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Classical $N = 1$ W-Superalgebras from Hamiltonian Reduction

José M. Figueroa-O'Farrill* and Eduardo Ramos**

Instituut voor Theoretische Fysica, Universiteit Leuven, Celestijnenlaan 200D, B-3001 Heverlee, Belgium

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Abstract. A combinatorial proof is presented of the fact that the space of supersymmetric Lax operators admits a Poisson structure analogous to the second Gel'fand–Dickey bracket of the generalized KdV hierarchies. This allows us to prove that the space of Lax operators of odd order has a symplectic submanifold – defined by (anty)symmetric operators – which inherits a Poisson structure defining classical W-superalgebras extending the $N = 1$ supervirasoro algebra. This construction thus yields an infinite series of extended superconformal algebras.

1. Introduction

The study of W-algebras is becoming increasingly relevant in two-dimensional conformal field theory, string theory, and quantum gravity and a lot of the progress in the study of both their classical and quantum versions arises from its connections with the theory of integrable models.

Although quantum W-algebras first make their appearance in the important paper [1] of A. B. Zamolodchikov on extensions of the conformal symmetry of two-dimensional statistical mechanics models, classical W-algebras had already appeared as somewhat exotic hamiltonian structures [2] for the generalized KdV hierarchies. In fact, it was Magri [3] who discovered that the KdV hierarchy was bi-hamiltonian: the second bracket defining a classical version of the Virasoro algebra. The analogous statement of the n^{th} order KdV hierarchy (KdV being $n = 2$) involves the so-called W_n algebra as the "second hamiltonian structure." Efforts to understand the second hamiltonian structure culminated with the discovery by Kupershmidt and Wilson [4, 5] (based on earlier work for the KdV equation by Adler and Moser [6]) of the fact that the second hamiltonian structure was induced

* e-mail: fgbda11@blekul11.BITNET, fgbda11@cc1.kuleuven.ac.be. Address after October 1991: Physikalisches Institut der Universität Bonn, FRG
** e-mail: fgbda06@blekul11.BITNET, fgbda06@cc1.kuleuven.ac.be

from a vastly simpler one via a generalisation of the celebrated Miura transformation of the KdV theory. This fact proved instrumental in the quantization, by Fateev and Lykyanov [7], of the W_n algebras, since the Miura transformation basically gives – at the classical level – a free field realization for the relevant W-algebra.

Further progress in W-algebras came from the seminal work of Drinfel'd and Sokolov [8] who related these integrable systems to affine algebras and who, following previous work of Reiman and Semenov–Tyan–Shanskiĭ [9], proved that the second hamiltonian structure was induced from the natural hamiltonian structure in the dual of an affine algebra via symplectic reduction by the Poisson action of a unipotent subgroup of the corresponding loop group. This result has had profound importance in the study of W-algebras. On the one hand, it has served to construct many other classical W-algebras: one associated to roughly each affine Lie algebra; but also to obtain important information about the quantum algebras as, for example, the discrete series associated to their minimal models [10]. Part of the work of Drinfel'd and Sokolov consists in the construction of an integrable hierarchy associated to the loop algebra of the general linear algebra out of which several other hierarchies could be obtained by reduction. In particular, the n^{th} order KdV hierarchy is the hierarchy associated to the A_n series of the Cartan classification. Other reductions yield hierarchies associated to the B_n and C_n series, and further reductions yield hierarchies associated to exceptional Lie algebras like G_2. All these hierarchies have their associated W-algebras which extend the Virasoro algebra by primary fields of dimensions equal to the exponents of the relevant Lie algebra. Not much is known in the quantum [11] case about these algebras, except for the B_2 and G_2 cases which correspond to the unique extensions of the Virasoro algebra by a primary field of spin 4 (see [12, 13]) and spin 6 (see [14]), respectively. It should be remarked that those algebras termed WB_n in the physics literature [15–17] are not the ones associated to B_n by the Drinfel'd–Sokolov scheme but contain an extra primary field of half-integral weight. These algebras do seem to be obtainable à la Drinfel'd–Sokolov, but from Lie superalgebras [18].

The connection between W-superalgebras and integrable systems is, however, mostly unexplored – most of the research done so far centering itself around the study of W-superalgebras as extended symmetries in superconformal field theory [19, 20]. Nevertheless, first steps in this direction have been made by the Komaba group [21], who – via super Toda field theory – has associated W-superalgebras to the Lie superalgebras $Sl(n+1|n)$ and $Osp(2n \pm 1|2n)$; and, more recently, by the authors [22] who have defined a hamiltonian structure in the space of supersymmetric Lax operators yielding a supersymmetric version of the Gel'fand–Dickey brackets for the generalized KdV hierarchies. This hamiltonian structure also arises from a much simpler one via a supersymmetric Miura transformation, a fact proven in [22]. The W-superalgebras arising in this way are the supersymmetric analogues of the ones Drinfel'd–Sokolov associated to the general linear algebra which prompts, naturally, the investigation of its possible reductions. A reduction similar to that yielding W_n in the bosonic case has been constructed in [23] via a small modification of the proof in [22] of the supersymmetric analogue of the Kupershmidt–Wilson theorem. The resulting algebras turned out to be the extensions of the $N=2$ supervirasoro algebra which the Komaba group had associated to $Sl(n+1|n)$. In this paper we investigate further reductions, analogous to the ones yielding the W-algebras of Drinfel'd–Sokolov associated to the B_n and C_n series. We find that they exist only for supersymmetric Lax operators of odd order and that they correspond to the extensions of the $N = 1$ supervirasoro algebra associated to $OSp(2n \pm 1|2n)$ by the Komaba group. In order to prove that these reductions exist, we make use of a combinatorial proof of the fact that the structure defined in [22] on the space of supersymmetric Lax operators is indeed hamiltonian. This proof follows closely the proof of Gel'fand and Dickey of the hamiltonian nature of the Adler mapping [2, 24].

This paper is organized as follows. In Sect. 2 we describe the formal geometry of the space of supersymmetric Lax operators and set up the formalism that will allow us to do differential calculus on it. In Sect. 3 we prove the hamiltonian nature of the supersymmetric version of the Adler mapping. In Sect. 4 we describe the symmetric reductions, after introducing the notion of an adjoint in the space of formal super-pseudo-differential operators. Finally in Sect. 5 we offer some concluding remarks.

2. The Space of Supersymmetric Lax Operators

In this section we review the general formalism introduced in [22] (see also [23]) concerning the differential calculus in the space of sypersymmetric Lax operators.

Supersymmetric Lax operators are special differential operators on a $(1|1)$ superspace with coordinates (x, θ). Let $\mathscr{F}$ denote the ring of superfields. It is a supercommutative $\mathbb{Z}_2$-graded ring. On $\mathscr{F}$ we can define an odd superderivation $D = \partial_\theta + \theta\partial$, the supercovariant derivative, which obeys $D^2 = \partial$. We define the ring $\mathscr{F}[D]$ of differential operators as polynomials in D with coefficients in $\mathscr{F}$. A differential operator $L \in \mathscr{F}[D]$ is called a supersymmetric Lax operator (of order n) if it is homogeneous (under the $\mathbb{Z}_2$ grading) and has the form $L = D^n + U_{n-1}D^{n-1} + \cdots + U_0$. The homogeneity condition simply states that $|U_i| \equiv n + i \bmod 2$. We let M_n denote the space of supersymmetric Lax operators of order n. When there is no chance for confusion we will simply call it M. This space can be given the structure of an infinite-dimensional supermanifold, but we will not need this machinery. It turns out that it is sufficient for our purpose to endow M with a "formal" geometry, i.e., an algebraization of the necessary geometric concepts. Since our ultimate aim is to define Poisson brackets on M, we need to specify several geometric objects: the class of functions on which we will define the Poisson brackets, the vector fields and 1-forms, and the map sending a function to its associated hamiltonian vector field.

We will define Poisson brackets on functions of the form:

$$F[L] = \int_B f(U), \tag{2.1}$$

where $f(U)$ is a homogeneous differential polynomial of the U and $\int_B$ is defined as follows: if $U_i = u_i + \theta v_i$, and $f(U) = a(u,v) + \theta b(u,v)$, then $\int_B f(U) = \int b(u,v)$, where the precise meaning of integration will depend on the context. It denotes integration over the real line if we take the u_i and v_i to be rapidly decreasing functions;

integration over one period if we take them to be periodic functions; or, more abstractly, a linear map annihilating derivatives so that we can "integrate by parts," i.e., denoting by $\mathscr{A}$ the differential polynomials in the coefficients of L, $\int_B$ is simply the natural surjection $\mathscr{A} \to \mathscr{A}/D\mathscr{A}$. It is worth remarking that whereas $\mathscr{A}$ is a graded supercommutative algebra, $D\mathscr{A}$ is not a subalgebra and hence the multiplication in $\mathscr{A}$ does not get induced in the quotient. This means, in particular, that it will not make sense to demand of our Poisson brackets to satisfy the usual derivation property. This, fortunately, does not affect the formalism.

The tangent space T_LM to M at L is isomorphic to the infinitesimal deformations of L. These are clearly the differential operators of order at most $n-1$ whose coefficients are in $\mathscr{A}$ or, more generally, in $\mathscr{F}$. If $A = \sum A_i D^i$ is one such operator and if $F[L] = \int_B f$ is a function, then the vector field D_A defined by A is given by

$$D_A F = (-1)^{|A|+n} \int_B \sum_{k=0}^{n-1} A_k \frac{\delta f}{\delta U_k}, \tag{2.2}$$

where the variational derivative is defined by

$$\frac{\delta}{\delta U_k} = \sum_{i=0}^{\infty} (-1)^{|U_k|i + i(i+1)/2} D^i \frac{\partial}{\partial U_k^{[i]}}, \tag{2.3}$$

with $U_k^{[i]} = D^i U_k$. One can check that D_A is well defined.

We expect the cotangent space T_L^*M to M at L is defined as the dual space of T_LM. To define this we introduce super-pseudo-differential operators (SΨDO's) [25]. We introduce a formal inverse D^{-1} to D and define the ring $\mathscr{F}((D^{-1}))$ of SΨDO's as formal Laurent series in D^{-1} where the composition law is given by

$$D^k \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} k \\ k-i \end{bmatrix} (-1)^{|\Phi|(k-i)} \Phi^{[i]} D^{k-i}, \tag{2.4}$$

where the superbinomial coefficients are defined by

$$\begin{bmatrix} k \\ k-i \end{bmatrix} \equiv \begin{cases} 0 & \text{for} \quad i < 0 \text{ or } (k,i) \equiv (0,1) \pmod 2; \\ \begin{pmatrix} \left[\frac{k}{2}\right] \\ \left[\frac{k-i}{2}\right] \end{pmatrix} & \text{for} \quad i \geqq 0 \text{ and } (k,i) \not\equiv (0,1) \pmod 2. \end{cases} \tag{2.5}$$

We shall abbreviate the ring of SΨDO's by $\mathscr{S}$. The space $\mathscr{S}_+ = \mathscr{F}[D]$ of differential operators forms a subring, and its complement is given by the subring $\mathscr{S}_- = D^{-1}\mathscr{F}[[D^{-1}]]$ of formal "integral" operators. We shall indicate by $\pm$ subscripts the projections of a SΨDO to these two complementary subrings, whence according to $\mathscr{S} = \mathscr{S}_+ \oplus \mathscr{S}_-$, a S$\Psi$DO P breaks up as $P_+ + P_-$. Given a SΨDO $P = \sum p_i D^i$ we define its super-residue as sres $P = P_{-1}$ and its (Adler) supertrace as Str $P = \int_B$ sres P. It can be shown that the supertrace vanishes on graded commutators: Str $[P, Q] = 0$, for $[P, Q] \equiv PQ - (-1)^{|P||Q|} QP$. This then defines a supersymmetric bilinear form on SΨDO's. $\operatorname{Str}(PQ) = (-1)^{|P||Q|} \operatorname{Str}(QP)$. This bilinear form pairs $\mathscr{S}_+$ with $\mathscr{S}_-$. In particular, T_LM is nondegenerately paired with $\mathscr{S}_-/D^{-n}\mathscr{S}_-$. In fact, if

$X = \sum_{k=0}^{\infty} D^{-k-1} X_k \in \mathscr{S}_-$ and $A = \sum_{k=0}^{n-1} A_k D^k \in \mathscr{S}_+$, their pairing is given by $\mathrm{Str}(AX) = \int_B \sum_{k=0}^{n-1} (-1)^k A_k X_k$. Therefore we define $T_L^* M$ as integral operators $X \in \mathscr{S}_-$ of the form $X = \sum_{k=0}^{n-1} D^{-k-1} X_k$ and, with a little abuse of notation, we also let X denote the 1-form it gives rise to at L. Thus if X and A are as above, the pairing between the vector field D_A and the 1-form X is given by

$$(D_A, X) \equiv (-1)^{|A|+|X|+n+1} \mathrm{Str}(AX) = (-1)^{|A|+n} \int_B \sum_{k=0}^{n-1} (-1)^k A_k X_k. \tag{2.6}$$

The strange choice of signs has been made to avoid undesirable signs later on. Given a function $F = \int_B f$ we define its gradient dF by $(D_A, dF) = D_A F$ whence, comparing with (2.2), yields

$$dF = \sum_{k=0}^{n-1} (-1)^k D^{-k-1} \frac{\delta f}{\delta U_k}. \tag{2.7}$$

So that the gradient of a function is a 1-form as expected.

To define Poisson brackets we need a linear map $J: T_L^* M \to T_L M$ inducing a map taking 1-forms to vector fields, so that for any two functions F and G, the bracket $\{F, G\}$, defined by

$$\{F, G\} \equiv D_{J(dF)} G = (D_{J(dF)}, dG) = (-1)^{|J|+|F|+|G|+n+1} \mathrm{Str}(J(dF) dG), \tag{2.8}$$

obeys the appropriate (anti)symmetric properties and the Jacobi identity. Such maps are often called "hamiltonian." This map is the formal analogue of the map taking the gradient of a function to the associated hamiltonian vector field in classical mechanics.

In [22] it was shown that the map $J: T_L^* M \to T_L M$ given by

$$J(X) = (LX)_+ L - L(XL)_+ = L(XL)_- - (LX)_- L \tag{2.9}$$

is hamiltonian. This was accomplished by showing that J is induced from a much simpler hamiltonian map in a different set of variables Φ_i defined by the factorization $L = (D - \Phi_n)(D - \Phi_{n-1}) \cdots (D - \Phi_1)$. In these variables, the hamiltonian map induced the following fundamental Poisson brackets

$$\{\Phi_i(X), \Phi_j(Y)\} = (-1)^i \delta_{ij} D\delta(X - Y), \tag{2.10}$$

where, if $X = (x, \theta)$ and $Y = (y, \omega)$, then $\delta(X - Y) = \delta(x - y)(\theta - \omega)$. The change of variables from the U_j to the Φ_i is called the Miura transformation.

3. Supersymmetric Gel'fand–Dickey Brackets

We now proceed to show, by direct computation, that the map given by (2.9) is indeed hamiltonian without involving the Miura transformation. This boils down to showing that the induced Poisson brackets have the appropriate symmetry properties and moreover that the Jacobi identities are satisfied.

Let Ω denote the map $X \mapsto D_{J(X)}$ from 1-forms to vector fields induced by the map J in (2.9). In analogy with the finite dimensional case it is convenient to introduce the symplectic form ω defined, on $\operatorname{Im}\Omega$, by

$$\omega(\Omega(X), \Omega(Y)) = (D_{J(X)}, Y). \tag{3.1}$$

Notice that in contrast with the usual case in classical mechanics, this 2-form is not defined for all vector fields since, in general, the map J will not be an isomorphism. It follows from the definition of ω that the Poisson brackets are given by

$$\{F, G\} = \omega(\Omega(dF), \Omega(dG)). \tag{3.2}$$

It is simple to check that for $J(X) \doteq (LX)_+ L - L(XL)_+$ this bracket has the correct symmetry properties. Explicitly,

$$\begin{aligned}
\{F, G\} &= (-1)^{|F|+|G|+n+1} \operatorname{Str}(J(dF)dG) \\
&= (-1)^{|F|+|G|+n+1} \operatorname{Str}((LdF)_+ LdG - L(dFL)_+ dG) \\
&= (-1)^{|F|+|G|+n+1} \operatorname{Str}((LdF)_+ (LdG)_- - (-1)^{n(|F|+|G|+n)}(dFL)_+ (dGL)_-) \\
&= (-1)^{|F|+|G|+n+1} \operatorname{Str}(LdF(LdG)_- - (-1)^{n(|F|+|G|+n} dFL(LdG)_-) \\
&= (-1)^{(n+1)(|F|+|G|+n)+1} \operatorname{Str}(dF((LdG)_- L - L(dGL)_-)) \\
&= (-1)^{(n+1)(|F|+|G|+n)} \operatorname{Str}(dFJ(dG)) \\
&= -(-1)^{|F|+|G|+n+1}(-1)^{|F||G|} \operatorname{Str}(J(dG)dF) \\
&= -(-1)^{|F||G|}\{F, G\}.
\end{aligned}$$

By analogy with the finite dimensional case, we define $d\omega$ by

$$\begin{aligned}
d\omega(D_{J(X)}, D_{J(Y)}, D_{J(Z)}) = D_{J(X)}\omega(D_{J(Y)}, D_{J(Z)}) \\
- \omega([D_{J(X)}, D_{J(Y)}], D_{J(Z)}) + \text{s.c.p.},
\end{aligned} \tag{3.3}$$

where s.c.p. is a shorthand notation for supercyclic permutation. As we will show below, closedness of ω, i.e. $d\omega = 0$, is equivalent to the Jacobi identities. But before proving this, notice that the second term in the right-hand side (RHS) of (3.3) is not well-defined unless $\operatorname{Im}\Omega$ forms a subalgebra of the vector fields. We now show that this is indeed the case.

Lemma. *For any X and $Y \in S_n^*$,*

$$[D_{J(X)}, D_{J(Y)}] = D_{J([\![X, Y]\!])},$$

where

$$[\![X, Y]\!] \equiv (-1)^{n|\bar{X}|} \hat{D}_{J(X)} Y + X(LY)_- - (XL)_+ Y - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y)$$

with $|\bar{X}| = |X| + n$ and $\hat{D}_A$ is defined by $D_A \int_B f = \int_B \hat{D}_A f$.

Proof. Using that $\hat{D}_{D(X)} L = J(X)$ we can write

$$[\hat{D}_{J(X)}, \hat{D}_{J(Y)}] L = \hat{D}_{J(X)} J(Y) - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y).$$

Expanding the RHS of this equation we find

$$\text{RHS} = -(-1)^{n|\bar{X}|} J(\hat{D}_{J(X)} Y) + ((LX)_+ LY)_+ L - (L(XL)_+ Y)_+ L$$

$$-(LX)_+L(YL)_+ + L(XL)_+(YL)_+ - (LX)_+(LY)_+L + (LX)_+L(YL)_+$$
$$+ L(X(LY)_+L) - L(XL(YL)_+)_+ - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y).$$

The fourth and seventh terms cancel, while the others rearrange to give

$$\begin{aligned}\text{RHS} = &-(-1)^{n|\bar{X}|}J(\hat{D}_{J(X)}Y) + (LX(LY)_- - L(XL)_+Y)_+L \\ &+ L((XL)_-YL + X(LY)_+L)_+ - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y).\end{aligned}$$

Now adding and subtracting $L(XLYL)_+ + -(-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y)$ we obtain

$$\begin{aligned}\text{RHS} &= -(-1)^{n|\bar{X}|}J(\hat{D}_{J(X)}Y) + (LX(LY)_- - L(XL)_+Y)_+L \\ &\quad - L(X(LY)_-L - (XL)_+YL)_+ - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y) \\ &= J((-1)^{n|\bar{X}|}\hat{D}_{J(X)}Y + X(LY)_- - (XL)_+Y - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y)) \quad \blacksquare\end{aligned}$$

We can now prove that the Jacobi identity of the Poisson brackets induced by J is equivalent to the closedness of ω.

Proposition. *For any functions F, G, and H,*

$$d\omega = -\text{Jacobi}(F, G, H),$$

where $\text{Jacobi}(F, G, H) \equiv \{F, \{G, H\}\} + \text{s.c.p.}$

Proof. Because of (3.2) and (3.3)

$$\begin{aligned}&d\omega(D_{J(dF)}, D_{J(dG)}, D_{J(dH)}) \\ &= D_{J(dF)}\{G, H\} - (D_{J([\![dF,dG]\!])}, dH) + \text{s.c.p.} \\ &= \{F, \{G, H\}\} - D_{J([\![dF,dG]\!])}H + \text{s.c.p.} \\ &= \text{Jacobi}(F, G, H) - ([D_{J(dF)}, D_{J(dG)}]H + \text{s.c.p.}) \\ &= \text{Jacobi}(F, G, H) - (D_{J(dF)}\{G, H\} - (-1)^{|G||F|}D_{J(dF)}\{G, H\} + \text{s.c.p.}) \\ &= \text{Jacobi}(F, G, H) - (\{F, \{G, H\}\} - (-1)^{|G||F|}\{G, \{F, H\}\} + \text{s.c.p.}) \\ &= \text{Jacobi}(F, G, H) - 2\,\text{Jacobi}(F, G, H) \\ &= -\text{Jacobi}(F, G, H). \quad \blacksquare\end{aligned}$$

We now have all the ingredients to prove the main result of this section.

Theorem. *For any three vector fields $D_{J(X)}$, $D_{J(Y)}$, and $D_{J(Z)}$ in* $\operatorname{Im}\Omega$,

$$d\omega(D_{J(X)}, D_{J(Y)}, D_{J(Z)}) = 0$$

i.e., *ω is a closed 2-form.*

Proof. From the definition of ω and (3.3), $dw(D_{J(X)}, D_{J(Y)}, D_{J(Z)})$ is given by

$$(-1)^{|\bar{X}|+|\bar{Y}|+|\bar{Z}|+n+1}\,\text{Str}\,[\hat{D}_{J(X)}(J(Y)Z) - J([\![X, Y]\!])Z + \text{s.c.p.}],$$

which, up to an irrelevant global sign, can be written as

$$\begin{aligned}&\text{Str}\,[(J(X)Y)_+LZ - J(X)(YL)_+Z + (-1)^{n|\bar{X}|}J(\hat{D}_{J(X)}Y)Z \\ &\quad + (-1)^{|\bar{X}||\bar{Y}|}((LY)_+J(X) + L(YJ(X))_+)Z + (-1)^{|\bar{X}||Y|}J(Y)\hat{D}_{J(X)}Z \\ &\quad - ((-1)^{n|\bar{X}|}J(\hat{D}_{J(X)}Y)Z - (-1)^{|X||\bar{Y}|}J(\hat{D}_{J(Y)}X)Z + J([X, Y]_L)Z) + \text{s.c.p.}],\end{aligned} \tag{3.4}$$

where

$$[X, Y]_L \equiv X(LY)_- - (XL)_+ Y - (-1)^{|\bar{X}||\bar{Y}|}(X \leftrightarrow Y).$$

Now we can show that the terms with $\hat{D}$'s in (3.4) cancel among themselves

$$\begin{aligned}&\operatorname{Str}[(-1)^{|\bar{X}||Y|}J(Y)\hat{D}_{J(X)}Z - (-1)^{|X||\bar{Y}|}J(\hat{D}_{J(Y)}X)Z + \text{s.c.p.}]\\ &\quad = \operatorname{Str}[(-1)^{|\bar{X}||Y|}J(Y)\hat{D}_{J(X)}Z - (-1)^{|X||\bar{Y}|+(|\bar{X}|+|\bar{Y}|)|\bar{Z}|}J(Z)\hat{D}_{J(Y)}X + \text{s.c.p.}]\end{aligned}$$

where we have used $\operatorname{Str}(J(X)Y) = -(-1)^{|\bar{X}||\bar{Y}|}\operatorname{Str}(J(Y)X)$. Writing the supercyclic permutations explicitly one sees that the terms cancel pairwise.

Up to a global sign $d\omega(D_{J(X)}, D_{J(Y)}, D_{J(Z)})$ can then be written as

$$\begin{aligned}&\operatorname{Str}[(J(X)Y)_+ LZ - J(X)YL)_+ Z\\ &\qquad + (-1)^{|\bar{Y}||\bar{Z}|}(J(X)Z(LY)_+ - J(X)(ZL)_- Y) - J([X, Y]_L)Z + \text{s.c.p.}].\end{aligned}$$

Using supercyclicity of the supertrace this equals $2\operatorname{Str} J(X)[Y, Z]_L + \text{s.c.p.}$ We now show that this is, in fact, zero. Indeed, we find

$$\operatorname{Str} J(X)[Y, Z]_L + \text{s.c.p.} = \operatorname{Str} J(X)(Y(LZ)_+ - (YL)_- Z) + \text{s.p.},$$

where s.p. stands for superpermutations. Expanding the RHS we have

$$\operatorname{Str}[(-(LX)_- L + \underline{L(XL)_-})Y(LZ)_+ + (\underline{-(LX)_+ L} + L(XL)_+)(YL)_- Z + \text{s.p.}].$$

The underlined terms cancel out as follows

$$\begin{aligned}&\operatorname{Str}[L(XL)_- Y(LZ)_+ - (LX)_+ L(YL)_- Z + \text{s.p.}]\\ &\quad = \operatorname{Str}[(-1)^{(|\bar{X}|+|\bar{Y}|)|\bar{Z}|}(LZ)_+ L(XL)_- Y - (LX)_+ L(YL)_- Z + \text{s.p.}] = 0\end{aligned}$$

because of supercyclicity of the supertrace. Now using the following fact $\operatorname{Str} AB_+C_- + \text{s.c.p.} = \operatorname{Str} ABC$ (see below), what is left can be written as

$$\begin{aligned}&\operatorname{Str}[(-1)^{|Z|(|\bar{X}|+|\bar{Y}|+n)}ZLXLYL - (-1)^{|\bar{X}|(|\bar{Y}|+|\bar{Z}|)}LYLZLX + \text{s.p.}]\\ &\quad = \operatorname{Str}[LXLYLX - (-1)^{|\bar{X}|(|\bar{Y}|+|\bar{Z}|)}LYLZLX + \text{s.p.}],\end{aligned}$$

which cancels because of supercyclicity. ∎

We finish this section by proving the fact we have just used:

Lemma. *For any three* SΨDO's *A*, *B*, *and C*

$$\operatorname{Str} AB_+C_- + \text{s.c.p.} = \operatorname{Str} ABC.$$

Proof.

$$\begin{aligned}\operatorname{Str} AB_+C_- + \text{s.c.p.} &= \operatorname{Str}(A_+B_+C_- + A_-B_+C_-) + \text{s.c.p.}\\ &= \operatorname{Str}(A_+B_+C + A_-BC_-) + \text{s.c.p.}\\ &= \tfrac{1}{3}\operatorname{Str}(2AB_+C_- + A_+B_+C + A_-BC_-) + \text{s.c.p.}\\ &= \tfrac{1}{3}\operatorname{Str}(2AB_+C_- + AB_+C_+ + AB_-C_-) + \text{s.c.p.}\\ &= \tfrac{1}{3}\operatorname{Str}(AB_+C + ABC_-) + \text{s.c.p.}\\ &= \tfrac{1}{3}\operatorname{Str}(AB_+C + AB_-C) + \text{s.c.p.}\\ &= \tfrac{1}{3}\operatorname{Str} ABC + \text{s.c.p.} = \operatorname{Str} ABC \quad ∎\end{aligned}$$

4. Symmetric Reduction of the Supersymmetric Gel'fand–Dickey Brackets

In this section we investigate the reduction of the supersymmetric Gel'fand–Dickey bracket induced by demanding that the Lax operator L have a definite adjointness property. To motivate the definition of the adjoint, let us think of differential operators as acting on superfields with inner product

$$(U, V) = \int_B UV. \tag{4.1}$$

If $L \in \mathscr{S}_+$ is a homogeneous differential operator, we define its adjoint L^* by $(LU, V) = (-1)^{|L||U|}(U, L^* V)$, for any homogeneous superfields U, V. The proof of the following proposition is routine.

Proposition. $*$ *extends to an involution in the space* $\mathscr{S}$ *of* SΨDO's *which obeys the following properties*:

(1) *For all* $P \in \mathscr{S}$, $(P^*)^* = P$.
(2) *For all homogeneous* $P, Q \in \mathscr{S}$, $(PQ)^* = (-1)^{|P||Q|} Q^* P^*$.
(3) *If* $P \in \mathscr{S}$ *is homogeneous and invertible*, $(P^{-1})^* = (-1)^{|P|}(P^*)^{-1}$.
(4) *For all* $p \in \mathbb{Z}$, $(D^p)^* = (-1)^{p(p+1)/2} D^p$.
(5) *For all* $P \in \mathscr{S}$, $(P_\pm)^* = (P^*)_\pm$.
(6) *For all* $P \in \mathscr{S}$, $\operatorname{sres} P^* = \operatorname{sres} P$ (*in particular*, $\operatorname{Str} P^* = \operatorname{Str} P$).

Since a supersymmetric Lax operator L is of the form $L = D^n + \cdots$, its adjointness property – if it has a definite one – is dictated by the first term. We shall say that L is "symmetric" if $L^* = (-1)^{n(n+1)/2} L$. We will show that the supersymmetric Gel'fand–Dickey bracket in the space M_{2k+1} of Lax operators of a given odd order induces a Poisson bracket in the submanifold of symmetric Lax operators and that the induced fundamental Poisson brackets defines a W-superalgebra extending the $N=1$ supervirasoro algebra.

We can understand what constraints symmetry imposes on the coefficients of L in two ways. Via the Miura transformation we see that these constraints are just linear constraints in the basic fields Φ_i. To see this let us factorize $L = D^n + \sum_j U_j D^j = (D - \Phi_n)(D - \Phi_{n-1}) \cdot (D - \Phi_1)$. Computing its adjoint we find $L^* = (-1)^{(n(n+1))/2} \times (D + \Phi_1)(D + \Phi_2) \cdots (D - \Phi_n)$, whence $\Phi_j = \Phi_{n+1-j}$. If n is even this imposes $\frac{n}{2}$ conditions, whereas if n is odd we get $\frac{n+1}{2}$. Thus in the even case $n = 2k$, a symmetric Lax operator has the following factorization $L = (D + \Phi_1) \cdots (D + \Phi_k)(D - \Phi_k) \cdots (D - \Phi_1)$; whereas in the odd case $n = 2k+1$ the factorization is $L = (D + \Phi_1) \cdots (D + \Phi_k) D (D - \Phi_k) \cdots (D - \Phi_1)$. In terms of the U_j, the constraints are more complicated and are best exemplified by writing L in a manifestly symmetric way. In general, a symmetric Lax operator has the form

$$L = D^n + \frac{1}{2} \sum_{j \in I_n} \{V_j, D^j\}, \tag{4.2}$$

where $\{V_j\} = V_j D^j + (-1)^{j(n+j)} D^j V_j$ is the graded anticommutator and the sum runs over the index set

$$I_n = \{ j \in \mathbb{Z} \mid 0 \leqq j < n \text{ and } (-1)^{j(j+1)/2} = (-1)^{n(n+1)/2} \}. \tag{4.3}$$

Equation (4.2) manifestly exhibits which of the fields U_j are independent; namely,

$$\begin{aligned} n &\equiv 0 \bmod 4: \quad U_j \text{ for } j \equiv 0,3 \bmod 4,\\ n &\equiv 1 \bmod 4: \quad U_j \text{ for } j \equiv 1,2 \bmod 4,\\ n &\equiv 2 \bmod 4: \quad U_j \text{ for } j \equiv 1,2 \bmod 4,\\ n &\equiv 3 \bmod 4: \quad U_j \text{ for } j \equiv 0,3 \bmod 4. \end{aligned}$$

Some general facts readily emerge. In the odd n case, there is always an independent field of weight $\frac{3}{2}$ and, moreover, this is the field of smallest weight. We will see that its Poisson bracket is that of the classical $N=1$ supervirasoro algebra. For even n the situation is radically different: there is never a field of weight $\frac{3}{2}$ but there is always a field of weight $\frac{1}{2}$. But besides the spectrum there is a more fundamental difference. Whereas in the odd n case, the constrained submanifold is symplectic – the matrix of Poisson brackets of the constraints being non-degenerate – in the even n case, the submanifold is coisotropic – the constraints being in involution. This is easy to see in the basic fields Φ_i of the factorized L. The constraints are given by $\varphi_i = \Phi_i + \Phi_{n+i-1}$ for $i = 1, \ldots, \left[\frac{n}{2}\right]$ and, in the odd n case the extra constraint $\Phi_{(n+1)/2}$. Using (2.10) one readily computes

$$\{\varphi_i, \varphi_j\} = (-1)^i \delta_{ij}(1-(-1)^n) D\delta(X-Y), \tag{4.4}$$

from which the previous observations readily follow. In particular it follows that the even n reduction collapses the algebra and, thus, from now on we will only consider the case of odd $n = 2k+1$.

Let M denote the space of Lax operators of the form $L = D^{2k+1} + \cdots$, and let M_0 denote the submanifold of symmetric operators $L^* = -(-1)^k L$. As we have seen this submanifold is symplectic and thus it inherits a well-defined Poisson bracket from that in M. To describe the bracket we first need to identify the vector fields and the 1-forms on M_0 as subobjects of the corresponding objects in M. The vector fields of M_0 will be parametrized by the deformations of L that remain in M_0. That is, deformations of a symmetric Lax operator L which keep it symmetric. These are clearly the differential operators of order at most $2k$ obeying the same symmetry property of L. As explained, for example, in [23], 1-forms on M_0 must be chosen to be those 1-forms on M which are mapped (via the hamiltonian map J) to vector fields tangent to M_0. In other words, 1-forms on M_0 are SΨDO's $X = \sum_l D^{-l-1} X_l$ satisfying that $J(X)^* = -(-1)^k J(X)$. Computing this we find

$$\begin{aligned} J(X)^* &= [(LX)_+ L - L(XL)_+]^*\\ &= (-1)^{|X|+1}[L^*(LX)^*_+ - (XL)^*_+ L^*]\\ &= -[L^*(X^*L^*)_+ - (L^*X^*)_+ L^*]\\ &= (LX^*)_+ L - L(X^*L)_+\\ &= J(X^*), \end{aligned}$$

where X must have the same symmetry properties of L, namely $X^* = -(-1)^k X$, for it to be a 1-form on M_0. It is easy to verify that these 1-forms are non-degenerately paired with the vector fields tangent to M_0. In fact, since $\operatorname{Str}(AX) = \operatorname{Str}(A^*X^*)$ we see that the supertrace pairs up 1-forms and vector fields of the same symmetry properties. Therefore the Poisson bracket of two functions $F = \int_B f$ and $G = \int_B g$ on M_0 is obtained from (2.8) (with J given by (2.9)) by simply requiring that dF and dG have the correct symmetry properties: $(dF)^* = -(-1)^k dF$ and the same for dG.

We now prove that the induced fundamental Poisson brackets on M_0 contains the $N = 1$ supervirasoro algebra as a subalgebra. We have already seen that the field of smallest weight is a field V_{2k-2} of weight $\frac{3}{2}$. We will now show that the induced fundamental Poisson bracket $\{V_{2k-2}(X), V_{2k-2}(Y)\}_0$ defines a $N = 1$ supervirasoro algebra. We find it more convenient to do this via the Miura transformation and for this we must first find the induced Poisson brackets in terms of the basic fields Φ_i. As we saw above the constraints imposed by the symmetry requirement on L turned out to give linear constraints among the Φ_i: $\varphi_i = \Phi_i + \Phi_{2k+2-i}$ for $i = 1, \ldots, k$; and $\varphi_{k+1} = \Phi_{k+1}$. We choose as independent fields in the constraint submanifold the Φ_i for $i = 1, \ldots, k$. It is an easy computation to find the induced Poison brackets of these fields

$$\{\Phi_i(X), \Phi_i(Y)\}_0 = \tfrac{1}{2}(-1)^i D\delta(X - Y). \tag{4.5}$$

Factorizing L in Eq. (4.2) as $L = (D + \Phi_1)\cdots(D + \Phi_k)D(D - \Phi_k)\cdots(D - \Phi_1)$ gives the V_j in terms of the Φ_i. Their fundamental Poisson brackets can be easily found in terms of those in (4.5) as follows. Define the Fréchet jacobian D_{ji} of V_j by Φ_i as the following differential operator:

$$D_{ji} \equiv \frac{\delta V_j}{\delta \Phi_i} = \sum_{p=0}^{\infty} (-1)^{p+p(p+1)/2} D^p \frac{\partial V_j}{\partial \Phi_i^{[p]}}. \tag{4.6}$$

Then, if we define the differential operators Ω_{ij} by

$$\{V_i(X), V_j(Y)\}_0 = \Omega_{ij}\delta(X - Y), \tag{4.7}$$

where the Ω_{ij} are taken at the point X, we find

$$\Omega_{ij} = \frac{1}{2}\sum_{l=1}^{k} (-1)^{i+j+l} D_{il}^* D D_{jl}, \tag{4.8}$$

where, for convenience, we write D_{il}^* explicitly

$$D_{il}^* = \sum_{p=0}^{\infty} (-1)^{pi} \frac{\partial V_i}{\partial \Phi_l^{[p]}} D^p. \tag{4.9}$$

A straightforward computation yields

$$V_{2k-2} = \sum_{j=1}^{k} (-1)^j [\Phi_j \Phi_j' + (k + 1 - j)\Phi_j''], \tag{4.10}$$

from where the Fréchet jacobian follows:

$$D_{2k-2,j} = (-1)^j [(k + 1 - j)D^2 + \Phi_j D + \Phi_j']. \tag{4.11}$$

It is now a simple matter to plug this into (4.8) to compute

$$\Omega_{2k-2,2k-2} = \frac{k(k+1)}{4} D^5 + \frac{3}{2} V_{2k-2} D^2 + \frac{1}{2} V_{2k-2}' D + V_{2k-2}'', \tag{4.12}$$

whence, if we let $\mathbb{T} = V_{2k-2,2k-2}$ gives rise to a classical version of the $N = 1$ supervirasoro algebra

$$\{\mathbb{T}(X), \mathbb{T}(Y)\}_0 = \left[\frac{k(k+1)}{4}D^5 + \frac{3}{2}\mathbb{T}(X)D^2 + \frac{1}{2}\mathbb{T}'(X)D + \mathbb{T}''(X)\right]\delta(X-Y). \quad (4.13)$$

5. Conclusion

In this paper we have analyzed a particular reduction of a recently constructed Poisson structure on the space of supersymmetric Lax operators. We have seen that for the case of Lax operators of odd order this Poisson structure admits a reduction yielding classical W-superalgebras containing the $N = 1$ supervirasoro algebra as a subalgebra. This reduction was obtained by imposing (anti)symmetry of the Lax operator with respect to a natural involution in the space of (pseudo) differential operators in a $(1|1)$ superspace. This is the supersymmetric analogue of the Drinfel'd–Sokolov reductions of the second Gel'fand–Dickey bracket which were associated to the B_n and C_n series of classical Lie algebras.

The spectrum of the resulting algebras consists of fields V_j for $j \in I_{2k+1}$, the index set defined by (4.3), of naïve weights $k-(j-1)/2$. In particular, V_{2k-2} and has been shown to generate a $N = 1$ supervirasoro algebra. It is then a natural conjecture to expect that each remaining field V_j gives rise to a superconformal primary field $\tilde{V}_j$ obtained by deforming V_j via the addition of differential polynomials in the $V_{i>j}$. This can be checked explicitly for the simplest examples. We have not proven it in general but we have no doubt of its validity. We hope to return to this point in a future publication.

As mentioned in the introduction, these algebras have been also obtained in [21] from the $Osp(2n \pm 1|2n)$ Toda field theories, where they appear as the algebra of conserved quantities. It is an interesting open question to obtain these algebras as hamiltonian reduction of the corresponding affine algebras. In this fashion one could recover the connection between Lie superalgebras and W-superalgebras and, in particular, the discrete series of the latter ones, as was done for the non-supersymmetric case in [10] following the work of Drinfel'd and Sokolov. Work on this is in progress.

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