Commun. Math. Phys. 138, 507–519 (1991)

Communications in Mathematical Physics

Particle Scattering in Euclidean Lattice Field Theories

J. C. A. Barata [1,*] **and K. Fredenhagen** [2]

[1] Instituto de Física a Universidade de São Paulo, Cep 05508, Caixa Postal 20516, São Paulo, SP, Brazil
[2] Freie Universität Berlin, Fachbereich Physik, Institut für Theorie der Elementarteilchen, Arnimallee 14, 1000 Berlin 33, FRG

Received June 20, 1990

Abstract. A Haag-Ruelle Scattering Theory for Euclidean Lattice Field Theories is developed.

1. Introduction

Euclidean lattice field theories are candidates for approximate models of particle physics. The particle aspects of these models, however, are usually analysed in a rather indirect way. One first considers the lattice model as an approximation to a continuum theory; by the Osterwalder-Schrader Theorem [9, 10], the continuum theory can be analytically extended to a Minkowski space quantum field theory. Then, provided there are single particle states, one finds by the methods of the general theory of quantized fields the corresponding incoming and outgoing multiparticle states (Haag-Ruelle theory [1, 2, 3]). According to Hepp [4, 5], the scattering amplitudes can be written in terms of the time-ordered functions by the LSZ reduction formulae [8]. In a last step the time ordered functions are approximated by lattice quantities.

This indirect description of the particle content of Euclidean lattice field theories has severe conceptual and practical problems which originate essentially in the nonuniqueness of the lattice approximation of continuum quantities. This becomes especially clear in theories with a trivial continuum limit which one would like to use as effective theories up to some high energy cutoff. The indirect particle interpretation described above does not lead in a natural way to non-zero scattering amplitudes.

Fortunately, as is well known, there is a quantum spin system which is associated directly to the Euclidean lattice model by the transfer matrix method. Moreover, in many cases these quantum spin systems have particle-like excita-

* Work supported by DAAD, Deutscher Akademischer Austauschdienst. Partially supported by CNPq

tions (see [11, 15] and references therein). One may therefore hope that these models can be interpreted as models of *interacting quasiparticles*. Actually Lüscher has shown that, provided free outgoing and incoming fields exist corresponding to these particles, and provided these fields satisfy an LSZ-asymptotic condition with respect to some interacting field, scattering amplitudes can be expressed in terms of Euclidean correlation functions [13, 14].

In this paper we prove that under very general conditions there are states in the quantum spin model associated to an Euclidean lattice model which can be interpreted as incoming, respectively outgoing multiparticle states. An LSZ-type asymptotic condition could not be found but LSZ reduction formulae were directly derived in [15].

The main difficulty which has to be overcome is the insufficient control on the locality properties of the real-time evolution of the quantum spin system.

Typically, the transfer matrix is of the form

$$T = e^{A/2} e^{B} e^{A/2}, \tag{1}$$

where A as well as B are sums of local operators. The Hamiltonian

$$H = -\ln T \tag{2}$$

however, has in addition to the local term $A+B$ contributions of multiple commutators of arbitrary high degree which are in general localized in large regions and therefore induce long range interactions. We show that this difficulty is essentially restricted to the high energy range. The high energy behavior of a lattice theory does not influence the formation of scattering states; it is not to be considered to be relevant for an approximated continuum theory.

The paper is organized as follows. In Sect. 2 we formulate our assumptions (reflection positivity, exponential clustering of Euclidean correlations, existence of single particle states) on the Euclidean lattice model and define the associated quantum spin system. In Sect. 3 we relate the real-time correlation functions to the Euclidean correlation functions. As an intermediate step we introduce a new type of correlation functions which we call Chebishev transformed correlation functions. These functions are finite linear combinations of Euclidean correlation functions and permit a rather direct representation of real-time correlation functions. In Sect. 4 clustering properties of the quantum spin system are derived which for observables with finite energy transfer are only slightly weaker than the corresponding clustering properties in a continuum quantum field theory. These results are then used to construct the scattering states by the methods of the Haag-Ruelle theory in Sect. 5. The paper is largely based on one of the author's thesis [15] where more details may be found, in particular LSZ reduction formulae.

2. Euclidean Lattice Field Theory and the Associated Quantum Spin System

We consider a classical statistical system on the hypercubic lattice $\mathbb{Z}^{d+1}$, $d \geqq 1$. The variables of the model are attached to finite subsets of $\mathbb{Z}^{d+1}$ (i.e. sites, bonds, plaquettes, etc.), and the observables are complex valued continuous bounded functions of finitely many of these variables. $\mathscr{E}(\Lambda)$ for $\Lambda \subset \mathbb{Z}^{d+1}$ denotes the set of observables depending only on variables located in Λ; it is an abelian normed $*$-algebra with respect to pointwise multiplication, complex conjugation and the supremum-norm $\|\cdot\|$. Lattice translations x shift the variables of the theory and induce automorphisms γ_x of $\mathscr{E}=\mathscr{E}(\mathbb{Z}^{d+1})$ such that

$$\gamma_x(\mathscr{E}(\Lambda))=\mathscr{E}(\Lambda+x), \tag{3}$$

and

$$\gamma_{x_1}\gamma_{x_2}=\gamma_{x_1+x_2}. \tag{4}$$

Lattice reflections with respect to some coordinate hyperplane

$$\vartheta_{\mu,a}: x\to x'|(x')^\nu=x^\nu, \quad \nu\neq\mu,\ (x')^\mu=2a-x^\mu \tag{5}$$

for $\mu=0,\ldots,d$ and $a\in\frac{1}{2}\mathbb{Z}$ induce antilinear automorphisms $\theta_{\mu,a}$ of $\mathscr{E}$ such that $\theta^2_{\mu,a}=\mathrm{id}$ and

$$\theta_{\mu,a}\gamma_x=\gamma_{\vartheta_{\mu,a}(x)}\theta_{\mu,a}=\gamma_x\theta_{\mu,a-x^\mu}. \tag{6}$$

The model is defined by the choice of a state $\langle\cdot\rangle$ on $\mathscr{E}$, i.e. a normalized, positive linear functional, typically the Gibbs state with respect to some Hamiltonian. The state $\langle\cdot\rangle$ is assumed to have the following three properties: reflection positivity (A.1), this permits us to define an associated quantum spin system; exponential clustering (A.2), so the associated quantum system has a mass gap; and existence of one particle excitations with an upper gap (A.3).

(A.1) Assumption. *Reflection positivity* [9, 10]:

$$\textit{For } f\in\mathscr{E}(\Lambda_{\mu,a}),\ \Lambda_{\mu,a}=\{x\in\mathbb{Z}^{d+1},\ x^\mu\geqq a\},$$
$$\langle\theta_{\mu,a}(f)f\rangle\geqq 0. \tag{7}$$

Relation (7) endowes $\mathscr{E}_+:=\mathscr{E}(\Lambda_{0,0})$ with a semidefinite scalar product. By factoring over the space of null vectors one obtains a Hilbert space $\mathscr{H}_0$ and a mapping $f\to\hat{f}$ from $\mathscr{E}_+$ onto a dense subspace of $\mathscr{H}_0$. The translation in 0-direction induces a positive contraction in $\mathscr{H}_0$,

$$T\hat{f}=\widehat{\gamma_{e_0}f}, \tag{8}$$

where e_0 is the unit vector in 0-direction. T is the transfer matrix. Translations $\underline{x}$ in the directions orthogonal to e_0 induce unitary operators

$$U(\underline{x})\hat{f}=\widehat{\gamma_{\underline{x}}f}. \tag{9}$$

The Hamilton operator can be defined by

$$H=-\ln T \tag{10}$$

provided T has no zero eigenvalue, a condition which is satisfied in typical cases [7]. Since it refers to the high energy behavior (absence of states with infinite energy) which will be eliminated from our considerations, we do not need this assumption. In the general case we define the subspace of finite energy

$$\mathscr{H}=(\mathrm{Ker}\,T)^\perp \tag{11}$$

and set

$$H=-\ln T\restriction\mathscr{H}. \tag{12}$$

In order to get a full quantum spin system we want to introduce local observables. Let $\Lambda^{(n)}=\{x\in\mathbb{Z}^{d+1},\ x^0\in[0,n]\}$. Then each $f\in\mathscr{E}(\Lambda^{(n)})$ induces an operator $\pi^{(n)}(f)$ on $T^n\mathscr{H}$ by

$$\pi^{(n)}(f)\hat{g}=\widehat{fg}. \tag{13}$$

It satisfies the estimate

$$\|\pi^{(n)}(f)T^n\| \leqq \|f\|. \tag{14}$$

For $n>0$ the operators $\pi^{(n)}(f)$ do not have an invariant domain of definition and therefore do not generate an algebra.

Better behaved are the operators with finite energy transfer

$$A = \sum_{\underline{x}} \int dt h(t, \underline{x}) e^{iHt} U(\underline{x}) \pi^{(n)}(f) U(-\underline{x}) e^{-iHt} \equiv f(h) \tag{15}$$

where the Fourier transform $\tilde{h}$ of h is smooth with compact support. A maps the dense subspace

$$D_\infty = \bigcap_n T^n \mathscr{H} \tag{16}$$

of $\mathscr{H}$ into itself; moreover, A is closable, and also its adjoint maps D_∞ into itself.

Let $\mathscr{A}$ denote the $*$-algebra which is generated by operators of the form (15). We consider $\mathscr{A}$ as the algebra of almost local observables of the quantum spin system. The time evolution acts as an automorphism group on $\mathscr{A}$ which is entire analytic, i.e. $t \to \alpha_t(A)$ is entire analytic for each $A \in \mathscr{A}$.

(A.2) Assumption. *Exponential clustering of Euclidean correlations: There exists* $m>0$ *such that for all* $f, g \in \mathscr{E}$,

$$|\langle f \gamma_x(g)\rangle - \langle f\rangle \langle g\rangle| \leqq \text{const} e^{-m|x|}, \qquad x \in \mathbb{Z}^{d+1}. \tag{17}$$

This assumption immediately implies that $\Omega \equiv \hat{1}$ is the (up to a phase) unique ground state vector of H and that

$$sp(H) \subset \{0\} \cup [m, \infty). \tag{18}$$

Using reflection positivity in all coordinate directions one obtains the following clustering properties of correlation functions:

$$\langle f_1 \dots f_n \rangle = \sum_{P \in \mathscr{P}\{1,\dots,n\}} \prod_{I \in P} \langle \{f_i, i \in I\}\rangle_T, \tag{19}$$

where $\mathscr{P}\{1,\dots,n\}$ is the set of all partitions of $\{1,\dots,n\}$ and where the truncated functions $\langle \cdot; \dots; \cdot\rangle_T$ satisfy the bound [15]

$$|\langle f_1; \dots; f_n\rangle_T| \leqq n^{n-1} \prod_{i=1}^n \|f_i\| e^{-mr(\Lambda_1,\dots,\Lambda_n)} \tag{20}$$

for $f_i \in \mathscr{E}(\Lambda_i)$, $i=1,\dots,n$, with

$$r(\Lambda_1,\dots,\Lambda_n) = \max_{\mu=0,\dots,d} \left\{ \text{diameter}^{(\mu)}(\Lambda_1 \cup \dots \cup \Lambda_n) - \sum_{i=1}^n \text{diameter}^{(\mu)}(\Lambda_i) \right\} \tag{21}$$

with

$$\text{diameter}^{(\mu)}(\Lambda) = \sup\{|x^\mu - y^\mu|, x, y, \in \Lambda\}. \tag{22}$$

In Sect. 4 we will show that assumption (A.2) also implies fast clustering of the real-time correlation functions of the associated quantum spin system. So the quantum spin system exhibits a behavior similar to a system with local interactions in spite of the presence of (at least in the moment) in general uncontrollable long range interactions induced by the finite lattice spacing in Euclidean time.

The third assumption concerns the existence of particle-like excitations. We restrict ourselves to the case of uncharged particles, i.e. particle states in the Hilbert space $\mathscr{H}$ (the "vacuum Hilbert space"). Charged particles, e.g., the charged particles in a $\mathbb{Z}(2)$ gauge Higgs model as constructed in [11] will be treated in a forthcoming paper [19].

(A.3) Assumption. *Existence of one particle excitations.*
There is an $f\in\mathscr{E}$, $f\neq 0$, such that for all $g\in\mathscr{E}$ the Fourier transform of the truncated 2-point function

$$\langle g, \gamma_x(f)\rangle_T \tag{23}$$

can be analytically extended for each $\underline{p}\in(-\pi,\pi]^d$ to a meromorphic function of p_0 in the region $\operatorname{Im} p_0<\hat{\omega}(\underline{p})$ with an isolated simple pole at $p_0=i\omega(\underline{p})$. $\omega(\underline{p})$ (the energy-momentum relation of the particle) is assumed to be smooth and $\hat{\omega}(\underline{p})$ is supposed to be continuous, $\hat{\omega}(\underline{p})>\omega(\underline{p})\geqq m$. ω and $\hat{\omega}$ are independent of g. The velocity $v(\underline{p})=\operatorname{grad}\omega(\underline{p})$ is nowhere constant.

Assumption (A.3) implies that there is a closed subspace $\mathscr{H}^{(1)}$ of $\mathscr{H}$ (the single particle subspace) on which the relation

$$(e^{-H}-e^{-\omega(\underline{P})})\restriction\mathscr{H}^{(1)}=0 \tag{24}$$

holds. Here $\underline{P}$ is the momentum operator, i.e. the infinitesimal generator of spatial translations,

$$e^{i\underline{P}\cdot\underline{x}}=U(\underline{x}), \qquad \operatorname{sp}(\underline{P})\subset(-\pi,\pi]^d \tag{25}$$

and $\mathscr{H}^{(1)}$ is the closure of the linear space

$$\mathscr{D}^{(1)}=\{A\Omega, A=f(h), \operatorname{supp}\tilde{h}\cap\operatorname{sp}(H,\underline{P})\subset\{(\omega(\underline{p}),\underline{p}),\underline{p}\in(-\pi,\pi]^d\}, \tilde{h}\in\mathscr{D}(\mathbb{R}^{d+1})\} \tag{26}$$

3. From Lattice Schwinger Functions to Wightman Functions; the Chebishev Transform

The Wightman functions, i.e. the real-time correlation functions of the quantum spin system constructed in the preceding section, can in principle be determined from the knowledge of the correlation functions of the Euclidean lattice field theory. It is the aim of this section to find an efficient formula for this connection.

The basic idea is to use the fact that continuous functions of the Hamilton operator H with compact support can be uniformly approximated by polynomials in $e^{-H}=T$ whose matrix elements are given in terms of Euclidean correlation functions. Chebishev polynomials turn out to be especially convenient for this purpose. On the interval $[-1,1]$ they are defined by

$$T_n(x)=\cos(n\arccos x) \tag{27}$$

and provide an orthogonal basis on $L^2([-1,1],(1-x^2)^{-1/2}dx)$, (see for instance [17]). After using the relation $T_n(2y-1)=T_{2n}(y^{1/2})$, $0\leqq y\leqq 1$, which follows from the identities $T_m(T_n(x))=T_{mn}(x)$ and $T_2(x)=2x^2-1$, one gets

$$f(H)=\sum_{n=0}^{\infty} b_n(f)T_{2n}(e^{-H/2}) \tag{28}$$

with the expansion coefficient

$$b_n(f)=\frac{2}{\pi}(2-\delta_{n,0})\int_0^{\pi/2} d\alpha f(-2\ln\cos\alpha)\cos(2n\alpha). \tag{29}$$

For smooth functions f with compact support $b_n(f)$ is strongly decreasing in n. Since $|T_n(x)|\leqq 1$ for $x\in[-1,1]$ this implies norm convergence of (28).

Now let $f_i\in\mathscr{E}(\Lambda^{(n_i)})$, $i=1,\dots,n$. The "Wightman function," formally given by

$$W_{f_1\dots f_n}(t_1,\dots,t_n)=(\Omega,\pi^{(n_1)}(f_1)e^{i(t_2-t_1)H}\dots e^{i(t_n-t_{n-1})H}\pi^{(n_n)}(f_n)\Omega) \tag{30}$$

is a distribution on test functions $h(t_1,\dots,t_n)$ with $\tilde{h}\in\mathscr{D}(\mathbb{R}^n)$, such that for $\tilde{h}_i\in\mathscr{D}(\mathbb{R})$, $i=1,\dots,n$

$$\int dt_1\dots dt_n W_{f_1\dots f_n}(t_1,\dots,t_n)h_1(t_1)\dots h_n(t_n)=(\Omega,f_1(h_1)\dots f_n(h_n)\Omega) \tag{31}$$

with

$$f_i(h_i)=\int dt h_i(t)e^{iHt}\pi^{(n_i)}(f_i)e^{-iHt}. \tag{32}$$

We use the fact that $\pi^{(n_i)}(f_i)e^{-n_iH}$ is a bounded operator, and expand

$$e^{(n+it)H}=\sum_{k=0}^{\infty} b_k(t-in)T_{2k}(e^{-H/2}). \tag{33}$$

The expansion coefficients $b_k(t-in)$ are distributions in t on $\tilde{\mathscr{D}}(\mathbb{R})$,

$$b_k(t-in)=\frac{2}{\pi}(2-\delta_{k,0})\int_0^{\pi/2} d\alpha(\cos\alpha)^{-2n}e^{-2it\ln\cos\alpha}\cos(2k\alpha) \tag{34}$$

and the series converges in the sense of distributions. Inserting these expansions into the Wightman function yields terms which are finite linear combinations of Euclidean correlation functions

$$\begin{aligned}W_{f_1\dots f_n}(t_1,\dots,t_n)=&\sum_{k_1,\dots,k_{n-1}} b_{k_1}(t_2-t_1-in_1)\dots b_{k_{n-1}}(t_n-t_{n-1}-in_{n-1})\\ &\times C_{f_1,\dots,f_n}(k_1,\dots,k_{n-1})\end{aligned} \tag{35}$$

where the "Chebishev transformed functions" $C_{f_1,\dots,f_n}$ are defined by

$$\begin{aligned}&C_{f_1,\dots,f_n}(k_1,\dots,k_{n-1})\\ &=(\Omega,\pi^{(n_1)}(f_1)e^{-n_1H}T_{2k_1}(e^{-H/2})\dots e^{-n_{n-1}H}T_{2k_{n-1}}(e^{-H/2})\pi^{(n_n)}(f_n)\Omega)\\ &=\sum_j a_{j_1}^{k_1}\dots a_{j_{n-1}}^{k_{n-1}}\langle f_1\gamma_{j_1+n_1}(f_2)\dots\gamma_{j_1+\dots+j_{n-1}+n_1+\dots+n_{n-1}}(f_n)\rangle\end{aligned} \tag{36}$$

with the a_j^k denoting the coefficients of the polynomials

$$T_{2k}(x^{1/2})=\sum_{j=0}^{k} a_j^k x^j. \tag{37}$$

The convergence of the expansion (35) may be seen as follows: since

$$\|T_{2k}(e^{-H/2})\|\leqq 1 \quad\text{and}\quad \|\pi^{(n_i)}(f_i)e^{-n_iH}\|\leqq\|f_i\|,$$

the Chebishev functions are uniformly bounded in $k_1,\dots,k_{n-1}$,

$$|C_{f_1,\dots,f_n}(k_1,\dots,k_{n-1})|\leqq\|f_1\|\dots\|f_n\|. \tag{38}$$

Now let $h \in \tilde{\mathscr{D}}(\mathbb{R}^n)$. Then after smearing each term on the right-hand side of (35) with h we get coefficients

$$\begin{aligned} & b_{k_1,\ldots,k_{n-1}}(h; n_1, \ldots, n_{n-1}) \\ & = \int dt_1 \ldots dt_n b_{k_1}(t_2 - t_1 + in_1) \ldots b_{k_{n-1}}(t_n - t_{n-1} + in_{n-1}) h(t_1, \ldots, t_n) \\ & = (2\pi)^{(n-1)/2} \prod_{i=1}^{n-1} \left[\frac{2}{\pi}(2 - \delta_{k_i,0}) \right] \\ & \quad \times \int d\alpha_1 \ldots d\alpha_{n-1} \tilde{h}\left(2 \ln \cos\alpha_1, 2 \ln\left(\frac{\cos\alpha_2}{\cos\alpha_1}\right), \ldots \right. \\ & \quad \left. \ldots, 2\ln\left(\frac{\cos\alpha_{n-1}}{\cos\alpha_{n-2}}\right), -2\ln\cos\alpha_{n-1}\right) \\ & \quad \times (\cos\alpha_1)^{-2n_1} \ldots (\cos\alpha_{n-1})^{-2n_{n-1}} \cos(2k_1\alpha_1) \ldots \cos(2k_{n-1}\alpha_{n-1}). \end{aligned} \tag{39}$$

Due to the support and smoothness properties of $\tilde{h}$ they decrease faster than any polynomial in $k_1, \ldots, k_{n-1}$, i.e. for each $N \in \mathbb{N}$ there is some $c > 0$ such that

$$|b_{k_1,\ldots,k_{n-1}}(h; n_1, \ldots, n_{n-1})| \leqq c(1 + \textstyle\sum k_i)^{-N}. \tag{40}$$

This yields the convergence of the right-hand side of (35).

4. Clustering of Wightman Functions

In view of the apparent nonlocality of the real time evolution it is not obvious whether the Wightman functions introduced in the last section exhibit any kind of clustering. Clustering is crucial for the construction of scattering states by the methods of Haag and Ruelle [1–5]. In this section we show that the existence of a mass gap (assumption (A.2)) implies a weak form of clustering of Wightman functions of fields with bounded energy transfer, which turns out to be sufficient for the purposes of the scattering theory.

Let $f_i \in \mathscr{E}([0, n_i] \times \Lambda_i)$, Λ_i finite subset of $\mathbb{Z}^d$, $n_i \geqq 0$ and let $h_i \in \tilde{\mathscr{D}}(\mathbb{R})$, $1 \leqq i \leqq n$. Consider the truncated Wightman function

$$\mathscr{W}_{h,f}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n)_T := (\Omega, f_{1,\underline{x}_1}(h_{1,t_1}), \ldots, f_{n,\underline{x}_n}(h_{n,t_n}), \Omega)_T, \tag{41}$$

where $h_{i,t_i}(t) = h_i\ (t - t_i)$ and $f_{i,\underline{x}_i} = \gamma_{\underline{x}_i}(f_i)$, $1 \leqq i \leqq n$. Our main result concerns the decay properties of these truncated functions.

Theorem 1. *Let $\mathscr{W}_{h,f}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n)_T$ be defined as above. There is for each $q \in \mathbb{N}$ a positive constant C_q, depending on h and f, so that*

$$|\mathscr{W}_{h,f}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n)_T| \leqq C_q \frac{(1 + \|t\|)^q}{(1 + \|\underline{x}\|)^{(q-1)}}, \tag{42}$$

where $\|t\| := \max\limits_{1 \leqq i \leqq n-1} |t_{i+1} - t_i|$ and $\|\underline{x}\| := r(\Lambda_1 + \underline{x}_1, \ldots, \Lambda_n + \underline{x}_n)$ is defined in (21).

Proof. The first step of the proof consists in expressing the truncated Wightman functions (41) in terms of truncated Euclidean functions. We use the fact that by Cartier's formula [16]) truncated functions can be written as expectations in a tensored theory, a fact which is familiar for the two point function. Hence the arguments leading to relations (35) and (36) between Wightman functions and

Euclidean functions hold equally well for the truncated functions,

$$\mathscr{W}_{h,f}(\underline{x}_1,t_1;\ldots;\underline{x}_n,t_n)_T=\sum_{k\in\mathbb{Z}_+^{(n-1)}} b_k(h_t;\mu)C^T_{f_x}(k)\,, \tag{43}$$

where

$$h_t(t'_1,\ldots,t'_n)=h_1(t'_1-t_1)\ldots h_n(t'_n-t_n)\,,\qquad \mu=(n_1,\ldots,n_{n-1})\,,$$

and

$$f_x=(f_{1,\underline{x}_1},\ldots,f_{n,\underline{x}})\,.$$

C^T denotes the Chebishev transform (36) applied to the truncated Euclidean function. We find the following estimates: $\left(|k|=\sum_{i=1}^{n-1}k_i\right)$

$$|b_k(h_t,\mu)|\leqq C_M\left(\frac{1+\|t\|}{1+\|k\|}\right)^M,\qquad M\geqq 0,\ C_M\geqq 0 \tag{44}$$

$$|C^T_{f_x}(k)|\leqq \text{const}\,, \tag{45}$$

$$|C^T_{f_x}(k)|\leqq \text{const}\, e^{A(m)|k|-m\|x\|}\,,\qquad A(m)=2\,\mathrm{Ar}\sinh(e^{-m/2})\,. \tag{46}$$

Inequality (44) follows directly from the definition of b_k in (39), and (45) follows as in (3.8). For the derivation of inequality (46) we use the exponential decay of the truncated Euclidean functions (20). Inserting inequality (20) into (36) we get

$$|C^T_{f_x}(k)|\leqq \sum_{j\in\mathbb{Z}_+^{(n-1)}} |a^{k_1}_{j_1}|\ldots|a^{k_{n-1}}_{j_{n-1}}|e^{-m(|j|+\|x\|)} \tag{47}$$

with $\|x\|=r(\Lambda_1+\underline{x}_1,\ldots,\Lambda_n+\underline{x}_n)$ and $|j|=j_1+\ldots+j_{n-1}$. Using the alternating nature of the coefficients of the polynomials $T_{2k}(x^{1/2})$ we find

$$\sum_{j=0}^{k}|a^k_j|(-e^m)^j=(-1)^kT_{2k}(ie^{-m/2})=\cosh(kA(m))\leqq e^{kA(m)} \tag{48}$$

with $A(m)=2\arg\sinh(e^{-m/2})$, thus establishing (46).

We now split the sum in Eq. (43) into two terms, one corresponding to $|k|<N$ and the other to $|k|\geqq N$ for some $N\in\mathbb{N}$. For the first term we use inequality (44) with $M=0$ and inequality (46), for the second term we choose $M=q(n-1)$ in (44), where q is the natural number appearing in the formulation of the theorem and apply the uniform bound (45). We obtain for the first term

$$\text{const}\sum_{k\in\mathbb{Z}_+^{n-1},\,|k|<N} e^{|k|A(m)-m\|x\|}\leqq\text{const}\frac{(N+n-1)^{n-1}}{(n-1)}e^{A(m)N-m\|x\|}\,, \tag{49}$$

and for the second term

$$\text{const}\sum_{k\in\mathbb{Z}_+^{n-1},\,|k|\geqq N}\left(\frac{1+\|t\|}{1+|k|}\right)^{q(n-1)}\leqq\text{const}\left(\frac{(1+\|t\|)^q}{(1+N)^{q-1}}\right)^{n-1}. \tag{50}$$

Finally we choose $N=\varepsilon\|x\|$ with

$$\varepsilon<m/A(m)\quad\text{for}\quad(1+\|t\|)<(1+\|x\|)^{1-1/q}\,.$$

For $1+\|t\|\geqq(1+\|x\|)^{1-1/q}$ the bound in the theorem follows from the uniform boundedness of the truncated Wightman functions. □

Remark. Theorem 1 admits the following generalization to expectation values of arbitrary elements of the algebra $\mathscr{A}$ (Sect. 2):

Theorem 2. *Let $A_1, \ldots, A_n \in \mathscr{A}$, $n \geqq 2$, and define the space-time translated operators $A_i(\underline{x}_i, t_i) = U(\underline{x}_1)e^{iHt_i}A_i e^{-iHt_i}U(-\underline{x}_i)$. Then for the Wightman functions*

$$\mathscr{W}^{(n)}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n) := (\Omega, A_1(\underline{x}_1, t_1) \ldots A_n(\underline{x}_n, t_n)\Omega) \tag{51}$$

the following clustering property holds: for each ε, $0 < \varepsilon < 1$ and for each $q \in \mathbb{N}$, there is a positive constant $C_{q,\varepsilon}$, depending on the A_i's, so that

$$|\mathscr{W}_T^{(n)}(\underline{x}_1, t_1; \ldots; \underline{x}_n, t_n)| \leqq C_{q,\varepsilon} \frac{(1 + \|t\|)^q}{(1 + \|\underline{x}\|)^{(q-1)-\varepsilon}}, \tag{52}$$

where

$$\|t\| := \max_{1 \leqq i \leqq n-1} |t_{i+1} - t_i| \tag{53}$$

and

$$\|\underline{x}\| := \max_{i,j \in \{1, \ldots, n\}} |\underline{x}_i - \underline{x}_j| . \tag{54}$$

$\mathscr{W}_T^{(n)}$ are the truncated Wightman functions.

The proof of this theorem follows the same ideas of that of Theorem 1 and we omit the details here (see [15]).

5. Existence of Scattering States

One of the most remarkable properties of quantum field theory is the fact that, once single particle states are present, automatically also the corresponding incoming and outgoing multiparticle scattering states exist. The crucial ingredient for the proof of this fact are cluster properties of vacuum expectation values [1, 2] which, due to Ruelle can be derived from spectrum condition and locality [3]. In the case of Euclidean lattice field theory we derived a somewhat weaker cluster property in the preceding section. In the present section we will show that the result still suffices for the construction of scattering states. Since the argument is almost identical to that of the continuum we will be rather brief (see [16, 18] for modern treatments in a form which is applicable in our case).

According to Assumption (A.3) there is a closed subspace $\mathscr{H}^{(1)} \subset \mathscr{H}$ on which a single particle dispersion relation holds, and a dense subspace $\mathscr{D}^{(1)}$ of $\mathscr{H}^{(1)}$ which is created from the vacuum by almost local operators $A \in \mathscr{A}$. Actually it suffices to take operators of the form

$$A = f(h), \quad f \in \mathscr{E}_+, \quad h \in \tilde{\mathscr{D}}([m - \varepsilon, \infty)), \ \varepsilon > 0, \tag{55}$$

as defined in (15). In particular we may require $A^*\Omega = 0$.

Let $\phi \in \mathscr{D}^{(1)}$ and $A \in \mathscr{A}$ such that $A\Omega = \phi$ and $A^*\Omega = 0$. Consider the solution of the wave equation corresponding to the dispersion relation ω in (A.3),

$$f(t, \underline{x}) = (2\pi)^{-d} \int_{[-\pi,\pi)^d} d^d p e^{i(\underline{p} \cdot \underline{x} - \omega(\underline{p})t)} h(\underline{p}) \tag{56}$$

with $h(\underline{p}) = 1$ for $\underline{p} \in sp_U \phi$ (the momentum spectrum of ϕ) and $h \in C^\infty([-\pi, \pi))$. We set

$$A(t) = \sum_{\underline{x}} f(t, \underline{x}) \alpha_{(t, \underline{x})}(A) . \tag{57}$$

Then $A(t)\Omega = A\Omega = \phi$. The localization properties of $A(t)$ follow from those of f.

Notation 1. Let Δ be a closed set in $[-\pi, \pi)^d$. Then the velocity content $V(\Delta)$ *of the set of momenta* Δ *is*

$$V(\Delta) = \{\operatorname{grad} \omega(\underline{p}) \mid \underline{p} \in \Delta\}. \tag{58}$$

The behavior of the solutions of the wave equation (56) may be summarized as follows:

Proposition 1 (See [16]).

$$\text{(i)} \qquad \sum_{\underline{x}} |f(t, \underline{x})| \leqq (1 + |t|)^{d/2}, \tag{59}$$

$$\text{(ii)} \qquad |f(t, \underline{x})| \leqq c_N (1 + \operatorname{dist}(\underline{x}, tV(\Delta)))^{-N}, \qquad N \in \mathbb{N}, \tag{60}$$

where Δ *is a neighborhood of* $\operatorname{supp} h$ *(*h *as in* (56)*).*

Now let $\phi_1, \ldots, \phi_n \in \mathscr{D}^{(1)}$ such that $V(sp_U \phi_i) \cap V(sp_U \phi_j) = \emptyset$ for $i \neq j$. Choose $A_i \in \mathscr{A}$ with $A_i \Omega = \phi_i$, $A_i^* \Omega = 0$, and $h_i \in C^\infty((-\pi, \pi]^d)$ with $V(\operatorname{supp} h_i) \cap V(\operatorname{supp} h_j) = \emptyset$ for $i \neq j$, $h_i(\underline{p}) = 1$ for $\underline{p} \in sp_U \phi_i$. Let f_i be the solution of the wave equation (56) with h replaced by h_i and $A_i(t)$ be defined as in Eq. (57). The Haag-Ruelle approximant on the scattering state of particles with single particle states $\phi_1, \ldots, \phi_n$ is now defined by

$$\phi(t) = A_1(t) \ldots A_n(t)\Omega. \tag{61}$$

Theorem 3. (i) *The Haag-Ruelle-Approximants* $\phi(t)$ *in* (61) *converge for* $t \to \pm\infty$. *The limits* $(\phi_1 \times \ldots \times \phi_n)_{\text{out, in}}$ *depend only on the single particle states* $\phi_i \in \mathscr{D}^{(1)}$, *and for all* $N \in \mathbb{N}$,

$$\|\phi(t) - (\phi_1 \times \ldots \times \phi_n)_{\text{out, in}}\| t^N \to 0, \qquad t \to \pm\infty. \tag{62}$$

(ii) *Let* $\psi_1, \ldots, \psi_k \in \mathscr{D}_{(1)}$ *with* $V(sp_u \psi_i) \cap V(sp_u \psi_j) = \emptyset$, $i \neq j$. *Then*

$$((\psi_1 \times \ldots \times \psi_k)_{\text{out, in}}, (\phi_1 \times \ldots \times \phi_n)_{\text{out, in}}) = \delta_{n,k} \sum_\sigma \prod_i (\psi_i, \phi_{\sigma(i)}), \tag{63}$$

where the sum is over elements σ *of the permutation group of* $\{1, \ldots, n\}$.

Proof. (i) As usual, (i) is derived from the fast decrease of the derivative $\frac{d}{dt}\phi(t)$. We have

$$\left\| \frac{d}{dt} \phi(t) \right\|^2 = \sum_{k,l=1}^{n} (\Omega, A_n(t)^* \ldots \dot{A}_k(t)^* \ldots A_1(t)^* A_1(t) \ldots \dot{A}_l(t) \ldots A_n(t)\Omega), \tag{64}$$

where

$$\dot{A}_k(t) = \frac{d}{dt} A_k(t) = \sum_{\underline{x}} \left\{ \left(\frac{\partial}{\partial t} f_k(t, \underline{x}) \alpha_{(t,\underline{x})}(A_k) + f_k(t, \underline{x}) \, [iH, \alpha_{(t,\underline{x})}(A_k)] \right) \right\}. \tag{65}$$

We represent the right-hand side of (64) as a sum of products of truncated functions. As $\dot{A}_k(t)\Omega = 0$ and $A_k^* \Omega = 0$ terms with only 2- and 1-point functions vanish. For a truncated k-point function with $k > 2$ we now use Theorem 2 together with Proposition 1. The expression to be estimated is of the form

$$\sum_{\underline{x}_1, \ldots, \underline{x}_k} g_1(t_1, \underline{x}_1) \ldots g_k(t_k, \underline{x}_k) W_T^{(k)}(\underline{x}_1, t; \ldots; \underline{x}_k, t), \tag{66}$$

where $W_T^{(k)}$ is the truncated vacuum expectation value for operators

$$B_1, \ldots, B_k \in \{A_1, \ldots, A_n, A_1^*, \ldots, A_n^*, [H, A_1], \ldots, [H, A_n^*]\} \subset \mathscr{A}, \tag{67}$$

and $g_1, \ldots, g_k$ are solutions of the wave equations from the set

$$\{f_1, \ldots, f_k, \overline{f_1}, \ldots, \overline{f_k}, \partial_t f_1, \ldots, \partial_t \overline{f_n}\}. \tag{68}$$

We split the sum in (66) in two pieces. In the first one we sum over all $\underline{x}_i \in tV(\Delta_i)$, $i=1, \ldots, k$, where Δ_i is a closed neighborhood of the set of velocities contained in g_i, $i=1, \ldots, k$ such that either $\Delta_i = \Delta_j$ (if g_i and g_j are obtained from the same solution f_l by complex conjugation and/or time derivative) or $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. For $k \geqq 3$ at least two different sets Δ_i occur. Hence Theorem 2 yields a strong decrease of $W_T^{(k)}$ in t which is uniform in $\underline{x}_i \in tV(\Delta_i)$, $i=1, \ldots, n$; as the sum over $|g_i|$ is only polynomially increasing according to (59), we find that the first contribution is strongly decreasing in t. For the second contribution we use the strong decrease of the wave functions g_i outside of the kinematically allowed range (60) together with the uniform boundedness of the truncated Wightman functions. We conclude that the expression (66) is strongly decreasing in t. As all other factors increase at most polynomially we conclude that $\frac{d}{dt}\phi(t)$ in (64) is fast decreasing, thus proving (i).

(ii) The proof of (ii) is similar. Let

$$\psi(t) = B_1(t) \ldots B_k(t)\Omega \tag{69}$$

denote the Haag-Ruelle-approximant for $(\psi_1 \times \ldots \times \psi_k)_{\mathrm{out,in}}$. Then

$$(\psi(t), \phi(t)) = (\Omega, B_k(t)^* \ldots B_1(t)^* A_1(t) \ldots A_n(t)\Omega) \tag{70}$$

may again be expanded into a sum of products of truncated functions. Terms with truncated l-point functions vanish for $l \geqq 3$ in the limits $t \to \pm\infty$ by analogous arguments as above, and 1-point functions vanish identically. Hence the only contributions to the limit come from 2-point functions

$$(\Omega, B_l(t)^* A_i(t)\Omega) = (\psi_l, \phi_i). \tag{71}$$

This implies (63). □

Note that relation (63) characterizes the statistics of particles as being bosonic, i.e. the vector states $(\phi_1 \times \ldots \times \phi_n)_{\mathrm{out,in}}$ are symmetric under permutations of the single particle state vectors. The translations act on the scattering state as expected

$$U(\underline{x})e^{iHt}(\phi_1 \times \ldots \times \phi_n)_{\mathrm{out,in}} = ((U(\underline{x})e^{iHt}\phi_1) \times \ldots \times (U(\underline{x})e^{iHt}\phi_n))_{\mathrm{out,in}}. \tag{72}$$

The full scattering spaces (including products of particle states with overlapping velocities) are obtained as closures of the linear span $\mathscr{D}_{\mathrm{out,in}}$ of the scattering state vectors with non-overlapping velocities. Let $\mathscr{F}(\mathscr{H}^{(1)})$ denote the bosonic Fock space over $\mathscr{H}^{(1)}$, and let $\mathscr{F}_0$ denote the linear subspace of $\mathscr{F}(\mathscr{H}^{(1)})$ spanned by tensor products of single particle vectors from $\mathscr{D}^{(1)}$ with non-overlapping velocities. Let

$$U_{\mathrm{out,in}}: \mathscr{F}_0 \to \mathscr{H} \tag{73}$$

be linear operators defined by

$$U_{\mathrm{out,in}}(\phi_1 \otimes \ldots \otimes \phi_n) = (\phi_1 \times \ldots \times \phi_n)_{\mathrm{out,in}}. \tag{74}$$

Because of (63) the operators $U_{\mathrm{out,in}}$ are isometries and extend therefore to the closure of $\mathscr{F}_0$. Provided the group velocity $\mathrm{grad}\,\omega(p)$ is nowhere constant as a function of p (which we assume in (A.3)) the subspace $\mathscr{F}_0$ is dense in the full Fock space, and we obtain two isometric images $\mathscr{H}_{\mathrm{out,in}} = \overline{\mathscr{D}_{\mathrm{out,in}}}$ of the Fock space in the physical Hilbert space $\mathscr{H}$.

The physical interpretation of $\mathscr{D}^{\mathrm{out,in}}$ as spaces of scattering states can be tested by looking at expectation values of operators representing counters in the sense of Haag and Araki [6]. By methods similar to those in the proof of Theorem 3 one obtains the expected results [15]. Whether the same results can be derived for $\mathscr{H}_{\mathrm{out,in}}$, as was done by Buchholz [12] for relativistic theories by using locality is unknown up to now. A solution of this problem may provide a step towards a proof of asymptotic completeness in lattice field theory.

Finally one may discuss whether the scattering amplitudes can be expressed in terms of time ordered correlation functions by LSZ relations. Formulae of this type have actually been found in [15]. It is an interesting problem whether these formulae can be used for perturbation theoretical and numerical calculations of scattering amplitudes. We hope to discuss this problem elsewhere.

Acknowledgements. We are indebted to Detlev Buchholz, Martin Lüscher, and Mihail Marcu for valuable discussions.

References

1. Haag, R.: Quantum field theories with composite particles and asymptotic conditions. Phys. Rev. **112**, 669–673 (1958)
2. Haag, R.: The framework of quantum field theory. (1959) Supplemento al vol. XIV, serie X del Nuovo Cimmento, 1563
3. Ruelle, D.: On the asymptotic condition in quantum field theory. Helv. Phys. Acta **35**, 147–163 (1962)
4. Hepp, K.: On the connection between the LSZ and the Wightman quantum field theory. Commun. Math. Phys. **1**, 95–111 (1965)
5. Hepp, K.: On the connection between the LSZ and the Wightman quantum field theory. In: Axiomatic field theory. Brandeis 1965. Chretien, M., Deser, S. (eds.). New York: Gordon and Breach 1966
6. Araki, H., Haag, R.: Collision cross sections in terms of local observables. Commun. Math. Phys. **4**, 77–91 (1967)
7. Fredenhagen, K.: On the existence of the real time evolution in Euclidean lattice gauge theories. Commun. Math. Phys. **101**, 579–587 (1985)
8. Lehmann, H., Symanzik, K., Zimmermann, W.: Zur Formulierung Quantisierter Feldtheorien. Il Nuovo Cimento **I**, 205–225 (1955)
9. Osterwalder, K., Schrader, R.: Axioms for Euclidean Green's functions. Commun. Math. Phys. **31**, 83–112 (1973)
10. Osterwalder, K., Schrader, R.: Axioms for Euclidean Green's functions. II. Commun. Math. Phys. **42**, 281–305 (1975)
11. Barata, J.C.A., Fredenhagen, K.: Charged particles in $\mathbb{Z}_2$ gauge theories. Commun. Math. Phys. **113**, 403–417 (1987)
12. Buchholz, D.: Harmonic analysis of local operators. Commun. Math. Phys. **129**, 631–641 (1989)
13. Lüscher, M.: Streutheorie für euklidische skalare Feldtheorien auf einem Raumzeitgitter. Unpublished notes (1985)
14. Lüscher, M.: Selected topics in lattice field theory. Desy-preprint 88–156. "Lectures given at the Summer School 'Fields, Strings, and Critical Phenomena', Les Houches, France (1988)"

15. Barata, J.C.A.: Scattering theory for Euclidean lattice field theories. PhD Thesis, F.U. Berlin, 1989
16. Reed, M., Simon, B.: Methods of modern mathematical physics. Scattering theory. Vol. III. New York: Academic Press 1972–1979
17. Fox, L., Parker, I.B.: Chebishev polynomials in numerical analysis. Oxford: Oxford University Press (1969)
18. Glimm, J., Jaffe, A.: Quantum Physics. A Functional Integral Point of View. Second Edition. Berlin, Heidelberg, New York: Springer 1987
19. Barata, J.C.A.: Scattering states of charged particles in the $\mathbb{Z}_2$ Gauge Higgs model, in preparation

Communicated by K. Gawedzki