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Index of Subfactors and Statistics of Quantum Fields

II. Correspondences, Braid Group Statistics and Jones Polynomial

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Abstract. The endomorphism semigroup End(M) of an infinite factor M is endowed with a natural conjugation (modulo inner automorphisms) $\bar{\rho} = \rho^{-1} \cdot \gamma$, where γ is the canonical endomorphism of M into $\rho(M)$. In Quantum Field Theory conjugate endomorphisms are shown to correspond to conjugate superselection sectors in the description of Doplicher, Haag and Roberts. On the other hand one easily sees that conjugate endomorphisms correspond to conjugate correspondences in the setting of A. Connes. In particular we identify the canonical tower associated with the inclusion $\rho(\mathscr{A}(\mathcal{O})) \subset \mathscr{A}(\mathcal{O})$ relative to a sector ρ. As a corollary, making use of our previously established index-statistics correspondence, we completely describe, in low dimensional theories, the statistics of a selfconjugate superselection sector ρ with 3 or less channels, in particular of sectors with statistical dimension $d(\rho) < 2$, by obtaining the braid group representations of V. Jones and Birman, Wenzl and Murakami. The statistics is thus described in these cases by the polynomial invariants for knots and links of Jones and Kauffman. Selfconjugate sectors are subdivided into real and pseudoreal ones and the effect of this distinction on the statistics is analyzed. The FYHLMO polynomial describes arbitrary 2-channels sectors.

1. Introduction

In a previous paper [19, 18] we established a link between the index theory of subfactors [12] and the statistics of a local quantum field [5]: if a superselection sector is represented by a localized endomorphism ρ of the quasi-local C^*-algebra $\mathscr{A} = \cup \mathscr{A}(\mathcal{O})^-$, then

$$\operatorname{Ind}(\rho)^{1/2} = d(\rho).$$

Here $\operatorname{Ind}(\rho)$ is the index of ρ that may be locally defined as the minimal index of $\rho(\mathscr{A}(\mathcal{O}))$ in $\mathscr{A}(\mathcal{O})$ as soon as ρ is localized in $\mathcal{O}$ and $d(\rho)$ is the statistical dimension

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of ρ, a physical invariant obtained by locality and henceforth reflecting the geometry of the Minkowski space-time and the localization region of ρ.

Our index formula relates an analytical quantity to a physical quantity and one may use this result to carry information from one structure to the other; in particular one immediately has new restrictions on the values of the statistical dimension in low dimensional theories[1].

This result was proposed as a first step in the structure analysis of the statistics of the sector ρ, further information being contained in the Jones tower associated with the inclusion $\rho(\mathscr{A}(\mathcal{O})) \subset \mathscr{A}(\mathcal{O})$, as in [20], and in the quantum field braid group representation given by the statistics, as indicated by the case of fields with a compact gauge group of the first kind where the braid group symmetry reduces to the usual permutation symmetry and the tower structure is easily seen to correspond to the tensor product tower $\pi \otimes \bar{\pi} \otimes \pi \ldots$, where π is the representation of the gauge group giving the sector and $\bar{\pi}$ the contragradient representation, cf. [5, 33].

An analysis of the corresponding structure in low dimensional theories requires however a clarification of the general picture. Also the similarity between the tower structure in [5] and in [12, 16] indicates that some deeper connection should exist.

Independently of our work two recent papers [6, 25] have provided a related analysis.

In the first and more closely related paper, Fredenhagen, Rehren and Schroer [6] have considered the braid group symmetry arising in low dimensional quantum field theory. They noticed in particular that in the special but significant case (as shown in particular by conformal models) of a "two-channel" sector ρ, namely ρ^2 has two irreducible components, since the statistics operator ε_ρ has at most two eigenvalues, the statistics is described by known analysis, see [24].

On the other hand Witten [25], see also [32], has proposed a 3-dimensional Lagrangian model that suggests a quantum field theory interpretation of the Jones link invariant polynomial [13] via a formal path integration.

In this paper we shall develop an analysis of the endomorphism semigroup of a factor that is directly motivated and applicable to the quantuam field theory context.

In the case of a selfconjugate two channel sector we shall use our index formula to identify the braid group $\mathbf{B}_n$ representation $\varepsilon_\rho^{(n)}$ giving the statistics of the sector ρ with the Jones braid group representation constructed from subfactors [14]. Since the latter is used to define the Jones polynomial [13], as a corollary we have a rigorous connection between the Jones polynomial V_L and the statistics of ρ:

$$V_L(q) = (-d(\rho))^{n-1}(-\omega_\rho)^{-l}\phi^{n-1}(\varepsilon_\rho^{(n)}(\alpha)).$$

Here ω_ρ is the phase of the statistics parameter λ_ρ, $d(\rho) = |\lambda_\rho|^{-1}$ is the statistical dimension of ρ, ϕ is the left inverse of ρ, L is the link represented by the element $\alpha \in \mathbf{B}_n$ with exponent sum l and $q \in \mathbf{T}$ satisfies $q + q^{-1} + 2 = d(\rho)^2$.

More generally, if ρ is selfconjugate with 3 channels we identify $\varepsilon_\rho^{(n)}$ with a representation of Birman–Wenzl and Murakami [26, 28]; as a consequence

$$K_L(t, s) = (-d(\rho))^{n-1}(-\omega_\rho)^{-1}\phi^{n-1}(\varepsilon_\rho^{(n)}(\alpha)),$$

[1] A certain analogy with the Atiyah–Singer index theorem may provide further insight here

where K is the Kauffman link invariant polynomial [29] at values t, s that depend only on λ_ρ, on the real-pseudoreal alternative, see below, and on a signature.

In particular we have a complete description of the statistics of sectors of the "minimal series," namely with statistical dimension $d(\rho) < 2$.

The right-hand side of the above formulas always define a link invariant associated with an irreducible sector [6]. Extensions of this formula needs more general link invariants that we only mention here. The general 2-channel sectors give the two variable polynomial [7].

Rather surprisingly the structure of the superselection sectors in quantum field theory is already present in a large portion in the abstract analysis of endomorphisms of factors; the essential point making the field theory structure richer being the appearance of the braid-permutation symmetry. A deeper role of locality, that makes the superselection semigroup abelian, ought to manifest itself.

We explain now our basic results. This paper is centered around a key formula for the conjugate sector by the canonical endomorphism [16, 17].

Let M be an infinite factor (always with separable predual) and $\rho \in \mathrm{End}(M)$ an irreducible endomorphism of M. We shall show that ρ admits an irreducible conjugate $\bar{\rho} \in \mathrm{End}(M)$, in the sense that both $\rho\bar{\rho}$ and $\bar{\rho}\rho$ contain the identity, namely there exists non-zero $v, w \in M$ such that

$$\begin{aligned} \rho\bar{\rho}(x)v &= vx, \\ \bar{\rho}\rho(x)w &= wx, \quad x \in M, \end{aligned} \tag{1.1}$$

if and only if ρ has finite index; $\bar{\rho}$ is unique modulo inner automorphisms and

$$\rho\bar{\rho} = \gamma,$$

where γ is the canonical endomorphism associated with the inclusion $\rho(M) \subset M$ (the latter is also defined modulo inner automorphisms of $\rho(M)$).

Conjugate endomorphisms in the sense of formulas (1.1) appear in the analysis of Doplicher, Haag and Roberts in quantum field theory [5] where they describe the particle–antiparticle symmetry and correspond to the contragradient map in the dual of a compact group. The formula

$$\bar{\rho} = \rho^{-1} \cdot \gamma \tag{1.2}$$

provides an intrinsic general contragradient map in $\mathrm{End}(M)$ (modulo inner automorphisms) that seems to cover all possible generalizations (locally compact groups, infinite statistics, quantum groups, braid group statistics). A better understanding of this point is obtained by the Connes concept of correspondence where the contragradient map may be defined by a natural flip on the coefficients, an operation equivalent to our one (1.2) in the endomorphism approach; most of this paper can be thus read from the correspondence point of view.

It follows that the tower (or tunnel) associated with the inclusion $\rho(M) \subset M$ is

$$M \supset \rho(M) \supset \rho\bar{\rho}(M) \supset \rho\bar{\rho}\rho(M) \cdots .$$

In particular for a self-conjugate sector the tower is

$$M \supset \rho(M) \supset \rho^2(M) \supset \rho^3(M) \cdots ,$$

providing a further link between inclusions of von Neumann algebras and quantum field theory.

The calculus of intertwiners associated with an action of a compact (gauge) group [5] is then extended to this general setting that specializes to the low dimensional field theoretical situation.

In particular one has the notion of real and pseudoreal sector, hence one has an extension of (the algebraic part of) Carruther's theorem. More generally we define invariants for endomorphisms that extend the Connes invariants for periodic automorphisms. The meaning of these invariants is explained in the normal statistics case.

As already mentioned, if we specialize to selfconjugate two channel sectors, the braid group representation of V. Jones coincides with the field theoretical braid group representation. Unitarity of the latter then implies that the statistical dimension is less or equal to 2 in this case; in other words a selfconjugate sector with statistical dimension greater than 2 must generate at least 3 sectors (with multiplicity) when composed with itself.

It follows that if ρ as above has statistical dimension $d(\rho)=\sqrt{n}$, $n\in\mathbf{N}$, the statistics is essentially described by finite groups, namely the braid group representations $\varepsilon_\rho^{(n)}$ giving the statistics, modulo tensoring with a one-dimensional representation, factors through finite groups [14]. These groups are in fact the monodromy groups of the Wightman functions that have been studied in conformal models, cf. [8].

2. Preliminaries on Connes Correspondences

Let M, N be von Neumann algebras, that we always assume to have separable preduals, and $\mathscr{H}$ a M–N correspondence [4], namely $\mathscr{H}$ is a (separable) Hilbert space, where M acts on the left, N acts on the right and the actions are normal. We denote by

$$x\xi y, \quad x\in M, \quad y\in N, \quad \xi\in\mathscr{H}$$

the relative actions.

Alternatively a correspondence may be defined as a binormal representation of $M\otimes_{\max}N^0$, where N^0 is the von Neumann algebra opposite to N and the concepts of representation theory makes sense: we shall use the relative terminology without mentioning it further.

The trivial M–M correspondence is the Hilbert space $L^2(M)$ with the standard actions given by the modular theory

$$x\xi y = xJy^*J\xi \quad x,y\in M, \quad \xi\in L^2(M),$$

where J is the modular conjugation of M; the trivial correspondence is well defined modulo unitary equivalence.

If ρ is a normal homomorphism of M into N we let $\mathscr{H}_\rho$ be the Hilbert space $L^2(N)$ with actions

$$x\cdot\xi\cdot y \equiv \rho(x)\xi y, \quad x\in M, \quad y\in N, \quad \xi\in L^2(N).$$

The following proposition, and likely all this section, is due to A. Connes.

2.1 Proposition. *If M and N are properly infinite, then any M–N correspondence $\mathscr{H}$ is equivalent to an $\mathscr{H}_\rho$ for some ρ as above. If the right action of N is faithful, then ρ is unital.*

Proof. Assume for simplcity that N acts faithfully on the right. Since M, N are properly infinite we may identify $\mathscr{H}$ with $L^2(N)$, N has its right action on $L^2(N)$. Since M acts on the left on $L^2(N)$, M acts as a subalgebra of N, namely M is embedded in N. Call this embedding ρ; then $\mathscr{H} = \mathscr{H}_\rho$. ■

Denote by $\mathrm{Corr}(M)$ the set of all $M - M$ correspondences.

2.2 Corollary. *Let M be an infinite factor. There exists a bijection between* $\mathrm{End}(M)$ *and* $\mathrm{Corr}(M)$. *Given $\rho, \rho' \in \mathrm{End}(M)$, $\mathscr{H}_\rho$ is equivalent to $\mathscr{H}_{\rho'}$ iff there exists a unitary $u \in M$ with $\rho'(x) = u\rho(x)u^*$.*

Proof. If u is a unitary that implements the equivalence of $\mathscr{H}_\rho$ with $\mathscr{H}_{\rho'}$ then u commutes with the right action of M, thus $u \in M$ (acting on the left). The rest is clear. ■

Let $\mathrm{Sect}(M)$ denote the quotient of $\mathrm{End}(M)$ modulo unitary equivalence in M as in Corollary 2.2. We call *sectors* the elements of the semigroup $\mathrm{Sect}(M)$; if $\rho \in \mathrm{End}(M)$ we denote by $[\rho]$ its class in $\mathrm{Sect}(M)$. By Corollary 2.2 $\mathrm{Sect}(M)$ may be naturally identified with $\mathrm{Corr}(M)/\sim$ the quotient of $\mathrm{Corr}(M)$ modulo unitary equivalence.

Given an M–N correspondence $\mathscr{H}$, the *conjugate correspondence* $\bar{\mathscr{H}}$ is the N–M correspondence obtained by considering the complex conjugate Hilbert space $\bar{\mathscr{H}}$ with actions

$$y \cdot \bar{\xi} \cdot x = \overline{x^* \xi y^*}, \quad x \in M, \quad y \in N, \quad \xi \in M,$$

where $\bar{\xi} \in \bar{\mathscr{H}}$ is the conjugate vector of $\xi \in \mathscr{H}$.

Recall that the coefficients of the M–N correspondence $\mathscr{H}$ are the functions

$$B = B_{\xi\eta}(x, y) = \langle x\xi y, \eta \rangle, \quad x \in M, \quad y \in N,$$

where $\xi, \eta \in \mathscr{H}$. The coefficient of $\bar{\mathscr{H}}$ are then given by

$$\bar{B}_{\bar{\xi}\bar{\eta}}(y, x) = \langle \overline{x^* \xi y^*}, \bar{\eta} \rangle = \langle \eta, x^* \xi y^* \rangle = \langle x\eta y, \xi \rangle = B_{\eta\xi}(x, y).$$

Thus we have:

2.3 Proposition. *$\bar{\mathscr{H}}$ is the unique N–M correspondence, modulo unitary equivalence, such that its coefficients $\bar{B}(y, x)$, $y \in N$, $x \in M$ are those ones given by $\bar{B}(y, x) = B(x, y)$, where B is a coefficient on $\mathscr{H}$.*

Proof. The coefficients of $\bar{\mathscr{H}}$ are flipped by those of $\mathscr{H}$ by the above elementary calculation. Now the coefficients of $\mathscr{H}$ correspond to the vector functionals of the associated representation of $M \otimes_{\max} N$, thus they characterize the representation modulo equivalence. ■

It follows that $\mathrm{Sect}(M)$, with M a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \to \bar{\theta}$ that commutes with all natural operations of direct sum, tensor product and other (the tensor product of correspondences corresponds to the composition of sectors). We shall discuss later how the conjugation may lift to a natural conjugation in End(M). If $\rho \in \mathrm{End}(M)$ and $\bar{\rho} \in \mathrm{End}(M)$ gives the conjugate sector, i.e. $\overline{[\rho]} = [\bar{\rho}]$, then we say that $\bar{\rho}$ is conjugate to ρ.

2.4 Corollary. *Let $\rho \in \mathrm{End}(M)$, where M is a properly infinite von Neumann algebra, and $\bar{\rho} \in \mathrm{End}(M)$ be conjugate to ρ. Then $\mathscr{H}_{\bar{\rho}}$ is given by $L^2(M)$ with actions*

$$x \cdot \xi \cdot y = x\xi\rho(y), \quad \xi \in \mathscr{H}.$$

Proof. We compute the coefficients B' of this correspondence according to Corollary 2.3. We have

$$\begin{aligned} B'(x,y) &\equiv \langle x \cdot \xi \cdot y, \eta \rangle \\ &= \langle x\xi\rho(y), \eta \rangle = \langle xJ\rho(y)^* J\xi, \eta \rangle \\ &= \langle \eta', JxJ\rho(y^*)\xi' \rangle = \langle \eta', \rho(y^*)\xi' x^* \rangle \\ &= \langle \rho(y)\eta' x, \xi' \rangle = B(y,x), \end{aligned}$$

where B is the coefficient of $\mathscr{H}_\rho$ associated with the vectors $\xi' \equiv J\xi$, $\eta' \equiv J\eta$. ∎

If we specialize to cyclic correspondences, we see by the above corollary that if $\Omega \in L^2(M)_+$ is a cyclic vector for $\rho'(M) \vee JMJ$ and $M \vee J\rho(M)J$, with $\rho, \rho' \in \mathrm{End}(M)$ (in particular if ρ and ρ' are irreducible), then ρ' is a conjugate of ρ iff the function $B(x,y) \equiv \langle \rho'(y)\Omega x, \Omega \rangle$ is a coefficient of $\mathscr{H}_\rho$.

Note that if M is a von Neumann algebra and $\omega \in M_*$ is a faithful state with modular group σ^ω, the bilinear form $B_\omega(\cdot,\cdot)$ in M and M^0 giving $L^2(M)$ is

$$B_\omega(x, y^0) \equiv \underset{t \to i/2}{\text{anal. cont.}}\, \omega(x\sigma_t^\omega(y)), \quad x, y \in M,$$

while $\mathscr{H}_\rho$, $\rho \in \mathrm{End}(M)$, is given by the bilinear form

$$B_\omega^\rho(x, y^0) = B_\omega(\rho(x), y^0). \tag{2.1}$$

It follows that in the cyclic case $\bar{\rho} \in \mathrm{End}(M)$ is conjugate to ρ iff

$$x, y \to B_\omega(x, \bar{\rho}(y)^0)$$

is a coefficient of $\mathscr{H}_\rho$.

3. The Basic Formula for the Conjugate Sector

Let $N \subset M$ be an inclusion of properly infinite von Neumann algebras and ω a bicyclic state for N, M namely ω is a faithful normal state of M_* represented by a vector $\Omega \in L^2(M)$ cyclic for both N and M. The canonical endomorphism $\gamma_\omega : M \to N$ is the endomorphism of M into N,

$$\gamma_\omega(x) = \Gamma x \Gamma^*, \quad x \in M,$$

where $\Gamma = J_N J_M$ is the product of the modular conjugations of N and M with repsect to Ω. Since $\gamma_\omega \in \mathrm{End}(M)$ it gives a class $[\gamma_\omega] \in \mathrm{Sect}(M)$ that does not depend on ω; in fact γ_ω depends on ω only up to a perturbation $\gamma_\omega \to u\gamma_\omega(\cdot)u^*$ with u a unitary of N [17]; the same applies if we choose J_N and J_M not necessarily with respect to the same vector and we work with canonical endomorphisms in this generality.

If $\theta \in \mathrm{Sect}(M)$ choose $\rho \in \mathrm{End}(M)$ with $[\rho] = \theta$ and ω a bicyclic state for $\rho(M)$ and M; then $\gamma_\omega : M \to \rho(M)$ gives a class $\gamma_\theta = [\gamma_\omega] \in \mathrm{Sect}(M)$ that does not depend on ω and ρ, in fact it depends only on the inner conjugacy class of $\rho(M)$ in M.

3.1 Theorem. *Let M be a properly infinite von Neumann algebra and $\theta \in \mathrm{Sect}(M)$. Then $\bar\theta$ is the unique sector $\bar\theta \in \mathrm{Sect}(M)$ such that*

$$\theta \cdot \bar\theta = \gamma_\theta .$$

More explicitly if $\rho \in \mathrm{End}(M)$ then all conjugates $\bar\rho \in \mathrm{End}(M)$ of ρ are given by

$$\bar\rho = \rho^{-1} \cdot \gamma ,$$

where γ is a canonical endomorphism of M into $\rho(M)$.

Proof. By Corollary 2.4 $\mathscr{H}_{\bar\rho}$ is given by $L^2(M)$ with left and right multiplications given by

$$x \cdot \xi \cdot y = x \xi \rho(y), \quad \xi \in L^2(M).$$

Let J be the modular conjugation of M with respect to a vector $\Omega \in L^2(M)_+$ and choose a unitary U implementing ρ

$$U x U^* = \rho(x), \quad x \in M.$$

Then $J_\rho \equiv UJU^*$ is a modular conjugation for $\rho(M)$. Set $\Gamma = J_\rho J$ and $\gamma(x) = \Gamma x \Gamma^*$, $x \in M$ so that $[\gamma] = \gamma_{[\rho]}$. We have

$$\begin{aligned}
(\rho^{-1}(\gamma(x))\Omega y, \Omega) &= (U^* J_\rho J x J J_\rho U J y^* J \Omega, \Omega) \\
&= (U^* J_\rho J x J U y^* U^* U J \Omega, \Omega) \\
&= (U^* J_\rho J x J \rho(y^*) J J U J \Omega, \Omega) = (J U^* J x J \rho(y^*) J J U J \Omega, \Omega) \\
&= (x \xi \rho(y), \xi)
\end{aligned}$$

with $\xi = JUJ\Omega$, hence $\rho^{-1} \cdot \gamma$ is a conjugate of ρ by Proposition 2.3.

Reversing the calculation we see that all conjugates arise in this way. ■

3.2 Corollary. *Let $\mathscr{S} \subset \mathrm{Sect}(M)$ be a selfconjugate semigroup ($\theta \in S \Rightarrow \bar\theta \in \mathscr{S}$) and $\tilde{\mathscr{S}} = \{\rho \in \mathrm{End}(M), [\rho] \in \mathscr{S}\}$. Suppose that one the following holds:*

a) *There exists a cyclic separating vector Ω for all $\rho(M)$, $\rho \in \tilde{\mathscr{S}}$, and M.*

b) *$\mathscr{S}$ is countable.*

Then there exists a canonical lifting (associated to Ω) for the conjugation operation

$$\begin{array}{ccc}
\rho \in \tilde{\mathscr{S}} & \longrightarrow & \bar\rho \in \tilde{\mathscr{S}} \\
\Big\downarrow & & \Big\downarrow \\
[\rho] \in \mathscr{S} & \longrightarrow & \overline{[\rho]} \in \mathscr{S}
\end{array}$$

Proof.

(a) Define

$$\bar\rho \equiv \rho^{-1} \cdot \gamma_\rho \quad \rho \in \tilde{\mathscr{S}},$$

where γ_ρ is the canonical endomorphism of M onto $\rho(M)$ with respect to Ω.
(b) We are in case (a) by the Dixmier–Marechal theorem. ■

Corollary 3.2, together with the next Theorem 5.2, applies in quantum field theory by the Reeh–Schlieder property of the vacuum Ω; after localizing sectors in a given region one obtains a canonical conjugation in the field bundle [5].

Before closing this section we mention that if $\rho: M \to M$ is a completely positive normal map then the formula (2.1) defines a M–M correspondence $\mathscr{H}_\rho$. It is easily seen that, since M is properly infinite, $\mathscr{H}_\rho$ is equivalent to $\mathscr{H}_{\tilde{\rho}}$, where $\tilde{\rho} \in \mathrm{End}(M)$ is the Stinespring dilation of ρ.

If $\rho \in \mathrm{End}(M)$, a left inverse ϕ of ρ is a completely positive map of M in M such that $\phi \cdot \rho = \mathrm{id}$.

3.3 Proposition. *If $N \subset M$ is an inclusion of infinite factors and $\varepsilon \in C(M, N)$ is a normal conditional expectation of M onto N, then $\mathscr{H}_\varepsilon$ is equivalent $\mathscr{H}_\gamma$, where γ is a canonical endomorphism of M in N.*

If $N = \rho(M)$ with $\rho \in \mathrm{End}(M)$ and $\phi = \rho^{-1} \cdot \varepsilon$ is a normal left inverse of ϕ then $\mathscr{H}_\phi$ is equivalent to $\mathscr{H}_{\bar{\rho}}$. In particular $\mathscr{H}_\varepsilon$ (respectively $\mathscr{H}_\phi$) does not depend on ε (respectively ϕ), but only on N and M (respectively and ρ).

Proof. The first statement is a consequence of Proposition 4.3 that will exhibit γ as the Stinespring dilation of ε. The second statement follows from the first one and Theorem 3.1. ■

Note that if M has a semifinite normal trace τ and $\rho \in \mathrm{End}(M)$ is irreducible, then, cf. [18],

$$\mathrm{mod}_\tau(\rho)\,\mathrm{mod}_\tau(\bar{\rho}) = \mathrm{mod}_\tau(\rho\bar{\rho}) = \mathrm{mod}_\tau(\gamma_\rho) = \mathrm{Ind}(\rho),$$

therefore $\mathrm{mod}_\tau(\bar{\rho}) = \mathrm{Ind}(\rho)\,\mathrm{mod}_\tau(\rho)^{-1}$. If ρ and $\bar{\rho}$ leaves τ invariant, then ρ is an automorphism; in particular in a II_1 factor $\bar{\rho}$ exists only as a completely positive map. Endomorphisms of II_1 factors are considered in [22].

4. Endomorphisms with Finite Index

Let $N \subset M$ be an inclusion of factors and denote by $C(M, N)$ the space of the normal conditional expectations of M onto N. Given a faithful $\varepsilon \in C(M, N)$ the index of N in M with respect to ε [11, 18] is denoted by $\mathrm{Ind}_\varepsilon(N, M)$ and $\mathrm{Ind}(N, M)$ denotes the minimum index. With $\rho \in \mathrm{End}(M)$ the index of ρ is defined as

$$\mathrm{Ind}(\rho) \equiv \mathrm{Ind}(\rho(M), M).$$

One immediately sees that $\mathrm{Ind}(\rho)$ depends only on the class $[\rho] \in \mathrm{Sect}(M)$. Denote by $\mathrm{End}_0(M)$ the semigroup of all $\rho \in \mathrm{End}(M)$ with finite index. If $\rho \in \mathrm{End}(M)$ we denote by

$$H(\rho) = \{v \in M / \rho(x)v = vx, \ \forall x \in M\}$$

the linear space of the intertwiners between ρ and the identity.

We now characterize the conjugate sector in analogy with the characterization of the contragradient representation in the dual of a compact group.

4.1 Theorem. *Let M be an infinite factor and $\theta \in \mathrm{Sect}(M)$ an irreducible sector. Then* $\mathrm{Ind}(\theta) < \infty$ *if and only if both $\bar{\theta}\cdot\theta$ and $\theta\cdot\bar{\theta}$ contain the identity sector.*

Moreover if $\theta' \in \mathrm{Sect}(M)$ is irreducible and $\theta'\cdot\theta$ and $\theta\cdot\theta'$ contain the identity, then $\theta' = \bar{\theta}$.

We have $\mathrm{Ind}(\bar{\theta}) = \mathrm{Ind}(\theta)$ *and $\bar{\theta}\cdot\theta$, $\theta\cdot\bar{\theta}$ contain the identity with multiplicity one.*

The analogous result for correspondences associated with finite factors is obtained by Theorem 4.1 by tensoring with a type I_∞ factor. As an immediate corollary of physical interest we have

4.2 Corollary. *Let $\theta \in \mathrm{Sect}(M)$ be irreducible. If $\bar{\theta}\cdot\theta = \theta\cdot\bar{\theta}$, then θ has finite index iff $\theta\cdot\bar{\theta}$ contains the identity.*

To prove the theorem we shall need a few facts.

In this case next propositions show that θ has finite index iff $C(M, \rho(M)) \neq \phi$ where $\theta = [\rho]$.

Let $N \subset M$ be an inclusion of infinite factors and $\gamma: M \to N$ a canonical endomorphism with respect to a bicyclic vector Ω. If $v \in H(\gamma|N)$, $\|v\| = 1$, then

$$\varepsilon_v(x) \equiv v^*\gamma(x)v, \quad x \in M$$

defines a conditional expectation $\varepsilon_v \in C(M, N)$ and all normal conditional expectations of M onto N arise in this way [18]. Since the canonical endomorphism of $M_1 = J_M N' J_M$ onto M with respect to Ω is implemented by $\Gamma = J_N J_M$, thus it extends γ, it follows that if $w \in H(\gamma)$, $\|w\| = 1$, then

$$\varepsilon'_w(x) = w^*\gamma(x)w, \quad x \in M$$

defines an element of $C(M_1, M)$ and $w \to \varepsilon'_w$ is surjective [18]. Since $M \subset M_1$ is antiisomorphic to $M' \subset N'$ we have also a surjective map

$$w \in H(\gamma),\ \|w\| = 1 \to j_M \cdot \varepsilon'_w \cdot j_M \in C(N', M'),$$

where $j_M = J_M \cdot J_M$.

4.3 Proposition. *With the above notations, every conditional expectation $\varepsilon \in C(M, N)$ is given by $\varepsilon = \varepsilon_v$ with $v \in H(\gamma|N)$ and every $\varepsilon' \in C(N', M')$ is given by $\varepsilon' = j_M \cdot \varepsilon'_w \cdot j_M$ with $w \in H(\gamma)$. If $N' \cap M = \mathbf{C}$ then $H(\gamma)$ and $H(\gamma|N)$ are at most one dimensional.*

Proof. By [18] and the above comments the first part is immediate. The uniqueness of v and w up to a phase factor in the irreducible case will follow directly by Corollary 5.8 when N is isomorphic to M, namely $N = \rho(M)$ for some $\rho \in \mathrm{End}(M)$; the general case then follows easily by considering the inclusion $N \otimes \gamma(M) \subset M \otimes N$. ■

4.4 Proposition. *Let $N \subset M$ be an inclusion of factors. The following are equivalent:*

(i) $\mathrm{Ind}(N, M) < \infty$.

(ii) (a) *$N' \cap M$ is finite dimensional.*

(b) *There exists a faithful expectation $\varepsilon \in C(M, N)$ and a faithful expectation $\varepsilon' \in C(N', M')$.*

Proof. If $\mathrm{Ind}(N, M) < \infty$ then by definition there exists a faithful $\varepsilon \in C(M, N)$ and a faithful $\varepsilon' \in C(M_1, N)$ [11] see also Propositions 2.3 and 2.4 of [18]. Also $\dim(N' \cap M) < \infty$ [12], see also [15].

On the other hand assume (ii) and let $\varepsilon \in C(M, N)$ and $\varepsilon' \in C(N', M')$ be faithful;

let ε'_0 be the operator valued weight of N' to M' dual to ε [3, 9]. Then the Connes Radon–Nikodym derivative $u_t = (D\varepsilon'_0 : D\varepsilon')_t$ belongs to $N' \cap M$. We may suppose that $\varepsilon | N' \cap M$ is a trace, see [1], so u_t is a one parameter group of unitaries in the finite dimensional algebra $N' \cap M$; thus u_t is norm continuous and ε'_0 is bounded. ■

4.5 Corollary. *Let $N_1 \subset N_2 \subset N_3$ be factors with $\mathrm{Ind}(N_1, N_3) < \infty$. Then $\mathrm{Ind}(N_1, N_2) < \infty$ and $\mathrm{Ind}(N_2, N_3) < \infty$ and we have $\mathrm{Ind}(N_1, N_3) \leqq \mathrm{Ind}(N_1 N_2) \cdot \mathrm{Ind}(N_2, N_3)$, where the equality holds if $N'_1 \cap N_3 = \mathbf{C}$.*

Proof. Since $\mathrm{Ind}(N_1, N_3) < \infty$ there exists a faithful $\varepsilon \in C\ (N_1, N_3)$ and by the extension of Pimsner–Popa inequality [18] we have $\varepsilon(x) \geqq \lambda x$, $x \in N_3^+$, where $\lambda^{-1} = \mathrm{Ind}_\varepsilon(N_1, N_3)$. The restriction ε_0 of ε to N_2 satisfies $\varepsilon_0(x) \geqq \lambda x$, $x \in N_2^+$, thus $\mathrm{Ind}_{\varepsilon_0}(N_1, N_2) \leqq \lambda^{-1}$. Consider now the inclusions $N'_3 \subset N'_2 \subset N'_1$; since $\mathrm{Ind}(N'_3, N'_1) = \mathrm{Ind}(N_1, N_3) < \infty$, by the above argument also $\mathrm{Ind}(N'_3, N'_2) < \infty$ therefore $\mathrm{Ind}(N_2, N_3) = \mathrm{Ind}(N'_3, N'_2) < \infty$.

The rest follows by [18] and by the unicity of ε if $N'_1 \cap N_3 = \mathbf{C}$. ■

Let now $\rho, \bar\rho \in \mathrm{End}(M)$ be irreducible, where M is an infinite factor, such that $\rho\bar\rho$ and $\bar\rho\rho$ contain the identity. To prove Theorem 4.1, we want to show that $\bar\rho$ is a conjugate of ρ. First note that by Proposition 4.3 and Theorem 3.1 $C(M, \rho(M)) \neq \phi$. Since $\rho(M)' \cap M = \mathbf{C}$, there exists a unique conditional expectation $\varepsilon \in C(M, \rho(M))$ and ε is faithful. Choose Ω a cyclic separating vector for $\rho(M)$ and M; we are in the situation covered by [18, Proposition 3.1] for the computation of the canonical endomorphism $\gamma_\Omega : M \to \rho(M)$.

Let $U \in B(L^2(M))$ be a unitary such that

$$UxU^* = \rho\bar\rho(x), \quad x \in M.$$

(M and $\rho\bar\rho(M)$ have properly infinite commutants thus $\rho\bar\rho$ is implemented by a unitary.) Let $v \in M$ be an isometry such that

$$\bar\rho\rho(x)v = vx, \quad x \in M.$$

4.6 Lemma. *The conditional expectation ε is given by*

$$\varepsilon(x) = \rho(v)^* \rho\bar\rho(x)\rho(v), \quad x \in M.$$

Proof. By the uniqueness of ε it suffices to show that ε defined as above is an expectation. Since $\varepsilon = \rho(v^* \bar\rho(\cdot) v)$ we have $\varepsilon(M) \subset \rho(M)$. If $x \in M$ then

$$\varepsilon(\rho(x)) = \rho(v^* \bar\rho\rho(x) v) = \rho(v^* v x) = \rho(x),$$

thus $\varepsilon | \rho(M)$ is the identity. ■

4.7 Lemma. *Let $\varphi \in M_*$ be given by $\varphi \equiv \omega \cdot \varepsilon$, where $\omega = (\cdot\,\Omega, \Omega)$. Then $\varphi = (\cdot\,\xi, \xi)$, where $\xi = U^* \rho(v) \Omega$.*

Proof. We have

$$\begin{aligned}(x\xi, \xi) &= (xU^*\rho(v)\Omega, U^*\rho(v)\Omega) = (\rho(v^*)UxU^*\rho(v)\Omega, \Omega) \\ &= (\rho(v^*)\rho\bar\rho(x)\rho(v)\Omega, \Omega) = (\varepsilon(x)\Omega, \Omega) = \varphi(x), \quad x \in M. \quad ■\end{aligned}$$

4.8 Lemma. *The vector ξ is separating for M.*

Proof. By Proposition 4.4 ε is faithful; since ω is faithful also $\varphi = \omega\cdot\varepsilon$ is faithful, thus ξ is separating. ■

Having fixed the unitary U implementing $\rho\bar{\rho}$ we may define

$$\bar{\rho}^{-1}(x) \equiv U^*\rho(x)U, \quad x \in M.$$

4.9 Lemma. *The projection $E \equiv [\rho(M)\xi]$ is given by*

$$E = \bar{\rho}^{-1}(e).$$

where $e = vv^*$. In particular $E \in \rho(M)' \cap \bar{\rho}^{-1}(M)$.

Proof. We have for $x \in M$,

$$\begin{aligned}\rho(x)\xi &= \rho(x)U^*\rho(v)\Omega = U^*U\rho(x)U^*\rho(v)\Omega\\ &= U^*\rho\bar{\rho}\rho(x)\rho(v)\Omega = U^*\rho(\bar{\rho}\rho(x)v)\Omega\\ &= U^*\rho(vx)\Omega = U^*\rho(v)\rho(x)\Omega.\end{aligned}$$

Since Ω is cyclic for $\rho(M)$ we then have

$$\begin{aligned}[\rho(M)\xi] &= \mathrm{range}(U^*\rho(v))\\ &= U^*\rho(v)\rho(v^*)U\\ &= U^*\rho(e)U = \bar{\rho}^{-1}(e).\end{aligned}$$

4.10 Sublemma. *Let $A \subset B$ be an inclusion of von Neumann algebras, $e \in B$ a projection and let E be the support of e in A'. Then $E \in B$.*

Proof. Let $\mathscr{H}$ be the underlying Hilbert space; then $E = [Ae\mathscr{H}]$. If $b' \in B'$, then

$$b'Ae\mathscr{H} = Aeb'\mathscr{H} \subset Ae\mathscr{H},$$

namely $Ae\mathscr{H}$ is an invariant subspace for B' thus $E \in B$. ■

4.11 Lemma. *The vector ξ is cyclic for M.*

Proof. By Lemma 4.9 we have

$$G \equiv [M\xi] \geqq [\rho(M)\xi] = \bar{\rho}^{-1}(e),$$

therefore

$$[M\xi] \geqq [M\bar{\rho}^{-1}(e)\mathscr{H}] \equiv P,$$

namely G majorizes the support P of $\bar{\rho}^{-1}(e)$ in M'. By the Sublemma 4.10 with $A = M$, $B = \bar{\rho}^{-1}(M)$, we have $P \in \bar{\rho}^{-1}(M)$. On the other hand $P \in M'$, thus $P \in M' \cap \bar{\rho}^{-1}(M)$ or $\bar{\rho}(P) \in \bar{\rho}(M)' \cap M = \mathbf{C}$ because $\bar{\rho}$ is irreducible. Thus $P = 1$. ■

Multiplying U by a unitary in M' if necessary, we may assume by Lemma 4.11 that $\xi \in L^2(M)_+$, so that we may apply the formula in [18, Prop. 3.1] for the computation of γ.

4.12 Proposition. $\mathrm{Ind}(\bar{\rho}) = \mathrm{Ind}(\rho)$.

Proof. Let $M_1 = \langle M, \bar{\rho}^{-1}(e)\rangle$ the von Neumann algebra generated by M and $\bar{\rho}^{-1}(e)$. By Lemma 4.9 and [12], M_1 is the extension of M by $\rho(M)$. By applying

$\bar{\rho}$ to M_1 we have

$$\bar{\rho}\rho(M) \subset \bar{\rho}(M) \subset \langle \bar{\rho}(M), e \rangle \subset M.$$

Now

$$\mathrm{Ind}(\bar{\rho}(M), \langle \bar{\rho}(M), e \rangle) = \mathrm{Ind}(\bar{\rho}\rho(M), \bar{\rho}(M)) = \mathrm{Ind}(\rho(M), M) = \mathrm{Ind}(\rho),$$

and $\mathrm{Ind}(\bar{\rho}(M), M) = \mathrm{Ind}(\bar{\rho})$, thus

$$\mathrm{Ind}(\bar{\rho}) = \mathrm{Ind}(\bar{\rho}(M), M) \geqq \mathrm{Ind}(\bar{\rho}(M), \langle \bar{\rho}(M), e \rangle) = \mathrm{Ind}(\rho).$$

Since $\bar{\rho}$ and ρ plays a symmetric role, we have $\mathrm{Ind}(\bar{\rho}) = \mathrm{Ind}(\rho)$ and $\langle \bar{\rho}(M), e \rangle = M$, *in fact* $\mathit{Ind}(\rho) < \infty$ by Propositions 4.3 and 4.4. ■

We have implicitly proved the following:

4.13 Corollary. $M = \langle \bar{\rho}(M), e \rangle$, *thus* M *is the extension of* $\bar{\rho}(M)$ by $\bar{\rho}\rho(M)$.

Proof of Theorem 4.1. By the above corollary we have

$$M_1 \equiv J\rho(M)'J = \bar{\rho}^{-1}(M),$$

where J is the modular conjugation of M with respect to Ω. Hence $\bar{\rho}^{-1}(v) \in M_1$ and

$$V_0 \equiv J\bar{\rho}^{-1}(v)J \in N', \quad N = \rho(M).$$

The standard implementation of the isomorphism $x \in N \to xE \in N_E$ with respect to Ω and ξ (here $E = \bar{\rho}^{-1}(e)$ as in Lemma 4.9) is given by the isometry $V = V_0 Z$ with Z a unitary in N' and by Proposition 3.1 of [18] we have

$$\Gamma = V^* J V J = Z^* V_0 J V_0^* J J Z J,$$

thus, to compute the class $[\gamma]$ of the canonical endomorphism of M into N, we may assume $V = V_0$; we then have

$$\begin{aligned}
\Gamma x \Gamma^* &= J\bar{\rho}^{-1}(v^*)J\bar{\rho}^{-1}(v)x\bar{\rho}^{-1}(v^*)J\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)JU^*\rho(v)\rho\bar{\rho}(x)\rho(v)^*UJ\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)JU^*\rho(v\bar{\rho}(x)v^*)UJ\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)JU^*\rho(e)\rho\bar{\rho}\rho\bar{\rho}(x)\rho(e)UJ\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)JU^*U\rho\bar{\rho}(x)U^*\rho(e)UJ\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)J\rho\bar{\rho}(x)EJ\bar{\rho}^{-1}(v)J \\
&= J\bar{\rho}^{-1}(v^*)J\rho\bar{\rho}(x)J\bar{\rho}^{-1}(v)J \\
&= \rho\bar{\rho}(x)
\end{aligned}$$

because $J\bar{\rho}(v)^{-1}J \in \rho(M)'$ and $JEJ = E$.

Hence the canonical endomorphism $\gamma : M \to \rho(M)$ is given by $[\gamma] = [\rho\bar{\rho}]$. Conversely if we define

$$\bar{\rho} = \rho^{-1} \cdot \gamma,$$

then $\bar{\rho}\rho = \gamma$ and $H(\gamma) \neq \{0\}$ by Proposition 4.3. It remains to show that $\rho\bar{\rho}$ contains the identity with multiplicity 1; we shall prove this fact in Corollary 5.8. ■

4.14 Corollary. *With* $\rho \in \mathrm{End}(M)$ *the tower associated with* $\rho(M) \subset M$ *is given by*

$$M \supset \rho(M) \supset \rho\bar{\rho}(M) \supset \rho\bar{\rho}\rho(M) \supset \cdots.$$

Proof. Immediate by Theorems 3.1 and 4.1 if we interchange ρ and $\bar{\rho}$. ■

4.15 Corollary. *If* $\rho\in\mathrm{End}(M)$ *then* $\mathrm{Ind}((\rho\bar{\rho})^n)=\mathrm{Ind}(\rho)^{2n}$ *and* $\mathrm{Ind}((\rho\bar{\rho})^n\rho)=\mathrm{Ind}(\rho)^{2n+1}$.

Proof. This corollary follows from Theorem 3.1 and the product formula for the minimal index given in [10]. ■

It would be interesting to know whether the product formula $\mathrm{Ind}(\rho_1\rho_2)=\mathrm{Ind}(\rho_1)\mathrm{Ind}(\rho_2)$ holds for all $\rho_1,\rho_2\in\mathrm{End}(M)$. However it holds in the case of supersection sectors in quantum field theory.

5. Reducible Sectors and Intertwiners

Let M be an infinite factor and $\rho\in\mathrm{End}_0(M)$ an endomorphism with finite index. Since $\rho(M)'\cap M$ is finite dimensional ρ decomposes as a finite direct sum of irreducible endmorphisms with finite index $\rho=\rho_1\oplus\cdots\oplus\rho_n$, thus $\bar{\rho}\equiv\bar{\rho}_1\oplus\cdots\oplus\bar{\rho}_n$ is a conjugate of ρ if $\bar{\rho}_i$ is a conjugate of ρ_i in the sense of Theorem 4.1, in particular $\bar{\rho}\rho$ contains the identity at least with multiplicity n and

$$\mathrm{Ind}(\rho)^{1/2}=\sum\mathrm{Ind}(\rho_i)^{1/2}=\sum\mathrm{Ind}(\bar{\rho}_i)^{1/2}=\mathrm{Ind}(\bar{\rho})^{1/2}$$

by the additivity of the square root of the index [18, Theorem 5.5].

By Theorem 3.1 we have

$$\rho\bar{\rho}=\gamma,$$

where $\gamma:M\to\rho(M)$ is a canonical endomorphism and

$$\rho\bar{\rho}(M)\subset\rho(M)\subset M$$

is a canonical extension.

We may thus assume that ρ is implemented by a unitary V so that $\Gamma=J_\rho J$, where J is the modular conjugation of $L^2(M)$ and $J_\rho=VJV^*$ and $\bar{\rho}=\rho^{-1}\cdot\gamma$, where $\gamma(x)=\Gamma x\Gamma^*$.

To each faithful expectation $\varepsilon\in C(M,\rho(M))$ there corresponds a dual expectation $\varepsilon'\in C(\rho(M)',M')$, thus a dual expectation $\bar{\varepsilon}\in C(M,\bar{\rho}(M))$ given by $\bar{\varepsilon}=\bar{\rho}\cdot j\cdot\varepsilon'\cdot j\cdot\bar{\rho}^{-1}$, where $j=J\cdot J$. If $e\in M$ is a projection giving the expectation $\bar{\rho}\cdot\varepsilon\cdot\bar{\rho}^{-1}\in C(\bar{\rho}(M),\bar{\rho}\rho(M))$ so that $M=\langle\bar{\rho}(M),e\rangle$, then $\bar{\varepsilon}\in C(M,\bar{\rho}(M))$ is characterized by the property that

$$\bar{\varepsilon}(e)=\mathrm{Ind}_\varepsilon(M,\rho(M))^{-1}$$

because elements in M are sums of monomials [12],

$$x_0+\sum x_i e y_i \quad x_i, y_i\in\bar{\rho}(M).$$

In particular if we interchange ρ and $\bar{\rho}$ we have $\bar{\bar{\varepsilon}}=\varepsilon$.

With ϕ a left inverse of ρ we have a left inverse $\bar{\phi}=\bar{\rho}^{-1}\cdot\bar{\varepsilon}$ of $\bar{\rho}$. We shall give below a different description of the map $\varepsilon\to\bar{\varepsilon}$.

Since ρ has finite index we have $H(\rho\bar{\rho})\neq\{0\}$ and $H(\bar{\rho}\rho)\neq\{0\}$.

5.1 Proposition. *ρ maps $H(\bar{\rho}\rho)$ onto $H(\rho\bar{\rho}|_{\rho(M)})$.*

Proof. If $v\in H(\bar{\rho}\rho)$, namely $\bar{\rho}\rho(x)v=vx$, $x\in M$, then if we apply ρ on both sides we have

$$\rho\bar{\rho}(y)\rho(v)=\rho(v)y$$

for any $y=\rho(x)\in\rho(M)$, thus $\rho(v)\in H(\rho\bar{\rho}|_{\rho(M)})$. Reversing the argument we obtain the surjectivity. ■

5.2 Theorem. *Let $\rho, \bar{\rho}\in \mathrm{End}_0(M)$ be conjugate endomorphisms with finite index. For every isometry $v\in H(\rho\bar{\rho})$ there exists a unique isometry $\bar{v}\in H(\bar{\rho}\rho)$ such that*

$$v^*\rho(\bar{v})=\lambda^{1/2},\quad \bar{v}^*\bar{\rho}(v)=\lambda^{1/2},$$

where $\lambda^{-1}\equiv \mathrm{Ind}_{\varepsilon_v}(\bar{\rho}(M),M)$.

Proof. Let $v\in H(\rho\bar{\rho})$ be an isometry and $\varepsilon_v\in C(M,\bar{\rho}(M))$ the corresponding expectation and $\varepsilon=\bar{\varepsilon}_v\in C(M,\rho(M))$ the dual expectation as before. Since $\rho\bar{\rho}(x)v=vx$, $x\in M$, if we apply ε to both members of this equality we have

$$\rho\bar{\rho}(x)\varepsilon(v)=\varepsilon(v)x,\quad x\in\rho(M),$$

namely $\varepsilon(v)\in H(\rho\bar{\rho}|_{\rho(M)})$, hence by Proposition 5.1,

$$\phi(v)=\rho^{-1}\cdot\varepsilon(v)\in H(\bar{\rho}\rho),$$

where ϕ is the corresponding left inverse of ρ.

We set

$$\phi(v)=\lambda^{1/2}\bar{v},$$

where $\lambda^{1/2}=\|\phi(v)\|$; since we shall show that $\lambda\neq 0$, $\bar{v}$ is an isometry.

We have

$$\begin{aligned}\bar{v}^*\bar{\rho}(v)&=\lambda^{-1/2}\phi(v^*)\bar{\rho}(v)=\lambda^{-1/2}\phi(v^*\rho\bar{\rho}(v))\\&=\lambda^{-1/2}\phi(vv^*)=\lambda^{-1/2}\phi(e),\end{aligned}$$

where $e=vv^*$. Now $\phi(e)=\rho^{-1}\cdot\varepsilon(e)$ and ε is dual to ε_v, thus

$$\phi(e)=\rho^{-1}(\varepsilon(e))=\mathrm{Ind}_\varepsilon(M,\rho(M))^{-1}.$$

Lemma 5.4 will show that $\lambda=\mathrm{Ind}_\varepsilon(M,\rho(M))^{-1}$, therefore we have

$$\bar{v}^*\bar{\rho}(v)=\lambda^{-1/2}\lambda=\lambda^{1/2}\tag{5.1}$$

as desired. By interchanging ρ and $\bar{\rho}$ it will follow that

$$v^*\rho(\bar{v})=\lambda^{1/2};\tag{5.2}$$

in fact (5.2) follows by the same argument leading to (5.1) as soon as we show that $\varepsilon_{\bar{v}}$ and ε_v are one another dual conditional expectations. Notice that we have made the following construction:

$$\begin{gathered}v\in H(\rho\bar{\rho})\\\downarrow\\\varepsilon_v\in C(M,\bar{\rho}(M)),\quad \varepsilon_v(x)=\bar{\rho}(v^*\rho(x)v)\\\downarrow\\\bar{\varepsilon}_v\in C(M,\rho(M))\\\downarrow\\\bar{\varepsilon}_v(v)\in H(\rho\bar{\rho}|_{\rho(M)})\\\downarrow\end{gathered}$$

$$\bar{v} \in H(\bar{\rho}\rho), \qquad \rho(\bar{v}) = \lambda^{-1/2}\bar{\varepsilon}_v(v)$$
$$\downarrow$$
$$\varepsilon_{\bar{v}} \in C(M, \rho(M)).$$

To show that $\varepsilon_{\bar{v}} = \bar{\varepsilon}_v$ let $w \in H(\bar{\rho}\rho)$ satisfy (5.1) with $w = \bar{v}$; we have

$$w^*\bar{\rho}(e)w = w^*\bar{\rho}(v)\bar{\rho}(v^*)w = \lambda,$$

thus

$$\varepsilon_w(e) = \rho(w^*\bar{\rho}(e)w) = \lambda,$$

which shows that ε_w is equal to $\bar{\varepsilon}_v$. The uniqueness of $\bar{v}$ satisfying (5.1) and (5.2) then follows as in the proof of Corollary 5.8. ■

5.3 Corollary. *Let v and $\bar{v}$ as in Theorem 5.2 and $e = vv^*, f = \bar{v}\bar{v}^*$. Then $\bar{\rho}(e), f$ are Jones projections for the inclusions $\bar{\rho}\rho(M) \subset \bar{\rho}(M) \subset M$, in particular*

$$\bar{\rho}(e) f \bar{\rho}(e) = \lambda\bar{\rho}(e), \quad f\bar{\rho}(e)f = \lambda f,$$

where $\lambda = \mathrm{Ind}(\rho(M), M)^{-1}$ as above. More generally setting

$$e_{2i+1} = (\bar{\rho}\rho)^i(f), \quad e_{2i} = (\bar{\rho}\rho)^{i-1}(\bar{\rho}(e)), \quad i = 1, 2, \ldots,$$

the projections e_i are Jones projections associated with the inclusion $\bar{\rho}(M) \subset M$, and therefore satisfy the Jones–Temperley–Lieb [12, 23] relations

$$e_i e_{i\pm 1} e_i = \lambda e_i, \quad e_i e_j = e_j e_i, \quad |i - j| \geqq 2.$$

Proof. Immediate by the proof of Theorem 5.2. ■

5.4 Lemma. *Let $v \in H(\rho\bar{\rho})$ be an isometry. Then $\| \bar{\varepsilon}_v(v) \| = \lambda^{1/2}$, where $\lambda^{-1} = \mathrm{Ind}_{\varepsilon_v}(\bar{\rho}(M), M)$.*

Proof. Set $\varepsilon = \bar{\varepsilon}_v$. Since ε is dual to ε_v we have $\varepsilon(e) = \lambda$, where $e = vv^*$. By Proposition 5.5 there exists $m \in \bar{\rho}(M)$ such that

$$v = ev = em,$$

in fact $m = \lambda^{-1}\varepsilon(v)$. Then

$$\| \varepsilon(v) \|^2 = \lambda^2 \| m \|^2 = \lambda^2 \| m^*m \| = \lambda \| m^*\varepsilon(e)m \| = \lambda \| \varepsilon(m^*em) \| = \lambda \| \varepsilon(v^*v) \| = \lambda.$$

■

The following proposition in an immediate extension of [21] as well as its corollary.

5.5 Proposition. *Let $N \subset M$ be an inclusion of infinite factors, $\varepsilon \in C(M, N)$ with $\mathrm{Ind}_\varepsilon(N, M) < \infty$ and $M_1 = \langle M, e \rangle$ the corresponding extension. For every $x \in M_1$ there exists a unique $m \in M$ with $ex = em$.*

Proof. The proof follows [21] with obvious modifications. ■

5.6 Corollary. *With the above notations let v be an isometry in M_1 with $vv^* = e$, and $m \in M$ with $v = em$. Every $x \in M$ can be written as $x = \varepsilon(xm^*)m$. Moreover $m^*m = \lambda^{-1}$ and $m^*em = 1$.*

Proof. With $\varepsilon' \in (M_1, M)$ the expectation dual to ε, we have

$$m^*m = \lambda^{-1} m^* \varepsilon'(e) m = \lambda^{-1} \varepsilon'(m^*em) = \lambda^{-1} \varepsilon'(v^*v) = \lambda^{-1}$$

and

$$m^*em = v^*v = 1,$$

thus

$$e\varepsilon(xm^*)m = exm^*em = exv^*v = ex,$$

which implies $\varepsilon(xm^*)m = x$. ∎

5.7 Corollary. *With the notations of Theorem 5.2, every $x \in M$ can be written as*

$$x = \lambda^{-1} \varepsilon_{\bar{v}}(xv^*)v.$$

Proof. Recall that $\varepsilon_{\bar{v}} = \rho\phi \in C(M, \rho(M))$, where ϕ is the left inverse of ρ given by $\phi = \bar{v}^* \bar{\rho}(\cdot)\bar{v}$, thus we have

$$\begin{aligned}\lambda^{-1}\varepsilon_{\bar{v}}(xv^*)v &= \lambda^{-1}\rho\phi(xv^*)v = \lambda^{-1}\rho(\bar{v}^*\bar{\rho}(xv^*)\bar{v})v = \lambda^{-1}\rho(\bar{v}^*\bar{\rho}(x)\bar{\rho}(v^*)\bar{v})v \\ &= \lambda^{-1/2}\rho(\bar{v}^*\bar{\rho}(x))v = \lambda^{-1/2}\rho(\bar{v})^*\rho\bar{\rho}(x)v = \lambda^{-1/2}\rho(\bar{v})^*vx = x,\end{aligned}$$

where we have used twice the formulas of Theorem 5.2. ∎

5.8 Corollary. *Let $\bar{\rho}, \rho \in \mathrm{End}(M)$ be irreducible conjugate endomorphisms with finite index. Then $H(\bar{\rho}\rho)$ is one dimensional.*

Proof. Let $v, w \in H(\bar{\rho}\rho)$ be isometries. Then $\bar{v}^*\rho(w) \in \rho(M)' \cap M$, thus $\bar{v}\rho(w) = z$ with $z \in \mathbf{C}$. We have

$$w = \lambda^{1/2}\bar{\rho}(\bar{v}^*)vw = \lambda^{1/2}\bar{\rho}(\bar{v}^*)\bar{\rho}\rho(w)v = \lambda^{1/2}\bar{\rho}(\bar{v}^*\rho(w))v = \lambda^{1/2}\bar{\rho}(z)v$$

with λ as in the theorem. ∎

5.9 Corollary. *Let $N \subset M$ be an inclusion of factors with finite index and $N' \cap M = \mathbf{C}$. The Jones projection $e \in M_1 (= J_M N' J_M)$ belongs to the center of $N' \cap M_1$.*

Proof. As in the proof of Proposition 4.3 we may suppose that $N = \rho(M)$ with $\rho \in \mathrm{End}(M)$. By Corollary 5.8 there exists a unique $v \in H(\rho\bar{\rho})$, $\|v\| = 1$, up to a phase factor. If $u \in \rho\bar{\rho}(M)' \cap M$ is a unitary then $uv \in H(\rho\bar{\rho})$, hence $uv = \beta v$ with $\beta \in \mathbf{T}$, so that $ueu^* = e$, where $e = vv^*$, namely e belongs to the center of $\rho\bar{\rho}(M)' \cap M$. The proof is complete by applying Corollary 5.8. ∎

6. Invariants for Endomorphisms. Real and Pseudoreal Sectors

Let $\rho \in \mathrm{End}(M)$ be an irreducible endomorphism of the factor M and suppose that, for some positive integer n, ρ^n contains the identity, namely

$$H_n \equiv \{v \in M \mid \rho^n(x)v = vx,\ x \in M\}$$

is a non-zero Hilbert space in M. By Corollary 4.2 ρ has finite index. Since $H_n H_n^*$ is contained in $\rho^n(M)' \cap M$ and the latter is finite-dimensional, also H_n is finite-dimensional.

6.1 Proposition. *If $\rho \in \mathrm{End}_0(M)$ is an irreducible endormorphism with finite index, then there exists a unique left inverse ϕ of ρ. Moreover ϕ is normal and faithful.*

Proof. By Lemma 7.2 of [18] in this setting, we have to show that there exists a unique conditional expectation ε of M onto $\rho(M)$. The existence follows by definition (see Proposition 4.4). By [18, Proposition 5.7] any such expectation is normal. It is then unique by a result of Connes or by Proposition 4.3. ■

If $H_n \neq \{0\}$ as above let ϕ be the unique left inverse of ρ, then with $v \in H_n$ we have

$$\rho^n(x)\phi(v) = \phi(\rho^{n+1}(x)v) = \phi(v\rho(x)) = \phi(v)x, \quad x \in M,$$

namely ϕ maps H_n into itself. Denote by T the restriction of ϕ to H_n; we regard T as an element of $B(H_n)$.

6.2 Proposition. *T^n is a positive non-singular operator.*

Proof. We have

$$\langle T^n v, v \rangle = \phi^n(v)^* v = \phi^n(v^* \rho^n(v)) = \phi^n(vv^*) = \phi^n(e) = \lambda \geqq 0,$$

since $e \in \rho^n(M') \cap M$, $\phi^{n-1}(e) \in \rho^n(M)' \cap \rho^{n-1}(M) = \mathbf{C}$, thus λ is a non-negative number. Moreover λ is non-zero because ϕ is faithful. ■

Proof. Immediate by Proposition 6.2. ■

By the above corollary the set of the eigenvalues of T with multiplicities form an invariant for the sector $[\rho]$. This invariant is a generalization of Connes outer invariant for periodic automorphisms of factors [2]. To understand the meaning of this invariant we mention that in the example of the subfactors associated with an irreducible representation π of a compact group G [18], thus in quantum field theory with normal statistics, if $\pi \otimes \cdots \otimes \pi$ (n-times) contains the identity, the restriction of the n-cycle of $\mathbf{P}_n$ to the subspace of the identity representation is equivalent to T. It is not difficult to construct examples with non-trivial invariants of all orders.

6.4 Corollary. *If $\rho \in \mathrm{End}(M)$ is an irreducible selfconjugate endomorphism, then H_2 is one-dimensional and $T = \pm \mathrm{Ind}(\rho)^{-1/2}$.*

Proof. H_2 is one dimensional by Corollary 5.8 and if $v \in H_2$ then $\|\phi(v)\|^2 = \mathrm{Ind}(\rho)^{-1}$ by Lemma 5.4. ■

If $\rho \in \mathrm{End}(M)$ is irreducible and selfconjugate we shall say that ρ is real or pseudoreal according to $\phi|_{H_2} = \pm \mathrm{Ind}(\rho)^{-1/2}$. In quantum field theory with normal statistics, this corresponds to the reality or pseudoreality of the associated representation of the gauge group G and, when G is $SU(2)$, to integral or half integral spin sectors. By the same argument we have a generalization of Carruther's theorem [5].

7. Braid Group Statistics and Link Invariant Polynomials

Some applications to Quantum Field Theory have been discussed or follow directly from the analysis of the previous sections. We give here an explicit analysis of the braid group representation giving the statistics for a class of sectors. We use the notations and assumptions in [18].

Let $\rho \in \mathscr{E}_c$ be a localized irreducible endomorphism of the quasi-local observable C^*-algebra $\mathscr{A} = \cup \mathscr{A}(\mathscr{O})^-$, and ε_ρ the statistics operator associated with ρ, namely $\varepsilon_\rho = u^*\rho(u)$, where $u \in \mathscr{A}$ and $u\rho(\cdot)u^* \in \mathscr{E}_c$ is localized (left) space-like to ρ (we deal with a Minkowski space of dimension $\geqq 2$). Then by duality

$$\varepsilon_\rho \in \rho^2(\mathscr{A})' \subset \rho^2(\mathscr{A}(\mathscr{O}))' \cap \mathscr{A}(\mathscr{O})$$

if ρ is localized in $\mathscr{O}$. Let $\mathbf{B}_n$ be the braid group with generators $\{\sigma_i, i = 1, \ldots, n-1\}$ satisfying the Artin relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i-j| \geqq 2.$$

An elementary calculation shows that

$$\rho(\varepsilon_\rho)\varepsilon_\rho\rho(\varepsilon_\rho) = \varepsilon_\rho\rho(\varepsilon_\rho)\varepsilon_\rho$$

(write $\varepsilon_\rho = u^*\rho(u)$ and use the equality $u\rho(\varepsilon_\rho)u^* = \varepsilon_\rho$ due to the fact that $u\rho(\cdot)u^*$ acts as the identity on $\mathscr{A}(\mathscr{O})$), henceforth the unitaries

$$\varepsilon_\rho^{(n)}(\sigma_i) \equiv \rho^{i-1}(\varepsilon_\rho)$$

have the above presentation of $\mathbf{B}_n$ and define the representation $\varepsilon_\rho^{(n)}$ of $\mathbf{B}_n$ giving the statistics of the sector $[\rho]$.

In [6] it has been observed, among other things, that if ρ^2 has two irreducible components, since

$$\varepsilon_\rho \in \rho^2(\mathscr{A})' = \mathbf{C}^2$$

ε_ρ has at most 2 eigenvalues, henceforth $\varepsilon_\rho^{(n)}$ belongs to the class of representations analyzed by Hecke algebras methods, see [24].

On the other hand V. Jones has constructed directly representations of $\mathbf{B}_n$ associated with subfactors of finite index I [13]. If $\{e_i, i = 1 \cdots n-1\}$ are projections that satisfy the relations $e_i e_{i\pm 1} e_i = I^{-1} e_i$, $e_j e_i = e_i e_j$ if $|i-j| \geqq 2$, then

$$\pi_{q,z}^{(n)}(\sigma_i) = z[qe_i - (1-e_i)]$$

is a representation of $\mathbf{B}_n$; here $q \in \mathbf{C}$ satisfies $q + q^{-1} + 2 = I$ and $z \in \mathbf{T}$ is arbitrary and set equal to 1 in [13].

Because of the equality

$$\operatorname{Ind}[\rho(\mathscr{A}(\mathscr{O})), \mathscr{A}(\mathscr{O})] = d(\rho)^2$$

it is natural to see when $\varepsilon_\rho^{(n)}$ is the representation $\pi_{q,z}^{(n)}$ associated with the inclusion $\rho(\mathscr{A}(\mathscr{O})) \subset \mathscr{A}(\mathscr{O})$.

We begin with the following proposition. We shall denote by ω_ρ the phase of the statistics parameter $\lambda_\rho = \phi(\varepsilon_\rho)$. Notice that here sectors refer to different (but obviously related) objects than in the previous sections.

7.1 Proposition. *Let ρ be an irreducible selfconjugate localized endomorphism and e the Jones projection given by Corollary 5.3, then*

$$\varepsilon_\rho e = \pm\,\omega_\rho^{-1} e,$$

where the $\pm$ sign is chosen according to whether the sector $[\rho]$ is real or pseudoreal respectively.

Proof. Assuming more generally that ρ is not necessarily selfconjugate let $v \in \mathscr{A}$ be an isometry with $\rho\bar{\rho}(x)v = vx$, $x \in \mathscr{A}$; by a calculation similar to that in [5], the isometry $w = \omega_\rho \varepsilon(\rho, \bar{\rho})v$ satisfies

$$v^* \rho(w) = w^* \bar{\rho}(v) = \frac{1}{d(\rho)},$$

where $\varepsilon(\rho, \bar{\rho})$ is the natural intertwiner between ρ and $\bar{\rho}$ (with $\bar{\rho}$ localized left spacelike to ρ). Restricting to a local von Neumann algebra and applying Theorem 5.2 we have $\bar{v} = w$, namely

$$\bar{v} = \omega_\rho \varepsilon(\rho, \bar{\rho})v.$$

Specializing this formula to the case $\rho = \bar{\rho}$, one has $\varepsilon_\rho = \varepsilon(\rho, \bar{\rho})$ and $\bar{v} = \pm v$ by Corollary 6.4, therefore

$$\varepsilon_\rho v = \pm \omega_\rho^{-1} v,$$

and multiplying both sides by v^* we have the thesis. ■

In the simplest case where ρ is an automorphism, Proposition 7.1 implies that ε_ρ is a fourth root of the unity: $\varepsilon_\rho = \pm 1$ (real sectors) or $\varepsilon_\rho = \pm i$ (pseudoreal sector) and the $\pm$ sign alternative are interpreted as corresponding to the Bose–Fermi statistics alternative. These possibilities appear also in the following cases.

7.2 Theorem. *Let ρ be an irreducible selfconjugate localized endomorphism. If ρ^2 has two irreducible components then $\varepsilon_\rho^{(n)}$ is equivalent to $\pi_{q,z}^{(n)}$, where $z = \pm iq^{-1/4}$ or $z = \pm q^{-1/4}$ according to $[\rho]$ is a real or pseudoreal sector, modulo the possible change $q \to q^{-1}$.*

Proof. Since ρ is selfconjugate by Theorems 3.1 and 4.1 the canonical tower associated with $\rho(\mathscr{A}(\mathcal{O})) \subset \mathscr{A}(\mathcal{O})$, where ρ is localized in $\mathcal{O}$, is given by

$$\mathscr{A}(\mathcal{O}) \supset \rho(\mathscr{A}(\mathcal{O})) \supset \rho^2(\mathscr{A}(\mathcal{O})) \supset \cdots,$$

hence ε_ρ belongs to $\rho^2(\mathscr{A})' = \rho^2(\mathscr{A})' \cap \mathscr{A}(\mathcal{O}) = \{\alpha e + \beta, \alpha, \beta \in \mathbf{C}\}$, where e is the Jones projection given by Theorem 4.1. The projections e_i defined in Corollary 5.3 with $\rho = \bar{\rho}$ are then the Jones projections. By definition

$$\pi_{q,z}^{(n)}(\sigma_i) = z(qe_i - (1 - e_i)) = \rho^{i-1}(z(qe - (1 - e)))$$

and

$$\varepsilon_\rho^{(n)}(\sigma_i) = \rho^{i-1}(\varepsilon_\rho) = \rho^{i-1}(\alpha e + \beta(1 - e)),$$

however the only choices for $\alpha, \beta \in \mathbf{C}$ that gives a representation of $\mathbf{B}_n$ are

$$\varepsilon_\rho = z(qe - (1 - e))$$

with $z \in \mathbf{C}$ [14]. It remains to determine the value of z. By Proposition 7.1 the projection e is a spectral projection of ε_ρ with spectral value $\pm \omega_\rho^{-1}$, thus $\pm \omega_\rho^{-1} = zq$. By the calculation in the Corollary 7.4 we have $\omega_\rho = -zq^{-1/2}$, hence

$$-zq^{-1/2} = \pm z^{-1} q^{-1},$$

namely $z = \pm iq^{-1/4}$ or $z = \pm q^{-1/4}$ as desired. ■

Note that explicitly we have found the possibilities, cf. [8],

$$\varepsilon_\rho = \mu[q^{3/4}e - q^{-1/4}(1-e)], \quad \varepsilon_\rho = \mu[q^{-3/4}e - q^{1/4}(1-e)]$$

with $\mu \in \mathbb{C}$, $\mu^4 = 1$.

7.3 Corollary. *Let ρ be as in Theorem 7.2. Then $d(\rho) \leqq 2$.*

Proof. $\varepsilon_\rho^{(n)}$ is unitary, therefore $\pi_{q,z}^{(n)}$ is unitary, but this entails $\mathrm{Ind}(\rho(\mathscr{A}(\mathcal{O})), \mathscr{A}(\mathcal{O})) \leqq 4$, thus $d(\rho) \leqq 2$ by [14]. ■

7.4 Corollary. *Let ρ be as in Theorem 7.2. The statistics parameter λ_ρ is given by $\lambda_\rho = -z(1+q)^{-1}$, where $q + q^{-1} + 2 = d(\rho)^2$ and z as in Theorem 7.2. Hence $\varepsilon_\rho^{(n)}$ is determined by $d(\rho)$ (modulo the above possibilities).*

Proof. Since $\varepsilon_\rho = z[qe - (1-e)]$ and $\phi(e) = d(\rho)^{-2}$, where ϕ is the left inverse of ρ (cf. [18]), we have

$$\lambda_\rho = \phi(\varepsilon_\rho) = z\left[\frac{(q+1)}{d(\rho)^2} - 1\right] = z\left(\frac{q+1}{q+q^{-1}+2} - 1\right) = -\frac{z}{1+q}.$$

In particular $\omega_\rho = -zq^{-1/2}$. ■

7.5 Corollary. *Let ρ be as above. If $d(\rho)^2$ is an integer, then up to tensor product by one dimensional representation of $\mathbf{B}_n$, $\varepsilon_\rho^{(n)}$ is given by a finite group representation, namely $\mathbf{B}_n | \ker \varepsilon_\rho^{(n)}$ is finite.*

Proof. Immediate by the analysis in [14]. ■

The groups $\mathbf{B}_n/\ker \varepsilon_\rho^{(n)}$ are discussed in [14]; for example the Hessian group of order 216 and the simple group of order 25,920 appear in this analysis. B. Schroer has called our attention to the fact that the group $\mathbf{B}_n/\ker \varepsilon_\rho^{(n)}$ are in fact the monodromy groups of the Wightman functions and their analysis is classical, cf. [8].

7.6 Corollary. *Let ρ be as above. If $d(\rho) = 2$, then $\varepsilon_\rho^{(n)}$ is the tensor product of a one dimensional representation of $\mathbf{B}_n$ (statistics of anyones) and the normal statistics representation of $\mathbf{P}_n$.*

Proof. This follows because when $I = 4$, $\pi_{q,1}^{(n)}$ factors through a representation of $\mathbf{P}_n$ as due. ■

Note now that the formula $\phi^\infty(\varepsilon_\rho^{(n)}(\alpha)) = \phi^{n-1}(\varepsilon_\rho^{(n)}(\alpha))$, $\alpha \in \mathbf{B}_n$, gives a Markov trace for the representations $\varepsilon_\rho^{(n)}$, $n \in \mathbf{N}$, [5, 6]

$$\phi^\infty(\varepsilon_\rho^{(n+1)}(\alpha\sigma_n)) = \phi(\phi^{n-1}(\varepsilon_\rho^{(n)}(\alpha)\rho^{n-1}(\varepsilon_\rho)) = \phi(\phi^{n-1}(\varepsilon_\rho^{(n)}(\alpha))\varepsilon_\rho) = \phi^\infty(\varepsilon_\rho^{(n)}(\alpha))\lambda_\rho,$$

the tracial property of ϕ^∞ follows directly by the tracial property of the minimal index expectation [18, Theorem 5.5].

7.7 Corollary. *Let ρ be as in Theorem 7.2, L a link represented by the element $\alpha \in \mathbf{B}_n$ and V_L the Jones polynomial. Then*

$$V_L(q) = (-d(\rho))^{n-1}(-\omega_\rho)^{-l}\phi^\infty(\varepsilon_\rho^{(n)}(\alpha)),$$

where l is the exponent sum of α a word in $\sigma_1, \ldots, \sigma_{n-1}$.

Proof. By definition [13]

$$V_L(q) = \left(-\frac{q+1}{\sqrt{q}}\right)^{n-1} (\sqrt{q})^l \operatorname{tr}(\pi_{q,1}^{(n)}(\alpha)),$$

where tr is the relative Markov trace, thus by Theorem 7.2,

$$V_L(q) = \left(-\frac{q+1}{\sqrt{q}}\right)^{n-1} (\sqrt{q})^l z^{-l} \operatorname{tr}(\pi_{q,z}^{(n)}(\alpha))$$

$$= \left(-\frac{q+1}{\sqrt{q}}\right)^{n-1} \left(\frac{\sqrt{q}}{z}\right)^l \phi^\infty(\varepsilon_\rho^{(n)}(\alpha)).$$

Now the above coefficient of ϕ^∞ is unaffected by the change $q \to q^{-1}$, $z \to zq^{-1}$, thus by Corollary 7.4 it is a function of λ_ρ. An elementary calculation of this function gives the formula in the statement. ■

In particular the right-hand side in the above corollary gives a link invariant associated with the sector $[\rho]$. This fact does not depend on the special class of sectors considered here [6], because the Markov property of ϕ^∞ readily entails its invariance under Markov moves of types I and II as in [13].

Let now $[\rho]$ be a general selconjugate sector. By Theorem 4.1 and our index-statistics correspondence $d(\rho)$ is finite and has n channels with $n \leqq d(\rho)^2$. Notice explicitly that if we set $g_i \equiv \rho^{i-1}(\varepsilon_\rho)$ and $e_i = \rho^{i-1}(e)$ as above the g_i are unitaries satisfying the Artin braid relations and the e_i are selfadjoint projections satisfying the Jones relations with index $d(\rho)^2$ and also

$$e_{i+1} g_i e_{i+1} = \lambda_\rho e_{i+1}, \tag{7.1}$$

$$g_i e_i = \pm \omega_\rho^{-1} e_i. \tag{7.2}$$

We shall discuss somewhere else the algebra generated by these relations (where (7.1) becomes $e_{i+1} g_i^k e_{i+1} = \lambda_\rho^{(k)} e_{i+1}$, $\lambda_\rho^{(k)} \equiv \phi^k(\varepsilon_\rho)$. In particular this algebra admits a natural Markov trace giving rise to link invariants and the statistics of ρ is described by finitely many parameters). Here we treat the case of a 3-channel sector where the description is complete.

7.8 Theorem. *Let $[\rho]$ be an irreducible (pseudo)-real self-conjugate 3-channel sector. Then ε_ρ has spectral decomposition of the form*

$$\varepsilon_\rho = (-)\omega_\rho^{-1} e \pm q^{-1} f_1 + q f_2,$$

where $q \in \mathbf{T}$ is given by $(q \pm q^{-1})(d(\rho) \pm 1) = \omega_\rho \pm \omega_\rho^{-1}$, (resp. $(q \pm q^{-1})(d(\rho) \mp 1) = \omega_\rho \pm \omega_\rho^{-1}$).

The statistics braid group representation $\varepsilon_\rho^{(n)}$ is equivalent to a representation of Birman–Wenzl and Murakami [26, 28] tensored by a one dimensional representation.

Proof. By Proposition 7.2 ε_ρ has a spectral decomposition of the form

$$\varepsilon_\rho = \omega_\rho^{-1} e + t^{-1} f_1 + q f_2$$

(we assume ρ is real, the pseudoreal case is analogous) for some t, $q \in \mathbf{T}$. We show now that $t = \pm q^2$. In fact

$$e = A\varepsilon_\rho + B\varepsilon_\rho^{-1} + C \tag{7.3}$$

for some constants A, B, C, because

$$(\omega_\rho - t)(\omega_\rho^{-1} - q)e = (\varepsilon_\rho^{-1} - t)(\varepsilon_\rho - q). \tag{7.4}$$

Evaluating the left inverse on both sides of (7.3) yields

$$\frac{1}{d(\rho)} = A\omega_\rho + B\omega_\rho^{-1} + Cd(\rho). \tag{7.5}$$

Moreover multiplying both sides of (7.3) by e and f_2 respectively one obtains

$$1 = A\omega_\rho^{-1} + B\omega_\rho + C, \tag{7.6}$$

$$0 = Aq + Bq^{-1} + C. \tag{7.7}$$

Since $\varepsilon_\rho^{\pm 1}\rho(\varepsilon_\rho)\varepsilon_\rho^{\mp 1} = \rho(\varepsilon_\rho^{\pm 1})\varepsilon_\rho\rho(\varepsilon_\rho^{\mp 1})$ one also has

$$\varepsilon_\rho^{\pm 1}\rho(e)\varepsilon_\rho^{\mp 1} = \rho(\varepsilon_\rho^{\pm 1})e\rho(\varepsilon_\rho^{\mp 1})$$

and multiplying both members of this equality by e from the left and by f_2 from the right one obtains

$$q^{\pm 1}e\rho(e)f_2 = \frac{1}{d(\rho)}e\rho(\varepsilon_\rho^{\mp 1})f_2$$

that, together with equation (7.3), yields

$$Aq^{-1} + Bq = \frac{1}{d(\rho)}. \tag{7.8}$$

Elementary calculations show that equations (7, 5–8) and the equation analogous to (7.8)

$$At + Bt^{-1} = \frac{1}{d(\rho)}$$

admit only the stated solutions (unless ρ degenerates to a 2-channel sector). It follows that the g_i, e_i in (7.1), (7.2) generate the Birman–Wenzl algebra [26]. The rest is now clear. ■

Restrictions on the possible values of the spectrum of ε_ρ are discussed in [27].

We consider now the Kauffman link invariant polynomial $K_L(t,s)$ [29] where we choose the variables as in [26].

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[2] We thank H. Wenzl for privately mentioning to us the validity of this equality in the course of our computations and K. H. Rehren for pointing out to us that our original proof was lacking equation (7.8) and for correcting signs together with B. Schroer, see also [36]

7.9 Corollary. *Let ρ be an irreducible (pseudo)-real self-conjugate 3-channel sector. Then*

$$K_L(t,s) = (-d(\rho))^{n-1}(-\omega_\rho)^{-l}\phi^\infty(\varepsilon_\rho^{(n)}(\alpha)),$$

where K_L is the Kauffman polynomial, $t = -\sqrt{\pm 1}\ \omega_\rho$ $(t = \sqrt{\pm 1}\omega_\rho)$ and $(d(\rho)-1)s = -(t+t^{-1})$.

Proof. By the analysis in [26] the Kauffman polynomial is given by

$$K_L(t,s) = (s^{-1}(t+t^{-1})-1)^{n-1}t^{-1}\,\mathrm{tr}(\pi(\alpha)),$$

where π is the representation of $\mathbf{B}_n$ given by $\pi(\sigma_i) = \sqrt{\pm 1}g_i$ (real case). An elementary calculation gives the formula in the statement. The pseudoreal case is analogous. ■

We mention that in the case of a two channel non-selfconjugate sector ρ, since ε_ρ satisfies a second order equation $\varepsilon_\rho^2 + r\varepsilon_\rho + s = 0$, one obtains by using the analysis in [7, 6] the formula (independent of Corollary 7.7)

$$P_L(t,x) = (-d(\rho))^{n-1}(-\omega_\rho)^{-l}\phi^\infty(\varepsilon_\rho^{(n)}(\alpha)),$$

where P_L is the two variable link invariant polynomial determined by the skein rule [7] at $t = -i\sqrt{s}\omega_\rho$ and $x = -i(r/\sqrt{s})$ (we use the notations in [13]).

Let now $[\rho]$ be an arbitrary 3-channel sector such that $\rho^2 = \alpha\oplus\rho_1\oplus\rho_2$ with α an automorphism. Then $\alpha^{-1}\rho^2 = id\oplus\alpha^{-1}\rho_1\oplus\alpha^{-1}\rho_2$ contains the identity, therefore the conjugate of ρ is $\bar\rho = \alpha^{-1}\cdot\rho$, cf. also [35]. If e is the projection in $\rho(\mathscr{A})'$ corresponding to α, as before the projections $e_i = \rho^{i-1}(e)$ satisfy the Jones relations and generate together to the $g_i \equiv \rho^{i-1}(\varepsilon_\rho)$ the Birman–Wenzl algebra by the argument in Theorem 7.8. Denoting by θ the spectral value of ε_ρ,

$$\varepsilon_\rho e = \theta e,$$

the representation of $\mathbf{B}_n$

$$\sigma_i \to z g_i$$

with $z = (\theta\omega_\rho)^{-1/2}$ is again equivalent to a Birman–Wenzl–Murakami representation and the statistics is given by the Kauffman polynomial.

Notice that if ρ is a sector with $d(\rho) < 2$ then ρ has at most 3 channels in which case one component of ρ^2 must have statistical dimension one, namely it is an automorphism. The same argument extends to the case of a larger, but sufficiently small, statistical dimension.

8. Comments

We briefly comment here on the structure of the inclusions of type III factors $\rho(\mathscr{A}(\mathscr{O})) \subset \mathscr{A}(\mathscr{O})$ considered in this paper, having the knowledge of the fusion rules matrices N_{ij}^l of the given QFT model. Notice that these inclusions are not built up from commuting squares, although there might be a different construction of this sort. The matrices N_{ii}^l determine the inclusion matrices of the relative commutant tower $\rho^n(\mathscr{A}(\mathscr{O}))' \cap \mathscr{A}(\mathscr{O})$ (and of the tower in Corollary 4.14) hence the inclusion matrices for the relative commutants in the Jones tower (Theorem 4.1). The corresponding Ocneanu principal graph [20] is an invariant for the sector. An illustrating example where fusion rules are formally known is the $SU(2)_k$ Wess–Zumino–Witten model: there are $k+1$ irreducible sectors $\rho_0, \rho_1, \ldots, \rho_k$ with fusion rules

$$\rho_l \rho_i = \bigoplus_j N^l_{ij} \rho_j = \rho_m \oplus \rho_{m+2} \oplus \cdots \oplus \rho_n$$

with $m = |l - i|$, $n = \min(i + l, 2k - i - l)$ [30]. The index of $\rho_l(\mathscr{A}(\mathcal{O})) \subset \mathscr{A}(\mathcal{O})$ is $d(\rho_l)^2$ [18] and $d(\rho_l) = \sin((l + 1/k + 2)\pi)/\sin(\pi/k + 2)$ [31]. WZW models based on compact lie groups G generalize this construction [31]. The emerging structure is similar to the one analyzed in [27]. We shall return with more details on this structure somewhere else. It is also a natural problem to explain in this context the A–D–E classification of minimal models.

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References

1. Combes, F., Delaroche, C.: Groupe modulaire d'une esperance conditionnelle dans une algèbre de un Neumann. Bull. Soc. Math. France **103**, 385–426 (1975)
2. Connes, A.: Periodic automorphisms of the hyperfinite factor of type II. Acta Sci. Math. **39**, 39–66 (1977)
3. Connes, A.: On the spatial theory of von Neumann algebras. J. Funct. Anal. **35**, 153–164 (1980)
4. Connes, A.: unpublished; see Connes, A., Jones, V. R. F.: Property T for von Neumann algebras. Bull. Lond. Math. Soc. **17**, 57 (1985); Popa, S.: Correspondences, preprint; Sauvageot, J. L.: Sur le produit tensoriel relatif de'espaces de Hilbert. J. Oper. Th. **9**, 237–252 (1983)
5. Doplicher, S., Haag, R., Roberts, J. E.: Local observables and particle statistics. I, Commun. Math. Phys. **35**, (1971), 199–230; II, Commun. Math. Phys. **35**, 49–85 (1974)
6. Fredenhagen, K., Rehren, K. H., Schroer, B.: Superselection sectors with braid group statistics and exchange algebras, I: General theory. Commun. Math. Phys. **125**, 201–226 (1989)
7. Freyd, P., Yetter, D., Hoste, J., Lickorish, W., Millet, K., Ocneanu, A.: A new polynomial invariant for knots and links. Bull. AMS **12**, 103–111 (1985)
8. Fröhlich, J.: Statistics of fields, the Yang–Baxter equation, and the theory of links and knots. In: Nonperturbative quantum field theory. New York: Plenum Press (1988); Rehren, K. H., Schroer, B.: Einstein causality and Artin braids. Nucl. Phys. **312**, 3 (1989) Tsuchiya, A., Kanie, Y.: Vertex operators in conformal field theory on $\mathbb{P}^1$ and monodromy representations of braid groups. Adv. Studies Pure Math. **16**, 297–372 (1988)
9. Haagerup, U.: Operator valued weights in von Neumann algebras. I, II, J. Funct. Anal. **32**, 175–206 (1979), **33**, 339–361 (1979)
10. Kosaki, H., Longo, R.: work in progress
11. Kosaki, H.: Extension of Jones theory on index to arbitrary factors. J. Funct. Anal. **66**, 123–140 (1986)
12. Jones, V. R. F.: Index for subfactors. Invent. Math. **72**, 1–25 (1983)
13. Jones, V. R. F.: Hecke algebra representations of braid groups and link polynomials. Ann. Math. **126**, 335–388 (1987)
14. Jones, V. R. F.: Braid groups, Hecke algebras and type II_1, factors. In: Geometric Methods in Operator Algebras. Proc. US–Japan seminar 242–273
15. Loi, P. H.: On the theory of index and type III factors. Penn. State Univ. Thesis (1988)

16. Longo, R.: Solution of the factorial Stone–Weierstrass conjecture. Invent. Math. **76**, 145–155 (1984)
17. Longo, R.: Simple injective subfactors. Adv. Math. **63**, 152–171 (1987)
18. Longo, R.: Index of subfactors and statistics of quantum fields. I. Commun. Math. Phys. **126**, 217–247 (1989)
19. Longo, R.: Inclusions of von Neumann algebras and quantum field theory. In IXth International Congress on Mathematical Physics. Simon, B., Truman, A., Davis, I. M. (eds.), Swansea, July 1988, Bristol, New York: Adam Hilger
20. Ocneanu, A.: Quantized groups, string algebras and Galois theory for algebras. Lond. Math. Soc., Lecture Notes vol. **136**. Evans, D., Takesaki., M. (eds.) pp. 119–172 (1989)
21. Pimsner, M., Popa, S.: Entropy and index for subfactors. Ann. Sci. Ec. Norm. Sup. **19**, 57–106 (1986)
22. Powers, R. T.: An index theory for semigroups of endomorphisms of B(H) and type II_1 factors. Can. J. Math. **40**, 86–114 (1988)
23. Temperley, H., Lieb, E.: Relation between the percolation and colouring problem. Proc. R. Soc. (London) 251–280 (1971)
24. Wenzl, H.: Hecke algebras of type A_n and subfactors. Invent. Math. **92**, 349–383 (1988)
25. Witten, E.: Quantum Field Theory and the Jones polynomial. Commun. Math. Phys. **121**, 351–399 (1989)
26. Birman, J. S., Wenzl, H.: Braids, link polynomials and a new algebra. Trans. AMS **313**, 249–273 (1989)
27. Wenzl, H.: Quantum groups and subfactors of type B, C and D, preprint
28. Murakami, J.: The Kauffman polynomial of links and representation theory. Osaka J. Math. **24**, 745–758 (1987)
29. Kauffman, L. H.: An invaraint of regular isotopy. Trans. AMS (to appear); Kauffman, L. H.: Statistical mechanics and the Jones polynomial. In: Braids. Cont. Math. **78**, 263–297 (1988)
30. Gepner, D., Witten, E.: String theory on group manifolds. Nucl. Phys. B278, 493–549 (1986)
31. Verlinde, E.: Fusion rules and modular transformations in 2D conformal field theory. Nucl. Phys. **B300**, 360 (1988)
32. Frohlich, J., King, C.: Two-dimensional conformal field theory and three-dimensional topology. Preprint
33. Wasserman, A. J.: Coactions and Yang–Baxter equations for ergodic actions and subfactors. London Math. Soc. Lecture Notes vol. 136. Evans, D., Takesaki, M. (eds.) pp. 173–193, 1988
34. Capelli, A., Itzykson, C., Zuber, J.B.: The A–D–E classification of minimal and $A_1^{(1)}$ conformal invariant theories. Commun. Math. Phys. **113**, 1–26 (1987)
35. Chode, M.: Entropy for canonical shifts. Preprint
36. Reheren, K.H.: Braid group statistics and their superselection structures. Preprint

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