). We take the scalar product of such a summed vertex with the coherent state

$$|t\rangle = \exp\left\{\sum_{n=1}^{\infty} t_n \alpha_{-n}\right\} |10,0\rangle \tag{62}$$

and define the function

$$\bar{\tau}(t) = \sum_{k \in S} \bar{V}(k)|t\rangle. \tag{63}$$

The function $\bar{\tau}(t)$ is then given by Eq. (58) with the substitution,

$$P(z) = \sum_{n=1}^{\infty} n t_n z^{-n} + \cdots, \tag{64}$$

where $+ \cdots$ stands for derivatives of t which do not contribute in $\bar{V}(k)$. We observe that the vertex of Eq. (61) in terms of coherent states is

$$V|t_1\rangle_1 |t_2\rangle_2 = \bar{\tau}(t_1)\bar{\tau}(t_2), \tag{65}$$

where the 1, 2 indices refer to the two components of α_n^μ. The operator $Q^\mu(z)$ in a coherent state basis takes the form

$$Q(z) = q - i\alpha_0 \ln z + i \sum_{n=1}^{\infty} t_n z^{-n} - i \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\partial}{\partial t_n}. \tag{66}$$

Taking into account momentum conservation, we saturate Eq. (60) with the state $\exp \sum_{n=1}^{\infty} \frac{\alpha_{-n} t_n}{n} |0, -p\rangle$ to produce its coherent state analogue:

$$\oint_{\infty} du \exp\left\{ \sum_{n=1}^{\infty} (t_n^2 - t_n^1) u^n \right\} \exp\left\{ - \sum_{n=1}^{\infty} \frac{u^{-n}}{n} \left(\frac{\partial}{\partial t_n^2} - \frac{\partial}{\partial t_n^1} \right) \right\} \bar{\tau}(t_1)\bar{\tau}(t_2) = 0. \tag{67}$$

This equation is clearly recognisable as the Hirota equation. In carrying out this last step we have made the replacement $u = \frac{1}{\xi}$.

In fact $\bar{\tau}(t)$ is not quite the τ function which is often defined by

$$\tau(t) = \exp\left\{ \frac{1}{2} \oint dz \oint dw\, t(z) \ln\left\{ \frac{E(z,w)}{z-w} \right\} t(w) \right\} \theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\oint dz \int_{z_0}^{z} w_m t(z) \right), \tag{68}$$

where $t(z) = \sum_{N=1}^{\infty} n t_n z^{-n}$. Recalling that

$$\theta\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \sum_{N^m \in \mathbb{R}} \exp\left\{ \frac{1}{2} \sum_{m,n=1}^{g} (N^m + \alpha^m)(2\pi i \tau_{mn})(N^n + \alpha^n) + \sum_{n=1}^{g} (N^n + \alpha^n)(z^n + 2\pi i \beta^n) \right\} \tag{69}$$

and identifying $k^m = N^m + \alpha^m$ we recognise that the object in the sum is $\bar{V}(k)$ times the additional factor

$$\exp \sum_{m=1}^{g} (N^m + \alpha^m) 2\pi i \beta^m \tag{70}$$

which does not depend on t_n or equivalently α_n^μ. Clearly, the overlaps of Eqs. (36) and (42) are still valid for the corresponding vertex with this additional factor, but when writing this latter equation in the form of Eq. (51) we must replace $\frac{\partial}{\partial k^n}$ by

$\frac{\partial}{\partial k^n} + 2\pi i\beta^n$. It is straightforward to verify that the conditions of Eqs. (55) and (56) are met and the Hirota equation also holds if we replace $\bar{\tau}$ by τ.

It has been recognised as an empirical observation that the one string vertex for the closed string [15, 12, 13] on a torus is related to the τ function. In general, we find a sum of terms bilinear in τ functions of different kinds [12], however for one particular choice of radius the sum reduces to one term. Within such a context the Hirota equation was discussed in Ref. [15] and also within the context of the relation between Sato's Grassmannian approach and free fermions on a Riemann surface of reference [16].

For the sake of completeness we sketch the well known path from the Hirota equation to the K–P hierarchy. Changing variables by $t_1 = x - y, t_2 = x + y$ and introducing the Schur polynomials S_k,

$$\sum_{k\in\mathbb{Z}} S_k(x)z^k = \exp \sum_{k=1}^{\infty} x_k z^k, \tag{71}$$

Eq. (67) becomes

$$\sum_{j=0}^{\infty} S_j(-2y)S_{j+1}(\bar{\partial}_y)\tau(x-y)\tau(x+y) = 0, \tag{72}$$

where

$$\bar{\partial}_y = \left\{\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \ldots\right\}. \tag{73}$$

Changing the y dependence in the τ functions to $y \to y + u$ and differentiating with respect to u instead of y we find

$$\sum_{j=0}^{\infty} S_j(-2y)\exp\left(\sum_{s=1}^{\infty} y_s \frac{\partial}{\partial u_s}\right)S_{j+1}(\bar{\partial}_u)\tau(x-u)\tau(x+u)|_{u=0} = 0. \tag{74}$$

This equation is a polynomial in $y_s; s = 1, 2, \ldots$, and each coefficient must therefore vanish leading to a hierarchy of equations. The first such non-trivial equation occurs as the coefficient of y_3, and upon substituting

$$u = 2\frac{\partial^2}{\partial x_1^2}\ln\tau, \tag{75}$$

it is found to be the K–P equation.

It is straightforward to define τ functions associated with vertices involving more than one external string. We begin by constructing the corresponding Hirota equation. Let us consider in two Euclidean dimensions a N string vertex V whose loop momentum are such that Eq. (57) holds.

Let us consider $\varphi = 1$, that is the equation

$$\sum_{j=1}^{N} \oint V e^{ip\cdot Q^j(\xi^j)} d\xi^j = 0, \tag{76}$$

and apply to V the coherent state $|t\rangle_1$ on leg one, but states $|\psi_j\rangle_j$ on the other legs such that

$$\oint d\xi^j{:}e^{ip\cdot Q(\xi^j)}{:}|\psi_j\rangle_j = 0. \tag{77}$$

A particular, example of such a state is the tachyon state $|0, p_j\rangle$, provided $p \cdot p_j \in \mathbb{Z}$ and $p \cdot p_j \geqq -1$. In view of the forthcoming decomposition of the two dimensional vertex into the one dimension vertex used to define the τ functions we let $p_j^\mu = (p_j, p_j)$ in which case $p \cdot p_j = 0$. Applying the states $|0, p_j\rangle, j = 2, \ldots, N+1$ and the coherent state on leg one we recover the Hirota equation (60) for the τ function,

$$\bar{\tau}_N(t) = V|t\rangle_1 \sum_{j=2}^{N+1} |0, p_j\rangle, \tag{78}$$

where V is the one dimensional vertex.

Momentum conservation implies that $\sum_{j=2}^{N} p_j = 0$. Examining the vertex of Eq. (31) which for convenience is simplest when $\xi^i = (V^i)^{-1}(z) = (z - z^i)$, one sees that having $2M$ tachyonic states $|0, p_i\rangle$ is equivalent to starting with a one leg vertex and inserting $\prod_{i=2}^{N} e^{ip \cdot Q^i(z_i)}$ on the external leg.

One way to ensure momentum conservation is to take $N = 2M + 1$ and add external states pairwise with momenta $p_{2j} = -1, p_{2j+1}, j = 1, \ldots, M$. Adding only one such pair is equivalent to inserting the operator

$$:e^{-iQ(z_2)}::e^{iQ(z_3)}: = \frac{1}{(z_2 - z_3)}:e^{i(Q(z_2) - Q(z_3))}: \tag{79}$$

and the new $\bar{\tau}_2$ function of equation (78) is related to $\bar{\tau}$ by

$$\bar{\tau}_2(t) = \frac{uv}{v-u} \exp\left\{\sum_{n=1}^{\infty} t_n(u^n - v^n)\right\} \exp\left\{-\sum_{n=1}^{\infty}\left(\frac{u^{-n}v^{-n}}{n}\right)\frac{\partial}{\partial t_n}\right\}\bar{\tau}(t).$$

One of the simplest M soliton solutions is to take the tree vertex of Eq. (31) where $N = 2M + 1$ and saturate pairwise. Taking the simple cycling $\xi^1 = (V^i)^{-1}z = (z - z^i)$ one readily finds that the resulting τ function is

$$\bar{\tau}_N = \exp \sum_{j=1}^{M} \sum_{n=1}^{\alpha} t_n(u_j^n - v_j^n) \prod_{1 \leq i < j \leq M} \frac{(u_j - u_i)(v_j - v_i)}{(v_j - u_i)(u_j - v_i)}, \tag{80}$$

where $z_{2j} = \frac{1}{u_j}, z_{2j+1} = \frac{1}{v_j} j = 1, \ldots, M$ in agreement with the well-known M soliton solutions corresponding to a sphere.

Since the τ functions are defined in terms of the g-loop N string vertex they will inherit the properties of the string vertex. In particular, the τ function will obey overlap equations and their integrated analogues for all conformal operators for which they are valid. The resulting equations are readily found by the replacements $\alpha_n^\mu \to nt_n$ and $\alpha_{-n}^\mu \to \frac{\partial}{\partial t_n}, n \geqq 1$ in the relevant expressions for $Q(z), P(z)$ and $L(z) \equiv \frac{1}{2}:P(z)^2:$. We can write such equations for the unintegrated overlap equations for the integrated vertex. Recalling that $k^m = N^{(m)} + \alpha^{(m)}$ and inserting the factor of β, Eqs. (51) and (36) for the integrated vertex are

$$V(Q^i(\xi^i) - Q^i(P_n^i \xi^i)) = i\left(\frac{\partial}{\partial k^n} + 2\pi\beta^n\right)V \tag{81}$$

and

$$V(Q^i(\xi^i) - Q^i(A^i_n\xi^i)) = -i\frac{\partial V}{\partial \beta^n}. \tag{82}$$

Taking one leg and inserting the coherent state we obtain the above two equations, but with the replacements mentioned above and $V \to \tau(t)$. From the point of view of this paper it is natural to take the above equations to define the τ functions. Differentiating with respect to ξ^i we find that $P^i(\xi^i)$ obeys

$$VP^i(\zeta^i) = VP^i(\zeta^i)\frac{d\zeta^i}{d\xi^i}, \tag{83}$$

where $\zeta^i = A^i_n\xi^i$ or $\zeta^i = P^i_n\xi^i$ depending on which equation we take. These equations in turn lead to an equation on the τ function. As a consequence, one can derive the overlap equations for $L(\xi)$ of Eq. (4), using the same techniques as we used for Eq. (21). One finds

$$VL^i(\xi^i) = VL^j(\zeta^j)\left(\frac{d\zeta^j}{d\xi^i}\right)^2 \tag{84}$$

as well as the analogous equations for A_n and B_n cycles and the integrated overlap equation

$$V\sum_{j=1}^{N} \oint d\xi^j L^j(\xi^j)\varphi\left(\frac{d\xi^j}{d\xi^i}\right)^{d-1} = 0. \tag{85}$$

Since $L(\xi)$ is not quite a conformal operator, the above equation can, for surfaces other than the sphere, acquire constants; we refer the reader to ref. [7] for a discussion of these overlap equations. Taking only one leg, and placing upon it a coherent state, we find the τ function obeys the Virasoro conditions

$$\tau\{\oint dz L(z)\varphi(z) + \text{constants}\} = 0, \tag{85}$$

where φ is a vector field which is analytic except at the point where the string is emitted. Such equations and their interpretation in terms of induced representations are the subject of Ref. [9]. The τ function will also inherit the modulus chainging equations of the one vertex. For string vertices the Virasoro condition equations are simplified by the introduction of ghosts into the vertex and L_n's which then become true conformal operators. One can imagine introducing Grassmann fields into the τ function by sandwiching the vertex with ghosts with a corresponding coherent state to obtain a τ function with simpler Virasoro properties.

In a situation for which a W algebra is present one will find by the same logic that the vertex, and so the τ function, will obey W conditions analogous to those of the Virasoro conditions.

4. Conclusion and Discussion

In this paper, we have shown that the string vertices obey the overlap equation (21), (53) and (54) for the conformal operator $:\exp ip\cdot Q(z):$. In view of the other known examples it would seem likely that the string vertex satisfies overlap equations for any conformal operator. The overlap equation for $:\exp ip\cdot Q(z):$ is likely to be useful in the many contexts in which this operator arises such as the coulomb gas representation of minimal models and the symmetries of compactified strings. The latter is discussed in Ref. [10].

As we have seen, a particular use of this overlap equation leads to the Hirota equation once we define τ functions in terms of certain string vertices. Although, we have given the explicit form of the vertices, the derivation of the Hirota equation does not rely on the explicit form of the vertex, but follows in a rather straightforward way from the overlap equations which define the vertex. Thus, given any Riemann surface, the overlap equations provide us with a string vertex which in turn allows us to define a τ function which automatically satisfies the Hirota equation.

The overlap equations for the g loop, N string vertex can be thought of as arising from the three string vertex out of which the former vertex can be constructed by repeated sewing. The overlap equations for the three vertex may be envisaged as a result of the fact that strings interact by joining at their end points and identifying the third string from the original two at the moment they join. The sewing procedure leads in an obvious way to overlap equations for the resulting vertex [7, 17]. Since the sewing procedure corresponds to the joining of "pants" to build up the Riemann surface, the relation between Riemann surfaces and the Hirota equation becomes intuitively self evident through the use of the overlap equations.

It might be hoped that the string vertices can play a useful means of deriving other results in Riemann surface theory. We observe that given any string vertex, we can derive its overlap equations and so construct the Riemann surface to which it corresponds. Certain aspects of the Riemann surface are readily apparent from the vertex point of view. One such aspect is the moduli of the surface which are a simple consequence of the sewing procedure. Another possible application may be a vertex derivation of the θ function identities using sewing and the overlap identities.

The derivation given in this paper provides a setting for the Hirota equation within a formalism which would seem more general than that of the Grassmannian approach, in that it involves more general vertex operators than those associated with fermions. One may hope that it may lead to generalisations of the Hirota equation. In view of the use of the g loop vertex with one arbitrary leg and an even number of tachyons, it might be desirable to consider the full N string g-loop vertex which for the appropriate internal momenta will satisfy Eq. (57) and define an extended "τ" function of $t_n^i, i = 1, \ldots, N$ and the momentum p^i. When saturating Eq. (57) with states, care must be taken with momentum conservation; in particular, the states momenta $p^j, j = 1, \ldots, N$ must sum to $-p$. It would seem that the resulting Hirota type of equation will involve "τ functions" at different momenta, although the number of different values of momenta involved may be chosen to be small.

Another approach would be to consider a compactified string with left moving states only on a lattice which is a direct sum of two identical lattices. Such a requirement is necessary for the vertex involved in the Hirota type of equation to split into the product of two vertices.

Yet one further avenue of development concerns relating other aspects of K–P equations to string theory. While the wave function $w(t_n, z^{-1})$ has a simple interpretation in terms of a two string vertex, one of which is saturated with a coherent state $|t_n, 0\rangle$ and the other is saturated with a tachyon state at point z. The wave function is normalised such that at $z = \infty$ it is one and so can be thought of as the partition function for the tachyon state in the presence of a background coherent state. It is obvious, from this viewpoint, that z is a point on the Riemann surface; less obvious is the role of the pseudo-differential operator Q and the corresponding formalism.

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