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Decay of Two-Point Functions for $(d+1)$-Dimensional Percolation, Ising and Potts Models with d-Dimensional Disorder

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Abstract. Let $\{J_{\langle x,y\rangle}\}_{\langle x,y\rangle\subset\mathbf{Z}^d}$ and $\{K_x\}_{x\in\mathbf{Z}^d}$ be independent sets of nonnegative i.i.d.r.v.'s, $\langle x,y\rangle$ denoting a pair of nearest neighbors in $\mathbf{Z}^d$; let $\beta,\gamma>0$. We consider the random systems: 1. A bond Bernoulli percolation model on $\mathbf{Z}^{d+1}$ with random occupation probabilities

$$q_{(x,t),(y,s)}=\begin{cases}1-e^{-2\beta J_{\langle x,y\rangle}} & \text{if } t=s,\ x,y \text{ nearest neighbors}\\ 1-e^{-2\gamma K_x} & \text{if } x=y,\ |t-s|=1\\ 0 & \text{otherwise}\end{cases}.$$

2. Ferromagnetic random Ising–Potts models on $\mathbf{Z}^{d+1}$; in the Ising case the Hamiltonian is

$$H=-\beta\sum_t\sum_{\langle x,y\rangle}J_{\langle x,y\rangle}\sigma(x,t)\sigma(y,t)-\gamma\sum_x\sum_t K_x\sigma(x,t)\sigma(x,t+1).$$

For such $(d+1)$-dimensional systems with d-dimensional disorder we prove:
(i) for any $d\geqq 1$, if β and γ are small, then, with probability one, the two-point functions decay exponentially in the d-dimensional distance and faster than polynomially in the remaining dimension,
(ii) if $d\geqq 2$, then, with probability one, we have long-range order for either any β with γ sufficiently large or β sufficiently large and any γ.

Introduction

Let $\mathscr{J}=\{J_{\langle x,y\rangle}\}_{\langle x,y\rangle\subset\mathbf{Z}^d}$ and $\mathscr{K}=\{K_x\}_{x\in\mathbf{Z}^d}$ be independent sets of nonnegative

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independent identically distributed random variables, where $\langle x, y\rangle$ denotes a pair of nearest neighbors (or bond) in $\mathbf{Z}^d$. We will use $\mathbf{E}$ and $\mathbf{P}$ to denote the underlying expectation and probability measure.

We will write $X=(x,t)$, where $X\in\mathbf{Z}^{d+1}, x\in\mathbf{Z}^d, t\in\mathbf{Z}$. We will use $|\cdot|$ to denote the sup-norm, both in $\mathbf{Z}^n$ or $\mathbf{R}^n$.

Let $\beta, \gamma>0$. We consider the following random systems:

1. A bond Bernoulli percolation model on $\mathbf{Z}^{d+1}$ with random occupation probabilities

$$q_{(x,t),(y,s)}=\begin{cases}1-e^{-2\beta J_{\langle x,y\rangle}} & \text{if } t=s, x, y \text{ nearest neighbors}\\ 1-e^{-2\gamma K_x} & \text{if } x=y, |t-s|=1\\ 0 & \text{otherwise}\end{cases}$$

The corresponding percolation model probability will be denoted by $\mathbf{Q}$. Notice that $\mathbf{Q}$ is random, it depends on the choice of $\mathscr{J}$ and $\mathscr{K}$.

The two point function of interest will be the connectivity function

$$G^{(1)}(X,Y)=\mathbf{Q}\{X\to Y\},$$

where by $X\to Y$ we mean that X is connected to Y by a path of occupied bonds.

2. A ferromagnetic random Ising model on $\mathbf{Z}^{d+1}$ with Hamiltonian

$$H=-\beta\sum_t\sum_{\langle x,y\rangle}J_{\langle x,y\rangle}\sigma(x,t)\sigma(y,t)-\gamma\sum_x\sum_t K_x\sigma(x,t)\sigma(x,t+1),$$

and two-point function

$$G^{(2)}(X,Y)=\langle\sigma(X)\sigma(Y)\rangle.$$

When $d=1$, this model has been studied by McCoy and Wu [1], in the case when K_x is constant, and by Shankar and Murthy [2] for $J_{\langle x,y\rangle}$ constant but with K_x not necessarily nonnegative.

3. Random Potts models on $\mathbf{Z}^{d+1}$ with Hamiltonian

$$H^{(q)}=-2\beta\sum_t\sum_{\langle x,y\rangle}J_{\langle x,y\rangle}(\delta_{\sigma(x,t)\sigma(y,t)}-1)-2\gamma\sum_x\sum_t K_x(\delta_{\sigma(x,t),\sigma(x,t+1)}-1),$$

where the spin variables $\sigma(X)$ are allowed to take $q\in\{2,3,\dots\}$ distinct values. The case $q=2$ coincides with the previously defined Ising model.

The quantity of interest is

$$G^{(q)}(X,Y)=\left\langle\frac{1}{q-1}(q\delta_{\sigma(X),\sigma(Y)}-1)\right\rangle.$$

These are all $(d+1)$-dimensional systems with d-dimensional disorder. In this article we show that all these systems undergo a phase transition (for $d\geqq 2$) in the following sense:

(i) Assume $\mathbf{E}(e^{\delta K_x})<\infty$ for some $\delta>0$. For any $d\geqq 1$, if β is small with γ sufficiently small (how small depends on β), we have, with probability one, that the two-point functions $G^{(q)}(X,Y)$ decay exponentially in the d-dimensional distance and faster than any polynomial in the remaining dimension.

(ii) Assume $\mathbf{P}(J_{\langle x,y\rangle}=0)=0$. If $d\geqq 2$, we have long-range order with probability one for either any β with γ sufficiently large or for β sufficiently large and any γ.

The fact that these are $(d+1)$-dimensional models with d-dimensional disorder makes them different in a nontrivial way from the corresponding models with $(d+1)$-dimensional disorder, as studied in, say [13, 8, 14, 5]. In the latter the singular sets that give raise to the Griffiths singularities are finite, in the models studied in this article those singular sets are infinite cylinders. This is the main difficulty we have to overcome in proving decay of the two-point functions and is reflected in a non-exponential estimate for the decay in the direction of the $(d+1)$-th coordinate axis.

Our original motivation to study such $(d+1)$-dimensional systems with d-dimensional disorder came from our joint work with J. F. Perez on localization in the ground state of an Ising model with a random transverse field. By the introduction of a path space the problem is reduced to the study of a model similar to the ferromagnetic random Ising model we study in this paper, except that the $(d+1)$ coordinate is now continuous. This model is studied in a companion paper [3].

2. Statement of Results

It follows from the Fortuin–Kasteleyn representation of Ising–Potts models and from Fortuin's comparison principles (see Aizenman, Chayes, Chayes and Newman [4]) that

$$G_{\mathbf{W}}^{(q)}(X,Y) \leqq G_{\mathbf{W}}^{(1)}(X,Y) \tag{2.1}$$

for all $X, Y \in \mathbf{W}, q = 2, 3, \ldots, \mathbf{W} \subset \mathbf{Z}^{d+1}$, where $G_{\mathbf{W}}^{(q)}$ denotes the two-point function of the system restricted to W with free boundary conditions. It thus suffices to prove decay for the percolation model; we will write $G(X,Y)$ for $G^{(1)}(X,Y)$. Our result is

Theorem 2.1. *Assume* $\mathbf{E}(e^{\delta K_x}) < \infty$ *for some* $\delta > 0$. *Then for any* $d = 1, 2, \ldots$ *there exists* $\beta_1 > 0$ *such that for* $0 < \beta < \beta_1$ *and any* $v > 1$ *we have* $\bar{m}(\beta) > 0$ *such that for if* $0 < m < \bar{m}(\beta)$ *there exists* $\gamma_1 = \gamma_1(\beta, v, m) > 0$ *such that if* $\gamma < \gamma_1$ *we have, with probability one,*

$$G((x,t),(y,s)) \leqq C_x e^{-m|(x-y,(\log|t-s|)^v)|}$$

for all $(x,t),(y,s) \in \mathbf{Z}^{d+1}$, *with*

$$C_x = C_x(\mathscr{J}, \mathscr{K}, \beta, \gamma, v, m) < \infty.$$

Theorem 2.1 is proven in Sect. 3 by a multiscale analysis; we follow the strategy of von Dreifus and Spencer [5, 6] and von Dreifus and Klein [7].

We can identify β_1 and $\bar{m}(\beta)$ in Theorem 2.1. Consider the d-dimensional percolation or Ising–Potts model we obtain by making all $K_x = 0$ and restricting ourselves to $\mathbf{Z}^d$, let $g^{(q)}(x,y)$ denote the corresponding two-point function. It is well known in the deterministic case (i.e., $J_{\langle x,y\rangle} \equiv J$), and it follows in the random case from [8, 5] that there exists $\beta_2^{(q)} > 0$ such that for $0 < \beta < \beta_2^{(q)}$ we can find $\bar{m} = \bar{m}(\beta) > 0$ such that $g^{(q)}(x,y) \leqq c_x e^{-\bar{m}|x-y|}$ with probability one. We can take $\beta_1 = \beta_2^{(1)}$.

Since our models are ferromagnetic, we always have

$$g^{(q)}(x,y) \leqq G^{(q)}((x,t),(y,t)) \tag{2.2}$$

for any choice of $\mathscr{K}$ and γ. Thus long-range order in the d-dimensional model always imply long-range order in the $(d+1)$-dimensional model for any γ. In particular, for $d \geqq 2$, if $J_{\langle x,y\rangle} \equiv J > 0$ we always have a phase transition for the $(d+1)$-dimensional model.

We have a more general result:

Theorem 2.2. *Assume* $\mathbf{P}\{J_{\langle x,y\rangle} = 0\} = 0$ *and let* $d \geqq 2$. *Let* $q = 1, 2, \ldots$. *For any* $\beta > 0$ *we can find* $\gamma_2 = \gamma_2(\beta, q) < \infty$ *such that for* $\gamma > \gamma_2$ *we have, with probability one,*

$$\overline{\lim_{|Y| \to \infty}}\, G^{(q)}(X, Y) > 0$$

for any $X \in \mathbf{Z}^{d+1}$. *Moreover, there exists* $\beta_3 = \beta_3(q) < \infty$ *such that we can take* $\gamma_2 = 0$ *for* $\beta > \beta_3$.

Theorem 2.2 is proved in Sect. 4. We use a percolation argument to find a sublattice of $\mathbf{Z}^{d+1}$ isomorphic to $\mathbf{Z}^2$ in which the model exhibits long-range order.

3. The Multiscale Analysis

Theorem 2.1 is proven in this section. We restrict ourselves to the case $v = 2$, the modifications for arbitrary v will be clear.

Let us consider the independent bond percolation model in $\mathbf{Z}^{d+1}$ whose occupation probabilities are given by (1.1). If $\mathbf{W} \subset \mathbf{Z}^{d+1}, X, Y \in \mathbf{W}$, we say that $X \xrightarrow[\mathbf{W}]{} Y$ if X is connected to Y in $\mathbf{W}$ by occupied bonds, i.e., there exists $Z_0, Z_1, \ldots, Z_k \in \mathbf{W}$ such that $Z_0 = X, Z_k = Y, |Z_{i+1} - Z_i| = 1$ for $i = 0, \ldots, k-1$, and the bonds $\langle Z_i, Z_{i+1}\rangle$ are occupied for $i = 1, \ldots, k-1$. We set

$$G_{\mathbf{W}}(X, Y) = \mathbf{Q}\{X \xrightarrow[\mathbf{W}]{} Y\}.$$

We have translation invariance in the t-axis for any realization of $\mathscr{J}$ and $\mathscr{K}$, in particular

$$G_{\mathbf{W}+(0,t)}(X + (0,t), Y + (0,t)) = G_{\mathbf{W}}(X, Y)$$

for any $t \in \mathbf{Z}$.

We start by reviewing some facts about independent bond percolation that will be needed (see Kesten [9], Durrett [10], Chayes and Chayes [11]).

Let w denote a configuration of occupied and vacant bonds in $\mathbf{Z}^{d+1}$, i.e.,

$$w : \mathscr{B} = \{\langle X, Y\rangle \subset \mathbf{Z}^{d+1}\} \to \{0, 1\},$$

where $\langle X, Y\rangle$ denotes a pair of nearest neighbors (or bond) in $\mathbf{Z}^{d+1}$, with $\langle X, Y\rangle$ being occupied if $w(\langle X, Y\rangle) = 1$ and vacant if $w(\langle X, Y\rangle) = 0$. The collection Ω of configurations comes with a natural partial order, $w \leqq w'$ if and only if $w(\langle X, Y\rangle) \leqq w'(\langle X, Y\rangle)$ for all bonds $\langle X, Y\rangle$. Functions on Ω which are nondecreasing (nonincreasing) with respect to this partial order are called positive (negative); events are positive (negative) when their characteristic functions are positive (negative). Events are measurable subsets of Ω.

We will use the following inequalities:

The Harris–FKG Inequality (e.g., [10]). *If A and B are both positive (negative) events,* *we have*

$$\mathbf{Q}(A \cap B) \geqq \mathbf{Q}(A)\mathbf{Q}(B).$$

The van der Berg–Kesten Inequality [12]. *Let A be an event, $\mathscr{C} \subset \mathscr{B}$. We set $A_{|\mathscr{C}} = \{w \in A; w' \in A$ for all w' such that $w' = w$ on each bond in $\mathscr{C}\}$.*
If A is a positive event, then $A_{|\mathscr{C}}$ consists of the configurations of $\mathscr{C}$ for which A occurs even if all bond in $\mathscr{B} \backslash \mathscr{C}$ are vacant.

If A, B are events, let $A \circ B$ be the event of A and B occurring disjointly, i.e., *$A \circ B = \{w \in A \cap B$; there exists $\mathscr{C}, D \subset \mathscr{B}, \mathscr{C} \cap D = \phi$, with $w \in A_{|\mathscr{C}} \cap B_{|D}\}$.*

The van der Berg–Kesten ($v-BK$) inequality states that, if both A and B are positive (negative) events, we have

$$\mathbf{Q}(A \circ B) \leqq \mathbf{Q}(A)\mathbf{Q}(B).$$

The Hammersley–Simon–Lieb (HSL) Inequality. *Let $\mathbf{W}, \mathbf{W}' \subset \mathbf{Z}^{d+1}$, if*

$$X \in \mathbf{W} \cap \mathbf{W}', \ Y \in \mathbf{W}' \backslash \mathbf{W},$$

we have

$$G_{\mathbf{W}'}(X, Y) \leqq \sum_{Z \in \partial(\mathbf{W}, \mathbf{W}')} G_{\mathbf{W} \cap \mathbf{W}'}(X, Z) G_{\mathbf{W}'}(Z, Y),$$

where

$$\partial(\mathbf{W}, \mathbf{W}') = \{Z \in \mathbf{W} \cap \mathbf{W}'; \text{ there exists } Z' \in \mathbf{W}' \backslash \mathbf{W} \text{ with } |Z - Z'| = 1\}.$$

The HSL inequality can be easily obtained from the $v-BK$ inequality.

We will actually use a simple consequence of the HSL inequality: if $X \in \mathbf{W} \cap \mathbf{W}'$, $Y \in \mathbf{W}' \backslash \mathbf{W}$, we have

$$G_{W'}(X, Y) \leqq G_{\mathbf{W}}(X, \partial) G_{\mathbf{W}'}(Z_1, Y) \tag{3.1}$$

for some $Z_1 \in \partial(\mathbf{W}, \mathbf{W}')$, where

$$G_{\mathbf{W}}(X, \partial) = \sum_{Z \in \partial \mathbf{W}} G_{\mathbf{W}}(X, Z)$$

with $\partial \mathbf{W} = \partial(\mathbf{W}, \mathbf{Z}^{d+1})$.

We now start to prepare for the multiscale analysis. For $x \in \mathbf{Z}^d$ and $L > 0$ we let

$$\Lambda_L(x) = \{y \in \mathbf{Z}^d; |y - x| \leqq L\}.$$

For $X = (x, t), L > 0, T > 0$, we set

$$B_{L,T}(X) = \Lambda_L(x) \times ([t - T, t + T] \cap \mathbf{Z})$$

and

$$B_L(X) = B_{L, e^{\sqrt{L}}}(X)$$

(for arbitrary v we should take $B_L(X) = B_{L, e^{L^{1/v}}}(X)$).

Definitions. *Let $m, L > 0$. We say that $x \in \mathbf{Z}^d$ is m-regular at scale L (or simply (m, L)-regular) if*

$$G_{B_L((x,0))}((x, 0), Y) \leqq e^{-mL}$$

for all $Y \in \partial B_L((x, 0))$. Otherwise we say that x is (m, L)-singular. $X = (x, t) \in \mathbf{Z}^{d+1}$ will also be called (m, L)-regular if x is (m, L)-regular.

$\Lambda \subset \mathbf{Z}^d$ will be called (m, L)-regular if every $x \in \Lambda$ is (m, L)-regular. Otherwise Λ is (m, L)-singular.

If x is (m, L)-regular we have

$$G_{B_L((x,0))}((x,0), \partial) \leqq e^{-(m - 2/\sqrt{L})L} \tag{3.2}$$

for all L sufficiently large.

We can now state the result of the multiscale analysis. We fix β, γ, d and assume $\mathbf{E}(e^{\delta K_x}) < \infty$ for some $\delta > 0$.

Theorem 3.1. *Let $p > 2d^2$. Suppose there exists $m_0 > 0$ and $L_0 > 0$ such that*

$$\mathbf{P}\{0 \text{ is } (m_0, L_0) - \text{regular}\} \geqq 1 - \frac{1}{L_0^p}.$$

Let $\alpha \in \left(2d, \frac{p}{d}\right)$, set $L_{k+1} = L_k^\alpha, k = 0, 1, \ldots$. Then, for any $0 < m_\infty < m_0$ there exists $\bar{L} = \bar{L}(p, d, m_0, \alpha, m, \gamma) < \infty$ such that if $L_0 > \bar{L}$ we have

$$\mathbf{P}\{0 \text{ is } (m_\infty, L_k) - \text{regular}\} \geqq 1 - \frac{1}{L_k^p}$$

for all $k = 0, 1, 2, \ldots$.

Corollary 3.2. *Assume the hypothesis and conclusion of Theorem 3.1. Then, given $0 < m < m_\infty$, we have, with probability one,*

$$G((x,t),(y,s)) \leqq C_x e^{-m|(x-y,(\log|t-s|)^2)|}$$

for all $x, y \in \mathbf{Z}^d, t, s \in \mathbf{Z}$, with $C_x = C_x(\mathscr{J}, \mathscr{K}, m) < \infty$.

Proof. Let $x \in \mathbf{Z}^d$. By Theorem 3.1

$$\mathbf{P}\{\Lambda_{L_{k+1}}(x) \text{ is } (m_\infty, L_k) - \text{singular}\} \leqq \frac{(2L_{k+1}+1)^d}{L_k^p} \leqq \frac{3^d}{L_k^{p-2d}}.$$

Since $p > \alpha d$ the above probabilities are summable so we can use the Borel Cantelli Lemma to conclude that, with probability one, we can find $k_1 = k_1(x, \mathscr{J}, \mathscr{K}) < \infty$ such that $\Lambda_{L_{k+1}}(x)$ is a (m_∞, L_k)-regular region for all $k \geqq k_1$.

Let $b > 2$, $Y = (y, s) \in \mathbf{Z}^{d+1}$. Except for finitely many Y's we can always choose $k > k_1$ such that

$$bL_k < |(x-y, (\log|s|)^2)| \leqq L_{k+1}.$$

In this case $Y \in B_{L_{k+1}}((x,0))$, and applying (3.1) repeatedly we get

$$G((x,0), Y) \leqq G_{B_{L_k}((x,0))}((x,0), \partial) G_{B_{L_k}(Z_1)}(Z_1, \partial) \cdots G_{B_{L_k}(Z_n)}(Z_n, \partial) G(Z_{n+1}, Y),$$

where $Z_{i+1} \in \partial B_{L_k}(Z_i), i = 0, 1, \ldots, n$ and $Z_0 = (x, 0)$.

As long as

$$n \leqq \min\left\{\frac{L_{k+1}}{L_k}, \max\left\{\frac{|y-x|}{L_k}, |s|e^{-\sqrt{L_k}}\right\}\right\}, \tag{3.3}$$

$Z_0, Z_1 \cdots Z_n \in B_{L_{k+1}}((x,0))$ and hence are (m_∞, L_k)-regular. Using (3.2) and

$G(Z_n, Y) \leqq 1$ we get

$$G((x,0), Y) \leqq \exp\left\{ -\left(m_\infty - \frac{2}{\sqrt{L_k}} \right) L_k n \right\}. \tag{3.4}$$

Since if $bL_k < (\log|s|)^2$ we always have $|s|e^{-\sqrt{L_k}} \geqq \frac{L_{k+1}}{L_k}$ for L_k large enough, the right-hand side of (3.3) is always either $\frac{|y-x|}{L_x}$ or $\frac{L_{k+1}}{L_k}$.

If $|y-x| \geqq (\log|s|)^2$, we can take $\frac{|y-x|}{L_k} \geqq n \geqq \frac{|y-x|}{L_k} - 1$; if $|y-x| < (\log|s|)^2$ we have $(\log|s|)^2 \geqq bL_k$ and hence

$$|s|e^{-\sqrt{L_k}} \geqq \frac{L_{k+1}}{L_k} \geqq \frac{|y-x|}{L_k}$$

for L_k large. In this case we can take

$$\frac{L_{k+1}}{L_k} \geqq n \geqq \frac{L_{k+1}}{L_k} - 1 \geqq \frac{(\log|s|)^2}{L_k} - 1.$$

Thus it follows from (3.4) that

$$\begin{aligned} G((x,0), Y) &\leqq \exp\left\{ -\left(m_\infty - \frac{2}{\sqrt{L_k}} \right) |(y-x, (\log|s|)^2)| - L_k \right\} \\ &\leqq \exp\left\{ -\left(m_\infty - \frac{2}{\sqrt{L_k}} - \frac{m_\infty}{b} \right) |(y-x, (\log|s|)^2)| \right\} \\ &\simeq \exp\{ -m|(y-x, (\log|s|)^2)| \} \end{aligned}$$

if b and L_k are sufficiently large. ■

We will now show that Theorem 3.1 implies Theorem 2.1. In view of Corollary 3.2 and the fact that $\bar{L}$ in Theorem 3.1 is nondecreasing in γ, it suffices to show that there exists $\beta_1 > 0$ such that for $0 < \beta < \beta_1$ we can find $\bar{m}(\beta) > 0$ with the property that given $0 < m_0 < \bar{m}(\beta)$ and $\bar{L} < \infty$ we can find $L_0 > \bar{L}$ and $\gamma_1 > 0$ such that for $0 < \gamma < \gamma_1$ we have

$$\mathbf{P}\{0 \text{ is } (m_0, L_0) - \text{regular}\} \geqq 1 - \frac{1}{L_0^p}.$$

If $J_{\langle x,y \rangle} \equiv J > 0$, it is well known that there exists $\beta_1 > 0$ such that for $\beta < \beta_1$ we have for the d-dimensional model

$$g(x,y) \leqq Ce^{-\bar{m}(x-y)}$$

for all $x, y \in \mathbf{Z}^d$ with $\bar{m} = \bar{m}(\beta) > 0$ and $C < \infty$. Keeping β fixed and making explicit the dependence of G on γ, we have

$$G_{\mathbf{W}}((x,0), (y,t); \gamma = 0) = g_\Lambda(x,y)\delta_{0,t}$$

for any $\mathbf{W} = \Lambda \times ([-T, T] \cap \mathbf{Z})$. If Λ is finite we also have

$$\lim_{\gamma \downarrow 0} G_{\mathbf{W}}((x,0),(y,t);\gamma) = G_{\mathbf{W}}((x,0),(y,t);\gamma = 0)$$

for any $x, y \in \Lambda, t \in [-T, T] \cap \mathbf{Z}$.

Since $g_{\Lambda}(x,y) \leqq g(x,y) \leqq Ce^{-\bar{m}(x-y)}$, we can conclude that, for any $L > 0$, we have

$$\lim_{\gamma \downarrow 0} \mathbf{P}\{G_{B_L((0,0))}((0,0), Y) \leqq 2Ce^{-\bar{m}L}\} = 1$$

for any $Y \in \partial B_L((0,0))$.

Thus, given $0 < m_0 < \bar{m}$, there exists $\bar{L} < \infty$ such that if $L_0 > \bar{L}$ we can find $\gamma_1 > 0$ such that

$$\mathbf{P}\{0 \text{ is } (m_0, L_0) - \text{regular}\} \geqq 1 - \frac{1}{L_0^p}$$

for $\gamma < \gamma_1$.

If $J_{\langle x,y \rangle}$ is random it follows from the multiscale analysis in [5] that we can find $\beta_1 > 0$ such that for $\beta < \beta_1$ we have $\bar{m} = \bar{m}(\beta) > 0$ for which

$$\mathbf{P}\{0 \text{ is } (\bar{m}, l_j) - \text{regular}\} \geqq 1 - \frac{1}{2l_j^p}$$

for some scale $l_j \to \infty$, where the notion of regularity for the d-dimensional model is defined in a similar way. Let us fix $\beta < \beta_1$ and choose $0 < m_0 < \bar{m}$; let $E_L^{(\gamma)}$ be the event that 0 is (m_0, L)-regular for the model with parameter γ. The previous deterministic argument implies that

$$\lim_{\gamma \downarrow 0} E_L^{(\gamma)} \supset \{0 \text{ is } (\bar{m}, L) - \text{regular}\}$$

if L is sufficiently large. Thus,

$$\lim_{\gamma \downarrow 0} \mathbf{P}(E_L^{(\gamma)}) \geqq \mathbf{P}\left(\liminf_{\gamma \downarrow 0} E_L^{(\gamma)}\right) \geqq \mathbf{P}\{0 \text{ is } (\bar{m}, L) - \text{regular}\}.$$

Thus given $\bar{L} < \infty$, we can always pick $L_0 = l_{j_0} > \bar{L}$ for some j_0, and find $\gamma_1 > 0$ such that

$$\mathbf{P}(E_{L_0}^{(\gamma)}) \geqq 1 - \frac{1}{L_0^p}$$

for all $\gamma < \gamma_1$.

Thus Theorem 2.1 follows from Theorem 3.1. ∎

We now turn to the proof of Theorem 3.1. We start with two lemmas.

Lemma 3.3. *Let Λ be a (m, L)-regular region, and let $\mathbf{W} \subset \Lambda \times \mathbf{Z}$. Then for any $X = (x,t)$, $Y = (y,s) \in \mathbf{W}$ we have*

$$G_{\mathbf{W}}(X, Y) \leqq \exp\left\{-\left(m - \frac{2}{\sqrt{L}}\right) L \left\{\left|\left(\frac{x-y}{L}, (t-s)e^{-\sqrt{L}}\right)\right| - 1\right\}\right\}.$$

Proof. Applying (3.1) repeatedly we get

$$G_{\mathbf{W}}(X,Y) \leqq G_{B_L(x)}(X,\partial)G_{B_L(Z_1)}(Z_1,\partial)\cdots G_{B_L(Z_n)}(Z_1,\partial)G_{\mathbf{W}}(Z_{n+1},Y),$$

where $Z_{i+1}\in\partial(B_L(Z_i),\mathbf{W}), i=0,1,\ldots,n$ with $Z_0=X_1$ as long as Y is not in $B_L(Z_i)$ for any $i=0,1,\ldots,n$, which we can guarantee for

$$\left|\left(\frac{x-y}{L},(t-s)e^{-\sqrt{L}}\right|\right)-1\leqq n<\left|\left(\frac{x-y}{L},(t-s)e^{-\sqrt{L}}\right)\right|.$$

The lemma follows. ∎

Lemma 3.4. *Let* $v=\log\mathbf{E}(e^{\delta K_x}), \sigma=\frac{4v}{\delta}$. *Then for any* $\Lambda\subset\mathbf{Z}^d$ *we have*

$$\mathbf{P}\left\{\exp\left(-2\gamma\sum_{x\in\Lambda}K_x\right)\leqq e^{-\sigma\gamma|\Lambda|}\right\}\leqq e^{v|\Lambda|}.$$

Proof.

$$\mathbf{P}\left\{\exp\left(-2\gamma\sum_{x\in\Lambda K_x}\right)\leqq\varepsilon\right\}\leqq\varepsilon^{\delta/2\gamma}[\mathbf{E}(e^{\delta K_x})]^{|\Lambda|}=\varepsilon^{\delta/2\gamma}e^{v|\Lambda|}.$$

So pick $\varepsilon=\exp\left(-\frac{4\gamma}{\delta}v|\Lambda|\right)$. ∎

Theorem 3.1 is proven by induction. The induction step is given by the following lemma.

Lemma 3.5. *Let* $p>2d^2, \alpha\in(2d,p/d)$, *and* $L=l^\alpha$. *Suppose*

$$\mathbf{P}\{0 \textit{ is } (m,l)-\textit{regular}\}\geqq 1-\frac{1}{l^p},$$

with $m\geqq\frac{3}{\sqrt{l}}$.

Then, if l is large enough,

$$\mathbf{P}\{0 \textit{ is } (M,L)-\textit{regular}\}\geqq 1-\frac{1}{L^p}$$

with

$$M\geqq m-\left(m-\frac{2}{\sqrt{l}}\right)\frac{5J}{l^{\alpha-1}}-\frac{2}{\sqrt{l}}-\frac{3J}{l^{\alpha/2}}\geqq\frac{3}{\sqrt{L}}. \tag{3.5}$$

Proof. Since $p>2d^2, \alpha\in(2d,p/d)$, we can pick a positive integer J such that

$$\alpha<\frac{(J+1)p}{p+(J+1)d}.$$

Thus, following [5–7], we have that

$\mathbf{P}\{$there exists $x_1,\ldots,x_{J+1}\in\Lambda_L(0)$ (m,l) – singular with $\Lambda_L(x_i)\cap\Lambda_L(x_j)=\phi$ if $i\neq j\}$

$$\leqq(2L+1)^{(J+1)d}\frac{1}{l^{(J+1)p}}<\frac{1}{2L^p}$$

if l is large enough. In this case,

$$\mathbf{P}\left\{\text{there exists } x_1,\ldots,x_J\in\Lambda_L(0) \text{ such that } \Lambda_L(0)\setminus\bigcup_{j=0}^{J}\Lambda_{2l}(x_j) \text{ is a } (m,l)\right.$$
$$\left.-\text{ regular region}\right\}\geqq 1-\frac{1}{2L^p}. \tag{3.6}$$

We now want to estimate $G_{B_L(0)}(0,Y)$ for $Y\in\partial B_L(0)$. We have two distinct cases that we must treat separately: either Y is in the lateral boundary or Y is in the top or bottom boundary of the parallelepiped $B_L(0)$. We can restrict ourselves to the case when the event described in (3.6) holds.

Sublemma 3.6. *Suppose there exists $x_1,\ldots,x_J\in\Lambda_L(0)$ such that*

$$\Lambda_L(0)\setminus\bigcup_{j=1}^{J}\Lambda_{2l}(x_j)$$

is a (m,l)-regular region. Let $Y=(y,s)\in\partial B_L(0)$ with $y\in\partial\Lambda_L(0)$. Then, if $m>\frac{3}{\sqrt{l}}$,

$$G_{B_L(0)}(0,Y)\leqq e^{-M_1L}$$

with

$$M_1\geqq m-\left(m-\frac{2}{\sqrt{l}}\right)\frac{5J}{l^{\alpha-1}}-\frac{2}{\sqrt{l}}-\frac{3J}{l^{\alpha/2}}\geqq\frac{3}{\sqrt{L}}$$

for l sufficiently large.

Proof. We can find $y_1,\ldots,y_{J'}\in\Lambda_L(0)$, with $J'\leqq J$, and $j_1,\ldots,j_{J'}\in\{1,2,\ldots,J\}$ with $j_1+\cdots+j_{J'}\leqq J$, such that the boxes $\Lambda_{2j_il}(y_i), i=1,\ldots,J'$ are disjoint with

$$\bigcup_{j=1}^{J}\Lambda_{2l}(x_j)\subset\bigcup_{i=1}^{J'}\Lambda_{2j_il}(y_i),$$

and hence $\Lambda'=\Lambda_L(0)\setminus\bigcup_{i=1}^{J'}\Lambda_{2j_i0}(y_i)$ is a (m,l)-regular region.

We set $B=B_L(0), B_i=B_{2j_il,e\sqrt{L}}((y_i,0)), \partial B_i=\partial(B_i\cap B,B), i=1,\ldots,J'$. Let $B'=B\setminus\bigcup_{i=1}^{J'}B_i$; if $0\in B'$ we set $\partial B_0=\{0\}$, otherwise $0\in B_{i'}$ for some i' and we set $\partial B_0=\partial B_{i'}$. Similarly if $Y\in B', \partial B_{J'+1}=\{Y\}$, otherwise $\partial B_{J'+1}=\partial B_{i''}$ with $Y\in B_{i''}$.

Let $C\xrightarrow[\mathbf{W}]{}D$ mean the existence of a connected occupied path from C to D in $\mathbf{W}$. We have

$$\{0\xrightarrow[B]{}Y\}=\bigcup_{r=0}^{J'}\bigcup_{\{i_1,\ldots,i_r\}\subset\{1,\ldots,J'\}}\{\text{there exist disjoint occupied paths}$$
$$\partial_0\xrightarrow[B']{}\partial B_{i_1},\partial B_{i_1}\xrightarrow[B']{}\partial B_{i_2},\ldots,\partial B_{i_r}\xrightarrow[B']{}\partial B_{J'+1}\}$$

$$= \bigcup_{r=0}^{J'} \bigcup_{\{i_1,\ldots,i_r\}\subset\{1,\ldots,J'\}} \{\partial B_0 \xrightarrow[B']{} \partial B_{i_2}\}\circ\cdots\circ\{\partial B_{i_r} \xrightarrow[B']{} \partial B_{J'+1}\}.$$

By Lemma 3.3 we have

$$\mathbf{Q}\{\partial B_i \xrightarrow[B']{} \partial B_k\} \leqq (2d(2Jl)^{d-1}2e^{\sqrt{L}})^2$$

$$\cdot\exp\left\{-\left(m-\frac{2}{\sqrt{l}}\right)l\left[\frac{\operatorname{dist}(\Lambda_{2j_il}(y_i),\Lambda_{2j_kl}(y_k))}{l}-1\right]\right\}.$$

Thus, it follows from the $v-BK$ inequality that

$$G_B(0,Y)=\mathbf{Q}\{0 \xrightarrow[B']{} Y\}$$

$$\leqq (J+1)!(2d(4Jl)^{d-1}2e^{\sqrt{L}})^{2J}\exp\left\{-\left(m-\frac{2}{\sqrt{l}}\right)l\left[\frac{L-4lJ}{l}-J\right]\right\}$$

$$\leqq e^{-M_1L},$$

where

$$M_1 \geqq \left(m-\frac{2}{\sqrt{l}}\right)-lm-\frac{5}{\sqrt{l}}\frac{5J}{l^{2-1}}-\frac{3J}{l^{2/2}}$$

$$\geqq \frac{1}{\sqrt{l}}\left(1-\frac{5J}{l^{2-1}}\right)-\frac{3J}{l^{2/2}}\geqq\frac{3}{\sqrt{L}}$$

if l large enough, since $m \geqq \dfrac{3}{\sqrt{l}}$ and $\alpha > 2d$. ■

Sublemma 3.7. *Suppose there exists* $x_1,\ldots,x_J\in\Lambda_L(0)$ *such that* $\Lambda'=\Lambda_L(0)\setminus\bigcup_{j=1}^{J}\Lambda_{2l}(x_j)$ *is a* (m,l)*-regular region. Let* $1<\kappa<\dfrac{\alpha}{2d}$, $\tilde{\Lambda}=\left(\bigcup_{j=1}^{J}\Lambda_{l^\kappa}(x_j)\right)\cap\Lambda_L(0)$ *and suppose*

$$\exp\left(-2\gamma\sum_{x\in\tilde{\Lambda}}K_x\right)\geqq\exp(-\sigma\gamma|\tilde{\Lambda}|). \tag{3.7}$$

Let $Y=(y,T)$ *with* $|T|=[e^{\sqrt{L}}]$, *and let* $\frac{1}{2}<\tau<\kappa-\frac{1}{2}$. *Then, if* $m\geqq\dfrac{3}{\sqrt{l}}$,

$$G_{B_L(0)}(0,Y)\leqq\exp(-M_2e^{1/4l^\tau})$$

for l sufficiently large, with

$$M_2\geqq m-e^{-1/4l^\tau}(m+1).$$

Proof. We take $T=[e^{\sqrt{L}}]$, where $[x]$ denotes the largest integer $\leqq x$, the other case being similar. Let also $T_1=2[\frac{1}{2}e^{l^\tau}]$. We define

$$S_j=B_{L,1/2T_1}((0,(j-\tfrac{1}{2})T_1))\quad\text{for}\quad j=1,2,\ldots,\left[\frac{T}{T_1}\right].$$

In addition, for

$$r = 1, 2, \ldots, R = \left[\frac{1}{3}\left[\frac{T}{T_1}\right]\right]$$

we set

$$\mathscr{S}_r = S_{3r-1}.$$

Notice that events in different $\mathscr{S}_r$'s are independent. Now let us define the event

$$D_s = \{\text{all bonds } \langle (x,s),(x,s+1)\rangle \text{ are vacant for all } x \in \tilde{\Lambda}\}.$$

D_s is the event that there exists a vertical disconnection at height s in a certain neighborhood of the singular region.

For each fixed s,

$$\mathbf{Q}(D_s) = e^{-2\gamma \sum_{x\in\tilde{\Lambda}} K_x} \geqq e^{-\sigma\gamma|\tilde{\Lambda}|} \geqq e^{-\gamma\xi l^{\kappa d}}$$

for l large enough by (3.7), with $\xi = 3^d \sigma J$.

Let $E_r = D_{(3r-1-1/2)T_1}$, the event that there is a vertical disconnection in the middle of $S_r \cap \tilde{\Lambda}$.

Now let $\hat{\Lambda} = \left(\bigcup_{j=1}^{J} \Lambda_{2l}(x_j)\right) \cap \Lambda_L(0)$, and for $\Lambda \subset \mathbf{Z}^d$ let $B_\Lambda = \Lambda \times ([e^{-\sqrt{L}}, e^{\sqrt{L}}] \cap \mathbf{Z})$.

We define the event F_r by

$$F_r^c = \{B_{\hat{\Lambda}} \cap \mathscr{S}_r \xrightarrow[\mathscr{S}_r \setminus B_{\hat{\Lambda}}]{} B_{\Lambda_L(0)\setminus\tilde{\Lambda}} \cap \mathscr{S}_r\},$$

i.e., F_r is the event that there is no connection inside $\mathscr{S}_r \setminus B_{\hat{\Lambda}}$ from $B_{\hat{\Lambda}}$ to $B_{\Lambda_L(0)\setminus\tilde{\Lambda}}$. But $S_r \setminus B_{\hat{\Lambda}} \subset B_{\Lambda'}$ and Λ' is a (m, l)-regular region. Hence we can use Lemma 3.3 to get, for l large,

$$\mathbf{Q}(F_r^c) \leqq [(T_1(2d)(4l+1)^{d-1})(T_1(2d)(2l^\kappa+1)^{d-1})]^J e^{-(m-2/\sqrt{l})l((l^\kappa-2l)/l-1)} \leqq e^{-cl^{\kappa-1/2}}$$

for some $c > 0$ independent of m and l, since $\kappa - \frac{1}{2} > \tau > \frac{1}{2}$ and $m \geqq \frac{3}{\sqrt{l}}$. Now let $A_r = E_r \cap F_r$. Since both E_r and F_r are negative events, we can apply the Harris–FKG inequality to get

$$\mathbf{Q}(A_r) \geqq \mathbf{Q}(E_r)\mathbf{Q}(F_r) \geqq {}^{-\xi\gamma l^{\kappa d}}(1 - e^{-cl^{\kappa-1/2}}) \geqq e^{-2\xi\gamma l^{\kappa d}}$$

for l large enough.

Let $A = \bigcup_{r=1}^{R} A_r$, since the A_r's are independent events, identically distributed, we have

$$\mathbf{Q}(A^c) = \prod_{r=1}^{R} (1 - \mathbf{Q}(A_r)) = (1 - \mathbf{Q}(A_1))^R \leqq (1 - e^{-2\xi\gamma l^{\kappa d}})^R \leqq e^{-Re^{-2\xi\gamma l^{\kappa d}}}.$$

But $R \geqq e^{1/2 l^{\alpha/2}}$ for l large, and $\kappa d < \frac{\alpha}{2}$, so

$$\mathbf{Q}(A^c) \leqq e^{-e^{1/4 l^{\alpha/2}}} \tag{3.8}$$

for l large enough. Now,

$$G_{B_L(0)}(0, Y) \leqq \mathbf{Q}\{\{0 \xrightarrow[B_L(0)]{} Y\} \cap A\} + \mathbf{Q}(A^c). \tag{3.9}$$

By the definition of the event A we have $\{0 \xrightarrow[B_L(0)]{} Y\} \cap A \subset C$, where C is the event that there exists a connection in $B_{\Lambda_L(0) \setminus \tilde{\lambda}}$ of vertical length $> T_1$, so

$$C \subset \bigcup_{\substack{(y_1,s_1),(y_2,s_2) \in B_{\Lambda_L(0)\setminus \tilde{\lambda}} \\ |s_1 - s_2| > T_1}} \{(y_1, s_1) \xrightarrow[B_{\Lambda_L(0)\setminus \tilde{\lambda}}]{} (y_2, s_2)\}.$$

Thus we can again use Lemma 3.3 to get

$$\begin{aligned} \mathbf{Q}(C) &\leqq \sum_{\substack{(y_1,s_1),(y_2,s_2) \in B_{\Lambda_L(0)\setminus \tilde{\lambda}} \\ |s_2 - s_1| > T_1}} G_{B_{\Lambda_L(0)\setminus \tilde{\lambda}}}((y_1, s_1), (y_1, s_2)) \\ &\leqq ((2L+1)^d 2e^{\sqrt{L}})^2 e^{-(m-2/\sqrt{l})l(T_1 e^{-\sqrt{l}}-1)} \leqq e^{M_2 e^{l/2l^\tau}} \end{aligned} \tag{3.10}$$

since $\tau > \frac{1}{2}$, with

$$M_2 \geqq m - e^{-1/4l^\tau}(m+1)$$

for l large.

Thus, it follows from (3.8), (3.9) and (3.10) that

$$\begin{aligned} G_{B_L(0)}(0, Y) &\leqq e^{-M_2 e^{l/2l^\tau}} + e^{-e^{l/4l^{\alpha/2}}} \leqq e^{-1/2 M_2 e^{l/2l^\tau}} \\ &\leqq e^{-M_2 e^{l/4l^\tau}} \end{aligned}$$

for l sufficiently large.

This proves the sublemma. ■

We can now finish the proof of Lemma 3.5. Since $M_1 \leqq M_2$ we only need to show that the hypothesis of Sublemmas 3.6 and 3.7 hold with probability $\geqq 1 - \dfrac{1}{L^p}$. But this follows from (3.6) and Lemma 3.4, for l large enough. ■

Theorem 3.1 now follows from Lemma 3.5 if we pick L_0 large enough so that $m_0 \geqq \dfrac{3}{\sqrt{L_0}}$ and L_0 is sufficiently large for Lemma 3.5 to hold, and, in addition

$$\sum_{k=0}^{\infty} \left[\left(-m_0 - \frac{2}{\sqrt{L_k}} \right) \frac{5J}{L_k^{\alpha-1}} + \frac{2}{\sqrt{L_k}} + \frac{3J}{L_k^{\alpha/2}} \right] \leqq m_0 - m.$$

This finishes the proof of Theorem 3.1. ■

4. Long-Range Order

We now prove Theorem 2.2. Let us fix $q \in \{1, 2, \ldots, \}$. Consider the corresponding deterministic model with $d = 1, J_{\langle x,y \rangle} \equiv J > 0, K_x \equiv K > 0, \beta = \gamma = 1$. It is well known that, given $J > 0$, we can always find $\bar{K} = \bar{K}(J, q)$ such that for $K > \bar{K}$ the model has long-range order, i.e., the two-point function does not go to zero at infinity.

Now we consider the random model with $d \geqq 2$, with $J_{\langle x,y \rangle} > 0$ with probability one. Fix $\beta > 0$. Then, given $0 < p_1 < 1$, we can find $J = J(\beta, p_1)$ such that

$$\mathbf{P}\{\beta J_{\langle x,y \rangle} \geqq J\} \geqq p_1.$$

Let us also choose $0 < p_2 < 1$, and find $\bar{K} = \bar{K}(J, q)$. We choose $\gamma_1 = \gamma_1(\beta, q, p_1, p_2)$ such that

$$\mathbf{P}\{\gamma K_x \geqq \bar{K}\} \geqq p_2.$$

Now let us consider the model of site and bond Bernoulli percolation on the d-dimensional hypercubic lattice with the occupation probability for bonds being p_1 and the occupation probability for sites being p_2. If p_1 and p_2 are sufficiently close to one, the event that there exist two disjoint infinite self-avoiding paths starting from some point $x \in \mathbf{Z}^d$ with both bonds and sites occupied has strictly positive probability (see Chayes and Chayes [11] for bond percolation, the same argument applies for bond and site percolation).

Thus, if p_1 and p_2 are chosen sufficiently close to one, then for any $x \in \mathbf{Z}^d$ is a strictly positive $\mathbf{P}$-probability that there exists a doubly infinite self-avoiding path $\{\varphi(n)\}_{n \in \mathbf{Z}}$ is $\mathbf{Z}^d$ such $\varphi(0) = x$ and

$$\beta J_{\langle \varphi(m), \varphi(n+1) \rangle} \geqq J, \gamma K_{\varphi(n)} \geqq \bar{K} \quad \text{for} \quad \gamma \geqq \gamma_1.$$

Let $G_\varphi^{(q)}(x, y)$ be the two-point function between X and Y in the sublattice $\mathbf{L}_\varphi = \{Z = (z, t); z = \varphi(n) \text{ for some } n \in \mathbf{Z}\}$ of $\mathbf{Z}^{d+1}$. Notice that $\mathbf{L}_\varphi$ is isomorphic to the lattice $\mathbf{Z}^2$, and on this sublattice we have long-range order (use ferromagnetism and Griffiths inequalities).

Thus, if $X = (x, s)$, we have $X \in \mathbf{L}_\varphi$, let also $Y \in \mathbf{L}_\varphi$. We then have, with strictly positive $\mathbf{P}$-probability, that

$$G(X, Y) \geqq G_\varphi(X, Y) \geqq C_1 > 0$$

for some $C_1 > 0$ independent of X and $Y \in \mathbf{L}_\varphi$.

It follows that

$$\overline{\lim_{|Y| \to \infty}} G^{(q)}(X, Y) > 0$$

for any $X = (x, s)$.

It is now easily seen by standard arguments that

$$\overline{\lim_{|Y| \to \infty}} G^{(q)}(X, Y) > 0$$

for any $X \in \mathbf{Z}^{d+1}$. Indeed, by ergodicity, with probability one we can find $x' \in \mathbf{Z}^d$ such that for each $S'_{s'}$

$$\overline{\lim_{|Y| \to \infty}} G^{(q)}((x', s'), Y) > 0.$$

By the Harris–FKG inequality,

$$G^{(q)}(X, Y) \geqq G^{(q)}(X, (x', s')) G^{(q)}((x', s'), Y)$$

so that

$$\overline{\lim_{|Y| \to \infty}} G^{(q)}(X, Y) > 0$$

with probability one since $G(X,(x',s')) > 0$.

This finishes the proof of Theorem 3.2. ■

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Note added in proof. Theorem 2.2 has been extended to $d=1$ by M. Aizenman and A. Klein if $\mathbb{E}(j_{\langle x,y\rangle}^{-\delta}) < \infty$ for some $\delta > 0$.