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Localization in the Ground State of the Ising Model with a Random Transverse Field

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Abstract. We study the zero-temperature behavior of the Ising model in the presence of a random transverse field. The Hamiltonian is given by

$$H = -J \sum_{\langle x,y \rangle} \sigma_3(x)\sigma_3(y) - \sum_x h(x)\sigma_1(x),$$

where $J > 0$, $x, y \in \mathbf{Z}^d$, σ_1, σ_3 are the usual Pauli spin $\frac{1}{2}$ matrices, and $\mathbf{h} = \{h(x), x \in \mathbf{Z}^d\}$ are independent identically distributed random variables. We consider the ground state correlation function $\langle \sigma_3(x)\sigma_3(y) \rangle$ and prove:

1. Let d be arbitrary. For any $m > 0$ and J sufficiently small we have, for almost every choice of the random transverse field $\mathbf{h}$ and every $x \in \mathbf{Z}^d$, that

$$\langle \sigma_3(x)\sigma_3(y) \rangle \leqq C_{x,\mathbf{h}} e^{-m|x-y|}$$

for all $y \in \mathbf{Z}^d$ with $C_{x,\mathbf{h}} < \infty$.

2. Let $d \geqq 2$. If J is sufficiently large, then, for almost every choice of the random transverse field $\mathbf{h}$, the model exhibits long range order, i.e.,

$$\overline{\lim_{|y| \mapsto \infty}} \langle \sigma_3(x)\sigma_3(y) \rangle > 0$$

for any $x \in \mathbf{Z}^d$.

1. Introduction

Quantum spin systems with random parameters have been introduced to study the effects of impurities in several physical systems (see for example, Halperin, Lee

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Examples of such systems are given by quantum $X-Y$, Heisenberg and Ising models in the presence of a random transverse field. In [1] it was argued that for such models localization should take place in the ground state of the system destroying the long-range order of the non-random component of the spin system, for sufficiently high disorder.

Klein and Perez [2] have studied the quantum $X-Y$ model with a random transverse field in one dimension and proved localization in the ground state of the system for any disorder. In particular they proved exponential decay for the two-point function, which is to be compared with the polynomial decay obtained by Lieb, Schultz and Mattis [3] for zero transverse field. Their method was to map the model into a free Fermi gas in the presence of a random external potential; the one-particle Hamiltonian for the Fermi gas turned out to be the one-dimensional Anderson Hamiltonian and exponential decay for the two-point function followed from Anderson localization.

In this article we study the Ising model in the presence of a random transverse field. The corresponding deterministic model appears in the pseudospin formulation of several phase transition problems and was used to study order disorder ferroelectrics with a tunneling effect by de Gennes [4] and magnetic ordering in materials with singlet crystal field ground state by Wang and Cooper [5]. The one-dimensional deterministic model was studied by Pfeuty [6] following Lieb, Schultz and Mattis [3].

The Ising model with a random transverse field is given, in a finite volume $\Lambda \subset \mathbf{Z}^d$, by the Hamiltonain

$$H_\Lambda = -J \sum_{\langle x,y\rangle \subset \Lambda} \sigma_3(x)\sigma_3(y) - \sum_{x\in\Lambda} h(x)\sigma_1(x)$$

acting on the Hilbert space $\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x$, with $\mathscr{H}_x = \mathbf{C}^2$ for all x, where $J>0$, $\langle x,y\rangle$ denote a pair of nearest neighbor sites, σ_1, σ_3 are the usual Pauli spin $\frac{1}{2}$ matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with $\sigma_i(x), i=1,3, x\in\Lambda$, the corresponding operator on $\mathscr{H}_\Lambda$ acting only on $\mathscr{H}_x$. The random transverse field is $\mathbf{h} = \{h(x), x\in\mathbf{Z}^d\}$, where the $h(x), x\in\mathbf{Z}^d$ are taken to be independent identically distributed random variables. Since for any $x_0\in\Lambda$ we have

$$\sigma_3(x_0)H_\Lambda\sigma_3(x_0) = H_\Lambda + 2h(x_0)\sigma_1(x_0),$$

we can take $h(x) \geqq 0$ without loss of generality.

If $h(x)>0$ for all $x\in\mathbf{Z}^d$, H_Λ has a unique ground state Ω_Λ for each Λ and, the correlation functions

$$\langle\sigma_3(x)\sigma_3(y)\rangle_\Lambda \equiv (\Omega_\Lambda, \sigma_3(x)\sigma_3(y)\Omega_\Lambda)$$

are monotone increasing in Λ and decreasing functions of each $h(x)$. These follow from the representation of H_Λ as the generator of a positivity improving semigroup plus correlation inequalities derived in the corresponding path space (see the discussion in Sect. 2). We can thus define the infinite volume ground state correlation function by

$$\langle \sigma_3(x)\sigma_3(y) \rangle = \lim_{\Lambda \nearrow \mathbf{Z}^d} \langle \sigma_3(x)\sigma_3(y) \rangle_\Lambda, \tag{1.1}$$

preserving the monotonicity in each $h(x)$. *We will always assume* $\mathbf{P}\{h(x) > 0\} = 1$.

The deterministic uniform model, i.e., $h(x) \equiv h > 0$ for all x, is one of the simplest quantum spin system with a non-trivial phase diagram, typical for a large class of models exhibiting discrete symmetry breakdown. The relevant parameter is $\zeta = h/J$. In any dimension there exists $0 < \zeta_1 \leqq \zeta_2 < \infty$ such that if $\zeta > \zeta_2$ the correlation function (1.1) decays exponentially and if $\zeta < \zeta_1$ there is long range order [7, 8]. In one dimension it is known that $\zeta_1 = \zeta_2 = 2$ [8].

It follows from the monotonicity of (1.1) in each $h(x)$ that, with $\zeta(x) = h(x)/J$, the random model correlation function decays exponentially if $\zeta(x) > \zeta_2$ with probability one, and exhibits long range order if $\zeta(x) < \zeta_1$ with probability one. Thus, if $0 < a \leqq h(x) \leqq b < \infty$ with probability one, the random model will exhibit a phase transition by varying J.

The interesting nontrivial case is thus when the events $\{\zeta(x) > \zeta_2\}$ and $\{\zeta(x) < \zeta_1\}$ both have nonzero probability, so the system exhibits Griffiths' singularities. Typical cases would be when each $h(x)$ is uniformly distributed or the interval $[0, 1]$ or exponentially distributed. Let $p_1 = \mathbf{P}\{\zeta(x) < \zeta_1), p_2 = \mathbf{P}\{\zeta(x) > \zeta_2)$, and recall $\mathbf{P}\{h(x) > 0\} = 1$. Then, we have $\lim_{J \downarrow 0} p_1 = 0$, $\lim_{J \downarrow 0} p_2 = 1$ so for J sufficiently small we should expect exponential decay of the correlation function (1.1) with probability one. On the other hand, $\lim_{J \to \infty} p_1 = 1$, $\lim_{J \to \infty} p_2 = 0$, so for sufficiently large the system should exhibit long range order.

Our results are

Theorem 1.1. *Suppose* $\mathbf{E}(h(x)^{-\delta}) < \infty$ *for some* $\delta > 0$. *Then for any* $d = 1, 2, \ldots$ *and* $m > 0$ *there exists* $J_1 > 0$ *such that for any* $J < J_1$ *and for almost every choice of the random transverse field* $\mathbf{h}$ *and every* $x \in \mathbf{Z}^d$ *we have*

$$\langle \sigma_3(x)\sigma_3(y) \rangle \leqq C_{x,\mathbf{h}} e^{-m|x-y|}$$

for all $y \in \mathbf{Z}^d$ *with* $C_{x,\mathbf{h}} < \infty$.

Theorem 1.2. *Let* $h(x)$ *have an arbitrary distribution. Then for any* $d \geqq 2$ *there exists* $J_2 < \infty$ *such that for all* $J > J_2$ *we have, for almost every choice of the random transverse field,*

$$\overline{\lim_{|y| \to \infty}} \langle \sigma_3(x)\sigma_3(y) \rangle > 0$$

for any $x \in \mathbf{Z}^d$.

Following Driessler, Landau and Perez [8] we write the correlation function (1.1) as the limit of two-point functions of $(d+1)$-dimensional classical Ising models with d-dimensional disorder. In Sect. 2 we show that

$$\langle \sigma_3(x)\sigma_3(y) \rangle = \lim_{n \to \infty} \langle \sigma(x, 0)\sigma(y, 0) \rangle^{(n)},$$

where $\langle\ \rangle^{(n)}$ is the expectation for the classical Ising model on $\mathbf{Z}^d \times \frac{1}{n}\mathbf{Z}$ with

Hamiltonian

$$H^{(n)} = \frac{-J}{n} \sum_{t} \sum_{\langle x,y \rangle} \sigma(x,t)\sigma(y,t) - \sum_{x} \sum_{t} K_n(x)\sigma(x,t)\sigma\left(x, t + \frac{1}{n}\right), \tag{1.2}$$

where $\tanh K_n(x) = e^{-(2/n)h(x)}$. Models of this type were studied by Campanino and Klein [9], who developed a multiscale expansion to prove exponential decay for the $(d+1)$-dimensional system with d-dimensional disorder, and gave a simple percolation argument to show long range order when $d \geqq 2$. They proved analogous results to Theorems 1.1 and 1.2 for such models, but their results can be applied directly only for n fixed. In this article we refine their methods to obtain estimates uniform in n for n large.

This paper is organized as follows. In Sect. 2 we discuss some general features of the deterministic model, construct the associated path space and the approximation by classical Ising models. In Sect. 3 we obtain mean field type bounds on the deterministic system which will give the initial step for the multiscale analysis. Section 4 contains the multiscale analysis and the proof of Theorem 1.1. In Sect. 5 we prove Theorem 1.2.

2. The Approximation by Classical Ising Models

Let $\mathscr{G}_\Lambda = \{1, -1\}^\Lambda$, if $\sigma \in \mathscr{G}_\Lambda$ we have $\sigma = \{\sigma(x), x \in \Lambda\}$ with each $\sigma(x) \in \{1, -1\}$.

If we identify $\mathbf{C}^2$ with $l^2(\{1, -1\})$ in the obvious way we can identify $\mathscr{H}$ with $l^2(\mathscr{G}_\lambda)$; notice that the matrices of linear operators with respect to the standard base in either $\mathbf{C}^2$ or $\mathscr{H}_\lambda$ are now the kernels of the same operators on $l^2(\{1, -1\})$ or $l^2(\mathscr{G}_\Lambda)$, respectively.

In this representation the operator $\sigma_3(x)$ is given by multiplication by the function $\sigma(x)$, for each $x \in \Lambda$. Thus, if we write

$$H_\Lambda = H^I_\Lambda + H^t_\Lambda$$

with

$$H^I_\Lambda = -J \sum_{\langle x,y \rangle \subset \Lambda} \sigma_3(x)\sigma_3(y),$$

H^I_Λ is given by multiplication by the function $-J \sum_{\langle x,y \rangle \subset \Lambda} \sigma(x)\sigma(y)$ and $H^t_\Lambda = -\sum_{x \in \Lambda} h(x)\sigma_1(x)$ generate a positivity improving semigroup since $h(x) > 0$ for all x and

$$e^{t\sigma_1} = \cosh t + (\sinh t)\sigma_1$$

has a strictly positive kernel for $t > 0$.

It follows from the general theory that H_Λ generates a positivity improving semigroup and hence H_Λ has a unique ground state Ω_Λ which is a strictly positive function. In particular, there exists a path space, i.e., a stochastic process $\{\sigma(x,t); x \in \Lambda, t \in \mathbf{R}\}$ taking value on $\{1, -1\}$, stationary and symmetric with respect to t, such that, for example,

$$\frac{(\Omega_\Lambda, \sigma_3(x)e^{-|t|H_\Lambda}\sigma_3(y)\Omega_\Lambda)}{(\Omega_\Lambda, e^{-|t|H_\Lambda}\Omega_\Lambda)} = \langle \sigma(x,0)\sigma(y,t) \rangle,$$

where $\langle\ \rangle$ denotes the expectation in the stochastic process (see, for instance, Klein and Landau [10] for a general discussion).

In our situation $\mathscr{H}_\Lambda$ is finite dimensional, H_Λ, H^I_Λ, H^t_Λ are therefore bounded self-adjoint operators, so it is possible to do everything more explicitly. For example, to show uniqueness of the ground, let $\lambda < H^I_\Lambda$, we have

$$(H_\Lambda - \lambda)^{-1} = (H^I_\Lambda - \lambda)^{-1} \sum_{n=0}^{\infty} (-H^t_\Lambda (H^I_\Lambda - \lambda)^{-1})^n,$$

the series being uniformly convergent for $\lambda \ll H^I_\Lambda$. It clearly follows that for such λ, $(H_\Lambda - \Lambda)^{-1}$ has strictly positive kernel and hence the Perron–Frobenius theorem applies so we can conclude that H_Λ has a unique ground state Ω_Λ which is a strictly positive function.

The operator H^t_Λ has the (normalized) unique ground state $\Omega^{(0)}_\Lambda$ given by

$$\Omega^{(0)}_\Lambda(\sigma) = \frac{1}{2^{|\Lambda|}}$$

for all $\sigma \in \mathscr{G}_\Lambda$. It follows immediately that

$$(\Omega_\Lambda, \Omega^{(0)}_\Lambda) > 0,$$

so for any operator A in $\mathscr{H}_\Lambda$ we have

$$\langle A \rangle_\Lambda \equiv (\Omega_\Lambda, A\Omega_\Lambda) = \lim_{\beta \to \infty} \frac{(\Omega^{(0)}_\Lambda, e^{-(\beta/2)H_\Lambda} A e^{-(\beta/2)H_\Lambda} \Omega^{(0)}_\Lambda)}{(\Omega^{(0)}_\Lambda, e^{-\beta H_\Lambda} \Omega^{(0)}_\Lambda)}.$$

Following Driessler, Landau and Perez [8] we can use Trotter's product formula to conclude that, if $B \subset \Lambda$,

$$\left\langle \prod_{x\in B} \sigma_3(x) \right\rangle_\Lambda \equiv \left(\Omega_\Lambda, \prod_{x\in B} \sigma_3(x) \Omega_\Lambda \right) = \lim_{\beta\to\infty} \lim_{n\to\infty} \left\langle \prod_{x\in B} \sigma(x,0) \right\rangle^{(n)}_{\Lambda,\beta} \tag{2.1}$$

where $\langle\ \rangle^{(n)}_{\Lambda,\beta}$ is the expectation for the classical Ising model on $\mathbf{Z}^d \times \frac{1}{n}\mathbf{Z}$ with Hamiltonian given by (1.2) restricted to the region $\Lambda \times \left(\left[-\frac{\beta}{2}, \frac{\beta}{2} \right] \cap \frac{1}{n}\mathbf{Z} \right)$, with free boundary conditions.

Since our classical Ising models are ferromagnetic and we are using free boundary conditions, we can apply correlation inequalities to obtain

$$\left\langle \prod_{X\in \mathbf{W}} \sigma(X) \right\rangle^{(n)}_{\Lambda,\beta} \leqq \left\langle \prod_{X\in \mathbf{W}} \sigma(X) \right\rangle^{(n)}_{\Lambda',\beta'} \tag{2.2}$$

for any $\Lambda \subset \Lambda'$, $\beta \leqslant \beta'$, $\mathbf{W} \subset \Lambda \times \left(\left[-\frac{\beta}{2}, \frac{\beta}{2} \right] \cap \frac{1}{n}\mathbf{Z} \right)$.

Thus we can interchange the limits in (2.1) to conclude

$$\left\langle \prod_{x\in B} \sigma_3(x) \right\rangle_\Lambda = \lim_{n\to\infty} \left\langle \prod_{x\in B} \sigma(x,0) \right\rangle^{(n)}_\Lambda,$$

so using again (2.2) we obtain the existence of the limit

$$\left\langle \prod_{x\in B} \sigma_3(x) \right\rangle \equiv \lim_{\Lambda\to\mathbf{Z}^d} \left\langle \prod_{x\in B} \sigma_3(x) \right\rangle_\Lambda = \lim_{n\to\infty} \left\langle \prod_{x\in B} \sigma(x,0) \right\rangle^{(n)}.$$

In particular we obtain (1.1).

More generally, for a fixed finite Λ we define

$$\sigma_3(x,t)=e^{-tH_\Lambda}\sigma_3(x)e^{tH_\Lambda},$$

so we have existence of

$$\langle\sigma_3(x,t)\sigma_3(y,s)\rangle\equiv\lim_{\Lambda\to\mathbf{Z}^d}\langle\sigma_3(x,t)\sigma_3(y,s)\rangle_\Lambda$$

and

$$\langle\sigma_3(x,t)\sigma_3(y,s)\rangle=\lim_{n\to\infty}\langle\sigma(x,t^{(n)})\sigma(y,s^{(n)})\rangle^{(n)},\tag{2.3}$$

where for $r\in\mathbf{R}$ we let $r^{(n)}=\frac{1}{n}[nr]$ if $r\geqq 0$ and $r^{(n)}=-|r|^{(n)}$ if $r<0$.

3. Percolation, Self-Avoiding Walks and Mean-Field Bounds

For each n we consider the bond Bernoulli percolation model on $\mathbf{Z}^d\times\frac{1}{n}\mathbf{Z}$ with occupation probabilities

$$q^{(n)}_{(x,t),(y,s)}=\begin{cases}1-e^{-2(J/n)} & \text{if } t=s,\ x,\ y \text{ nearest neighbors}\\ 1-e^{-2K_n(x)} & \text{if } x=y,\ |t-s|=\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}$$

The corresponding percolation probability will be denoted by $\mathbf{Q}^{(n)}$; notice that it depends on the choice of the random transverse field $\mathbf{h}$.

If $\mathbf{W}\subset\mathbf{Z}^d\times\mathbf{R}$, we set $\mathbf{W}^{(n)}=\mathbf{W}\bigcap\left(\mathbf{Z}^d\times\frac{1}{n}\mathbf{Z}\right)$. If $X, Y\in\mathbf{W}^{(n)}$, by $X\underset{\mathbf{W}}{\longrightarrow}Y$ we mean that X is connected to Y by a path of occupied bonds in $\mathbf{W}^{(n)}$. We set

$$G^{(n)}_{\mathbf{W}}(X,Y)=\mathbf{Q}^{(n)}\{X\underset{\mathbf{W}}{\longrightarrow}Y\}.$$

It follows from the Fortuin–Kasteleyn representation of Ising models and from Fortuin's comparison principles (see Aizenman et al. [11]) that

$$\langle\sigma(X)\sigma(Y)\rangle^{(n)}_{\mathbf{W}}\leqq G^{(n)}_{\mathbf{W}}(X,Y),\tag{3.1}$$

where the left-hand-side denotes the two-point of the classical Ising model in $\mathbf{Z}^d\times\frac{1}{n}\mathbf{Z}$ with Hamiltonian given by (1.2), restricted to the region $\mathbf{W}^{(n)}$ with free boundary condition.

Following Campanino and Klein [9], we will prove Theorem 1.1 by showing decay for $G^{(n)}(X,Y)$.

Since if $X\underset{\mathbf{W}}{\longrightarrow}Y$ we can always find a self-avoiding walk in $\mathbf{W}^{(n)}$ starting at X and ending at Y, we also have

$$G^{(n)}_{\mathbf{W}}(X,Y)\leqq S^{(n)}_{\mathbf{W}}(X,Y),\tag{32}$$

where

$$S^{(n)}_{\mathbf{W}}(X,Y)=\sum_w q^{(n)}_w,\tag{3.3}$$

the summation being taken over all nearest neighbors self-avoiding walks w in $\mathbf{W}^{(n)}$ that go from X to Y, i.e., $W:\{0,1,\ldots,|w|\}\to\mathbf{W}^{(n)}$ with $w(0)=X, w(|w|)=Y$, and $w(i)$, $w(i+1)$ nearest neighbors, $w(i)\neq w(j)$ if $i\neq j$. Here

$$q_W^{(n)}=\prod_{i=0}^{|w|-1} q_{w(i),w(i+1)}^{(n)}.$$

Another similar bound for $\langle\sigma(X)\sigma(Y)\rangle^{(n)}$ (but not for $G^{(n)}(X,Y)$) was obtained by Fisher [12]:

$$\langle\sigma(X)\sigma(Y)\rangle_{\mathbf{W}}^{(n)}\leqq\tilde{S}_{\mathbf{W}}^{(n)}(X,Y), \tag{3.4}$$

where $\tilde{S}_{\mathbf{W}}^{(n)}$ is defined also by (3.3) but with $q_{x,y}^{(n)}$ replaced by

$$\tilde{q}_{(x,t),(y,s)}^{(n)}=\begin{cases}\tanh\dfrac{J}{n} & \text{if } t=s,\ x,y \text{ nearest neighbors}\\ \tanh K_n(x) & \text{if } x=y,\ |t-s|=\dfrac{1}{n}\\ 0 & \text{otherwise}\end{cases}.$$

Let $\mathbf{W}=\Lambda\times[-T,T]$, with $\Lambda\subset\mathbf{Z}^d$. Following Fisher [12] we estimate (3.3) by replacing the self-avoiding condition by the weaker requirement of no immediate return after a vertical step, obtaining

$$S_{\Lambda\pm[-T,T]}^{(n)}((x,t),(y,s))\leqq\sum_{J:x\to y}\ \sum_{k_0,\ldots,k_{|\tau|}}{}'\ \prod_{i=0}^{|\tau|}(1-e^{-2K_n(\tau(i))})^{|k_i|}(1-e^{-2(J/n)})^{|\tau|}, \tag{3.5}$$

where the first summation runs over all walks τ in Λ from x to y, the second summation being over the number of vertical steps k_i taken by τ after the i^{th} horizontal step, with $k_i>0$ if the steps are upwards, $k_i<0$ if downwards, the prime in the summation accounts for the restriction

$$\sum_{i=0}^{|\tau|}k_i=(s-t)^n. \tag{3.6}$$

In particular,

$$\sum_{i=0}^{|\tau|}|k_i|\geqq|s-t|n. \tag{3.7}$$

Recall $e^{-2K_n(x)}=\tanh\dfrac{h(x)}{n}$, and set $h_\Lambda=\min_{x\in\Lambda}h(x)$. We get

$$\begin{aligned}
&S_{\Lambda\times[-T,T]}^{(n)}((x,t),(y,s))\\
&\leqq\sum_{\tau:x\to y}(1-e^{-(2J/n)})^{|\tau|}\sum_{k_0,k_1,\ldots,k_{|\tau|}}{}'\left(1-\tanh\frac{h_\Lambda}{n}\right)^{\sum_{i=0}^{|\tau|}|k_i|}\\
&\leqq\left(1-\tanh\frac{h_\Lambda}{n}\right)^{\delta n|t-s|}\sum_{\tau:x\to y}(1-e^{-(2J/n)})^{|\tau|}\sum_{k_0,k_1,\ldots,k_{|\tau|}}{}'\left(1-\tanh\frac{h_\Lambda}{n}\right)^{(1-\delta)\sum_{i=0}^{|\tau|}|k_i|}\\
&\leqq\left(1-\tanh\frac{h_\Lambda}{n}\right)^{\delta n|t-s|}\sum_{\tau:x\to y}\ \sum_{k_0,k_1,\ldots,k_{|\tau|-1}}{}'\left(1-\tanh\frac{h_\Lambda}{n}\right)^{(1-\delta)\sum_{i=0}^{|\tau|-1}|k_i|}
\end{aligned}$$

for any $0 \leqq \delta < 1$, using (3.7) and (3.6); where the sum over $k_0, \ldots, k_{|\tau|-1}$ is now unrestricted.

Thus, we have

$$S^{(n)}_{\Lambda \times [-T,T]}((x,t),(y,s)) \leqq \left(1 - \tanh\frac{h_\Lambda}{n}\right)^{\delta n|t-s|} \sum_{\tau: x \to y} \zeta^{|\tau|}_{n,\delta}, \tag{3.8}$$

where

$$\zeta_{n,\delta} = \frac{2(1 - e^{-2(J/n)})}{1 - \left(1 - \tanh\frac{h}{n}\right)^{1-\delta}}.$$

If $0 < 2dR < 1$, we have

$$\sum_{\tau: x \to y} R^{|\tau|} = (-R\Delta_\Lambda + 1)^{-1}(x,y) \leqq \frac{(2dR)^{|x-y|}}{1 - 2dR}, \tag{3.9}$$

where Δ_Λ is the centered Laplacian in $\Lambda \subset \mathbf{Z}^d$, i.e., $\Delta_\Lambda(x,y) = 1$ if x and y in Λ are nearest neighbors, and equals zero otherwise. Thus it follows from (3.8) and (3.9) that

$$\begin{aligned} S^{(n)}_{\Lambda \times [-T,T]}((x,t),(y,s))) &\leqq \left(1 - \tanh\frac{h_\Lambda}{n}\right)^{\delta n|t-s|} (-\zeta_{n,\delta}\Delta_\Lambda + 1)^{-1}(x,y) \\ &\leqq \left(1 - \tanh\frac{h_\Lambda}{n}\right)^{\delta n|t-s|} \frac{(2d\zeta_{n,\delta})^{|x-y|}}{1 - 2d\zeta_{n,\delta}}. \end{aligned} \tag{3.10}$$

We have

$$\lim_{n\to\infty} \zeta_{n,\delta} = \frac{4J}{(1-\delta)h_\Lambda}.$$

It follows that, if $\frac{8dJ}{(1-\delta)h_\Lambda} < 1$, we have

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, S^{(n)}_{\Lambda \times [-T,T]}((x,t^{(n)}),(y,s^{(n)})) &\leqq e^{-\delta h_\Lambda |t-s|}\left(-\frac{4J}{(1-\delta)h_\Lambda}\Delta_\Lambda + 1\right)^{-1}(x,y) \\ &\leqq \left(1 - \frac{8dJ}{(1-\delta)h_\Lambda}\right)^{-1} e^{-\delta h_\Lambda |t-s|}\left(\frac{8dJ}{(1-\delta)h_\Lambda}\right)^{|x-y|} \end{aligned} \tag{3.11}$$

for any $0 < \delta < 1$.

Similarly, if $\frac{2dJ}{(1-\delta)h_\Lambda} < 1$, we have

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, \tilde{S}^{(n)}_{\Lambda \times [-T,T]}((x,t^{(n)}),(y,s^{(n)})) &\leqq e^{-2\delta h_\Lambda |t-s|}\left(-\frac{J}{(1-\delta)h_\Lambda}\Delta_\Lambda + 1\right)^{-1}(x,y) \\ &\leqq \left(1 - \frac{2dJ}{(1-\delta)h_\Lambda}\right)^{-1} e^{-2\delta h_\Lambda |t-s|}\left(\frac{2dJ}{(1-\delta)h_\Lambda}\right)^{|x-y|}. \end{aligned} \tag{3.12}$$

In particular, from (1.1), (2.3), (3.4) and (3.12) we immediately get

Theorem 3.1. *Suppose* $\bar{h} = \inf_{x \in \mathbf{Z}^d} h(x) > 0$.

Then, if $2dJ < \bar{h}$ *we have*

$$\langle \sigma_3(x)\sigma_3(y) \rangle \leqq \left(-\frac{J}{\bar{h}} \Delta + 1 \right)^{-1} (x, y) \leqq \left(1 - \frac{2dJ}{\bar{h}} \right)^{-1} \left(\frac{2dJ}{\bar{h}} \right)^{|x-y|}$$

for all $x, y \in \mathbf{Z}^d$.

More generally, we have

$$\langle \sigma_3(x,t)\sigma_3(y,s) \rangle \leqq e^{-2\delta\bar{h}|t-s|} \left(\frac{-J}{(1-\delta)\bar{h}} \Delta + 1 \right)^{-1} (x, y)$$

$$\leqq \left(1 - \frac{2dJ}{(1-\delta)\bar{h}} \right)^{-1} e^{-2\delta\bar{h}|t-s|} \left(\frac{2dJ}{(1-\delta)\bar{h}} \right)^{|t-y|}$$

for any $x, y \in \mathbf{Z}^d, t, s \in \mathbf{R}$ *and all* $0 \leqq \delta < 1$ *such that* $2dJ < (1-\delta)\bar{h}$.

4. The Multiscale Analysis

We will now prove Theorem 1.1. Our proof follows the proof of Theorem 2.1 in [9], the main difference is that we need to control the limit as $n \to \infty$ in (1.1) so we must perform a multiscale analysis uniformly in n for n large.

In view of (3.1) Theorem 1.1 will follows from

Theorem 4.1. *Suppose* $\mathrm{E}(h(x)^{-\delta}) < \infty$ *for some* $\delta > 0$. *Then for any* $d \geqq 1, m > 0$ *and* $v > 1$ *there exists* $J_1 > 0$ *and* $n_1 < \infty$ *such that if* $J < J_1$, *then for almost every choice of* $\mathbf{h}$ *we have*

$$G^{(n)}((x,t),(y,s)) \leqq C_{x,\mathbf{h}} e^{m|(x-y,(\log|t-s|)^v)|}$$

for all $n \geqq n_1, x, y \in \mathbf{Z}^d, t, s \in \mathbf{R}$, *with* $C_{x,\mathbf{h}} = C_{x,\mathbf{h}}(J, m, v) < \infty$.

We will restrict ourselves to the case $v = 2$, the modification for arbitrary $v > 1$ will be clear. Notice we use the notation $|(x,t)| = \max\{|x|, |t|\}$, where $|x| \equiv \|x\|_\infty$ for $x \in \mathbf{Z}^d$.

The proof of Theorem 4.1 will use properties of independent bond percolation, including the Harris-FKG, van der Berg–Kesten (v – BK) and Hammersley–Simon–Lieb (HSL) inequalities. The first two will be used as described in [9], but we will need a slightly different form of the HSL inequality which follows from the v – BK inequality.

Let $\Lambda, \Lambda' \subset \mathbf{Z}^d, I, I' \subset \mathbf{R}, \mathbf{W} = \Lambda \times I$, $\mathbf{W}' = \Lambda' \times I'$. Let $\mathbf{W}^{(n)} = \mathbf{W} \cap \left(\mathbf{Z}^d \times \frac{1}{n} \mathbf{Z} \right)$, similarly for $\mathbf{W}'^{(n)}$. We set

$$\partial_H^{(n)}(\mathbf{W}, \mathbf{W}') = \left\{ (y,s) \in \mathbf{W}^{(n)} \cap \mathbf{W}'^{(n)};\ \text{where} \left(y, s + \frac{1}{n} \right) \text{or} \left(y, s - \frac{1}{n} \right) \in \mathbf{W}'^{(n)} \backslash \mathbf{W}^{(n)} \right\},$$

$$\partial_V^{(n)}(\mathbf{W}, \mathbf{W}') = \{ \langle (y,s), (y',s) \rangle; (y,s) \in \mathbf{W}^{(n)} \cap \mathbf{W}'^{(n)}, (y,s') \in \mathbf{W}'^{(n)} \backslash \mathbf{W}'^{(n)} \}.$$

If $\mathbf{W}' = \mathbf{Z}^d \times \mathbf{R}$, we omit $\mathbf{W}'$. We also write $\partial^{(n)}\mathbf{W} = \partial_H^{(n)}\mathbf{W} \cup \{Z; \langle Z, Z'\rangle \in \partial_V^{(n)}\mathbf{W}$ for some $Z'\}$.

Now let $X \in \mathbf{W}^{(n)} \cap \mathbf{W}'^{(n)}$, $Y \in \mathbf{W}'^{(n)} \setminus \mathbf{W}^{(n)}$, it follows from the HSL inequality that

$$G_{\mathbf{W}'}^{(n)}(X, Y) \leqq \sum_{Z \in \partial_H^{(n)}(\mathbf{W}, \mathbf{W}')} G_{\mathbf{W} \cap \mathbf{W}'}^{(n)}(X, Z) G_{\mathbf{W}'}^{(n)}(Z, Y) + (1 - e^{-2(J/n)}) \sum_{\langle Z, Z'\rangle \in \partial_V^{(n)}(\mathbf{W}, \mathbf{W}')} G_{\mathbf{W} \cap \mathbf{W}'}^{(n)}(X, Z) G_{\mathbf{W}'}^{(n)}(Z', Y). \tag{4.1}$$

For large n, $1 - e^{-2J/n} \approx \frac{2J}{n}$; this factor is needed in (4.1) since a vertical line of length T contains nT points of $\mathbf{Z}^d \times \frac{1}{n}\mathbf{Z}$.

We will use the following consequence of (4.1). If $X \in \mathbf{W}^{(n)}$, let

$$G_{\mathbf{W}}^{(n)}(X, \partial) = \sum_{Z \in \partial_H^{(n)}\mathbf{W}} G_{\mathbf{W}}^{(n)}(X, Z) + (1 - e^{-2J/n}) \sum_{\langle Z, Z'\rangle \in \partial_V^{(n)}\mathbf{W}} G_{\mathbf{W}}^{(n)}(X, Z).$$

Then, for $X \in \mathbf{W}^{(n)} \cap \mathbf{W}'^{(n)}$, $Y \in \mathbf{W}'^{(n)} \setminus \mathbf{W}^{(n)}$, we have

$$G_{\mathbf{W}'}^{(n)}(X, Y) \leqq G_{\mathbf{W}}^{(n)}(X, \partial) G_{\mathbf{W}'}(Z_1, Y) \tag{4.2}$$

for some

$$Z_1 \in \partial_H^{(n)}(\mathbf{W}, \mathbf{W}') \cup \{Z'; \langle Z, Z'\rangle \in \partial_V^{(n)}(\mathbf{W}, \mathbf{W}') \text{ for some } Z\}.$$

We will now start the multiscale analysis. For $x \in \mathbf{Z}^d$ and $L > 0$ let us consider the hypercube

$$\Lambda_L(x) = \{y \in \mathbf{Z}^d; |x - y| \leqq L\}.$$

For $X = (x, t) \in \mathbf{Z}^d \times \mathbf{R}$, $L > 0$, $T > 0$, we consider the cylinder

$$B_{L,T}(X) = \Lambda_L(x) \times [t - T, t + T]$$

and, in particular

$$B_L(X) = B_{L, e^{\sqrt{L}}}(X)$$

Definitions. Let $m > 0$, $L > 0$, $\bar{n} > 0$. A site $x \in \mathbf{Z}^d$ is said to be $(m, L, \bar{n})$-regular if

$$G_{B_L((x,0))}^{(n)}((x, 0), Y) \leqq e^{-mL}$$

for all $n \geqq \bar{n}$ and $Y \in \partial^{(n)} B_L((x, 0))$. Otherwise x is said to be $(m, L, \bar{n})$-singular. A set $\Lambda \subset \mathbf{Z}^d$ is called $(m, L, \bar{n})$-regular if every $x \in \Lambda$ is $(m, L, \bar{n})$-regular.

If x is $(m, L, \bar{n})$-regular we have

$$G_{B_L((x,t))}^{(n)}((x, t), \partial) \leqq e^{-(m - 2/\sqrt{L})L} \tag{4.3}$$

for all $n > \bar{n}$, $t \in \mathbf{R}$ and all L sufficiently large.

Theorem 4.2. *Assume $\mathbf{E}(h(x)^{-\delta}) < \infty$ for some $\delta > 0$. Fix $J > 0$, and let $p > 2d^2$. Suppose there exists $m_0 > 0$, $L_0 > 0$ and $n_0 > 0$ such that*

$$\mathbf{P}\{0 \textit{ is } (m_0, L_0, n_0) - \textit{regular}\} \geqq 1 - \frac{1}{L_0^p}.$$

Let $\alpha \in \left(2d, \frac{p}{d}\right)$, set $L_{k+1} = L_k^\alpha$, $k = 0, 1, \ldots$. Then for any $0 < m_\infty < m_0$ there exists

$\bar{L}(p, d, m_0, m_\infty, \alpha, J) < \infty$, nondecreasing in J, such that if $L_0 > \bar{L}$ there exists $\bar{n} < \infty$ such that

$$\mathbf{P}\{0 \textit{ is } (m_\infty, L_k, \bar{n}) - \textit{regular}\} \geqq 1 - \frac{1}{L_k^p}$$

for all $k = 0, 1, 2, \ldots$.

Theorem 4.2 implies Theorem 4.1. For given $m > 0$ let $m_0 = 2m$, $m_2 = m$, take $\Lambda = \Lambda_L(0)$ and suppose that

$$h_\Lambda > 16dJe^{2m_0} \tag{4.4}$$

and

$$h_\Lambda > 4m_0 Le^{\sqrt{L}}. \tag{4.5}$$

It follows from (3.2) and (3.11) that if (4.4) and (4.5) hold we have that 0 is (m_0, L, n_1)-regular for some $n_1 > \infty$ if J is sufficiently small and L sufficiently large.

Let $E_{J,L}$ be the event that (4.4) and (4.5) hold. We have

$$\begin{aligned}\mathbf{P}(E^c_{J,L}) &\leqq (2L+1)^d[\mathbf{P}\{h(0) \leqq 16dJe^{2m_0}\} + \mathbf{P}\{h(0) \leqq 4m_0 Le^{\sqrt{L}}\}]\\ &\leqq (2L+1)^d \mathbf{E}(h(0)^{-\delta})[(16dJe^{2m_0})^\delta + (4m_0 Le^{-\sqrt{L}})^\delta].\end{aligned}$$

Thus there exist $J_\alpha > 0$, $\tilde{L} < \infty$ such that if $J \leqq J_\alpha$, $L > \tilde{L}$ we have

$$\mathbf{P}(E_{J,L}) \geqq 1 - \frac{1}{L^p}.$$

Now pick $\bar{L}(J_2)$ from Theorem 4.2, take $L_0 > \max\{\bar{L}(J_\alpha), \tilde{L}\}$, and pick $0 < J_1 \leqq J_2$ such that $\mathbf{P}(E_{J,L_0}) \geqq 1 - \frac{1}{L_0^p}$ for all $J \leqq J_1$. Since $\bar{L}(J)$ is nondecreasing in J we have $\bar{L}(J) < L_0$ and hence we can apply Theorem 4.1 for $J \leqq J_1$.

Theorem 4.1 now follows from Theorem 4.2 by the proof of Corollary 3.2 in [9]. Notice that under the conclusions of Theorem 4.2 the estimates can be done uniformly in n for $n \geqq \bar{n}$.

Theorem 4.2 is proved in a similar way to Theorem 3.1 in [9]. Again, the main difference is that the estimates have to be done uniformly in n for n large enough. This has been built in our definitions of regular sites and regular regions, which include the uniformity in n for all n large enough.

For the benefit of the reader we will sketch the proof stating clearly the main steps in the framework of this paper and highlighting the differences from [9].

Theorem 4.2 is proven by induction. The induction step is given by the following lemma.

Lemma 4.3. *Let $p > 2d^2$, $\alpha \in (2d, p)$ and $L = l^\alpha$. Suppose*

$$\mathbf{P}\{0 \textit{ is } (m, l, \bar{n}) - \textit{regular}\} \geqq 1 - \frac{1}{l^p}$$

with $m \geqq \frac{3}{\sqrt{l}}$. Then, we have

$$\mathbf{P}\{0 \textit{ is } (M, L, \bar{n}) - \textit{regular} \geqq 1 - \frac{1}{L^p}$$

with

$$M \geqq m - (a_1 m + a_2)\frac{1}{\sqrt{l}} \geqq \frac{3}{\sqrt{L}}$$

for some constants a_1, a_2 independent of l, in case l and $\bar{n}$ are sufficiently large.

As in the proof of Lemma 3.5 in [9], one starts by picking a positive integer R such that

$$\alpha < \frac{(R+1)p}{p+(R+1)d}.$$

For l large enough we can show that

$$\mathbf{P}\left\{\text{there exists } x_1, \ldots, x_R \in \Lambda_L(0) \text{ such that } \Lambda_L(0) \setminus \bigcup_{j=0}^{R} \Lambda_{2l}(x_j) \right.$$
$$\left. \text{is a } (m, l, \bar{n}) - \text{regular region}\right\} \geqq 1 - \frac{1}{2L^p}. \tag{4.6}$$

We now want to estimate $G^{(n)}_{B_{L}(0)}(0, Y)$ for $Y \in \partial^{(n)} B_L(0)$ and $n \geqq \bar{n}$. There are two distinct cases: either Y is in the vertical boundary $\partial^{(n)}_V B_L(0)$ or in the horizontal boundary $\partial^{(n)}_H B_L(0)$. We can restrict ourselves to the case when the event described in (4.6) holds.

Sublemma 4.4. *Suppose there exist $x_1, \ldots, x_R \in \Lambda_L(0)$ such that*

$$\Lambda_L(0) \setminus \bigcup_{j=1}^{R} \Lambda_{2l}(x_j)$$

is $(m, l, \bar{n})$-regular region. Then, if l is sufficiently large and $m > \frac{3}{\sqrt{l}}$, we have

$$G^{(n)}_{B_L(0)}(0, Y) \leqq e^{-M_1 L}$$

with

$$M_1 \geqq m - (a_3 m + a_4)\frac{1}{\sqrt{l}} \geqq \frac{3}{\sqrt{L}}$$

for all $Y \in \partial^{(n)}_V B_L(0)$ and $n \geqq \bar{n}$, for some constant, a_3, a_4 independent of l and n.

Proof. Same as Sublemma 3.6 in [9].

Sublemma 4.5. *Suppose there exist $x_1, \ldots, x_R \in \Lambda_L(0)$ such that $\Lambda' = \Lambda_{L(0)} \setminus \bigcup_{j=1}^{R} \Lambda_{l^\kappa}(x_j)$ is a $(m, l, \bar{n})$-regular for $1 < \kappa < \frac{\alpha}{2d}$,*

$$\tilde{\Lambda} = \left(\bigcup_{j=1}^{R} \Lambda_{l^\kappa}(x_j)\right) \cap \Lambda_L(0),$$

and suppose

$$e^{2Jd|\tilde{\Lambda}|} \prod_{x \in \tilde{\Lambda}} \left(1 - \left(1 - \tanh\frac{h(x)}{n}\right)^n\right) \leqq e^{-\sigma|\tilde{\Lambda}|} \tag{4.7}$$

for all $n \geqq \bar{n}$, *where* $\sigma > 0$ *is a given constant. Then, if* $m \geqq \frac{3}{\sqrt{l}}, \frac{1}{2} < \tau < \kappa - \frac{1}{2}$, *we have*

$$G^{(n)}_{B_L(0)}(0, Y) \leqq e^{-M_2 e^{l^{\tau}/4}}$$

for $Y \in \partial^{(n)}_H B_L(0)$, $n \geqq \bar{n}$, *with*

$$M_2 \geqq m - e^{-l^{\tau}/4}(m+1),$$

for l *sufficiently large.*

Proof. The proof proceeds as in the proof of Sublemma 3.7 in [9] with one important modification. The main difficulty in the proof is how to control the percolation inside the cylinder based on the singular region. This was done in [9] by introducing the event D_s of the existence of a vertical disconnection at height s in a certain neighborhood of the singular region. In this paper we replace D_s by the events

$$D^{(n)}_s = \Big\{\text{all horizontal bonds } \langle (x,t),(y,t)\rangle \text{ are vacant for } x,y \in \tilde{\Lambda}, \text{ nearest neighbors, } t \in [s, s+1] \cap \frac{1}{n}\mathbf{Z}, \text{ and for each } x \in \tilde{\Lambda} \text{ at least one vertical bond of the type } \left\langle (x,t), \left(x, t+\frac{1}{n}\right)\right\rangle, t, t+\frac{1}{n} \in [s, s+1] \cap \frac{1}{n}\mathbf{Z} \text{ is vacant}\Big\}.$$

We have

$$D^{(n)}_s \subset \Big\{\text{there is no connection from } \tilde{\Lambda} \times \{s^{(n)}\} \text{ to } \tilde{\Lambda} \times \{(s+1)^{(n)}\}, \text{ contained in } \tilde{\Lambda} \times \frac{1}{n}\mathbf{Z}\Big\},$$

and

$$\mathbf{Q}^{(n)}(D^{(n)}_s) \geqq (e^{-2J/n})^{nd|\tilde{\Lambda}|} \prod_{x \in \tilde{\Lambda}} \left(1 - \left(1 - \tanh\frac{h(x)}{n}\right)^n\right).$$

By (4.7) we have

$$\mathbf{Q}^{(n)}(D^{(n)}_s) \geqq e^{-\sigma(2l^{\kappa}+1)^d R} \geqq e^{-\xi l^{\kappa d}},$$

where $\xi = 3^d \sigma R$, for all $n \geqq \bar{n}$. Apart from this modification the proof is identical to that of Sublemma 3.7 in [9].

To finish the proof of Lemma 4.3 we need only to show that

$$\mathbf{P}\{(4.7) \text{ holds for all } n \geqq \bar{n}\} \geqq 1 - \frac{1}{2L^p}.$$

This follows from the following lemma.

Lemma 4.6. *Let* $\mu = \log \mathbf{E}((1 - e^{-h(x)})^{-\delta})$,

$$\sigma = 2\delta^{-1}(2dJ\delta + \mu + \log 2), \quad \nu = \delta\sigma.$$

Then, there exists $\bar{n} < \infty$ such for all $n \geqq \bar{n}$ and all $\Lambda \subset \mathbf{Z}^d$ finite we have

$$\mathbf{P}\left\{e^{-2dJ|\Lambda|} \prod_{x \in \Lambda}\left(1-\left(1-\tanh\frac{h(x)}{n}\right)^n\right) \leqq e^{-\sigma|\Lambda|}\right\} \leqq e^{-\nu|\Lambda|}.$$

Proof. Notice $\mu < \infty$ since $\mathbf{E}(h(x)^{-\delta}) < \infty$. Using Chebychev's inequality we get

$$\mathbf{P}\left\{e^{-2dJ|\Lambda|} \prod_{x \in \Lambda}\left(1-\left(1-\tanh\frac{h(x)^n}{n}\right) \leqq \varepsilon\right\}\right.$$

$$\leqq \varepsilon^{\delta} e^{2dJ|\Lambda|}\left[\mathbf{E}\left[\left(\left(1-\left(1-\tanh\frac{h(x)}{n}\right)^n\right)^{-\delta}\right]\right]^{|\Lambda|}.$$

Since $\lim_{n\to\infty}\left(1-\tanh\frac{h}{n}\right)^n = e^{-h}$, there exists $\bar{n}$ such that if $n \geqq \bar{n}$ we have

$$\mathbf{E}\left[\left(\left(1-\left(1-\tanh\frac{h}{n}\right)^n\right)^{-\delta}\right] \leqq 2\mathbf{E}[(1-e^{-h})^{-\delta}].$$

Choosing $\varepsilon = e^{-\sigma|\Lambda|}$, the result follows.

This finishes the proof of Theorem 4.2.

5. Long Range Order

We first discuss the existence of long-range order and spontaneous magnetization in the ground state for the uniform deterministic model. Related results may be found in the literature (see Ginibre [7] for a finite temperature discussion, Pfeuty [6] for an explicit solution in $d=1$ with periodic boundary conditions, and also Driessler, Landau and Perez [8]), but none are in the form needed in this work.

Our Peierls' argument is performed in the classical Ising model approximation coupled with Fisher's trick [12] for summing over contours, which allows estimates uniform in n.

Let, for $x \in \mathbf{Z}^d$.

$$P_{\pm}(x) = \frac{1 \pm \sigma_3(x)}{2}.$$

Then

$$\langle P_+(x)P_-(y)\rangle = \lim_{n\to\infty}\langle P_+(x,0)P_-(y,0)\rangle^{(n)},$$

where

$$P_{\pm}(X) = \frac{1 \pm \sigma(X)}{2}, \quad X \in \mathbf{Z}^d \times \frac{1}{n}\mathbf{Z}.$$

We now apply Peierls' contour argument to the right-hand side to obtain

$$\langle P_+(x,0)P_-(y,0)\rangle^{(n)} \leqq \sum_{\gamma \supset (x,0)} e^{-E^{(n)}(\gamma)},$$

where the sum is performed over all closed contours γ in the dual lattice of $\mathbf{Z}^d \times \frac{1}{n}\mathbf{Z}$

enclosing the point $(x,0)$. For the deterministic model with $h(x)=h$ for $x\in\mathbf{Z}^d$,

$$E^{(n)}(\gamma)=|\gamma_h|2K_n+|\gamma_v|2\frac{J}{n},$$

where γ_h are the horizontal elements of γ and γ_v the vertical elements of γ. We now perform the sum over γ by fixing $|\gamma_h|=L$, and an upper bound is obtained by summing over all possible numbers of vertical steps (with no immediate returns) after each horizontal step:

$$\langle P_+(x,0)P_-(y,0)\rangle^{(n)}\times\sum_{L=2}^{\infty}L^d(2d)^L e^{-2K_nL}\times\left(\frac{2(2d-1)}{1-e^{-2J/n}}\right)^L$$

$$=\sum_{L=2}^{\infty}L^d\left(\frac{4d(d-1)\tanh\frac{h}{n}}{1-e^{-2J/n}}\right)^L.$$

Now

$$\lim_{n\to\infty}\frac{\tanh\frac{h}{n}}{1-e^{-2J/n}}=\frac{h}{2J}$$

and for $n>J$, $\dfrac{\tanh\frac{h}{n}}{1-e^{-2J/n}}<\dfrac{h}{2J}$ and therefore,

$$\langle P_+(x)P_-(y)\rangle\leqq\sum_{L=2}^{\infty}L^d\left(\frac{4d(2d-1)h}{J}\right)^L$$

$$\leqq\left(\frac{4d(2d-1)h}{J}\right)^2c_d\left(\frac{h}{J}\right)\quad\text{if}\quad\left(\frac{4d(d-1)h}{J}\right)<1,$$

where: $c_d(x)$ is monotonically increasing in x for $x\geqq0$, $c_d(0)=2d$. Therefore for all $x,y\in\mathbf{Z}^d$, $\langle\sigma_3(x)\sigma_3(y)\rangle\geqq a>0$ for some $a>0$, provided $\left(\frac{4d(2d-1)h}{J}\right)^2c_d<\frac{1}{2}$. The above discussion may be summarized as follows.

Theorem 5.1. *Let $d\geqq1$ and consider the d-dimensional deterministic model with $h(x)\equiv h$. Then there exists $h_c(J,d)>0$, monotonically increasing in J with $\lim_{J\to\infty}h_c(J,d)=\infty$, such that if $h<h_c(J,d)$, there exists $a(h,J,d)>0$ with*

$$\langle\sigma_3(x)\sigma_3(y)\rangle\geqq a(h,J,d)$$

for all $x,y\in\mathbf{Z}^d$.

Proof. From the above discussion $h_c(J,d)\geqq\bar{h}(J,d)$, where

$$\left(\frac{4d(2d-1)\bar{h}}{J}\right)c_d\left(\frac{\bar{h}}{J}\right)=\frac{1}{2}.$$

Monotonicity of $h_c(J,d)$ follows from Griffiths inequalities. ∎

Remark 5.2. If we consider the deterministic model restricted to a half space $\mathbf{Z}_+^d = \{(x_1, \ldots, x_d) \in \mathbf{Z}^d, x_1 \geqq 0\}$ the same results with essentially the same proof hold true with $h_c^+(J,d)$ and $a^+(d, Jh)$ substituting for the corresponding quantities. From Griffiths inequalities it follows that

$$h_c^+(J,d) < h_c^d(J).$$

To prove long range order for the random system we first introduce the independent *site* percolation model in $\mathbf{Z}^d$, where a site $x \in \mathbf{Z}^d$ is said to be occupied if $h(x) \leqq (1-\varepsilon) h_c^+(J,d)$, for $0 < \varepsilon < 1$. Therefore the probability of occupation of a site is

$$p(J) = \mathbf{P}(h \leqq (1-\varepsilon) h_c^+(J,d)).$$

Now, from Theorem 5.1 and Remark 5.2 $h_c^+(J,d) \to \infty$ as $J \to \infty$ and therefore there exists J_2 such that $p(J) > p_c^{(d)}$ for all $J > J_2$, where $p_c^{(d)}$ is the critical value for the d dimensional site percolation problem. So, if $J > J_2$ with strictly positive $\mathbf{P}$-probability there exists an infinite selfavoiding path w of occupied sites starting at the origin:

$$\begin{aligned} &w: \{0, 1, 2, \ldots\} \to \mathbf{Z}^d \\ &i \to w_i; w_0 = 0, \quad w_i \neq w_j \quad \text{if} \quad i \neq j, \quad |w_{i+1} - w_i| = 1. \end{aligned}$$

We then consider the model in $\mathbf{Z}^d$ given by

$$H_w = -J \sum_{i=0}^{\infty} \sigma_3(w_i)\sigma_3(w_{i+1}) - \sum_{x \in \mathbf{Z}^d} h(x)\sigma_1(x).$$

Notice that the points $x \in \mathbf{Z}^d$, $x \neq w_i$ for all i, are completely decoupled from the points in w. Therefore, the corresponding correlation functions are given by:

$$\langle \sigma_3(w_i)\sigma_3(w_j)\rangle_w = \langle \sigma_3(i)\sigma_3(j)\rangle^{(1)},$$

where the right-hand side is the two-point function of a one dimensional model in the half line with $h(i) \leqq (1-\varepsilon)h_c(J)$ for every $i \in \mathbf{Z}_+$. Therefore, from Theorem 5.1 and Remark 5.2

$$\langle \sigma_3(w_0)\sigma_3(w_i)\rangle_w \geqq a > 0.$$

From Griffiths inequalities it follows that

$$\langle \sigma_3(w_0)\sigma_3(w_j)\rangle \geqq \langle \sigma_3(w_0)\sigma_3(w_j)\rangle_w \geqq a > 0$$

which implies:

$$\overline{\lim_{y \to \infty}} \langle \sigma_3(0)\sigma_3(y)\rangle \geqq a > 0.$$

Ergodicity then implies that, with probability one, there exists $z \in \mathbf{Z}^d$ such that

$$\overline{\lim_{|y| \to \infty}} \langle \sigma_3(z)\sigma_3(y)\rangle \geqq a > 0.$$

As in [9] we now use the Harris-FKG [13] inequality, whose validity is guaranteed by the path-space approximation, to get

$$\langle \sigma_3(x)\sigma_3(y)\rangle \geqq \langle \sigma_3(x)\sigma_3(z)\rangle \langle \sigma_3(z)\sigma_3(y)\rangle.$$

This implies that if $J > J_2$, with probability one

$$\overline{\lim_{|y|\to\infty}} \langle \sigma_3(x)\sigma_3(y) \rangle > 0$$

for every $x \in \mathbf{Z}^d$, thus proving Theorem 1.2. ■

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Note added in proof. Theorem 1.2 has been extended to $d = 1$ by M. Aizenman and A. Klein if $\mathbf{E}(e^{\delta h(x)}) < \infty$ for some $\delta > 0$.