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Modular Invariants for Affine $\widehat{SU}(3)$ Theories at Prime Heights

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Abstract. A proof is given for the existence of two and only two modular invariant partition functions in affine $\widehat{SU}(3)_k$ theories at heights $n=k+3$ which are prime numbers. Arithmetic properties of the ring of algebraic integers $\mathbb{Z}(\omega)$ which is related to $SU(3)$ weights are extensively used.

1. Introduction

The classification of all modular invariant partition functions of a rational conformal field theory is obviously an important problem. In the case of affine theories, a complete answer is known, at present, only for $\widehat{SU}(2)$ at all levels [1] and for $\widehat{SU}(n)$ at level one [2]. For $\widehat{SU}(3)$ theories two modular invariants have been constructed at all levels [3] and, for exceptional cases, additional invariants are known as well [4]. However, as far as we know, there is no proof, at any level except $k=1$, that these invariants actually exhaust all possibilities.

In this paper we take a little step towards setting up the complete classification of modular invariants for $\widehat{SU}(3)_k$: we will prove that at prime heights $n=k+3$ there are indeed two and only two modular invariant partition functions. The proof makes extensive use of arithmetic properties of the (quadratic) ring of algebraic integers $\mathbb{Z}(\omega)$ which is naturally related to $SU(3)$ weights.

In an affine $\widehat{SU}(3)$ theory, the Hilbert space splits into two chiral parts, each of which decomposes into a finite sum of subspaces corresponding to integrable

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representations of the chiral algebra. The partition functions

$$Z(\tau)=\sum_{p,p'\in B_n}\bar{\chi}_{\bar{p}}(\tau)M_{p,p'}\chi_{p'}(\tau) \tag{1.1}$$

is a sum over characters $\chi_p(\tau)$ of these representations labelled by $SU(3)$-weights which lie in a fundamental domain B_n. Physical constraints on the partition functions, Eq. (1.1), require that the coefficients $M_{p,p'}$ be non-negative integers and that M_{00} be equal to one.

Modular invariance of the partition functions, Eq. (1.1), requires the matrix M to commute with the matrices of the representation of the modular group to which the characters belong. Since the modular group is generated by the transformations T, $(\tau\to\tau+1)$, and S, $\left(\tau\to-\frac{1}{\tau}\right)$, a necessary and sufficient condition for modular invariance is that the matrix M belongs to the *commutant* of S and T, i.e., commutes with the matrices representing the modular transformations S and T.

To determine all partition functions which satisfy the physical constraints and are modular invariant we follow the strategy outlined ref. [1]: using the $SU(3)$ Weyl group we first unfold the domain B_n to a larger set, namely the weight lattice modulo n times the root lattice. We then construct for prime heights n an explicit basis of the commutant $\hat{M}$ of the unfolded (or extended) modular transformations $\hat{S}$ and $\hat{T}$. Folding back these matrices $\hat{M}$ we obtain all modular invariant $Z(\tau)$'s for prime heights n. Imposing the physical constraints leaves us finally with two and only two partition functions.

The paper is organized as follows: in Sect. 2 we briefly review the arithmetic properties of the ring of algebraic integers $\mathbb{Z}(\omega)$ which plays an essential role in implementing the strategy we have just sketched. We also recall some symmetry properties of the affine $\widehat{SU}(3)$ characters and establish the connection between $\mathbb{Z}(\omega)$ and $SU(3)$ weights, roots and Weyl operations. In Sect. 3 we use $\mathbb{Z}(\omega)$ to construct an explicit basis for the (extended) commutant of the modular representation on the extended set of characters. Folding back these matrices we determine the general form of the matrix M in Eq. (1.1) which guarantees modular invariance of the partition functions $Z(\tau)$.

The $\mathbb{Z}(\omega)$ parametrization used here is useful for the construction of modular invariant M matrices but turns out to be less convenient for the implementation of the positivity constraints. This is discussed in Sect. 4, where we prove an arithmetic lemma which eventually allows us to impose the positivity condition. This is worked out in Sect. 5, where we show that, as a result of all physical constraints on the matrix M, there remains, at each prime height $n>5$, two and only two modular invariant partition functions. At present, it is not completely clear to us how to extend our analysis for arbitrary heights.

2. $\mathbb{Z}(\omega)$ and Characters of $\widehat{SU}(3)$

In this section we exploit the fact that the weight lattice of $SU(3)$ can be identified with the ring of algebraic integers $\mathbb{Z}(\omega)$ to re-express basic properties of $\widehat{SU}(3)$ characters in a form particularly well suited for the construction of modular invariants [5].

For the sake of completeness, we first recall some well-known arithmetic features of $\mathbb{Z}(\omega)$ [6].

2.1. The Ring $\mathbb{Z}(\omega)$

Let $\omega = \frac{-1+\sqrt{-3}}{2}$ be a third root of unity which satisfies the algebraic equation $1+\omega+\omega^2=0$. The ring of algebraic integers $\mathbb{Z}(\omega)$ is the set

$$\mathbb{Z}(\omega)=\{z=q_1-q_2\omega;\, q_1,q_2\in\mathbb{Z}\}, \tag{2.1}$$

with the usual addition and multiplication rules.

For $z=q_1-q_2\omega$, its complex conjugate $\bar{z}$ is given by $\bar{z}=q_1-q_2\bar{\omega}=q_1-q_2\omega^2 =q_1+q_2+q_2\omega$; it also belongs to $\mathbb{Z}(\omega)$.

The norm and the trace of an element of $\mathbb{Z}(\omega)$ are defined by

$$N(z)=z\bar{z}=q_1^2+q_1q_2+q_2^2, \tag{2.2}$$

$$\mathrm{Tr}(z)=z+\bar{z}=2q_1+q_2. \tag{2.3}$$

Obviously $N(z)$ and $\mathrm{Tr}(z)$ belong to $\mathbb{Z}$ (henceforth called the ring of rational integers). Elements of $\mathbb{Z}(\omega)$ with norm one are called *units*. There are precisely six such elements, namely ± 1, $\pm\omega$, $\pm\omega^2$.

Divisibility properties of the algebraic integers are analogous to those of the rational integers. In particular, an algebraic prime (or $\mathbb{Z}(\omega)$-prime) is an algebraic integer whose only divisors are itself and the units. $\mathbb{Z}(\omega)$-primes whose ratio is a unit are called associated primes.

$\mathbb{Z}(\omega)$ is a *unique factorization domain*, i.e., every non-unit of $\mathbb{Z}(\omega)$ can be written uniquely (up to trivial reorderings) as a product of (non-associated) algebraic primes. From the multiplicative properties of the norm, Eq. (2.2), the study of the algebraic prime decomposition in $\mathbb{Z}(\omega)$ boils down to the $\mathbb{Z}(\omega)$-prime decomposition of rational primes, p. The following results hold:

1. $p\equiv 2 \pmod 3$. Then p remains prime in $\mathbb{Z}(\omega)$ and $N(p)=p^2$.
2. $p\equiv 1 \pmod 3$. Then p *splits* in $\mathbb{Z}(\omega)$ as $p=\pi\bar{\pi}$, where $N(\pi)=N(\bar{\pi})=p$. π and $\bar{\pi}$ are *not* associated primes in $\mathbb{Z}(\omega)$. An example is given by $19=(3-2\omega)\,(5+2\omega)$.
3. $p=3$ is not a $\mathbb{Z}(\omega)$-prime since $3=-\omega^2(1-\omega)^2$. The algebraic number $(1-\omega)$ is a $\mathbb{Z}(\omega)$-prime and $N(1-\omega)=3$.

Congruences modulo an element t of $\mathbb{Z}(\omega)$ are straightforwardly defined

$$z\equiv w \pmod t \quad \text{iff} \quad t|(z-w).$$

The congruence ring $\frac{\mathbb{Z}(\omega)}{t\mathbb{Z}(\omega)}$ has $N(t)$ elements. It is a field ($\mathbb{F}_{N(t)}$) iff t is a $\mathbb{Z}(\omega)$-prime. When t decomposes as $t=\pi_1\pi_2$ with π_1 and π_2 coprime, the congruence ring modulo t factorizes as

$$\frac{\mathbb{Z}(\omega)}{t\mathbb{Z}(\omega)} \cong \frac{\mathbb{Z}(\omega)}{\pi_1\mathbb{Z}(\omega)} \times \frac{\mathbb{Z}(\omega)}{\pi_2\mathbb{Z}(\omega)}$$

leading to the decomposition

$$z=z_1\frac{t}{\pi_1}+z_2\frac{t}{\pi_2}, \tag{2.4}$$

where z_1 and z_2 are taken modulo π_1 and π_2 respectively.

For simplicity, let us now restrict our discussion to the case of congruences modulo a *rational prime* $n (n \neq 3)$. As representatives of the congruence ring mod n, one may choose the set

$$\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)} \cong \left\{ z = a - b\omega;\; a, b \in \frac{\mathbb{Z}}{n\mathbb{Z}} \right\}. \tag{2.5}$$

Note that $z \in \frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ implies $\mathrm{Tr}(z)$, $N(z) \in \frac{\mathbb{Z}}{n\mathbb{Z}}$. Since the trace is additive, i.e., $\mathrm{Tr}(z_1 + z_2) = \mathrm{Tr}(z_1) + \mathrm{Tr}(z_2)$, it can be used to define

$$\chi_w(z) = \exp\left(-2\pi i \frac{\mathrm{Tr}(z\bar{w})}{n} \right), \tag{2.6}$$

which is an additive character on the ring $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$. When w runs over $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$, these characters $\chi_w(z)$ form a *complete* set.

An important multiplicative subgroup of Eq. (2.5) is the cyclic group of units modulo n, $U^{(n)}$ $(n > 3)$:

$$U^{(n)} = \left\{ u \in \frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)};\; N(u) = u\bar{u} \equiv 1 \pmod{n} \right\}. \tag{2.7}$$

The order of this group depends on the algebraic prime factorization of the rational prime n. Furthermore $U^{(n)}$ necessarily contains the six units of $\mathbb{Z}(\omega)$ as a subgroup. Standard methods [6] lead to the result

$$\operatorname{card} U^{(n)} = n + 1 \quad \text{if} \quad n \equiv 2 \pmod{3}, \tag{2.8}$$

$$\operatorname{card} U^{(n)} = n - 1 \quad \text{if} \quad n \equiv 1 \pmod{3}. \tag{2.9}$$

Since $\operatorname{card} U^{(n)}$ is a multiple of 6, we define the integer q by

$$\operatorname{card} U^{(n)} = 6q. \tag{2.10}$$

$U^{(n)}$ being a cyclic group, every unit modulo n can be written as u^m, with u some generator of $U^{(n)}$ and $m \in \{1, 2, \ldots, 6q\}$. The six units of $\mathbb{Z}(\omega)$ then correspond to m a multiple of q.

Since any non-zero element of $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is the norm of an element in the ring, Eq. (2.5), one finally arrives at the following detailed description of elements of $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$.

$n \equiv 2 \pmod{3}$. $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ is then a field. $z = 0$ is the only element with zero norm. The remaining $n^2 - 1$ elements subdivide into $(n-1)$ subsets each containing the $n+1$ elements which have the same norm mod n. Since exactly half the non-zero elements of $\frac{\mathbb{Z}}{n\mathbb{Z}}$ are squares mod n, the following parametrization is obtained using Eqs. (2.7), (2.8), and (2.10):

$$x = v\varrho^\alpha u^j \quad 1 \leqq v \leqq \frac{n-1}{2};\; \alpha = 0, 1;\; 1 \leqq j \leqq n + 1 = 6q;$$

$$\varrho\bar{\varrho} \text{ not a square mod } n. \tag{2.11}$$

When $n \equiv 1 \pmod 4$ we will use the explicit value $\varrho = \omega - \bar{\omega}$ with $\varrho\bar{\varrho} = 3$. 3 is not a square mod n because $n \equiv 2 \pmod 3$ implies that -3 is not a square mod n while $n \equiv 1 \pmod 4$ implies that -1 is a square mod n. When $n \equiv 3 \pmod 4$ it is sufficient for our purposes to remark that since ϱ and $\bar{\varrho}$ have the same norm mod n one can always choose a ϱ such that $\bar{\varrho} = u\varrho$ with u a generator of $U^{(n)}$.

$n \equiv 1 \pmod 3$. As already mentioned, n now splits in $\mathbb{Z}(\omega)$, i.e., $n = \pi\bar{\pi}$. Among the non-zero elements of the congruence ring, there are now $2(n-1)$ of them which have zero norm mod n. They can be written as

$$x = \alpha u^j \pi + (1-\alpha) u^j \bar{\pi} \quad \alpha = 0, 1;\ 1 \leqq j \leqq n-1 = 6q\,. \tag{2.12}$$

The remaining $(n-1)^2$ elements are not divisible by either π or $\bar{\pi}$. According to our previous discussion, they can be parametrized as

$$x = v\varrho^\alpha u^j \quad 1 \leqq v \leqq \frac{n-1}{2};\ \alpha = 0, 1;\ 1 \leqq j \leqq n-1 = 6q;$$
$$\varrho\bar{\varrho} \text{ not a square mod } n\,. \tag{2.13}$$

When $n \equiv 3 \pmod 4$ we may again choose $\varrho = \omega - \omega^2$, while for $n \equiv 1 \pmod 4$ we pick a ϱ such that $\bar{\varrho} = u\varrho$.

As a final remark on congruence rings in $\mathbb{Z}(\omega)$, we note that $\dfrac{\mathbb{Z}(\omega)}{(1-\omega)\mathbb{Z}(\omega)}$ is a field with three elements (isomorphic to $\mathbb{F}_3$). We will choose the representatives of the congruence classes as 0, ± 1. Clearly the units $\bmod (1-\omega)$ are ± 1 and the congruences on the norm and the trace of elements of $\dfrac{\mathbb{Z}(\omega)}{(1-\omega)\mathbb{Z}(\omega)}$ are now modulo 3.

2.2. *$\widehat{SU}(3)$ Characters*

We first recall some definitions and properties of affine $\widehat{SU}(3)$ characters and then proceed to express these properties in terms of the algebraic integers $\mathbb{Z}(\omega)$.

The affine $\widehat{SU}(3)$ characters [7] are labelled by a highest weight $\boldsymbol{\lambda} = \lambda_1 \boldsymbol{\Lambda}_1 + \lambda_2 \boldsymbol{\Lambda}_2$ with $\boldsymbol{\Lambda}_i$ the fundamental weights and λ_i non-negative rational integers. If the height of affine $\widehat{SU}(3)$ is $n = k+3$ with k the level ($k \geqq 0$), the highest weights corresponding to unitary representations satisfy the condition $\lambda_1 + \lambda_2 \leqq k$. There are thus $\dfrac{(k+1)(k+2)}{2} = \dfrac{(n-1)(n-2)}{2}$ independent affine characters. To give their explicit form, it is convenient to define a shifted weight $\mathbf{p} = \boldsymbol{\lambda} + \boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_2 = p_1 \boldsymbol{\Lambda}_1 + p_2 \boldsymbol{\Lambda}_2$. Unitarity of the representations implies that $\mathbf{p}$ belongs to the fundamental domain

$$B_n = \{\mathbf{p} \in M^*;\ p_1, p_2 \geqq 1 \text{ and } p_1 + p_2 \leqq n-1\}\,, \tag{2.14}$$

where M^* is the weight lattice generated by $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$. With $q = \exp 2\pi i\tau$, $Im\,\tau > 0$, the restricted affine character labelled by a shifted weight $\mathbf{p}$ is given by

$$\chi_{\mathbf{p}}(\tau) = \tfrac{1}{2}[\eta(\tau)]^{-8} \sum_{\mathbf{t} \in M} \prod_{\boldsymbol{\alpha} > 0} [\boldsymbol{\alpha} \cdot (\mathbf{p} + n\mathbf{t})]\, q^{\frac{(\mathbf{p}+n\mathbf{t})^2}{2n}} \tag{2.15}$$

with M the root lattice and the product taken over positive roots. $\eta(\tau)=q^{1/24}\prod_{n\geqq 1}(1-q^n)$ is the usual Dedekind function. From Eq. (2.15), it is clear that characters have the periodicity property

$$\chi_{\mathbf{p}+n\mathbf{t}}(\tau)=\chi_{\mathbf{p}}(\tau)$$

for any root $\mathbf{t}$.

Furthermore, under Weyl transformations $w\in S_3$,

$$\chi_{w(\mathbf{p})}(\tau)=(\det w)\chi_{\mathbf{p}}(\tau). \tag{2.16}$$

Because of Eq. (2.16), it is convenient to consider an extended set (i.e., unfolded under the Weyl group) of affine characters $\chi_{\mathbf{p}}$ for $\mathbf{p}$ in $\frac{M^*}{nM}$, namely a set of $3n^2$ characters. Antisymmetry under odd Weyl transformations implies that this extended set will contain null characters. There are precisely $9n-6$ such null characters and the remaining ones fall into orbits of length 6 under the Weyl group so that $3n^2=9n-6+6\frac{(n-1)(n-2)}{2}$.

The modular transformations of the unfolded set of characters are well known [8]:

$$\chi_{\mathbf{p}}(\tau+1)=\sum_{\mathbf{p}'\in\frac{M^*}{nM}}\hat{T}_{\mathbf{p},\mathbf{p}'}\chi_{\mathbf{p}'}(\tau);\qquad \hat{T}_{\mathbf{p},\mathbf{p}'}=\left[\exp 2\pi i\left(\frac{\mathbf{p}^2}{2n}-\frac{1}{3}\right)\right]\delta_{\mathbf{p},\mathbf{p}'}, \tag{2.17}$$

$$\chi_{\mathbf{p}}\left(-\frac{1}{\tau}\right)=\sum_{\mathbf{p}'\in\frac{M^*}{nM}}\hat{S}_{\mathbf{p},\mathbf{p}'}\chi_{\mathbf{p}'}(\tau);\qquad \hat{S}_{\mathbf{p},\mathbf{p}'}=\frac{-i}{\sqrt{3n^2}}\exp 2\pi i\left(-\frac{\mathbf{p}\cdot\mathbf{p}'}{n}\right). \tag{2.18}$$

These equations imply that the unfolded characters transform in a unitary representation of $SL(2,\mathbb{Z})$.

To relate these properties of characters to the algebraic integers $\mathbb{Z}(\omega)$, the following correspondence is made: to each weight $\mathbf{p}=(p_1,p_2)$ one associates the algebraic number $z=p_1-p_2\omega$ thus identifying the weight lattice with $\mathbb{Z}(\omega)$. The positive roots are then represented by the algebraic integers $\alpha_1=2+\omega$, $\alpha_2=-1-2\omega$ and $\alpha_1+\alpha_2=1-\omega$. It follows that an arbitrary root $t=m\alpha_1+n\alpha_2$ is represented as $t=(2m-n)-\omega(2n-m)=[m+\omega(m-n)](1-\omega)$. The root lattice M is thus identified with the (prime) ideal $(1-\omega)\mathbb{Z}(\omega)$ of $\mathbb{Z}(\omega)$. The triality of a weight is related in $\mathbb{Z}(\omega)$ to its residue mod$(1-\omega)$.

The fundamental domain B_n, Eq. (2.14), corresponds to the set

$$B_n=\{z=a-b\omega;\ a,b\geqq 1 \text{ and } a+b\leqq n-1\}. \tag{2.19}$$

Charge conjugation which exchanges the two components of a weight

$$z=a-b\omega\to Cz=b-a\omega=-\omega\bar{z} \tag{2.20}$$

leaves B_n invariant and is a symmetry of the restricted characters

$$\chi_z(\tau)=\chi_{-\omega\bar{z}}(\tau). \tag{2.21}$$

B_n is also invariant under the order 3 automorphism

$$z\to\sigma(z)=\omega^2 z+n;\qquad \sigma^2(z)=\omega z-\omega n. \tag{2.22}$$

Since the weight space metric is given by the inverse Cartan matrix $\frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$, we obtain immediately

$$\mathbf{p}\cdot\mathbf{p}'=\tfrac{1}{3}\mathrm{Tr}\,\bar{z}z', \tag{2.23}$$

while

$$\mathbf{p}^2=\tfrac{1}{3}\mathrm{Tr}\,\bar{z}z=\tfrac{2}{3}N(z). \tag{2.24}$$

It is then easy to check that Eq. (2.15) becomes

$$\chi_z(\tau)=\frac{1}{18}[\eta(\tau)]^{-8}\sum_{t\in\mathbb{Z}(\omega)}\mathrm{Tr}[(1+2\omega)\{(z+nt(1-\omega))^3\}]\,q^{\frac{N(z+nt(1-\omega))}{3n}} \tag{2.25}$$

with z an element of the congruence ring $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ which is of order $N(n(1-\omega))=3n^2$.

Up to irrelevant numerical factors, the modular transformation matrices $\hat{T}$ and $\hat{S}$ now read

$$\hat{T}_{z,z'}=\left[\exp\left\{2\pi i\frac{N(z)}{3n}\right\}\right]\cdot\delta_{zz'}, \tag{2.26}$$

$$\hat{S}_{z,z'}=\frac{1}{\sqrt{3n^2}}\exp\left\{-2\pi i\frac{\mathrm{Tr}(z\bar{z}')}{3n}\right\}. \tag{2.27}$$

In view of Eq. (2.6), $\hat{S}_{z,z'}$ is simply the Fourier kernel on the ring $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$.

Finally, the Weyl group operations are easily seen to have a simple representation in $\mathbb{Z}(\omega)$: even Weyl transformations act on $\mathbb{Z}(\omega)$ as

$$z\xrightarrow{w}z,\ \omega z,\ \omega^2 z, \tag{2.28}$$

while the odd ones correspond to

$$z\xrightarrow{w}\bar{z},\ \omega\bar{z},\ \omega^2\bar{z}. \tag{2.29}$$

3. The Commutant of Modular Transformations

The dimension of the commutant of $\hat{S}$ and $\hat{T}$, Eqs. (2.26), (2.27), has been computed by Bauer and Itzykson [9] [1]: at prime heights $n(\neq 3)$ it is equal to $2(2n+1)$. Using $\mathbb{Z}(\omega)$ we are now in a position to construct an explicit basis of this commutant.

Since $(n,3)=1$, the ring $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ factorizes as in Eq. (2.4), i.e.,

$$z=x(1-\omega)+tn \tag{3.1}$$

with $x\in\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ and $t\in\frac{\mathbb{Z}(\omega)}{(1-\omega)\mathbb{Z}(\omega)}$ $(\approx\mathbb{F}_3=\{0,\pm 1\})$.

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[1] For $\widehat{SU}(N)$, Bauer and Ityzkson give an abstract construction of the commutant of $\hat{S}$ and $\hat{T}$. Its dimension is computed in [10]

The modular matrices $\hat{S}$ and $\hat{T}$ factorize similarly and the construction of matrices in the commutant can be handled separately in each sector of Eq. (3.1):

$$\hat{M}_{z,z'} = \hat{M}_{x,x'}\mathscr{M}_{t,t'} . \tag{3.2}$$

It is straightforward to check that there are two independent solutions in the "triality sector" $\frac{\mathbb{Z}(\omega)}{(1-\omega)\mathbb{Z}(\omega)}$. They are given by

$$(\mathscr{M}_{\pm})_{t,t'} = \delta_{t',\pm t} . \tag{3.3}$$

In the $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ or x-sector, commutativity with $\hat{T}$ requires vanishing matrix elements $\hat{M}_{x,x'}$ when labelled by algebraic integers which have different norms mod n, i.e.,

$$\hat{M}_{x,x'} = 0 \quad \text{if} \quad N(x) \not\equiv N(x') \,(\bmod\, n) . \tag{3.4}$$

We are thus naturally led to consider the cyclic group of units, Eq. (2.7), resulting in the following matrices of the commutant:

$$(\hat{M}_{u^m})_{x,x'} = \delta_{x',u^m x} , \qquad m = 1, \ldots, 6q , \tag{3.5}$$

$$(\hat{\bar{M}}_{u^m})_{x,x'} = \delta_{x',u^m \bar{x}} , \qquad m = 1, \ldots, 6q . \tag{3.6}$$

We note that these matrices simply permute elements of the ring $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ which have the same norm.

When $n \equiv 2 \pmod 3$, i.e., when n remains prime in $\mathbb{Z}(\omega)$, the $2(n+1)$ matrices given by Eqs. (3.5) and (3.6) are readily seen to satisfy one linear relation

$$\sum_{m=1}^{n+1} (\hat{M}_{u^m} - \hat{\bar{M}}_{u^m}) = 0 . \tag{3.7}$$

There is no other linear relation among these matrices. Looking, first at the sector of units, $x = u^j$, $x' = u^j$ [Eq. (2.11)], it is easily seen that the matrices $\hat{M}_{u^m}$ (respectively $\bar{M}_{u^m}$) realize all cyclic (respectively anticyclic) permutations on j, j'. Hence there exist at most two linear relations. Using $\bar{\varrho} - u^k \varrho$, k odd, the explicit form of the matrices for $x = \varrho u^j$, $x' = \varrho u^{j'}$ concludes the argument.

Combining this with the two solutions given by Eq. (3.3) we have thus a basis of the commutant, namely $2[2(n+1)-1]$ independent matrices commuting with $\hat{S}$ and $\hat{T}$.

When n splits in $\mathbb{Z}(\omega)$, i.e., $n \equiv 1 \pmod 3$ and $n = \pi\bar{\pi}$ we are still missing solutions in the x-sector because of Eq. (2.9). They can be obtained from the zero-norm elements of $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ namely from multiples of π or $\bar{\pi}$ [Eq. (2.12)]. The following four matrices with $\beta = \pi, \bar{\pi}$:

$$(\hat{M}_{\beta})_{x,x'} = \begin{cases} 1 & \text{if } \beta | x \text{ and } \beta | x' , \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

$$(\hat{\bar{M}}_{\beta})_{x,x'} = \begin{cases} 1 & \text{if } \beta | x \text{ and } \bar{\beta} | x' , \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

are easily checked to belong to the commutant. Again one finds a single linear relation

$$\sum_{m=1}^{n-1} (\hat{M}_{u^m} - \hat{M}_{u^m}) = \sum_{\beta=\pi,\bar{\pi}} (\hat{M}_\beta - \hat{M}_\beta) \tag{3.10}$$

and the commutant basis is thus complete in this case as well.

Our next task is to "fold back" the modular matrices $\hat{S}$, $\hat{T}$ and their commutant. This folding back is defined from the action of the $SU(3)$ Weyl group. Since the t-sector in Eq. (3.1) is invariant under Weyl transformations, the action of the Weyl group is non-trivial in the x-sector only where it reads [Eqs. (2.28) and (2.29)]

$$x \to w(x) = x,\ \omega x,\ \omega^2 x \quad \text{if} \quad \det w = +1\,, \tag{3.11}$$

$$x \to w(x) = -\bar{x},\ -\omega\bar{x},\ -\omega^2\bar{x} \quad \text{if} \quad \det w = -1\,. \tag{3.12}$$

The folded commutant matrices are indexed by Weyl inequivalent algebraic integers excluding those which correspond to null characters. They are explicitly constructed from the "Weyl orbits of these integers," namely

$$M_{x,x'} = \sum_{w_1,w_2} \det(w_1 w_2) \hat{M}_{w_1(x),\, w_2(x')}\,. \tag{3.13}$$

Equation (3.13) can be written in matrix form as

$$M = R\hat{M}R^T\,, \tag{3.14}$$

where the "folding matrix" R is a rectangular $\frac{(n-1)(n-2)}{6} \times n^2$ matrix with entries

$$R_{x,y} = \begin{cases} \det w & \text{when } w(y) = x, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

To proceed, we need the explicit parametrizations of the x-sector given by Eqs. (2.11)–(2.13). It is convenient to group the non-zero elements of $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ in three distinct sectors $(+,-,0)$ according to the quadratic class of their norm mod n:

$$\hat{C}_+ = \left\{ x = vu^j;\ v = 1, \ldots, \frac{n-1}{2};\ j = 1, \ldots, 6q \right\}, \tag{3.16}$$

$$\hat{C}_- = \left\{ x = \varrho v u^j;\ v = 1, \ldots, \frac{n-1}{2};\ j = 1, \ldots, 6q \right\}, \tag{3.17}$$

$$\hat{C}_0 = \{ x = \alpha\pi u^j + (1-\alpha)\bar{\pi}u^j;\ \alpha = 0, 1;\ j = 1, \ldots, 6q \}\,. \tag{3.18}$$

The $\hat{C}_0$ sector is of course empty for $n \equiv 2(3)$.

An explicit form of the folding matrix, Eq. (3.15) is obtained once a choice of representatives of Weyl orbits, excluding null characters, is made. We consider the various cases in succession.

In the $\hat{C}_0$ sector, the Weyl orbit of the $\alpha = 1$ subsector is given by

$$\begin{array}{ll} \det w = +1\,, & \pi u^j \to \pi u^{j+2q} \to \pi u^{j+4q}\,, \\ & \quad\downarrow \\ \det w = -1\,, & \bar{\pi} u^{q-j} \to \bar{\pi} u^{3q-j} \to \bar{\pi} u^{5q-j}\,, \end{array} \tag{3.19}$$

hence it covers $\hat{C}_0$ three times. There are no null characters in this sector and a possible choice of representatives whose orbits cover $\hat{C}_0$ once is the following:

$$C_0 = \{x = \pi u^j;\, j = 1, \ldots, 2q\}\,. \tag{3.20}$$

The folding matrix R then reads

$$R^{(0)}_{j;\alpha' j'} = \delta_{1,\alpha'}\delta^{(2q)}_{j',j} - \delta_{0,\alpha'}\delta^{(2q)}_{j',q-j}, \tag{3.21}$$

where $\delta^{(2q)}_{j,j'}$ stands for the Kronecker symbol modulo $2q$. The line index of $R^{(0)}$ labels an element of the set C_0, Eq. (3.20), while the column indices label $\hat{C}_0$, Eq. (3.18).

In the $\hat{C}_\pm$ sectors, we must consider separately the cases q odd and q even as pointed out in the remarks following Eqs. (2.11) and (2.13). When *q is odd*, representatives of Weyl orbits can be taken as

$$C^{\text{odd}}_{+} = \left\{x = \nu u^{2i};\, i = 1, \ldots, q;\, \nu = 1, \ldots, \frac{n-1}{2}\right\}, \tag{3.22}$$

$$C^{\text{odd}}_{-} = \left\{x = \varrho \nu u^{i};\, i = 1, \ldots, q-1;\, \nu = 1, \ldots, \frac{n-1}{2}\right\}. \tag{3.23}$$

Null characters occur in $\hat{C}_-$ when $x = \pm\nu\varrho\omega^l$ $(l = 0, 1, 2)$. The R matrix elements are now given by

$$R^{(+)}_{(\nu,i);(\nu',j')} = (\delta^{(2q)}_{j',2i} - \delta^{(2q)}_{j',q-2i})\delta_{\nu,\nu'}, \tag{3.24}$$

$$R^{(-)}_{(\nu,i);(\nu',j')} = (\delta^{(2q)}_{j',i} - \delta^{(2q)}_{j',-i})\delta_{\nu,\nu'} \tag{3.25}$$

in an obvious notation for the labels.

When *q is even*, null characters occur in $\hat{C}_+$ for $x = \pm\nu u^{(2l+1)q/2}$ $(l = 0, 1, 2)$. We take as representatives

$$C^{\text{even}}_{+} = \left\{x = \nu u^{q/2+i};\, i = 1, \ldots, q-1;\, \nu = 1, \ldots, \frac{n-1}{2}\right\}, \tag{3.26}$$

and

$$C^{\text{even}}_{-} = \left\{x = \nu\varrho u^{2i};\, i = 1, \ldots, q;\, \nu = 1, \ldots, \frac{n-1}{2}\right\}, \tag{3.27}$$

and the R matrix reads

$$R^{(+)}_{(\nu,i);(\nu',j')} = (\delta^{(2q)}_{j',q/2+i} - \delta^{(2q)}_{j',q/2-i})\delta_{\nu,\nu'}, \tag{3.28}$$

$$R^{(-)}_{(\nu,i);(\nu',j')} = (\delta^{(2q)}_{j',2i} - \delta^{(2q)}_{j',q+1-2i})\delta_{\nu,\nu'}\,. \tag{3.29}$$

Folding the commutant Eq. (3.14), is now straightforward. For the matrices $\hat{M}_{u^m}$, Eq. (3.5), the result is the following: for *q odd*

$$M^{(+)}_{u^m} = 3\left[\delta^{(2q)}_{m,2(i-i')} - \delta^{(2q)}_{m+q,2(i+i')}\right]\delta_{\nu,\nu'} + (m \leftrightarrow -m) \tag{3.30}$$

$$M^{(-)}_{u^m} = 3\left[\delta^{(2q)}_{m,i-i'} - \delta^{(2q)}_{m,i+i'}\right]\delta_{\nu,\nu'} + (m \leftrightarrow -m) \tag{3.31}$$

$$M^{(0)}_{u^m} = 3\delta^{(2q)}_{m,j-j'} + (m \leftrightarrow -m) \tag{3.32}$$

while for *q even*

$$M^{(+)}_{u^m} = 3\left[\delta^{(2q)}_{m,i-i'} - \delta^{(2q)}_{m,i+i'}\right]\delta_{\nu,\nu'} + (m \leftrightarrow -m), \tag{3.33}$$

$$M^{(-)}_{u^m} = 3\left[\delta^{(2q)}_{m,2(i-i')} - \delta^{(2q)}_{m+q+1,2(i+i')}\right)\delta_{\nu,\nu'} + (m \leftrightarrow -m), \tag{3.34}$$

$$M^{(0)}_{u^m} = 3\delta^{(2q)}_{m,j-j'} + (m \leftrightarrow -m)\,. \tag{3.35}$$

Folding of the matrices $\hat{M}_{u^m}$, Eq. (3.6), brings nothing new, since

$$\bar{M}_{u^m} = -M_{u^{m+q}}. \tag{3.36}$$

Finally, as is easily seen, the four matrices given by Eqs. (3.8) and (3.9) fold into a single matrix which, in the C_0 sector, has *all* its entries equal to 9 while in the $C_\pm$ sectors they all vanish:

$$\begin{aligned} M_\pi^{(0)} = M_{\bar{\pi}}^{(0)} = -\bar{M}_\pi^{(0)} = -\bar{M}_{\bar{\pi}}^{(0)} = 9 \text{ (everywhere)}, \\ M_\pi^{(\pm)} = M_{\bar{\pi}}^{(\pm)} = \bar{M}_\pi^{(\pm)} = \bar{M}_{\bar{\pi}}^{(\pm)} = 0. \end{aligned} \tag{3.37}$$

Using $M_{u^m} = M_{u^{m+2q}} = M_{u^{-m}}$, Eqs. (3.7) and (3.10) now read

$$\begin{aligned} M_{u^0} + M_{u^q} + 2 \sum_{m=1}^{q-1} M_{u^m} &= 0 \quad \text{if} \quad n \equiv 2 \,(\mathrm{mod}\, 3) \\ &= \tfrac{2}{3} M_\pi \quad \text{if} \quad n \equiv 1 \,(\mathrm{mod}\, 3). \end{aligned} \tag{3.38}$$

Eliminating, e.g., M_{u^q} through this equation we obtain, for the x-sector of Eq. (3.1), that any matrix in the commutant of the (folded) modular transformations S and T can be written as

$$M = \frac{1}{6} \sum_{m=0}^{q-1} c_m M_{u^m} + \frac{1}{9} c_\pi M_\pi \tag{3.39}$$

with $c_\pi = 0$ when $n \equiv 2(3)$.

With Eq. (3.3), one obtains the general form of $M_{z,z'}$:

$$M_{z,z'} = \begin{pmatrix} M & 0 & M' \\ 0 & M+M' & 0 \\ M' & 0 & M \end{pmatrix}, \tag{3.40}$$

where the three lines and columns label $t = +1, 0, -1$ (in that order) and M and M' are two independent copies of the x-sector matrix given by Eq. (3.39).

The above discussion proves in particular that the dimension of the (folded) commutant of the modular group representation is

$$2q = \frac{n+1}{3} \quad \text{for} \quad n \equiv 2 \,(\mathrm{mod}\, 3)$$

or

$$2(q+1) = \frac{n+5}{3} \quad \text{for} \quad n \equiv 1 \,(\mathrm{mod}\, 3).$$

4. Positivity Conditions

Modular invariants expressed in terms of characters with *weights in the fundamental domain* B_n must have positive coefficients.

The matrices $M_{z,z'}$ written down at the end of the previous section are indexed by a set of numbers in $\dfrac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$. Each z in this set is mapped by a Weyl transformation onto one and only one element of B_n. Since

$$\chi_{w(z)}(\tau) = (\det w)\, \chi_z(\tau) \tag{4.1}$$

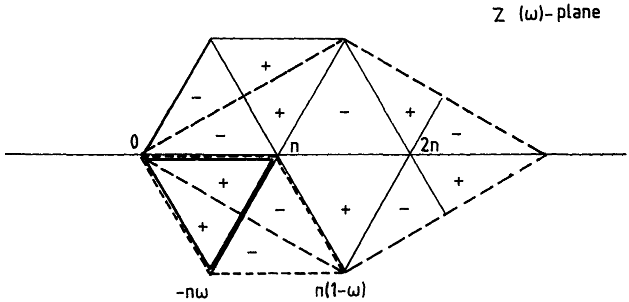

Fig. 1. Tesselation of the plane $\mathbb{Z}(\omega)$ with triangles of alternating parity. The fundamental domain B_n is bounded by solid lines, the congruence ring $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ by dotted lines and $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ by dashed lines

the positivity conditions have to be imposed on the matrix $\det w_1(z) w_2(z') M_{z,z'}$ with $w_1(z)$ and $w_2(z') \in B_n$. The problem with our parametrization is that, in general, the parities of these Weyl transformations are not easily determined.

Let us define the parity, $\mathscr{P}(z)$, of an element z in the ring $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ as the parity of the Weyl transformation which maps z into B_n:

$$\mathscr{P}(z) = \det w(z) \quad \text{with} \quad w(z) \in B_n. \tag{4.2}$$

As will be shown in Sect. 5, we do not really need to know the parity of each index z: symmetries of the ring $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ and of B_n, together with a necessary change of parity in a sequence of multiples of z, which we will prove shortly, turn out to be sufficient to implement all positivity constraints.

In Fig. 1 we display the three sets B_n, $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$ and $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ with the parity of the various regions indicated. This illustrates the following properties of $\mathscr{P}(z)$:

$$\mathscr{P}(w(z)) = (\det w)\mathscr{P}(z), \tag{4.3}$$

$$\mathscr{P}(Cz) = \mathscr{P}(-\omega\bar{z}) = \mathscr{P}(z), \tag{4.4}$$

$$\mathscr{P}(-z) = -\mathscr{P}(z). \tag{4.5}$$

The change of parity in a sequence of multiples mentioned earlier is made precise with the following:

Lemma. *Let $n > 3$ be a prime integer. Then for every z in $\frac{\mathbb{Z}(\omega)}{n(1-\omega)\mathbb{Z}(\omega)}$ such that* $\mathrm{Tr}(z^3(1+2\omega)) \neq 0 \pmod{n}$, *except for one of the eighteen numbers* $z = \pm\omega^k(1-\omega) + ln (k, l = 0, 1, 2)$, *at least one of the multiples* $2z, 3z, \ldots \frac{n-1}{2}z$, *also taken* $\bmod n(1-\omega)$, *has a parity opposite to that of* z.

We note first that the condition $\mathrm{Tr}(z^3(1+2\omega)) \neq 0 \pmod{n}$ simply ensures that z does not label a null character so that parity is well defined [see Eq. (2.25)]. Furthermore, it follows from Eqs. (2.28) and (2.29) that the parity of z does not

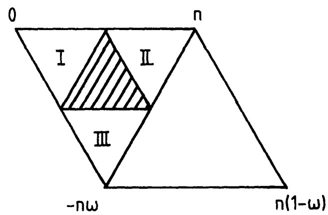

Fig. 2. The congruence ring $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$

depend on its residue mod $(1-\omega)$. This is illustrated in Fig. 1. The lemma must thus be proved for z's belonging to $\frac{\mathbb{Z}(\omega)}{n\mathbb{Z}(\omega)}$. As representatives of this congruence ring we take the n^2 numbers $y=\alpha-b\omega$ with $a,b=0,\ 1\dots(n-1)$. From $\mathrm{Tr}(y^3(1+2\omega))\neq 0 \pmod n$ follows that neither a, nor b, nor $a+b$ vanish modulo n and it is clear from Fig. 1 that y is even for $1\leqq a+b\leqq n-1$ and odd for $n+1\leqq a+b\leqq 2n-1$. Using Eq. (4.5) we are thus left with the proof of the lemma for y's with $1\leqq a+b\leqq n-1$.

From Fig. 2 it is obvious that when y lies in the shaded region, y is even but $2y$ is odd which proves the lemma for these cases.

The regions II and III in Fig. 2 are related to the region I by the automorphisms, Eq. (2.22), which, modulo n, reduce to even Weyl transformations and thus preserve parity.

Hence it remains to prove the lemma for algebraic integers lying in region I, namely for $y=a-b\omega$ with $1\leqq a,\ b,\ a+b\leqq\frac{n-1}{2}$. We will show that there exists at least one v between 2 and $\frac{n-1}{2}$ such that $\overline{(va)}+\overline{(vb)}>n$ (where the bar indicates the congruence class modulo n chosen between 1 and $n-1$). The proof differs slightly in the cases $n\equiv 1 \pmod 3$ and $n\equiv 2 \pmod 3$. We give the details of the proof for $n\equiv 1 \pmod 3$ indicating along the way the modifications needed in the other case.

Let us write $a=3\sigma_1+\tau_1$, $b=3\sigma_2+\tau_2$ with $\tau_1,\tau_2=0, 1$ or 2. Since $n\equiv 1 \pmod 6$, the condition $a+b\leqq\frac{n-1}{2}$ implies $\sigma_1+\sigma_2\leqq\frac{n-1}{6}$ for $\tau_1=0$ and $\sigma_1+\sigma_2\leqq\frac{n-7}{6}$ for $\tau_1\neq 0$. When τ_1 and τ_2 are not both equal to 1, one then easily checks that $\overline{(va)}+\overline{(vb)}>n$ for either $v=\frac{n-1}{3}$ or $v=\frac{n+2}{3}$. The same type of argument holds in the case $n\equiv 2 \pmod 3$ but with $v=\frac{n-2}{3}$ or $\frac{n+1}{3}$. We are thus left with $a=3\sigma_1+1$ and $b=3\sigma_2+1$. We remark that σ_1 and σ_2 cannot both be larger than $\frac{2n-8}{15}$ since this would imply $\sigma_1+\sigma_2>\frac{4n-16}{15}$ in contradiction with the hypothesis $\sigma_1+\sigma_2\leqq\frac{n-7}{6}$. We have thus two possibilities to consider:

1. either σ_1 or σ_2 is larger than $\frac{2n-8}{15}$ and, since the problem is symmetric in σ_1 and σ_2, we will take $\sigma_1>\frac{2n-8}{15}$;
2. both σ_1 and σ_2 are smaller than or equal to $\frac{2n-8}{15}$.

In Case 1, $\frac{2n-8}{15}<\sigma_1\leqq\frac{n-7}{6}$ implying $0\leqq\sigma_2<\frac{n-19}{30}$. Considering the multiple $v=\frac{n-4}{3}$ one immediately obtains

$$\frac{2n+10}{3}\leqq\overline{(va)}=n+\frac{n-4}{3}-4\sigma_1<4\frac{n+1}{5},$$

$$\frac{n+6}{5}<\overline{(vb)}=\frac{n-4}{3}-4\sigma_2\leqq\frac{n-4}{3},$$

hence $\overline{(va)}+\overline{(vb)}=n+2\frac{n-4}{3}-4(\sigma_1+\sigma_2)\geqq n+2\frac{n-4}{3}-4\frac{n-7}{6}=n+2>n.$

In case 2, we are going to show that the only value of σ_1 and σ_2 for which the parities of y, $2y$, $3y\ldots\frac{n-1}{2}y$ are the same is $\sigma_1=\sigma_2=0$, namely $y=1-\omega$.

Take first $v=\frac{n+5}{3}$. Then $\overline{(va)}=\frac{n+5}{3}+5\sigma_1$, and $\overline{(vb)}=\frac{n+5}{3}+5\sigma_2$ since, e.g., $\frac{n+5}{3}+5\sigma_1\leqq\frac{n+5}{3}+\frac{2n-8}{3}=n-1$.

Hence $\overline{(va)}+\overline{(vb)}=2\frac{n+5}{3}+5(\sigma_1+\sigma_2)$ which is smaller than or equal to $n-1$ (i.e., vy has the same parity as y) only if $\sigma_1+\sigma_2\leqq\frac{n-13}{15}$.

If we next take $v=\frac{n+8}{3}$, we find that y, $\frac{n+5}{3}y$ and $\frac{n+8}{3}y$ have the same parity only if $\sigma_1+\sigma_2\leqq\frac{n-19}{24}$. Iterating the process, it is found that y, $\frac{n+5}{3}y$, $\frac{n+8}{3}y,\ldots,$ $\frac{n+3r-1}{3}y$ have the same parity if $\sigma_1+\sigma_2\leqq\frac{n-6r-1}{3(3r-1)}$. Since v is bounded by $\frac{n-1}{2}$, the greatest value that r can take is $r_{\max}=\frac{n-1}{6}$, for which one obtains the bound

$$\sigma_1+\sigma_2\leqq\frac{n-6r_{\max}-1}{3(3r_{\max}-1)}=0\quad\text{i.e.}\quad\sigma_1=\sigma_2=0.$$

The proof of the lemma is thus complete for $n\equiv1$ (mod 3). When $n\equiv2$ (mod 3) and $\tau_1=\tau_2=1$ there is no need to distinguish the cases 1 and 2: the recurrence argument used in 2 shows that the numbers y, $\frac{n+4}{3}y$, $\frac{n+7}{3}y\ldots\frac{n-1}{2}y$ have the same parity iff $\sigma_1+\sigma_2\leqq0$.

We will see in the next section how this lemma in fact allows us to impose all positivity constraints.

5. Modular Invariant Partition Functions

Besides the positivity and integrality requirements whose precise content was given at the beginning of Sect. 4, the physical constraints on the commutant matrix

$M_{z,z'}$ require that the entry $M_{1-\omega,1-\omega}$ be equal to $+1$ [remembering that $z=1-\omega$ corresponds to the identity representation of $SU(3)$].

First of all we now solve the positivity and integrality constraints.

From the general form of $M_{z,z'}$, Eq. (3.40), it is clear that these constraints have to be imposed independently on $M_{x,x'}$ and $M'_{x,x'}$, Eq. (3.39). For definiteness, let us consider $M_{x,x'}$ (i.e., the "trialities" are $t=t'=\pm 1$).

We begin with the C_+ sector. In this case the coefficient c_π does not appear in any matrix element $M_{x,x'}$. When *q is odd*, using Eqs. (3.22) and (3.30), half the non-zero entries of the $\frac{n-1}{2}$ columns indexed by $x'=v'u^{2q}$ (i.e., $i'=q$) for $v'=1$ to $\frac{n-1}{2}$ simply read

$$M^{(+)}_{i,q}=\tfrac{1}{2}(c_{2i}-c_{q-2i})\delta_{v,v'} \qquad i=1,\dots,\frac{q-1}{2}. \tag{5.1}$$

The index $x'=v'u^{2q}$ corresponds to $z'=v'u^{2q}(1-\omega)\pm n$ which is even in the sense of Sect. 4 [Eq. (4.2)]. Following the discussion of that section, the matrix element $M^{(+)}_{i,q}$ must be positive (respectively negative) for line indices $x=v'u^{2i}$ related to a z-label which is even (respectively odd). The Lemma of Sect. 4 now guarantees that this parity will change at least once when v' varies from 1 to $\frac{n-1}{2}$. It follows that the entries in Eq. (5.1) must vanish, i.e.:

$$c_{2i}=c_{q-2i}, \qquad i=1,\dots,\frac{q-1}{2}. \tag{5.2}$$

On the other hand, $M^{(+)}_{q,q}=c_0$ must be a positive integer. Indeed, $u^{2q}=\omega$ or ω^2 [see Eq. (2.10)] and the "triality" of z' does not affect its parity, hence $\mathscr{P}(z')=\mathscr{P}(\omega z')=\mathscr{P}(\omega^2 z')=\mathscr{P}(v'(1-\omega))=+1$ for all v'. Thus

$$c_0\in\mathbb{Z}^+. \tag{5.3}$$

We next focus our attention on the lines $x=vu^2$ ($i=1$):

$$M^{(+)}_{1;i'}=\tfrac{1}{2}(c_{2(i'-1)}-c_{q-2(i'+1)})\delta_{v',v}, \qquad i'=2,\dots,\frac{q-3}{2}, \tag{5.4}$$

$$M^{(+)}_{1;(q-1)/2}=\tfrac{1}{2}(c_{q-3}-c_1)\delta_{v',v}, \tag{5.5}$$

$$M^{(+)}_{1;(q+1)/2}=\tfrac{1}{2}(c_{q-1}-c_3)\delta_{v',v}, \tag{5.6}$$

$$\begin{aligned} M^{(+)}_{1;q-i'}&=\tfrac{1}{2}(c_{2(i'+1)}-c_{q-2(i'-1)})\delta_{v',v}, \qquad i'=2,\dots,\frac{q-3}{2}\\ &=\tfrac{1}{2}(c_{q-2(i'+1)}-c_{2(i'-1)})\delta_{v',v}, \end{aligned} \tag{5.7}$$

where Eq. (5.2) has been used in the last line. From charge conjugation symmetry, Eq. (4.4), we have $\mathscr{P}(vu^{2i'}(1-\omega))=\mathscr{P}(vu^{2(q-i')}(1-\omega))$ implying that $M^{(+)}_{i,i'}$ and $M^{(+)}_{i,q-i'}$ must be simultaneously positive or negative. Hence, Eqs. (5.4)–(5.7) lead to

$$\begin{aligned} &c_{2k-2}=c_{2k+2}, \qquad k=2,\dots,\frac{q-3}{2},\\ &c_{q-3}=c_1,\\ &c_{q-1}=c_3, \end{aligned}$$

which, together with Eq. (5.2), finally give

$$c_1 = c_2 = \ldots = c_{q-1} = c\,. \tag{5.8}$$

The remaining non-zero entries in the C_+ sector are

$$M^{(+)}_{i,i} = \left(c_0 - \frac{c}{2}\right)\delta_{v',v}\,, \tag{5.9}$$

$$M^{(+)}_{i,q-i} = \frac{c}{2}\,\delta_{v',v}\,, \tag{5.10}$$

and positivity requires

$$c_0 - \frac{c}{2} \in \mathbb{Z}^+\,; \quad \frac{c}{2} \in \mathbb{Z}^+\,. \tag{5.11}$$

When q *is even*, one repeats essentially the same steps starting from Eqs. (3.26) and (3.33). Since each v-block of $M^{(+)}$ is now $(q-1)\times(q-1)$, the resulting constraints are slightly weaker, yielding

$$\begin{gathered} c_2 = c_4 = \ldots = c_{q-2} = c \in 2\mathbb{Z}^+\,, \\ c_1 = c_3 = \ldots = c_{q-1}\,, \\ c_0 \in \mathbb{Z}^+\,, \\ c_0 - \frac{c}{2} \in \mathbb{Z}^+\,. \end{gathered}$$

The C_-^{even} sector leads to the additional constraint $c_1 = c$ as can be seen from the line $x = v\varrho u^2$ $(i=1)$:

$$M^{(-)}_{1,1} = \left(c_0 - \frac{c_1}{2}\right)\delta_{v',v}\,; \qquad M^{(-)}_{1,i'} = \frac{1}{2}(c - c_1)\delta_{v',v}\,, \qquad i' = 2, \ldots, \frac{q}{2}\,,$$

$$M^{(-)}_{1,1+q/2} = \left(-\frac{c_1}{2}\right)\delta_{v',v}\,; \qquad M^{(-)}_{1,i'+q/2} = \frac{1}{2}(c - c_1)\delta_{v',v}\,.$$

Remembering that $\bar{\varrho} = u\varrho$ and using Eqs. (4.3) and (4.5), $x' = v'\varrho u^{2i'}$ and $x'' = v'\varrho u^{2(i'+q/2)}$ are of opposite parity and the result thus follows.

When $n \equiv 1\,(3)$, the coefficient c_π appears in the null sector C_0. In that case, the linear relation among the matrices M_{u^m}, Eq. (3.38), and the constraints from the $C_\pm$ sectors, Eq. (5.8), allow us to write

$$M = \left(c_0 - \frac{c}{2}\right)\frac{M_{u^0}}{6} - \frac{c}{2}\,\frac{M_{u^q}}{6} + \left(c_\pi + \frac{c}{2}\right)\frac{M_\pi}{9}\,. \tag{5.12}$$

Looking at the first line $(x = \pi u,\ x' = \pi u^{j'})$ gives

$$M^{(0)}_{1,1} = c_0 - \frac{c}{2}\,; \qquad M^{(0)}_{1,j'} = c_\pi + \frac{c}{2}\,, \qquad j' = 2, \ldots, q\,,$$

$$M^{(0)}_{1,q+1} = -\frac{c}{2}\,; \qquad M^{(0)}_{1,j'+q} = c_\pi + \frac{c}{2} \tag{5.13}$$

but, from Eqs. (4.3)–(4.5), $M^{(0)}_{1,j'}$ and $M^{(0)}_{1,j'+q}$ have to be of opposite sign, hence

$$c_\pi = -\frac{c}{2}\,. \tag{5.14}$$

The results obtained so far leave us with the matrices

$$M_{x,x'}=\left(c_0-\frac{c}{2}\right)\frac{M_{u^0}}{6}-\left(\frac{c}{2}\right)\frac{M_{u^q}}{6}, \tag{5.15}$$

where $c_0, \frac{c}{2}$ and $c_0-\frac{c}{2}\in\mathbb{Z}^+$. With Eqs. (4.4) and (4.5), $M_{u^0}=M_1$, $M_{u^q}=M_{-\omega}$ one has

$$\tfrac{1}{6}(M_1)_{x,x'}=\delta_{x',x}, \tag{5.16}$$

$$\tfrac{1}{6}(M_{-\omega})_{x,x'}=\begin{cases}-\delta_{x',\bar{x}} & \text{in } C^{\pm}\\ \delta_{x',\bar{x}} & \text{in } C^0.\end{cases} \tag{5.17}$$

Mutatis mutandis the same form is obtained for the matrix M' with coefficients c'_0 and c'. With these explicit forms for M and M', it is straightforward to check that $M_{z,z'}$, Eq. (3.40), indeed satisfies *all* positivity and integrality constraints.

The last condition to be imposed is the correct normalization of $M_{z,z'}$: the coefficient of the characters of the singlet representation must be one. This gives at once

$$c_0+c'_0=1\,.$$

Since c_0 and $c'_0\in\mathbb{Z}^+$, there are but two solutions to this equation. Equation (5.11) then restricts $\frac{c}{2}$ to be 0 or 1.

The final result of this lengthy discussion is thus that there are four physically acceptable solutions to all constraints. They are

$$M_{z,z'}=\begin{pmatrix}M&0&0\\0&M&0\\0&0&M\end{pmatrix}\quad\text{or}\quad M_{z,z'}=\begin{pmatrix}0&0&M\\0&M&0\\M&0&0\end{pmatrix}, \tag{5.18}$$

with

$$M_{x,x'}=\tfrac{1}{6}(M_1)_{x,x'}\quad\text{or}\quad -\tfrac{1}{6}(M_{-\omega})_{x,x'}\,. \tag{5.19}$$

These four solutions are pairwise related by the charge conjugation symmetry of the (specialized) Kac-Moody characters, Eq. (2.21), which is equivalent to

$$\chi_{x,t}(\tau)=\chi_{\bar{x},-t}(\tau)\,.$$

Using this symmetry we reach, at last, the end of the proof that for all prime heights n of affine $\widehat{SU}(3)$, there exist two and only two modular partition functions, namely

$$Z(\tau)=\sum_{x,t}\chi_{x,t}(\tau)\bar{\chi}_{x,t}(\tau), \tag{5.20}$$

or

$$Z(\tau)=\sum_{x,t}\chi_{x,t}(\tau)\bar{\chi}_{x,-t}(\tau), \tag{5.21}$$

where the summation is over all x and t such that the corresponding $z=x(1-\omega)+tn$ cover once a domain isomorphic to B_n (e.g., $t=0, \pm 1$; $x\in C_+$, C_- and C_0).

As concluding remarks, we would like to point out that the arithmetic approach developed in this paper is immediately applicable to $\hat{G}_2$. Moreover, the rings $\mathbb{Z}(i)$ of Gaussian integers and $\mathbb{Z}(\zeta_p)$, p prime, of cyclotomic integers play an analogous role for $\hat{B}_2$ and $\widehat{SU}(p)$ as $\mathbb{Z}(\omega)$ for $\widehat{SU}(3)$ or $\hat{G}_2$.

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