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Multiloop Calculations in P-adic String Theory and Bruhat-Tits Trees

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Abstract. We treat the open p-adic string world sheet as a coset space $F = T/\Gamma$, where T is the Bruhat-Tits tree for the p-adic linear group $GL(2, \mathbb{Q}_p)$ and $\Gamma \subset PGL(2, \mathbb{Q}_p)$ is some Schottky group. The boundary of this world sheet corresponds to a p-adic Mumford curve of finite genus. The string dynamics is governed by the local gaussian action on the coset space F. The tachyon amplitudes expressed in terms of p-adic θ-functions are proposed for the Mumford curve of arbitrary genus. We compare them with the corresponding usual archimedean amplitudes. The sum over moduli space of the algebraic curves is conjectured to be expressed in the arithmetic surface terms. We also give the necessary mathematical background including the Mumford approach to p-adic algebraic curves. The connection of the problem of closed p-adic strings with the considered topics is discussed.

1. Introduction

The idea of a non-archimedean string proposed in the papers [1–4] has stimulated great activity in this field [5–10]. Different approaches were suggested. One of them treats both the string coordinates (and momenta) and the string amplitudes as complex- (or real-) valued functions, but the string world sheet variables as the p-adic numbers [3, 4]. This approach seems to be the most fruitful. At least, it was the only one which allows to obtain some non-trivial results and to compare them with the archimedean ones. For example, Freund, Olson and Witten [3, 4] have interpreted bosonic string amplitudes at the tree level of perturbation theory over the non-archimedean local field $\mathbb{Q}_p$ (p is a prime number) as integrals of some combinations of multiplicative characters on $\mathbb{Q}_p$ (it is very close logically to the definition of the corresponding amplitudes for the usual open string over the real field). They discovered a remarkable property of the theory, namely, the so-called "product formula" (see also [11]). That is, they have calculated the 4-point tachyon amplitude, $A_4^{(p)}$, which is given in the archimedean case by the Veneziano formula for $A_4^{(\infty)}$:

$$A_4^{(p)}(k_1, k_2, k_3, k_4) = \int_{\mathbb{Q}_p} dx |x|_p^{k_1 k_2} |1-x|_p^{k_1 k_3} , \tag{1.1}$$

where k_i are d-dimensional vectors, $k_i k_j$ denote the corresponding scalar products, and the condition $\sum_{i=1}^{4} k_i = 0$ is implied, together with the constraints $k_i^2 = 2$. Then it was demonstrated that (with appropriate regularization involved)

$$A^{(\infty)} \prod_p A^{(p)} = 1 , \tag{1.2}$$

where p runs over all prime numbers. Some non-trivial extensions of this formula also have been obtained [7].

Now some questions arise:

1. Whether it is possible to obtain the p-adic amplitudes (1.1) within the Polyakov approach?
2. Whether there exist any more formulas like Eq. (1.2)?
3. Whether there exists some formulation of the string theory taking into account the adelic (or arithmetic) structure of Eq. (1.2), i.e. the formulation which includes string theories over all p simultaneously?

The first question was answered at the tree level of perturbation theory. That is, the non-local action was proposed [8] which reproduces the Freund-Olson amplitudes:

$$S^{(p)}[\varphi] = \frac{p(p-1)}{4(p+1)\ln p} \int_{\mathbb{Q}_p} dx dx' \frac{[\varphi(x) - \varphi(x')]^2}{|x-x'|_p^2} . \tag{1.3}$$

Here dx is the additive Haar measure on $\mathbb{Q}_p$ and we only write the single scalar field $\varphi(x)$ for the simplicity. We know a similar object in archimedean string theory. It is "the effective action" governing the field dynamics on the boundary of the open string world sheet. Actually, this object is a secondary one, as it originates from the world sheet local action upon integrating out the field fluctuations in the interior of the world sheet.

It turns out that the situation is just the same in the non-archimedean case. Indeed, a natural analog of the p-adic world sheet was proposed by one of us [9, 10]. There was demonstrated that a discrete homogeneous space T (the so-called Bruhat-Tits tree) whose boundary is $\mathbb{Q}_p$, yields the correct analog of the interior of the open string world sheet. It was shown too that there exists a simple "lattice" local action (a kind of gaussian model) on the tree which produces the correct p-adic string amplitudes.

In this paper we propose a multiloop generalization of the above results. We consider the Bruhat-Tits tree which is an infinite homogeneous graph without cycles, each vertex being connected with exactly $p+1$ neighbours by edges of unit length, as p-adic zero genus Riemann surface. (We use the term "surface" in the Bruhat-Tits construction as it is a direct non-archimedean analog of the open string world sheet.) To produce p-adic Riemann surfaces of higher genera we should factorize the tree by some discrete (Schottky) groups. The surface obtained is the graph with cycles (their number is equal to the genus of the surface), the properties of this graph can be described by means of the so-called reduced graph, which is the finite subgraph containing only the cycles with crosspieces between them. It permits us to introduce the p-adic counterpart of the Jacobian, period matrix etc. All this machinery as well as the detailed description of the Bruhat-Tits tree is contained in Sect. 2.

Having a local action on the tree we can calculate the p-adic string amplitudes. This action turns out to be the discrete analog of the usual quadratic one. Thus in order to find the amplitudes one needs to construct the solutions to the Neumann problem. The Neumann boundary condition is imposed by the following reasons: The archimedean case teaches us that an algebraically non-closed local field plays the role of a boundary of an open string world sheet. We know that the open string is characterized by the Neumann boundary condition. The situation in the non-archimedean case is to be quite similar with the exception of one point: there exist a number of algebraic extensions of the field $\mathbb{Q}_p$ in contrast to the field $\mathbb{R}$, each extension corresponding to a different type of the open string.

The problem of finding the Green function with the Neumann boundary condition can be solved by two methods. The first is to find the solution to the Laplace equation. One of us followed this way in [12] where slightly modified (in comparison to this paper) notations have been used. The second way to calculate the Neumann function is to use the path integral approach. This method was developed in the general case of arbitrary lattice theory by Zinov'ev [13], and we apply it to produce the tachyon string amplitudes in Sects. 3.2 and 3.3.

The obtained results appear to have a very natural structure, Namely, the archimedean answer turns out to be just the same, with the usual norms, abelian differentials, period matrix, etc. being substituted by their p-adic counterparts (Sect. 3.3). Certainly, it is consistent with precedent predictions for genus 1 [14].

In this paper we calculate only the tachyon p-adic amplitudes, with no accounting of corresponding determinants which are to be introduced for the correct normalization. The way to correctly determine them as well as the p-adic amplitudes for the emission of the states with higher spins is unknown at this moment. The difficulties appearing here are discussed in Sect. 4. In particular the crucial role of the p-adic analog of the closed string (given by a string model over the complete algebraically closed field $\bar{\Omega}$) is pointed out.

We know that the archimedean Riemann surfaces, parametrized by the Schottky groups (over $\mathbb{R}$ or $\mathbb{C}$) uniformize all algebraic curves over archimedean fields ($\mathbb{R}$ or $\mathbb{C}$). This is not the case in p-adic theory. That is, the p-adic Schottky parametrized Riemann surfaces uniformize only some class of algebraic curves over the p-adic field, namely, the so-called Mumford curves. These curves are placed close (in p-adic sense) to that part of the moduli space boundary which corresponds to highly degenerate algebraic curves. This fact is of great importance and it is not caused by unfortunate parametrization (indeed, the Schottky parametrization is in a sense unique). Certainly, such a situation should be clarified anyway. A sketch of the resolution of this issue is also contained in Sect. 4. Finally, we briefly discuss a possible way to integrate over the moduli space of algebraic curves in order to obtain a kind of "arithmetic string partition function" and correctly normalized amplitudes. We formulate a conjecture in the spirit of the adelic string viewpoint and the product formula [6]. Indeed, by this consideration we tried to answer partially the third question above.

At last, it is interesting to generalize the relation (1.2) to the string amplitudes at higher levels of the perturbation theory proposed in this paper. It would answer the second question on the above list of problems. The work on this topic is in progress now. Some comments on these problems with several concluding remarks can be found in Sect. 5.

There are also two Appendices. The first contains the necessary facts about identification of the tree boundary with p-adic numbers. Appendix B is devoted to some analytic objects over $\mathbb{Q}_p$-fields like θ-functions. The properties of p-adic θ-functions and associated Prime forms for the Schottky groups are briefly reviewed.

This paper is though of as the second one in a series which began with the publication [10]. The rapid version of this work was presented [15].

2. Schottky Parametrization of the String World Sheet

In this section we briefly describe Riemann surfaces over the $\mathbb{Q}_p$ field. We only formulate the statements with some comments. The reader can easily find the necessary proofs and more material concerning this subject in Refs. [15–19]. We begin with the description of the Riemann surface over $\mathbb{R}$ in order to work an insight, as this case appears to be logically close to the $\mathbb{Q}_p$ one.

2.1. Schottky Groups

To begin with, we describe "real Riemann surfaces" ($\mathbb{R}RS$ for brevity) which are closely connected with algebraic curves over $\mathbb{R}$. The word "surface" may seem to be inadequate in this context, because here we deal with one dimensional objects over $\mathbb{R}$. Nevertheless, we shall use such terminology in order to attain an unification when working with different number fields. Besides it, the word "surface" refers to the algebraic curve uniformized by a Schottky group in an analytic domain. To obtain $\mathbb{R}RS$ one has to start with natural action of the $SL(2, \mathbb{R})$ group on the real axis (we always imply the compactified real axis $\hat{\mathbb{R}}$, or equivalently, projective line $P^1(\mathbb{R})$): $x \to \frac{ax+b}{cx+d}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ being $SL(2, \mathbb{R})$ matrix, $ad-bc=1$. We call $P^1(\mathbb{R})$ the zero genus $\mathbb{R}RS$; it determines all properties of the string model in the zero loop approximation. For example, one easily obtains correct expressions for the string amplitudes using the well-known non-local action (throughout this paper we consider for the simplicity the single string coordinate):

$$S=\frac{1}{2} \iint_{\mathbb{R}} dx dy \, \frac{[\varphi(x)-\varphi(y)]^2}{(x-y)^2} \,. \tag{2.1.1}$$

However, a more fruitful approach exists: we would obtain these amplitudes from a local action on some extended object which we call *extended* $\mathbb{R}RS$ ($E\mathbb{R}RS$), the $\mathbb{R}RS$ being $E\mathbb{R}RS$ boundary. The terms $\mathbb{R}RS$ and $E\mathbb{R}RS$ are not standard and are introduced in this paper for convenience. We shall also use the notations $\mathbb{Q}_pRS$

and $E\mathbb{Q}_pRS$ in a similar sense. To obtain $E\mathbb{R}RS$ one has to extend the $SL(2,\mathbb{R})$ action to upper half plane by clear means: $z \to \frac{az+b}{cz+d}$, $z \in \hat{\mathbb{C}}$, $\operatorname{Im} z \geqq 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Then the upper half plane is just $E\mathbb{R}RS$ (the open string world sheet). To make this subject more evident one can provide $E\mathbb{R}RS$ with $SL(2,\mathbb{R})$-invariant hyperbolic metric and treat it as homogeneous space of $SL(2,\mathbb{R})$ factorized by its maximal compact subgroup $SO(2)$. This object is called the hyperbolic plane. The local action on $E\mathbb{R}RS$ generating (2.1.1) is written as

$$S = \tfrac{1}{2} \int \partial\phi \bar{\partial}\phi d^2 z, \ z \in \text{hyperbolic plane} . \tag{2.1.2}$$

The same technique works for a genus one Riemann surface. In fact, one needs to fix two arbitrary real discs D_1 and D_2 at first stage. We choose them to be:

$$D_1 = \{x : |x| < \sqrt{q} | x \in \hat{\mathbb{R}}\}, \ D_2 = \{x : |x| > \sqrt{1/q} | x \in \hat{\mathbb{R}}\}, \ 0 < |q| < 1, q > 0, q \in \mathbb{R} . \tag{2.1.3}$$

Another choice of these discs corresponds to another fundamental domain of the Schottky group (see below). The complement of these discs to the whole $\hat{\mathbb{R}}$: $\hat{\mathbb{R}} \backslash \{D_1 \cup \bar{D}_2\}$ ($\bar{D}_2$ is closure of D_2) is a fundamental domain for the subgroup of $SL(2,\mathbb{R})$ generated by the matrix $\begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{1/q} \end{pmatrix} \in SL(2,\mathbb{R})$, which multiplies x by q and is the simplest example of the Schottky group Γ. So the coset space $\hat{\mathbb{R}}/\Gamma$ consists of two segments, both with identified endpoints, or, equivalently, of two circles S^1 whose lengths are equal to $\sqrt{1/q} - \sqrt{q}$ and are related to the moduli space parameter q. This set is just an example of $\mathbb{R}RS$. In this case $E\mathbb{R}RS$ having this $\mathbb{R}RS$ as a boundary is a cylinder. The extended fundamental domain of Γ is depicted in Fig. 1. (In fact, one has to identify the boundaries of fundamental domain, resulting in the cylinder.) Indeed, the $E\mathbb{R}RS$ is the coset of hyperbolic plane by Γ, so $SL(2,\mathbb{R})$ acts naturally on $E\mathbb{R}RS$.

Now we generalize the above construction to higher genera. At first, we define the Schottky group over a local field $\mathbb{K}$ (for convenience we don't specialize the field at this moment). Instead of $SL(2,\mathbb{Q}_p)$ we would like to consider a slightly modified group, namely $PGL(2,\mathbb{K})$ which is determined to be equivalence classes of $GL(2,\mathbb{K})$ matrices with respect to multiplication by a non-zero element. (This

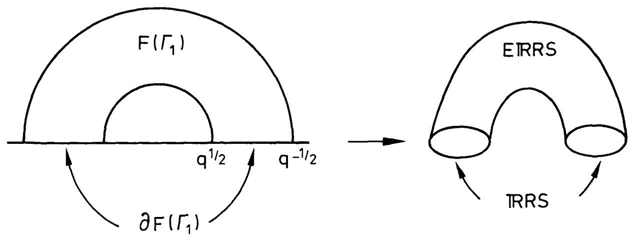

Fig. 1. Fundamental domains $F(\Gamma_1)$, $\partial F(\Gamma_1)$ and the half-torus ($E\mathbb{R}RS$) in the archimedean case

extension corresponds to a pair of uncoupled planes in the archimedean case and doesn't affect the results.) Certainly, we can consider the $SL(2,\mathbb{K})$ construction as well, but we prefer $PGL(2,\mathbb{K})$-group as a more natural one from the Bruhat-Tits viewpoint.

The natural $PGL(2,\mathbb{K})$ action on $P^1(\mathbb{K})$ (which is just zero genus $\mathbb{K}RS$) is given in homogeneous coordinates (x_0,x_1) by $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,\mathbb{K})$, or, equivalently, $z \to \dfrac{az+b}{cz+d}$, $z \in \hat{\mathbb{K}}$. Note that $PGL(2,\mathbb{K})$ exhausts all possible automorphisms of $P^1(\mathbb{K})$.

An abstract Schottky group Γ is, by definition, a free discrete subgroup of $PGL(2,\mathbb{K})$ with a finite number of generators $\gamma_i \in PGL(2,\mathbb{K})$, $i=1,\dots,g$. In fact, all non-unit elements of Γ are hyperbolic, i.e. corresponding $PGL(2,\mathbb{K})$-matrices have to have different moduli of their eigenvalues. Each generator γ_i is defined by three parameters. We choose them to be two (distinct) fixed points u_i, v_i ($\gamma_i(u_i)=u_i$, $\gamma_i(v_i) = v_i$) and the coefficient $\mathcal{K}_i$. They define $\gamma(z)$ by the equation:

$$\frac{\gamma_i(z)-u_i}{\gamma_i(z)-v_i} = \mathcal{K}_i \frac{z-u_i}{z-v_i} \ . \tag{2.1.4}$$

Explicit parametrization of γ_i is

$$\gamma_i = \frac{1}{v_i-u_i} \begin{pmatrix} v_i\mathcal{K}_i - u_i & v_i u_i(1-\mathcal{K}_i) \\ \mathcal{K}_i - 1 & v_i - u_i\mathcal{K}_i \end{pmatrix} .$$

For any local field $\mathbb{K}$ one can extend the $PGL(2,\mathbb{K})$-action on $\hat{\mathbb{K}} \cong P^1(\mathbb{K})$ to the homogeneous space $\mathcal{H} = PGL(2,\mathbb{K})/G$, G being the maximal compact subgroup of $PGL(2,\mathbb{K})$. For example, $PGL(2,\mathbb{C})$-action can be extended from $\mathbb{C}$ to the upper half space (see for ex. [18]). A fundamental domain for the Schottky group Γ acting on the "extended" space $\mathcal{H}$ is denoted by $F(\Gamma)$ and a corresponding fundamental domain on $\Xi \equiv P^1(\mathbb{K})\backslash\Sigma(\Gamma)$ is denoted by $\partial F(\Gamma)$, $\Sigma(\Gamma)$ being the set of all limit points of Γ, i.e. the closure of limit points of all non-unit elements $\gamma \in \Gamma$. These fundamental domains are imbedded into the covering spaces $\mathcal{H}$ and Ξ respectively. We shall deal with the coset spaces $F \equiv \mathcal{H}/\Gamma$ and $\partial F \equiv \Xi/\Gamma$ which can be produced from $F(\Gamma)$ and $\partial F(\Gamma)$ by an appropriate glueing. The notation ∂F refers to the fact that ∂F can be realized as the boundary of F (see the above example for the archimedean case and the constructions of Subsect. 2.2 for non-archimedean local fields).

Let us present an explicit construction of the so-called classical Schottky groups acting on $\hat{\mathbb{K}}$. Consider $2g$ open discs $\mathcal{G}_{1i}, \mathcal{G}_{2i} \in \hat{\mathbb{K}}$ ($i=1,\dots,g$) such that their closures $\bar{\mathcal{G}}_{1i}$ and $\bar{\mathcal{G}}_{2i}$ do not intersect and have the same radius r_i. Then we can construct a Schottky group Γ_g with g generators γ_i corresponding to the pairs $\bar{\mathcal{G}}_{1i}$, $\bar{\mathcal{G}}_{2i}$ as follows: $\gamma_i^{-1}(\hat{\mathbb{K}}\backslash\bar{\mathcal{G}}_{2i}) = \mathcal{G}_{1i}$, $\gamma_i(\hat{\mathbb{K}}\backslash\bar{\mathcal{G}}_{1i}) = \mathcal{G}_{2i}$, $r_1 = 1/|\mathcal{K}_i^{1/2} - \mathcal{K}_i^{-1/2}|$ and $u_i \in \mathcal{G}_{1i}$, $v_i \in \mathcal{G}_{2i}$. Then

$$\partial F(\Gamma_g) = \hat{\mathbb{K}} \backslash \bigcup_{i=1}^{g} \{\bar{\mathcal{G}}_{2i} \cup \mathcal{G}_{1i}\} \ , \tag{2.1.5}$$

is a fundamental domain of the classical Schottky group Γ_g. For $g>1$ we suppose that the discs do not contain the point ∞. Throughout this paper we only deal with groups of this type, though we know that non-classical groups could be important in the uniformization problem when $\mathbb{K}$ is specialized as $\mathbb{C}$.

Thus any $\mathbb{K}RS$ is represented by ∂F. In particular, $\mathbb{R}RS$ is a collection of $g+1$ disconnected circles S^1, the connection between those and moduli space being far from obvious. In fact, in the cylinder case we have observed that two circles are described by a unique moduli parameter q. But one can cut the cylinder in its middle part by a proper closed curve (the so-called invariant axis [21], see also below), which immediately describes moduli space. Such a procedure can be easily generalized to higher genus extended Riemann surface. In $\mathbb{Q}_p$ case we shall introduce a "reduced graph" (1-chain complex) corresponding to the moduli space in a more direct way, this graph being just the analogue of the cut described above. Now extend the action of Γ from $\mathbb{R}$ to the upper half plane, the boundaries of $\mathcal{G}_{1i}$, $\mathcal{G}_{2i}$ being extended to half circles on it. Then we produce the fundamental domain $F(\Gamma)$ which is just $E\mathbb{R}RS$ (after glueing) and it can be interpreted as the coset of the hyperbolic plane by Γ.

From the one hand, the life on $E\mathbb{R}RS$ has an essential advantage, namely, the locality of the string action which has the form (2.1.2) on $E\mathbb{R}RS$. But $E\mathbb{R}RS$ itself is a rather complicated object. On the other hand, $\mathbb{R}RS$, being the simple one-dimensional object, should be provided with non-local action in order to reproduce the correct string amplitudes [22]. In Sect. 4 we shall present some more arguments in favour of an "extended" viewpoint. It seems to be quite natural since it is the extended object that is analogous to the string world sheet in the archimedean case.

2.2. Riemann Surface Over $\mathbb{Q}_p$ and Bruhat-Tits Tree

It is just the time to describe the construction of the Riemann surfaces over $\mathbb{Q}_p$ field. To obtain the extended zero genus $\mathbb{Q}_pRS$ one has to factorize $PGL(2, \mathbb{Q}_p)$ by its maximal compact subgroup, that is $PGL(2, \mathbb{Z}_p)$, being determined as 2×2 matrices with p-adic integer entries and invertible determinant in $\mathbb{Z}_p$. It is this homogeneous space that is called the Bruhat-Tits tree T. It is manifestly determined to be the connected infinite graph with no loops; each vertex of T being connected with $p+1$ neighbour vertices by edges. Obviously, any two vertices z_1, z_2 in the tree are connected by exactly one path $z_1\to z_2$. We define the *distance* $d(z_1, z_2)$ between these vertices to be the number of edges in the path $z_1\to z_2$.

There are half-axes in the tree which are infinite subtrees with no branch points but with a single starting point (Fig. 2). We introduce an equivalence relation for half-axes: two half-axes given by an infinite sequence of vertices $\{z_1, z_2, ..\}$ and $\{z'_1, z'_2, ...\}$ are equivalent if $\exists k, n\in\mathbb{Z}$: $z_j=z'_{j+n}\,\forall j\geqq k$. We call the equivalence classes the rays. Then the tree T can be compactified by adding the set of "infinitely far points" ∂T defined as the set of all rays. In fact, ∂T can be canonically identified with $P^1(\mathbb{Q}_p)$. On the other hand, the $PGL(2, \mathbb{Q}_p)$-action can be naturally extended to ∂T from the tree T, so we shall consider ∂T as the boundary of a compactified tree $T\cup\partial T$ with $PGL(2, \mathbb{Q}_p)$-action on it.

In order to "coordinatize" $P^1(\mathbb{Q}_p)$ we fix a point C (the "origin") in T. This vertex corresponds to three half-axes starting at C whose endpoints in $\partial T\cong P^1(\mathbb{Q}_p)$ are $(0, 1, \infty)$, by definition. Then we can identify $P^1(\mathbb{Q}_p)$ with $\hat{\mathbb{Q}}_p$ and after fixing C only $PGL(2, \mathbb{Z}_p)$-freedom remains (for details see [16, 17] and Appendix A). Now C

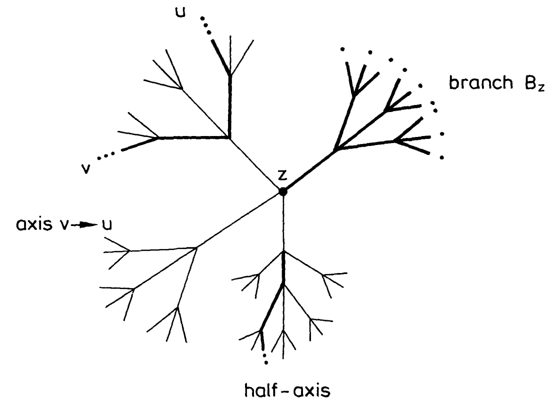

Fig. 2. Subgraphs in the tree: half-axis, axis $v \to u$ and branch B_z

is the fixed point of $PGL(2, \mathbb{Z}_p)$ and we describe the $PGL(2, \mathbb{Q}_p)$-action on T manifestly as follows: $PGL(2, \mathbb{Q}_p)$ acts on $T = PGL(2, \mathbb{Q}_p)/PGL(2, \mathbb{Z}_p)$ transitively and isometrically (i.e. the distances $d(\cdot, \cdot)$ are conserved); vertices correspond to $\mathrm{g}PGL(2, \mathbb{Z}_p)\mathrm{g}^{-1}$ subgroups, $\mathrm{g} \in PGL(2, \mathbb{Q}_p)$, and edges correspond to $\mathrm{g}H\mathrm{g}^{-1}$ subgroups, $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{Z}_p$, $c \in \mathscr{M}$ and $ad \notin \mathscr{M}$, where $\mathscr{M}$ is the unique maximal ideal in the ring $\mathbb{Z}_p$ (i.e. $\mathscr{M} = \{x \in \mathbb{Q}_p : |x|_p < 1\}$).

Let us define a *branch* B_z to be an entire subgraph of T with the only boundary point z in the interior of T. The graph B is called entire if $B \backslash \partial B$ is a connected graph (Fig. 2) (this definition differs slightly from the one given in [10]). In what follows, we assume that the branches contain no cycles (in the case of the factorized tree also). Then the set of rays contained in B_z corresponds to an open domain ∂B_z in ∂T and induce a natural topology on $\hat{\mathbb{Q}}_p$.

Thus we have described the zero genus $E\mathbb{Q}_pRS$. Note that as above the correct string amplitudes may be produced from the non-local action on $\mathbb{Q}_pRS$ [8] (1.3) and from the local action on $E\mathbb{Q}_pRS$ [9] which shall be described in Sect. 3.1, see formula (3.1.12). It is also valid for higher genus surfaces [23].

Now introduce $\mathbb{Q}_pRS$ and $E\mathbb{Q}_pRS$ of higher genera. We consider a Schottky group Γ over $\mathbb{Q}_p$ acting on T. Then for any hyperbolic element $\gamma \in \Gamma$ the only γ-invariant axis in T exists which is called the γ-axis (by definition, an axis is an infinite connected subtree with no branch vertices and terminating vertices inside T (Fig. 2)). Each element γ acts by shifts along the corresponding γ-axis, and for any γ conjugated to the element $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, $0 < |q|_p \equiv p^{-\mathrm{ord}_p q} < 1$, the shift is equal to $\mathrm{ord}_p q$. So the hyperbolic element has no invariant vertices and edges. Another description of the γ-axis uses the fact that axis endpoints in ∂T are fixed points of γ, and any two such points are connected by the only path in the tree. Obviously, this path is the invariant γ-axis. More precisely, the action of γ on the whole tree is the shift along γ-axes defined by $|q|_p$ with a simultaneous "rotation" around γ-axes defined by the "phase" of q, i.e. by $q|q|_p$.

We define the Schottky tree $T(\Gamma)$, which is a union of axes of all elements of Γ and crosspieces between them (crosspiece is the unique finite path which has common vertices but not edges with two chosen axes), or, equivalently, a minimal connected subgraph containing the axes of all elements of Γ (let us recall that the *composition of any two hyperbolic elements* $\gamma_1, \gamma_2 \in \Gamma$ *is again a hyperbolic element* unless $\gamma_1 = \gamma_2^{-1}$). Then $T(\Gamma) \subset T$, $\partial T(\Gamma) \subset \partial T$ and $T(\Gamma)/\Gamma \equiv F^R$ is a *finite* graph, which is called the *reduced graph* (sometines we add index g which labels the number of loops in the corresponding graph). Let us demonstrate a way in which the reduced graph permits us to construct $E\mathbb{Q}_pRS$ from the Bruhat-Tits tree. As the first step we consider the simplest example, namely, the torus. We choose an element γ generating Γ_1 in the form:

$$\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 < |q|_p < 1 \; . \tag{2.2.1}$$

The γ-axis passes through the origin C, the fixed points $u, v \in \hat{\mathbb{Q}}_p = \partial T$ are zero and infinity and any element of Γ_1 acts on the axis $\infty \to 0$ by Δ-shifts, $\Delta = 0 \bmod \mathrm{ord}_p q$ (see Fig. 3a). So the Schottky tree is a single axis, F_1^R is a cycle (ring) consisting of $m \equiv \mathrm{ord}_p q$ edges. To obtain $E\mathbb{Q}_pRS$ we factorize T by $\Gamma_1 : F_1 \equiv T/\Gamma_1$ (Fig. 3b). The

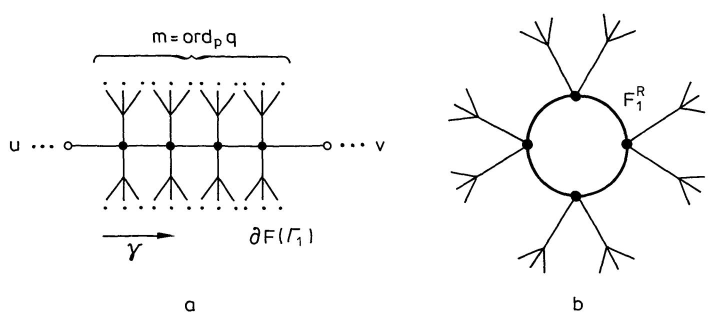

Fig 3 a–b. A typical fundamental domain of the Schottky group Γ_1 in the tree. **b.** The "extended" p-adic torus (in the case of $p = 3$)

result may be realized as a fundamental domain for Γ_1 glued into a ring. Correspondingly, it induces the isomorphism $\partial F_1 \equiv \partial(T/\Gamma_1) \simeq \partial T/\Gamma_1$. So far we have seen that F_1^R can be produced from F_1 by truncating all branches with origins at the reduced graph. It appears to be the general procedure for the surface of arbitrary genus. The inverse operation is clear as well: to construct $E\mathbb{Q}_pRS$ $F_g \equiv T/\Gamma_g$ it is necessary to draw the reduced graph with a given number of loops (this number g is equal to genus of the surface) and after that to add all the necessary branches with origins at the vertices of the reduced graph in an evident way (Fig. 4a and b). If one treats identity transformation as the trivial Schottky group Γ_0, then F_0^R is merely a single vertex (denoted above as C), and $T/\Gamma_0 = T$.

Now one can see that the reduced graph consists of $3g-3$ or less segments [segment in F_g^R is the line containing only 2-vertices and connecting two branching vertices in F_g^R (see Fig. 4a)]. We denote s_i the lengths of these segments. In fact, they are "*the moduli*" of the corresponding p-adic surface. Strictly speaking, these

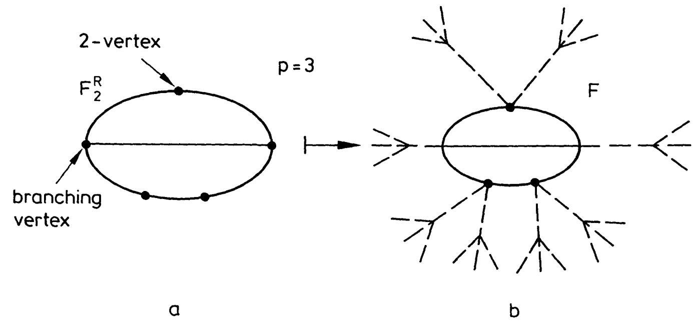

Fig. 4 a–b. A reduced graph for g=2. **b.** The whole factorized tree (g=2, $p=3$)

parameters are p-adic orders of the moduli of an algebraic curve, but we shall call them merely moduli. Thus F_g^R provides a good description of the moduli space of Riemann surfaces over $\mathbb{Q}_p$. The structure of F_g^R contains all necessary information about ∂F_g which consists of several different components, each of them being isomorphic to an open set in $P^1(\mathbb{Q}_p)$ (i.e. the boundary ∂B_z of a branch B_z in F_g) which in turn is defined as a boundary ∂B of some branch B in F_g. So non-trivial dependence on moduli is gathered into F_g^R, but all properties at small distances on $P^1(\mathbb{Q}_p)$ (moduli space of the punctures) are determined by geometry of a single boundary component. Thus if we wish to restrict string amplitudes produced from the local action on F to ∂F we should take into account only the reduced graph when all points belong to different components of ∂F and, quite contrary, only structure of the boundary component is important in the case when points belong to one component of ∂F. In particular, for zero genus surface the reduced graph shrinks into the point and we deal with the boundary of the whole tree (the non-local action of [8] is given on the whole $\hat{\mathbb{Q}}_p$!).

To conclude this section we introduce some analytic constructions on $E\mathbb{Q}_pRS$. Generally speaking, they are well-defined only over algebraically closed field $\bar{\Omega} \supset \mathbb{Q}_p$ but we shall restrict them to $\mathbb{Q}_p$. Here we consider the simplest objects (as the period group and the Jacobi map) leaving θ-functions and Prime forms to Appendix B.

Given a Schottky group Γ, we define the abelian group $H=\Gamma/[\Gamma,\Gamma]$, where $[\Gamma,\Gamma]$ is the commutant of Γ. Then the scalar product in H is given by:

$$(\chi_i,\chi_j)=\prod_{\gamma\in G_i\backslash G/G_j}\{u_i,v_i,\gamma u_j,\gamma v_j\}\ ,\qquad i\neq j\ , \tag{2.2.2a}$$

$$(\chi_i,\chi_i)=K_i\prod_{\gamma\in G_i\backslash G/G_i}\{u_i,v_i,\gamma u_i,\gamma v_i\}\ . \tag{2.2.2b}$$

Here χ_i,χ_j are classes of γ_i and γ_j in H ($(\cdot,\cdot)$ depends only on classes of elements) and classes of equivalence $G_i\backslash G/G_j$ can be parametrized by the elements $\gamma=\gamma_{i_1}^{j_1}\cdot\ldots\cdot\gamma_{i_k}^{j_k}, j_l\neq 0, i_1\neq i, i_k\neq j$; $\{a,b,c,d\}\equiv\frac{a-c}{a-d}\cdot\frac{b-d}{b-c}$ is the cross-ratio. Now we can construct the period group B by the isomorphism:

$$\varphi\colon H\to B\ ,\qquad \varphi(\chi)\equiv(\cdot,\chi)\ . \tag{2.2.3}$$

Here B is a discrete subgroup of the p-adic g-dimensional torus $(\mathbb{Q}_p^*)^g$ ($\mathbb{Q}_p^*$ is the multiplivative group of $\mathbb{Q}_p$) and the elements of H are naturally identified with the functions on $(\mathbb{Q}_p^*)^g$. The factor $(\mathbb{Q}_p^*)^g/B$ is the "p-adic Jacobian". One can check the correctness of these definitions [16].

At last, we observe the connection between H, B and the reduced graph F^R. It is important that these objects depend only on F^R. It again confirms the separating of moduli dependence from the dependence connected with the geometry of each component of ∂F. Let us introduce on T the "F^R-restricted intersection index" of two paths $\langle x_1 \to x_2, y_1 \to y_2 \rangle_R$, where the notation $x \to y$ denotes the oriented path from x to y. This index is merely the number of common edges (accounting for the orientation) which, besides it, belong to the reduced graph F^R. Every Schottky generator γ corresponds to a cycle $\mathscr{X}(\gamma)$ in F^R. Thus we have a natural scalar product $\langle \mathscr{X}(\chi_i), \mathscr{X}(\chi_j) \rangle_R$ on the abelian group of 1-chains which is just $\mathrm{ord}_p(\chi_i, \chi_j)$ and it is a direct analog of the imaginary part of period matrix $\mathrm{Im}\, \tau_{ij}$ in the archimedean case. Note that the choice of Schottky generators γ_i determines the basis of cycles $\{\mathscr{X}(\gamma_i)\}$ in F^R and the period matrix as well:

$$\Lambda_{ij} = \langle \mathscr{X}(\gamma_i), \mathscr{X}(\gamma_j) \rangle_R \; . \tag{2.2.4}$$

(This definition differs from the one adopted in [15] by a sign.)

Finally, we have to describe the Jacobi map. It can be written in the Poincaré product form like (2.2.2) and its manifest expression is not necessary for us. But the p-adic order of this map, which is the counterpart of $\int_{z_0}^{z} \omega_i$ in the archimedean case ($\{\omega_i\}$ is a basis of the holomorphic sections of canonical line bundle on the surface), will be important in Sect. 3. So consider $E\mathbb{Q}_p RSF$ and fix a point $z_0 \in \partial F$ which maps into zero of the Jacobian. If $\{\gamma_i\}$ are Schottky generators which define the basis of cycles $\mathscr{X}(\gamma_i)$ in F^R then the order of the Jacobi map is given by:

$$\mathrm{ord}_p j_{z_0}(z)_i = \langle z \to z_0, \mathscr{X}(\gamma_i) \rangle_R \; , \qquad z \in \partial F \; . \tag{2.2.5}$$

This definition is correct as (2.2.5) is defined up to the choice of cycles. which results into p-adic order of the element of the period group, and Jacobi map is defined up to the period group torus, as in the ordinary case. This ambiguity does not affect the amplitudes (see Sect. 3). It is easy to check that $\mathrm{ord}_p j_{z_0}(z)_i$ really depends only on F^R. Thus $\mathrm{ord}_p B = \mathrm{ord}_p \varphi(\chi) = \mathrm{ord}_p(\cdot, \chi) = \langle \cdot, \chi \rangle_R$ [cf. (2.2.3)], the Jacobi map has the order given by (2.2.5). All these quantities are non-archimedean counterparts of the usual ones, and the string amplitudes are expressed in these terms.

3. Tachyon Emission Amplitudes from the Mumford Curve

In this section we obtain the N-point amplitude for arbitrary multiloop graph. The plan is the following: the section falls into three subsections. In the first one we fix notations and give general definitions, in the second one the 1-loop case will be considered in detail; the third part is devoted to the mutiloop case and to the comparison with the archimedean case.

3.1. The Basic Definitions

Passing in this section to unified notations we shall use the standard language of algebraic topology. Let us consider a factorized tree $F = T/\Gamma$ with the corresponding reduced graph F^R. We introduce the space $C_n (n = 0, 1, \ldots)$ of n-chains which are the formal linear combinations of the oriented elementary n-symplexes $u_i^{(n)}$ in the graph F, $u_i^{(0)}$ being the vertices z_i, $u_i^{(1)}$ being the edges e_i (generally speaking, one may consider symplexes of higher dimension): $\sum_i \xi_i u_i^{(n)}$, $\xi_i \in \mathbb{R}$. Further, one can define the linear space C_n^* of n-cochains which are the functions on C_n. We choose the basis $\{\eta_i^{(n)}\}$ in C_n^* such that $\eta_i^{(n)} \equiv \eta_j^{(n)}(u_i^{(n)}) = \delta_{ij}$ and define the scalar product:

$$\langle \eta_j^{(n)}, u_i^{(n)} \rangle = \delta_{ij} \ .$$

Now for arbitrary $\psi = \prod_i \gamma_i \eta_i \in C_n^*$, $\chi = \sum_i \xi_i u_i \in C_n$ (we omit index n) one obtains by linearity:

$$\langle \psi, \chi \rangle = \sum_i \xi_i \gamma_i \ . \tag{3.1.1}$$

Index i runs over all vertices in F for C_0 and over all edges for C_1. Later we shall approximate the graph F by finite subgraphs, so these sums are well-defined. Having these scalar products we identify the spaces C_n^* and C_n and denote elements of C_0 and C_1 as $\varphi(z)$ and $\psi(e)$ respectively.

Note that F^R-restricted intersection index (see Sect. 2.2) is indeed restricted from the scalar product (3.1.1):

$$\langle \psi_1, \psi_2 \rangle_R = \sum_{e_i \in F^R} \psi_1(e_i)\psi_2(e_i) \ . \tag{3.1.2}$$

To clarify this construction let us consider two examples: the first is the path $x \to y$ which we shall also denote $\mathscr{X}_{xy} \in C_1$, the corresponding function $\psi(e)$ equals ± 1 (the sign depends on the mutual orientation of the path $x \to y$ and each edge) for the edges contained in this path, and zero otherwise. A cycle $\mathscr{Z}(\gamma)$ is a function $\psi \in C_1$ which equals $+1$ for edges contained in this cycle with, say, clockwise orientation, -1 for opposite orientation and zero for edges outside of the cycle. Obviously, for cycles the scalar products (3.1.1) and (3.1.2) are identical.

For each vertex $z \in F$ we define the *distance* $d(z, F^R)$ (or simply $d(z)$) by the formula:

$$d(z) = \inf_{\omega \in F^R} d(z, \omega) \ . \tag{3.1.3}$$

In the case of the trivial Schottky group Γ_0 $d(z)$ coincides with $d(C, z)$ of Ref. [10]. Further, we should define a measure on the boundary ∂F. This measure μ will be defined completely as soon as the measure on the basis of open sets ∂B_z in ∂F is given:

$$\mu(\partial B_z) = p^{-d(z)-1} \ , \tag{3.1.4}$$

for any branch B_z.

Example. For g-loop graph F_g^R ($g>1$) with the moduli s_i, $i=1,\dots,3g-3$ the full measure of the boundary ∂F_g is:

$$\mu(\partial F_g)=\left(1-\frac{1}{p}\right)S-\left(1+\frac{1}{p}\right)(g-1)\ , \qquad S\equiv\sum_{i=1}^{3g-3} s_i\ . \tag{3.1.5}$$

Consider now the function $\varphi(z)\in C_0$. Suppose the limit

$$\partial_n\varphi(x)=\lim_{z\to x}\,(\varphi(x)-\varphi(z))p^{d(z)} \tag{3.1.6}$$

exists for a point $x\in\partial F$. In this case we shall call $\partial_n\varphi(x)$ the normal derivative of φ at the boundary point x.

Now we shall introduce more notations originating from (lattice) chain complexes theory. One can define the *coboundary map* $\partial^*: C_0\to C_1$ such that

$$\partial^*\varphi(e_r)=\varphi(z_1^{(r)})-\varphi(z_2^{(r)})\ . \tag{3.1.7}$$

$z_1^{(r)}$ and $z_2^{(r)}$ are endpoints of the edge e_r, the order in (3.1.7) depends on the orientation of the edge. The operator conjugated to ∂^* is the *boundary map* $\partial: C_1\to C_0$ which is defined as follows:

$$(\partial\psi)(z_0)=\sum_{i=1}^{p+1}(\pm 1)\,\psi(e_i^{(0)})\ , \tag{3.1.8}$$

the sign is plus when arrows on edges enter the vertex z_0 and minus otherwise, the sum runs over all edges $e_i^{(0)}$ terminating in z_0. For any two functions with appropriate boundary conditions imposed (see below) we have:

$$\langle\partial\psi,\varphi\rangle=\langle\psi,\partial^*\varphi\rangle\ , \tag{3.1.9}$$

so the operators ∂ and ∂^* are indeed conjugated to each other.

The Laplace operator on F acts locally as follows [20, 9]:

$$\Delta\varphi(z)=\sum_{i=1}^{p+1}\varphi(z_i)-(p+1)\varphi(z)\ . \tag{3.1.10}$$

Here z_i are all neighbours of the vertex z. We can rewrite this operator simply as

$$\Delta=-\partial\partial^*\ . \tag{3.1.11}$$

The string action on F (the free Gaussian model action) is the following [9]:

$$S[\varphi]=\frac{1}{2\ln p}\langle\partial^*\varphi,\partial^*\varphi\rangle\ . \tag{3.1.12}$$

It is natural to impose the Neumann boundary condition at infinity [see (3.1.5)]:

$$\partial_n\varphi(x)=0\ , \qquad x\in\partial F\ , \tag{3.1.13}$$

as it should be expected in case of the open string. Then the action (3.1.12) acquires the form:

$$S[\varphi]=-\frac{1}{2\ln p}\langle\varphi,\Delta\varphi\rangle\ . \tag{3.1.14}$$

Consider now the scattering process for N identical tachyons attached to the boundary of F. Let $E_r \subset F$ denote the "sphere", i.e. the set of vertices: $E_r = \{z \in F | d(z) = r\}$. The definition of the amplitude under consideration is the direct generalization of the one from [10]:

$$A_N(k_1,\ldots,k_N) = \lim_{r\to\infty} \sum_{\{z_i\}\in E_r} \frac{\int D\varphi \exp\left\{-S[\varphi] + i\sum_{j=1}^{N} k_j \varphi(z_j)\right\}}{\int D\varphi \exp\{-S[\varphi]\}}, \tag{3.1.15}$$

where k_j are momenta constrained by

$$\sum_{j=1}^{N} k_j = 0 \text{ (momentum conservation law)} \tag{3.1.16a}$$

and

$$k_j^2 = 2 \text{ (projective invariance condition)} . \tag{3.1.16b}$$

The sum in (3.1.15) runs over all possible placements of N points z_i on E_r. There exist two slightly different methods for calculating the Gaussian integral (3.1.15). Consider the first one, the second method will be used in Subsect. 3.3. We should find a solution φ_{cl} to a classical equation of motion which can be obtained from the exponential in (3.1.15):

$$\Delta\varphi_{\mathrm{cl}}(z) = -i \ln p \sum_{j=1}^{N} k_j \delta_{z,z_j} , \tag{3.1.17}$$

here $\delta_{z,w} = \{1, z = w; 0 \text{ elsewhere}\}$. The Laplacian (3.1.10) has exactly one zero mode in $C_0 : \varphi_0 = \text{const}$. Integrating out this zero mode yields the infinite factor $\delta(\sum k_j)$ which will be omitted in the following. Substituting φ_{cl} into (3.1.15) gives:

$$A_N(k_1,\ldots,k_N) = \lim_{r\to\infty} \sum_{\{z_i\}\in E_r} \exp\left\{i/2 \sum_{j=1}^{N} k_j \varphi_{\mathrm{cl}}(z_j)\right\} . \tag{3.1.18}$$

[The condition (3.1.16a) ensures that zero mode dependence is excluded from (3.1.18).]

The Neumann condition (3.1.13) being imposed, the solution φ_{cl} exists only if the constraint (3.1.16a) is satisfied. Simultaneously, it guarantees that amplitudes do not depend on ambiguities in the determination of the Green function (see Subsect. 2). The limit $r \to \infty$ is correct and, moreover, does not depend on the order in which the points z_i tend to the boundary if the condition (3.1.16b) is imposed. (The last proposition implies that we may consider a more general case: each tachyon may live on its own sphere E_{r_i} and the limits $r_i \to \infty$ can be taken in an arbitrary order.) Let us write φ_{cl} in the form:

$$\varphi_{\mathrm{cl}}(z) = -i \ln p \sum_{j=1}^{N} k_j N(z, z_j) , \tag{3.1.19}$$

where $N(z, w)$ is a Neumann function for the graph F. This function is not uniquely defined but the answer (3.1.18) does not depend on its concrete form. In fact, we define $N(z,w)$ as follows:

$$N(z,w)=N(w,z) \ , \tag{3.1.20a}$$

$$\partial_n N(z,x)=0 \ , \quad x\in\partial F \ , \tag{3.1.20b}$$

$$\Delta_{(z)}N(z,w)=\delta_{z,w}+\kappa(z) \ , \tag{3.1.20c}$$

where $\kappa(z)$ is an auxiliary "source" depending only on z. It follows from (3.1.20b) that

$$\sum_{z\in F}\kappa(z)=-1 \ . \tag{3.1.21}$$

If (3.1.16a) is valid $\kappa(z)$ does not give a contribution to (3.1.18). In what follows we choose $\kappa(z)$ to be concentrated at the vertices of the reduced graph F^R. We shall apply this method to the simplest case of one-loop graph. For the general case of the g-loop graph this technique has been developed in [12] where the answer for the Neumann function for an arbitrary genus graph was presented. It expresses the Neumann function in terms of moduli s_i so the answer is more complicated. The method we shall use further in Subsect. 3 is more geometric and it allows us to express the Neumann function in proper terms of period matrix determinant and the Jacobi map. In fact, our formulas can be produced by integrating abelian differentials of the third kind [23, 24]. Moreover, the reader can find the answers for Green functions are very similar to Sect. 12 of the book [24]. They were produced there on absolutely different grounds.

3.2. The One-Loop Case

In this section we treat the one-loop case in detail. This case is the simplest to deal with and it provides convenient tools for studying more complicated cases. The plan of this subsection is the following: given a Schottky group Γ_1, we find the invariant expression for the N-point tachyon amplitude as an integral over ∂F_1. After that, in order to compare our result with the archimedean one we transform the answer into an integral over the appropriate fundamental domain in $\mathbb{Q}_p$. This transformation will be done for an arbitrary choice of the generator γ_1.

It was already pointed out in Sect. 2 that for an invariant description in terms of the graph F neither the position of γ-axis nor the detailed information about q in (2.1.4) are essential. The only information we need is the p-adic norm of q that defines the length m of the shift along the γ-axis. So the reduced graph F_1^R is the ring consisting of m vertices and m edges. (Fig. 3b).

The Neumann function for the graph F_1 may be found by elementary methods [12]. The result is:

$$N(z,w)=\begin{cases}\dfrac{n_{z,w}(m-n_{z,w})}{2m} \ , & 0<n_{z,w}<m \ , \\[2ex] -l_{z,w} \ , & n_{z,w}=0\end{cases} \tag{3.2.1}$$

where $n_{z,w}$ is the minimal overlap of the path $z\to w$ and the ring F_1^R (Fig. 5a), $l_{z,w}$ is the minimal distance between the shortest path $z\to w$ and F_1^R (Fig. 5b). Obviously,

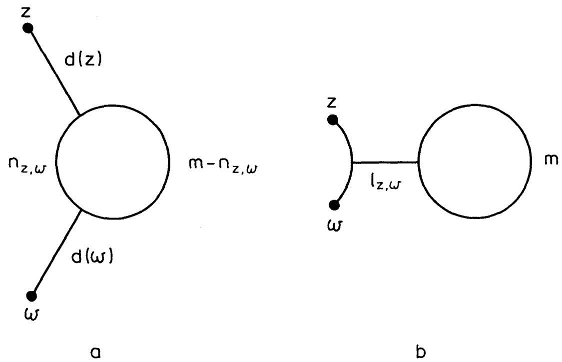

Fig. 5. The various mutual positions of the reduced graph F_1^R and the path $z \to w$

$n_{z,w}$ and $l_{z,w}$ cannot be nonzero simultaneously. This follows from the fact that the correlation function for two points lying on the same branch of F is just the same as the Neumann function for the tree T. The properties (3.1.20) can be easily verified. The auxiliary source $\kappa(z)$ is nonzero only for the vertices of the ring F_1^R and has an uniform density $-m^{-1}$. The answer (3.2.1) can be rewritten in a more convenient form:

$$N(z,w) = \tfrac{1}{2}\langle \mathscr{X}_{wz}, \mathscr{X}_{wz}\rangle_R - \tfrac{1}{2}\langle \mathscr{X}_{wz}, \mathscr{L}_1\rangle \Lambda_{11}^{-1}\langle \mathscr{X}_{wz}, \mathscr{L}_1\rangle - d(w \to z, F_1^R) \ . \tag{3.2.2}$$

Here $\mathscr{X}_{wz}$ is an arbitrary path between the points w and z, and $\mathscr{L}_1 \in C_1$ is the cycle, Λ_{11} is the only matrix element of period matrix (2.2.4), $\Lambda_{11} = \langle \mathscr{L}_1, \mathscr{L}_1\rangle = m$. We define (not only for the 1-loop but also for any multiloop case):

$$d(w \to z, F^R) = \sup_{\mathscr{X}_{wz}} \inf_{\substack{y \in \mathscr{X}_{wz} \\ v \in F^R}} d(y,v) \ . \tag{3.2.3}$$

This "distance to reduced graph" is non-zero only when w, z belong to the same branch in F. Actually, the answers (3.2.1–3) make sense in the limit $z, w \to \partial F_1$. Namely, as $r \to \infty$, the sum over E_r in (3.1.15) transforms into an integral over ∂F_1 with the measure:

$$\sum_{B_z^{(r)} \subset E_r} \to p^r d\mu(B_z) \ ,$$

$B_z^{(r)} \subset B_z$ is the subset $\{w \in B_z : d(z) = r\}$.

For the Gaussian integral (3.1.15) we obtain the expression:

$$A_N(k_1, \dots, k_N) = \lim_{r_i \to \infty} \int d\mu(x_1) \dots d\mu(x_N) p^{r_1} \dots p^{r_N}$$
$$\cdot \exp\left\{\ln p \sum_{\substack{i,j=1 \\ i<j}}^{N} k_i k_j \left(N(x_i, x_j) + \frac{r_i}{2} + \frac{r_j}{2}\right)\right\} \ . \tag{3.2.4}$$

The prefactors $p^{r_1} \dots p^{r_N}$ should be cancelled by the singular terms in the exponential $(r_i \to \infty)$. The conditions (3.1.16) do ensure these cancellations and the final answer has the form:

$$A_N(k_1,\ldots,k_N)=\int_{\partial F_1}\prod_{j=1}^{N}d\mu(x_j)\prod_{i<j}^{N}[\psi(x_i,x_j)]^{k_ik_j}\ , \qquad (3.2.5)$$

where

$$\log_p\psi(x,y)=N(x,y)=\begin{cases}\dfrac{n_{x,y}(m-n_{x,y})}{2m}\ , & 0<n_{x,y}<m\\ -l_{x,y}\ , & n_{x,y}=0\end{cases}\qquad (x,y\in\partial F_1). \qquad (3.2.6)$$

Let us rewrite this integral as the one over the fundamental domain $\partial F(\Gamma_1)\subset\partial T$. We can continue φ periodically to the whole tree boundary (or, to the complete set of integers $n_{x,y}$):

$$\psi(x,\gamma(y))=\psi(x,y)$$

for any $\gamma\in\Gamma_1$. The form (3.2.5) is invariant because it does not depend on the choice of the fundamental domain. To compare our results with p-adic θ-functions (Appendix B) it is necessary to choose some concrete domain $\partial F(\Gamma_1)\subset\partial T=\hat{\mathbb{Q}}_p$. This is equivalent to the choice of the generating element γ_1 for Γ_1 in $PGL(2,\mathbb{Q}_p)$. Let us begin with the canonical form (2.1.3) of the generator. Then

$$\partial F(\Gamma_1)=\{x\in\mathbb{Q}_p|-m<\mathrm{ord}_p x\leqq 0\}\ . \qquad (3.2.7)$$

The invariant measure (3.1.4) transforms into

$$d\mu(x)=dx/|x|_p\ , \qquad (3.2.8)$$

where dx is the standard Haar measure on $\mathbb{Q}_p$. On the left-hand side of (3.2.8) $x\in\partial F_1$ and on the right-hand side $x\in\partial F(\Gamma_1)\subset\hat{\mathbb{Q}}_p$, the isomorphism being implied in what follows. For $p^m\cdot|y|_p>|x|_p>|y|_p$ we have

$$p^{n_{x,y}}=|x/y|_p\ , \qquad (3.2.9a)$$

and for $|x|_p=|y|_p$:

$$p^{-l_{x,y}}=\left|1-\frac{x}{y}\right|_p \qquad (3.2.9b)$$

(see Appendix A). In the second case the points x and y belong to the same branch growing from F_1. Suppose $1\leqq|x/y|_p<p^m$, then (3.2.6) can be rewritten using (3.2.9) as follows:

$$\psi(x,y)=p^{-\frac{\mathrm{ord}_p^2(x/y)}{2\,\mathrm{ord}_p q}}\left|\frac{x}{y}\right|_p^{1/2}\left|1-\frac{y}{x}\right|_p\ . \qquad (3.2.10)$$

This formula can be periodically extended to all values of x and y if one performs the infinite product:

$$\begin{aligned}\psi(x,y)=\psi(x/y)&=p^{-\frac{\mathrm{ord}_p^2(x/y)}{2\,\mathrm{ord}_p q}}\left|\frac{x}{y}\right|_p^{1/2}\left|1-\frac{y}{x}\right|_p\cdot\prod_{n=1}^{\infty}\left|1-q^n\frac{y}{x}\right|_p\left|1-q^n\frac{x}{y}\right|_p\\ &=p^{-\frac{\mathrm{ord}_p^2(x/y)}{2m}}|\theta_p(x/y,q)|_p\ ,\qquad \psi(qx)=\psi(x)\ .\end{aligned} \qquad (3.2.11)$$

Here $m=\mathrm{ord}_p q>0$, $q\in p\mathbb{Z}$, the periodicity with respect to $x\to\gamma(x)=qx$ being clear. $\theta_p(\xi,q)$ is the one-dimensional odd p-adic θ-function (B.1). As for the amplitude (3.2.5), we obtain:

$$A_N(k_1,\dots,k_N)=\int\limits_{\partial F(\Gamma_1)}\prod_{j=1}^{N}\frac{dx_j}{|x_j|_p}\prod_{i<j}^{N}[\psi(x_i/x_j)]^{k_ik_j}=\int\limits_{1\leq|x_j|_p<|1/q|_p}\prod_{j=1}^{N}dx_j$$

$$\cdot\prod_{i<j}^{N}\left[|E(x_i,x_j)|_p\exp\left\{-\frac{\ln p}{2\,\mathrm{ord}_p q}\,\mathrm{ord}_p^2(x_i/x_j)\right\}\right]^{k_ik_j},\quad(3.2.12)$$

where $E(x,y)$ is the Prime form for the genus 1 (B.5):

$$E(x,y)=(x-y)\prod_{n=1}^{\infty}\frac{(1-q^nx/y)(1-q^ny/x)}{(1-q^n)^2},\qquad x,y\in\mathbb{Q}_p\ .\qquad(3.2.13)$$

The exponential in (3.2.12) can be treated as the p-adic norm of the integral over zero modes which together with the Prime form give a contribution to the Green function for the open string. This integral in the archimedean case gives the following contribution (in standard notations [25]):

$$\exp\left\{-\frac{\ln^2(x_i/x_j)}{(-2\ln q)}\right\},\qquad q=e^{2\pi i\tau},\ \mathrm{Im}\,\tau>0\ .\qquad(3.2.14)$$

This expression may be identically rewritten as

$$p^{-\frac{\log_p^2(x_i/x_j)}{-2\log_p q}}\ .\qquad(3.2.15)$$

The p-adic modulus of this expression can be correctly defined only in an algebraic extension of $\mathbb{Q}_p$ (note however that it is always sufficient to consider finite extensions since the power value in (3.2.15) is a rational number). Thus, the p-adic norm in (3.2.15) being taken, we obtain the exponential term in (3.2.12). Therefore, in order to obtain the p-adic amplitudes one may simply perform the integration over $\mathbb{Q}_pRS$ with the additive Haar measure instead of the standard integration over $\mathbb{R}RS$ and replace all real moduli by the p-adic ones. It confirms the proposal of the paper [14] in which this trick has been claimed ad hoc for $g=1$. This important observation appears to be valid with slight modifications for higher genera as we shall demonstrate in the next subsection.

Now let us consider for completeness the case of Γ_1 generated by arbitrary hyperbolic γ:

$$\gamma(x)=\frac{ax+b}{cx+d}\subset PGL(2,\mathbb{Q}_p)\ .\qquad(3.2.16)$$

The attractive and repulsive points v,u together with the multiplier q can be easily found and we have for $n_{x,y}$, $l_{x,y}$:

$$p^{n_{x,y}}=\left|\frac{(x-u)(y-v)}{(x-v)(y-u)}\right|_p,\qquad p^{-l_{x,y}}=\left|\frac{(x-y)(u-v)}{(x-u)(y-v)}\right|_p\ .$$

Let $\tilde{\gamma}=\begin{pmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{pmatrix}\in PGL(2,\mathbb{Q}_p)$ be an element transforming the axis $v\to u$ into $\infty\to 0$ such that

$$\tilde{\gamma}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\tilde{\gamma}^{-1}=\begin{pmatrix}q & 0\\ 0 & 1\end{pmatrix}.$$

$\left(\text{A possible choice of } \tilde{\gamma} \text{ is } \tilde{\gamma}=\begin{pmatrix}1 & -u\\ -1 & v\end{pmatrix}\right)$. Then the measure (3.1.4) transforms as

$$d\mu(x)=\frac{dy}{|y|_p}=\frac{|\det\tilde{\gamma}|_p dx}{|\tilde{c}x+\tilde{d}|_p|\tilde{a}x+\tilde{b}|_p}=\frac{|u-v|_p}{|x-u|_p|x-v|_p}dx\,. \tag{3.2.17}$$

Here $y=\tilde{\gamma}(x)=\dfrac{x-u}{v-x}$. So we have

$$p^{(\frac{1}{2}n_{ij}-l_{ij})}=|x_i-x_j|_p\left|\frac{(u-v)^2}{(x_i-u)(x_j-u)(x_i-v)(x_j-v)}\right|_p^{1/2}.$$

(Here $n_{ij}=n_{x_i,x_j}$ and $l_{ij}=l_{x_i,x_j}$ for brevity.) Hence

$$\prod_{j=1}^{N}d\mu(x_j)\prod_{i<j}^{N}p^{(\frac{1}{2}n_{ij}-l_{ij})k_ik_j}\to\prod_{j=1}^{N}dx_j\prod_{i<j}^{N}|x_i-x_j|_p^{k_ik_j}.$$

(Here the constraints (3.1.16) has been used). This relation is valid for a special fundamental domain [see (2.1.4–5)], which is in fact the $\hat{\mathbb{Q}}_p$ with two discs removed. The Prime form is performed by the following infinite product:

$$E_\gamma(x,y)=(x-y)\prod_{n\neq 0}\left[\frac{(x-\gamma^n(y))(y-\gamma^n(x))}{(x-\gamma^n(x))(y-\gamma^n(y))}\right]^{1/2}. \tag{3.2.18}$$

So the periodicity of the whole expression holds and we obtain:

$$A_N(k_1,\dots,k_N)=\int_{\partial F(\Gamma_1)}\prod_{j=1}^{N}dx_j\prod_{i<j}^{N}\left[|E_\gamma(x_i,x_j)|_p p^{-\frac{n_{ij}}{2m}}\right]^{k_ik_j}, \tag{3.2.19}$$

where $m=\mathrm{ord}_p q$.

In the case of general orientation this comparison teaches us one trivial but rather important lesson; the amplitudes do not depend on the orientation of the γ-axis. It is also true for the multiloop case: the $PGL(2,\mathbb{Q}_p)$-transformations do not affect the structure of the answer, they change the fundamental domain only. Thus, it is more instructive to describe the scattering processes in invariant terms of the factorized tree F.

3.3. The Amplitudes for Arbitrary Genus

In this subsection we consider correlation functions for tachyons on arbitrary homogeneous space $F_g=T/\Gamma_g$. The invariant description of F_g implies that we treat it as an infinite lattice of some special kind. In fact, any such lattice consists of:

a) A finite closed g-loop reduced graph F_g^R;

b) The branches B_{z_i}, $z_i\in F_g^R$, which should be added in order to fill all $p+1$ bounds of any vertex z_i.

Thus there arises a technical problem of finding the Green functions with the Neumann boundary condition (3.1.13) in the Gaussian (free) lattice field theory with the action (3.1.12). The general method for the solving of such a problem has been developed by Zinov'ev [14] who proposed the general geometric formulation based on chain complexes. (This issue has been also raised independently in [28]). Here we calculate the amplitudes using this method.

We begin with the Gaussian integral (3.1.15) with the constraints (3.1.16) imposed. Consider now the regularized functional integral on a *finite* lattice $\mathscr{K}_r$. It consists of all vertices $z \in F$ with $d(z) \leq r$ and edges between them. For clarity we also assume, though it is not necessary, that all sources are attached to the boundary of the graph $\mathscr{K}_r$. Using the operator ∂^* (3.1.7) we can rewrite the integral for $A_N(k_1, \ldots, k_N)$ (3.1.15) as

$$A_N(\cdot) = \lim_{r\to\infty} \sum_{\{z_i\}\in E_r} \frac{\int\limits_{B_g(\mathscr{K}_r)} Db_i \exp\left\{-\frac{1}{2\ln p}\langle b, b\rangle + i\sum_{j=1}^{N} k_j \langle \mathscr{X}_{z_i C}, b\rangle\right\}}{\int\limits_{B_g(\mathscr{K}_r)} Db_i \exp\left\{-\frac{1}{2\ln p}\langle b, b\rangle\right\}} . \tag{3.3.1}$$

This formula needs few comments. Firstly, we choose in graph $\mathscr{K}_r$ some point C, "the center", but the answer does not depend on this choice due to the condition (3.1.16a). Secondly, the integration goes over the space of functions $\partial^*\varphi \in B_g(\mathscr{K}_r) \subset C_1(\mathscr{K}_r)$. This space is called "the space of coboundaries", i.e. it is the image of the operator ∂^*. This operator generates a short exact sequence:

$$0 \to C_0(\mathscr{K}_r) \xrightarrow{\partial^*} C_1(\mathscr{K}_r) \xrightarrow{\partial^*} Z_g \to 0 . \tag{3.3.2}$$

Here $Z_g \ni \{\mathscr{Z}_i\}$ is a space of cycles in F_g^R. Moreover, the whole space of functions $C_1(\mathscr{K}_r)$ splits into a direct sum:

$$C_1(\mathscr{K}_r) = B_g(\mathscr{K}_r) \oplus Z_g . \tag{3.3.3}$$

So the integration in (3.3.1) goes over functions $b(e_i)$ which are orthogonal to any cycle $\mathscr{Z}_i$: for $i = 1, \ldots, g$ $\langle b, \mathscr{Z}_i\rangle = 0$. The quadratic form in (3.3.1) is very simple and the gaussian integral can be easily done. The answer has the form:

$$A_N(k_1, \ldots, k_N) = \lim_{r\to\infty} \sum_{\{z_i\}\in E_r} \exp\left\{\frac{1}{2}\ln p \langle \mathscr{R}, P_B \mathscr{R}\rangle\right\} , \tag{3.3.4}$$

where $\mathscr{R} = \sum_{i=1}^{N} \mathscr{X}_{z_i C}$. P_B is the projector to the space $B_g(F_g)$. In fact $P_B = \mathrm{Id} - P_z$, P_z being a projector to the space of cycles Z_g:

$$P_z \psi = \sum_{i,j=1}^{g} \mathscr{Z}_i \Lambda_{ij}^{-1} \langle \mathscr{Z}_j, \psi\rangle . \tag{3.3.5}$$

Here Λ_{ij}^{-1} is the inverse period matrix (2.2.4). This answer can be rewritten in terms of two point Neumann functions [cf. (3.1.20), (3.2.1)]:

$$\frac{\ln p}{2} \langle \mathscr{R}, P_B \mathscr{R}\rangle = \ln p \sum_{\substack{i,j=1 \\ i<j}}^{N} k_i k_j N(z_i, z_j) . \tag{3.3.6}$$

Here $N(z_i, z_j)$ is the following:

$$N(x,y)=\frac{\det\begin{vmatrix}\langle\mathscr{X}_{xy},\mathscr{X}_{xy}\rangle & \langle\mathscr{X}_{xy},\mathscr{Z}_1\rangle & \dots & \langle\mathscr{X}_{xy},\mathscr{Z}_g\rangle\\ \langle\mathscr{Z}_1,\mathscr{X}_{xy}\rangle & \langle\mathscr{Z}_1,\mathscr{Z}_1\rangle & \dots & \langle\mathscr{Z}_1,\mathscr{Z}_g\rangle\\ \vdots & \vdots & & \vdots\\ \langle\mathscr{Z}_g,\mathscr{X}_{xy}\rangle & \langle\mathscr{Z}_g,\mathscr{Z}_1\rangle & \dots & \langle\mathscr{Z}_g,\mathscr{Z}_g\rangle\end{vmatrix}}{\det\begin{vmatrix}\langle\mathscr{Z}_1,\mathscr{Z}_1\rangle & \dots & \langle\mathscr{Z}_1,\mathscr{Z}_g\rangle\\ \vdots & & \vdots\\ \langle\mathscr{Z}_g,\mathscr{Z}_1\rangle & \dots & \langle\mathscr{Z}_g,\mathscr{Z}_g\rangle\end{vmatrix}} \tag{3.3.7}$$

It is clear from this expression that the answer does not depend on the choice of the path $\mathscr{X}_{xy}$ since any two paths differ by a linear combination of the cycles which does not affect the determinant (3.3.7).

The answer (3.3.7) expresses the Neumann function in terms of period matrices determinants. Note that the upper matrix in (3.3.7) may also be treated as a period matrix for some reduced graph F^R_{g+1}. This graph can be obtained if one identifies the points x and y resulting in a $g+1$-loop graph, the path $\mathscr{X}_{xy}$ being the $g+1$'th vector in the new basis of cycles $\{\mathscr{Z}_i, i=1,\dots,g, \mathscr{X}_{xy}\}$. The one-loop graph in Fig. 5a gives us the simplest example. The identification of the points z and w leads to a two-loop graph (Fig. 6). A question arises how one may express these determinants through the moduli s_i of the corresponding surfaces. Acttually, the formula (15) of the paper [12] gives this answer.

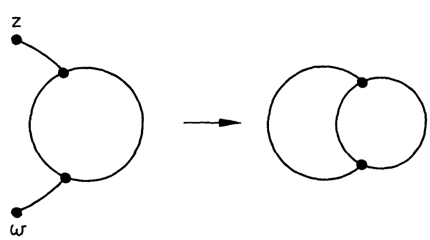

Fig. 6. The "auxiliary" cycle

Now we present the tachyon amplitudes. Again we replace the sum in the limit $r\to\infty$ by the integral over ∂F with the invariant measure (3.1.4). The details are identical to the above consideration for the 1-loop case (3.2.3–4) and we give only the final expression. For any two points $x_1, x_2 \in \partial F$ we define $d(x_1\to x_2, F^R)$ by the formula (3.2.3) with $z\to x_1$ and $w\to x_2$. We also choose an arbitrary path $\mathscr{X}_{x_1x_2}$. Then one can define the function $\varphi(x_1, x_2)$:

$$\log_p \varphi(x_1,x_2)=\tfrac{1}{2}\langle\mathscr{X}_{12},\mathscr{X}_{12}\rangle_R - d(x_1\to x_2, F^R) \tag{3.3.8}$$

and the function $\Phi(x_1, x_2)$:

$$\log_p \Phi(x_1,x_2)=-\frac{1}{2}\sum_{i,j=1}^{g}\langle\mathscr{X}_{12},\mathscr{Z}_i\rangle\Lambda^{-1}_{ij}\langle\mathscr{Z}_j,\mathscr{X}_{12}\rangle\,. \tag{3.3.9}$$

Here as above $\mathscr{X}_{12}=\mathscr{X}_{x_1x_2}$ for brevity. Certainly, the Neumann function $N(z,w)$ (3.3.7) is connected with φ and Φ by the relation:

$$\log_p(\varphi(x_1,x_2)\Phi(x_1,x_2))=\lim_{\substack{z\to x_1\\ w\to x_2}}\left(\frac{1}{2}N(z,w)-\frac{1}{2}d(z)-\frac{1}{2}d(w)\right). \qquad (3.3.10)$$

The result for the amplitude (3.1.15) acquires the form:

$$A_N(k_1,\dots,k_N)=\int_{\partial F_g}\prod_{j=1}^{N}d\mu(x_j)\prod_{i<j}^{N}[\varphi(x_i,x_j)\Phi(x_i,x_j)]^{k_ik_j}. \qquad (3.3.11)$$

In order to compare the result with the archimedean one we should "coordinatize" ∂F. For the canonical choice of the fundamental domain $\partial F(\Gamma_g)$ of the corresponding group Γ_g the measure appears to be the additive Haar measure dx on $\mathbb{Q}_p$, and the final answer is the following:

$$A_N(k_1,\dots,k_N)=\int_{\partial F(\Gamma_g)\in\hat{\mathbb{Q}}_p}\prod_{j=1}^{N}dx_j\prod_{i<j}^{N}[|E(x_i,x_j)|_p\Phi(x_i,x_j)]^{k_ik_j}. \qquad (3.3.12)$$

The $E(x,y)$ is the p-adic Prime form ((B.5), cf. [29]). One can always choose the fundamental domain $\partial F(\Gamma_g)\subset\hat{\mathbb{Q}}_p$, such that $|E(x,y)|_p=|x-y|_p$ (see Appendix B).

We conclude this section by the comparison of the answer (3.3.12) with the archimedean one. The archimedean amplitude has the form [29] (up to normalization factors):

$$A_N(\cdot)\propto\int_{\partial F_g\in\hat{\mathbb{R}}}\prod_{j=1}^{N}dx_j\prod_{i<j}^{N}\left|E(x_i,x_j)\exp\left\{-\pi\sum_{r,s}^{g}\int_{x_i}^{x_j}\omega_r(\operatorname{Im}\tau)^{-1}_{rs}\int_{x_i}^{x_j}\omega_s\right\}\right|^{k_ik_j}. \qquad (3.3.13)$$

The Neumann function on the $\mathbb{R}RS$ is expressed in terms of the real Prime form $E(x,y)$ $(x,y\in\mathbb{R})$. The exponential in (3.3.14) being in fact the nonholomorphic part of the Neumann function resulting from zero modes integral. In the case of $\mathbb{R}RS$ τ_{ij} is pure imaginary. We also have the analog of the Jacobi map $\int_x^y\omega$ which is now defined as the intersection index between the cycle basis $\{\mathscr{L}_i\}$ and the path $x\to y$. So in order to obtain the p-adic expression we should replace the real norm of the Prime form by p-adic one and besides it substitute new (p-adic) definitions of the period matrix and Jacobi map:

$$2\pi\operatorname{Im}\tau_{rs}\to A_{rs}\,,\qquad \int_{x_i}^{x_j}\omega_r\to\langle\mathscr{X}_{ij},\mathscr{L}_r\rangle\,. \qquad (3.3.14)$$

Thus we do reproduce the formula (3.3.12) from (3.3.13).

4. The Mumford Curves and Other Issues

In this section we intend to answer some more questions related to the mathematical background of our treatment. In particular, we discuss the role of algebraic extensions of $\mathbb{Q}_p$. Then we describe the Mumford curves and formulate a conjecture about the general construction of the sums over moduli space which appear, for example, when calculating the p-adic string partition function for different values of p.

4.1. The Role of Algebraic Extensions

The open string theory over $\mathbb{R}$-field is well-known to be self-consistent only if closed strings (corresponding to the algebraically closed field $\mathbb{C}$) are included. More precisely, the open strings without closed ones are suitable to describe only tachyon amplitudes in higher loops (or arbitrary amplitudes at the tree level) [25]. The main reason for such a situation is the absence of a profound analyticity notion in algebraically non-closed fields. Generally speaking, the constructions of algebraic geometry are well-defined only over algebraically closed fields [28]. Just the same situation appears in the non-archimedean case. But there exist serious complications connected with the involved structure of algebraic extensions, or, equivalently, with a huge Galois group.

The field $\mathbb{Q}_p$ has an infinite number of algebraic extensions, each finite extension corresponding to a proper type of the open string (in contrast to the statements of some earlier papers [3–6]). Therefore the "p-adic closed string" should be connected with the complete algebraically closed field $\bar{\Omega}$. Again the self-consistent "finite extension string" is possible only if $\bar{\Omega}$ is included. In particular, the analytic expressions exist only on $\bar{\Omega}$.

The following interesting example is the determinant calculation [29]. We can easily obtain the part determined by zero modes, that is $[\det \Lambda_{ij}]^{-1/2}$, which is just the same as in the archimedean case. This expression can be obtained using only the open string framework, but the rest should depend on the metric and diverges as for the archimedean non-compact hyperbolic domains due to the conformal factor (any natural regularization makes the answer rather complicated). The string partition function for a given surface is a finite combination, which does not depend on the metric, and is equal to $\det(\Delta_{-1})(\det(\Delta_0))^{-13}$ where Δ_j is the Laplace-Beltrami operator acting on j-differential space. This definition requires the notion of metric, which can be hoped to exist only on $\bar{\Omega}$. Thus, in contrast to the amplitudes, the determinant calculation can be done only over $\bar{\Omega}$. To all appearance, the answer for the properly defined partition function should be equal to p^S [30] (for the multiplier connected with the non-zero modes), S being defined in (3.1.5) (see also below).

As we have seen in the previous section, the Neumann function is natural in the $\mathbb{Q}_p$ case in contrast to the Dirichlet one (though the latter can also be obtained [23]). It follows from the fact that the Dirichlet function corresponds to the semi-off-shell closed string amplitudes [31] (i.e. it relates to $\bar{\Omega}$, not to $\mathbb{Q}_p$).

At last we point out one more example when the necessity of algebraic extensions is obvious. Consider a Schottky parametrized Riemann surface over $\mathbb{Q}_p$. It is presented by $\hat{\mathbb{Q}}_p$ with a number of discs removed (see Sect. 2), whose radii may be equal to p^r, r being half-integer, which corresponds to a valuation on an extension of $\mathbb{Q}_p$ [17].

Now we would like to describe finite extensions manifestly in terms of the Bruhat-Tits tree [17]. Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$ of degree n with the ramification index e. Set $f = n/e$. A Bruhat-Tits tree $T(\mathbb{K})$ can be drawn for any such $\mathbb{K}$ by the following procedure: 1) insert into each edge of T $e-1$ new vertices separated by equal distances; 2) draw new branches in such a way that each vertex has exactly $p^f + 1$ neighbours. All the constructions may be developed for such trees and the answers can be obtained with only slight modifications. However, $T(\bar{\Omega})$ gives us a less trivial example of the tree which contains infinitesimal edges and vertices with an infinite number of neighbours. So the tree language seems to be adequate only if some limiting procedure is implied. In any case we expect that our answers for the amplitudes turn out to be the same for arbitrary extensions (up to slight modifications) with the lengths of reduced graph segments s_i (the moduli) being continuous in $\bar{\Omega}$-case.

The only trouble we should note is the special case of $\mathbb{Q}_p$-tree with small p and $s_i = 1$. Let us consider the example in Fig. 7 for $p = 2$ and $g = 2$. Then the whole measure of the boundary of this world sheet is zero [see (3.1.5)] because no branches go to infinity in this case. So we cannot define the important quantities like the Jacobi map and so on. But the problem can be resolved if we turn to the proper finite extension.

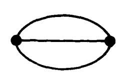

Fig. 7. An example of $E\mathbb{Q}_pRS$ with the empty $\mathbb{Q}_pRS$ ($g=2$, $p=2$)

4.2. The Schottky Parametrization and Mumford Curves

The next problem to be discussed is the integration (summation) of the amplitudes obtained in the previous section over the moduli space. The key question is the uniformization of algebraic curves. Namely, in the archimedean case one can uniformize any algebraic curve over $\mathbb{C}$. To treat the open string theory it is necessary to uniformize the curves over $\mathbb{R}$. This problem can be reduced to the uniformization of the curves over $\mathbb{C}$ as follows. Given a Riemann surface with complex antilinear involution such that the fixed points of this involution form a set of circles (i.e. $\mathbb{R}$-points of the curve, or $\mathbb{R}RS$) which divide the surface into two pieces ($E\mathbb{R}RS$) without handles. Then we manifestly describe the embedding of $E\mathbb{R}RS$ into a Riemann surface over $\mathbb{C}$, with the corresponding equation admitting the involution. Certainly, it is not an isomorphism. For example, the curve given by the equation $y^2 = (x^2+1)\prod(x-\alpha_i)$, $\alpha_i \in \mathbb{R}$, has a handle which is not cut by the fixed contour since it lies in a non-real domain.

A more complicated situation may be expected in the p-adic case since there exists a lot of different algebraic extensions. But the general scheme seems to be the same, and the uniformization problem should be resolved for Ω and $\mathbb{Q}_p$ simultaneously. In the non-archimedean case in contrast to the archimedean one, curves exist which do not admit any uniformization even over an algebraically closed field $\bar{\Omega}$. Namely, only the so-called Mumford curves admit the Schottky uniformization. Now we discuss these curves in more detail [17, 18].

To begin with, let us concentrate on the elliptic curve case [32]. One can write the elliptic curve equation in Legendre form:

$$y^2 = x(x-1)(x-\lambda) \ , \tag{4.2.1}$$

where the ramification points are placed in 0, 1, ∞ and λ (we can fix three of them by using of the global $PGL(2)$-invariance). There exists the relation between λ and the modular parameter $q = e^{2\pi i\tau}$:

$$\Delta(q) = 16\,[\lambda(1-\lambda)]^2 \propto q \prod_{n>0} (1-q^n)^{24} \ . \tag{4.2.2}$$

Here $\Delta(q)$ is the Jacobi Δ-function, which is just the discriminant of Eq. (4.2.1). Such relations also do not depend on the number field. Since we are interested in the norms of both sides of relation (4.2.2) and the constant multiplier on the right-hand side has unit norm when $p \neq 2$, we shall consider (4.2.2) as the precise relation and suppose $p \neq 2$ in the elliptic and hyperelliptic cases. Then we have

$$|\Delta(q)|_p = |q|_p = |\lambda(1-\lambda)|_p^2 \quad \text{for} \quad |q|_p < 1 \ . \tag{4.2.3}$$

The parameter q describes a Schottky parametrized Riemann surface. It is natural to consider Eq. (4.2.1) over a quadratic extension of $\mathbb{Q}_p$. Then $\mathrm{ord}_p\lambda$ and $\mathrm{ord}_p(1-\lambda)$ may be half-integer, with $\mathrm{ord}_p q$ being integer. If one wishes to consider odd $\mathrm{ord}_p q$, then it is impossible to represent the cubic Weierstrass equation [32] in the form (4.2.1) with $\lambda \in \mathbb{Q}_p$. Consider two possibilities: $|\lambda|_p < 1$ and $|1-\lambda|_p < 1$, which cannot be satisfied simultaneously. In both cases one can determine $\mathrm{ord}_p q$ using (4.2.3) and it is a kind of modular parameter (see Sect. 2) with the condition $\mathrm{ord}_p q > 0$ being analogous to $\mathrm{Im}\,\tau > 0$. However, the case $|\lambda|_p \geqq 1$ destroys this inequality. It is an illustration of Tate's well-known result [18]: only elliptic curves with non-integer j-invariants (as $j = 1/q +$ regular terms) can be uniformized. These "Tate's curves" are just the Mumford curves for genus 1.

A less trivial example of a higher genus curve is the hyperelliptic curve defined by an equation of the form:

$$y^2 = \prod_{i=1}^{2g+2} (x-\alpha_i) \ , \tag{4.2.4}$$

where three arbitrary ramification points α_i can be fixed. Again it is sufficient to work within a quadratic extension. Let us construct a concrete example of the hyperelliptic Mumford curve. Choose the set $\{\alpha_i\}$ satisfying the following conditions:

1. $|\alpha_{2i-1} - \alpha_{2i}|_p < 1$, $i = 1, \ldots, g+1$;
2. all other pairwise differences are nonzero modulo p. In fact, we can describe this "degenerating" curve in terms of the reduced graph F_g^R. That is, pinch the handles corresponding to each pair $\{\alpha_{2i-1}, \alpha_{2i}\}$. Then we obtain a set of zero genus Riemann surfaces (spheres) with punctures. Each sphere corresponds to a vertex in F_g^R, and each pinching – to a segment with the length $s_i = 2\,\mathrm{ord}_p(\alpha_{2i-1} - \alpha_{2i})$ [cf. (4.2.3)].

Now we are ready to define a general Mumford curve [18]. Let a set $\mathscr{C}(\mathbb{Q}_p)$ of $\mathbb{Q}_p$-points of a curve $\mathscr{C}$ be given by a set of algebraic equations:

$$P_k(\{x_s\}) = 0 \ , \qquad x_s \in \mathbb{Q}_p \ . \tag{4.2.5}$$

Here $P_k(\{x_s\})$ are irreducible polynomials in x_s with p-adic coefficients (we do not specialize them somehow). Strictly speaking, it should be "homogenized" to obtain the projective curve. By appropriately replacing $(\{x_s\}) \to (\{x_s'\})$ one can cause each coefficient of (4.2.5) to become a p-adic integer. There are many ways to do so; we choose a "minimal" way ("a minimal model," see below).

Then one can immediately define a reduction of $\mathscr{C}$ to be a curve $\bar{\mathscr{C}}$ over a finite field $\mathbb{F}_p$ given by (4.2.5) modulo p. By definition, the Mumford curve is a non-singular curve $\mathscr{C}$, whose reduction $\bar{\mathscr{C}}$ is a set of components isomorphic to $P^1(\mathbb{F}_p)$ and containing only double singular points with separated tangent lines (such curves are also called degenerating with split reduction). Obviously, the above examples fall under this definition. In the first case $\lambda = 0 \bmod p$ implies that a degenerate curve $\bar{\mathscr{C}}$ is just the "sphere" [i.e. $P^1(\mathbb{F}_p)$] with two punctures (Fig. 8a). In the hyperelliptic case condition 2) means that the cusp singularities (Fig. 8b) are forbidden.

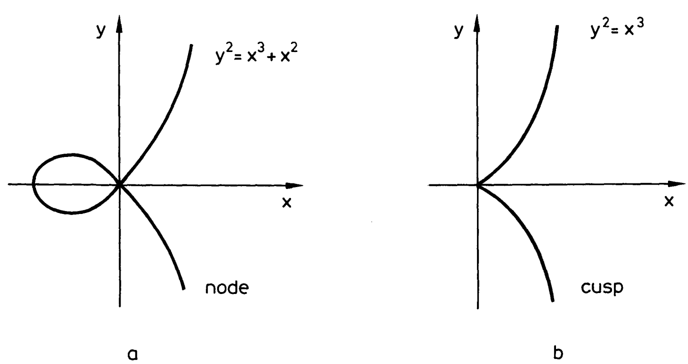

Fig. 8. Examples of singularities (elliptic curves over $\mathbb{R}$)

Mumford has shown [18] that one can establish one-to-one correspondence between factorized Bruhat-Tits trees $F(\Gamma)$ and the Mumford curves (see Fig. 9). That is, ∂F (all rays contained in the branches B_z, $z \in F^R$) correspond to $\mathbb{Q}_p$-rational points of the curve. All edges with the only endpoint belonging to F^R correspond to the $\mathbb{F}_p$-rational non-singular points of the curve (in particular, the reduced zero genus curve is $P^1(\mathbb{F}_p)$, i.e. $p+1$ edges with the common origin [17]). Each branching

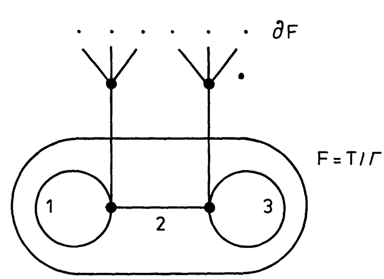

Fig. 9. The correspondence between rays in $F = T/\Gamma$ and $\mathbb{Q}_p$-rational points of an algebraic curve ($g = 2$, $p = 3$). The dots correspond to $\mathbb{F}_p$-rational points of the curve while the segments 1, 2, 3 correspond to double singular points (stable reduction)

vertex in the reduced graph corresponds to a "sphere" $P^1(\mathbb{F}_p)$ in $\bar{\mathscr{C}}$ and, eventually, all segments in F^R correspond to double singular points in $\bar{\mathscr{C}}$. Thus the map $\mathscr{C}(\mathbb{Q}_p) \to \bar{\mathscr{C}}(\mathbb{F}_p)$ is described by the natural map:

$$\partial F \to F^R .$$

A simple example is depicted in Fig. 10.

Up to now we dealt with the so-called stable reduction of the curve (i.e. any $\bar{\mathscr{C}}$-component without self-intersections has more than two double points), when the map f:

$$f: \{\text{edges of } F^R\} \to \{\text{double singular points}\} \qquad (4.2.6)$$

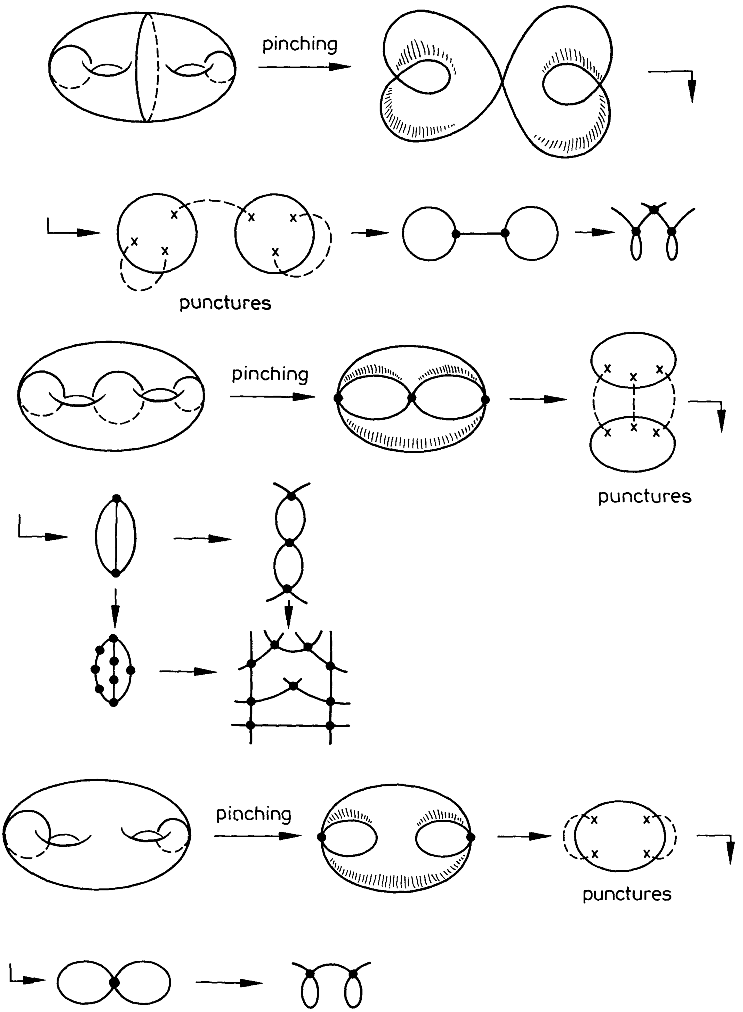

Fig. 10. Different types of degeneration for $g=2$ and corresponding reduced graphs. In **b** an example of semistable reduction is depicted. Dots in the reduced graph refer to the irreducible components P^1 which are depicted schematically as lines at the last picture of each sequence. Their intersections correspond to the edges in the reduced graph

is surjective (but in general non-isomorphic). A finer reduction is the semistable one, when any $\mathscr{C}$-component without self-intersections has at least two double points (Fig. 10). It makes the map f isomorphic and can be described as follows: Let us associate with the curve (4.2.5) a surface given by the equation:

$$P_k(\{x_s\};\{a_i(t)\}) \equiv P_k(\{x_s\},t)=0 \ , \tag{4.2.7}$$

where all the coefficients a_i are represented as formal polynomials in t, namely, the integer number $a=\sum_{i\geqq 0} a_i p^i$ must be replaced by $a=\sum_{i\geqq 0} a_i t^i$. Then the reduction $\bmod p$ tends t to zero (t plays the role of "the arithmetical coordinate" [33]). Thus we have a surface (4.2.7) with the singularities which should be resolved by a sequence of σ-processes [28]. For example, the simplest singularity of the surface

$$y^2=x(x-1)(x-p) \to y^2=x(x-1)(x-t)$$

at the point $(x,y,t)=(0,0,0)$ can be resolved by the single σ-process resulting in two components $P^1(\mathbb{F}_p)$ of the section $t=0$ of the surface (which is nothing else but the reduced curve). A stronger singularity of

$$y^2=x(x-1)(x-p^2) \to y^2=x(x-1)(x-t^2)$$

requires one more σ-process resulting in four components $P^1(\mathbb{F}_p)$. Given a curve one can choose the integer coefficients in (4.2.5) in many ways. We constrain them to give a "minimal model" of the curve. By definition, the minimal model should have a minimal number of irreducible components in any fiber of the surface (4.2.7) after resolving all the singularities [28]. For instance, the minimal model of an elliptic curve has the discriminant with the minimal p-adic order (for more details see [32]). In fact, the number of components originating from a singular point of the minimal model is equal to the length of the corresponding F^R-segment under the map (4.2.6). Thus, now all irreducible components are mapped to the edges of the segment bijectively.

For elliptic curves this description is part of the Neron-Kodaira classification [32]. For higher genera, there are different patterns of degeneration corresponding to topologically inequivalent reduced graphs. See examples for genus two in Fig. 10. So the moduli space is naturally divided into distinct domains.

Finally, note that the quantity S introduced in (3.1.5) is nothing else but the number of all components $P^1(\mathbb{F}_p)$ of the whole reduced curve, i.e. it coincides with the number δ_p in Ref. [30], which is hoped to be closely related to the Mumford measure on the moduli space [29].

4.3. *The Mumford Curves and the Moduli Space*

Thus we have observed that the main information about the moduli space is contained in the reduced graphs. Certainly, one would like to sum various quantities over the moduli space. Working with fixed p, it is natural to expect that this sum should go over only Mumford curves, since the Riemann surfaces but not algebraic curves provide an adequate description in the fixed p framework. So we introduce a sum $Z_p=\sum_{\mathscr{M}_{\mathrm{RS}}} f_p(\mathscr{C})$, where f_p is a function on the "moduli space" $\mathscr{M}_{\mathrm{RS}}$ of Riemann surfaces over $\mathbb{Q}_p$. In particular, a proper density on the moduli space should give the p-adic string partition function (the notation Z_p refers to this case).

On the other hand, working with arithmetic surfaces [30] requires us to consider all the prime numbers simultaneously, i.e. this treatment implies summing some "adelic quantities" over the moduli of the algebraic curves: $A = \sum_{\substack{\text{Arithmetic}\\ \text{surfaces}}} \psi^{\text{adelic}}(\mathscr{C})$. The quantity ψ^{adelic} is expected to be the product like $\prod_p \psi_p$ in analogy with Eq. (1.2). The p-component ψ_p of ψ^{adelic} is quite similar to f_p. Moreover, these two quantities are supposed to be the same, when f_p is restricted to the subspace of the Mumford curves space. It should be noted that the idea of product formulas was proposed by Manin [11].

We would like to say some words about the notion of arithmetic surfaces [30, 34]. Let a curve $\mathscr{C}$ be given over a global field, say, the rational number field $\mathbb{Q}$ (for simplicity we consider the curve embedded into $\mathbb{P}^2$ here):

$$P(x,y)=0 \ , \qquad x,y\in\mathbb{Q} \ , \qquad P\in\mathbb{Z}[T_1,T_2] \ . \tag{4.3.1}$$

Here $\mathbb{Z}[T_1,T_2]$ is the ring of polynomials with the integer coefficients [cf. (4.2.5)]. Then one can consider the reductions ("fibres") $\bar{\mathscr{C}}^{(p)}$ of $\mathscr{C}$ at each place p of the field $\mathbb{Q}$. In order to "compactify" this construction Arakelov has introduced [30, 33, 34] a "non-existing" fiber over the archimedean place $\{\infty\}$. The number of degenerate fibres $\bar{\mathscr{C}}^{(p^*)}$ over p^* is always finite, with the archimedean fibre assumed to be highly degenerate (in a rather sophisticated sense). Such p^*'s are called the places of bad reduction of the curve. All these data can be arranged into the "arithmetic surface" $\mathfrak{C}$:

$$\begin{array}{c} \mathfrak{C} \\ {\scriptstyle\pi}\big\downarrow \\ B\equiv \operatorname{Spec}\mathbb{Z}\cup\{\infty\} \ . \end{array} \tag{4.3.2}$$

Here Spec $\mathbb{Z}$ denotes the set of all prime ideals of the ring $\mathbb{Z}$, with the maximal ideals being generated by the prime numbers (i.e. the non-archimedean places of $\mathbb{Q}$). This construction may be considered as a fibre bundle over the base B (spread in an arithmetic direction) with the fibres over $p\in B$ being the reduced curves $\bar{\mathscr{C}}^{(p)}$. There exists also a generic fibre which is the initial curve over $\mathbb{Q}$. It grows over zero ideal in Spec $\mathbb{Z}$.

After these preliminaries are done we are ready to formulate our conjecture (only this conjecture can justify the summing over moduli space under fixed p [35]):

Conjecture.

$$A=\sum_{\substack{\text{Arithmetic}\\ \text{surfaces}}} \prod_p \psi_p(\mathscr{C}) \to \prod_p Z_p = \prod_p \sum_{\mathscr{M}_{\mathbf{RS}}} f_p(\mathscr{C}) \ . \tag{4.3.3}$$

At the present time we are not able to give any precise meaning to the quantities entering this formula. In particular, we hope that the arrow should imply an equality provided all the quantities are correctly defined. We would like to consider the formula (1.1) as a very special case of (4.3.3). The left-hand side of (4.3.3) is also reminiscent of the proper adelic expression for the string measure $\prod_p \mu_p \equiv \prod_p e^{\delta_p}$ (up to possible zero mode factors) which can be extracted from [30, 35].

We immediately observe from (4.3.3) that non-Mumford curves give the unit contribution to Z_p. It follows that the expression for ψ_p is non-trivial only at the places of split reduction; this makes both sides of (4.3.3) consistent.

The first problem to be resolved is to check the above conjecture for genera 1 and 2 when a convenient parametrization in terms of ramification points exists. In these cases the sums in (4.3.3) can be transformed into the sums over integer numbers.

5. Concluding Remarks

Thus we have demonstrated the considerable resemblance of the p-adic strings and the usual ones. The underlying reasons of this similarity can be only guessed at this moment. For instance, it is rather favourable to think about both p-adic and archimedean strings as two faces of the unique object associated with the arithmetic surface [30]. This viewpoint is rather close to the adelic formulas like (1.2) (see e.g. [3–7]). It is likely that further investigation of the conjecture (4.3.3) will be helpful in clarifying this point.

In any case a number of important questions remains beyond the scope of this paper [23, 29]. In particular, it is not evident how one should define a string model over $\bar{\Omega}$, and, moreover, it seems that there is no adequate language for the description of $\bar{\Omega}$ itself. A related question is to obtain the p-adic analog (if any exists) of the usual conformal metric.

There are some more questions; we point out the following:

1. A natural formulation of the p-adic string is expected to be given by using the moduli space for all genera (an universal moduli space, or grassmannian [36]). So questions arise whether a maximal unification exists which is a grassmannian related to arithmetic surfaces? Should one sum in (4.3.3) over the universal moduli space or over the moduli for a fixed genus?

2. It is an absolutely unclear question how we can describe a fermionic string on $E\mathbb{Q}_pRS$ (in contrast to the archimedean case). Its formulation on $\mathbb{Q}_pRS$ provided with the corresponding non-local action was proposed by A. Marshakov and one of us (A.Z.) [37], but the extension to $E\mathbb{Q}_pRS$ remains an open question.

To all appearance, all these problems are connected with the absence of an analyticity notion on the tree. Probably, a good understanding of $\bar{\Omega}$ should clarify these points.

All the above refer to the tree with a fixed prime p. But there exists another viewpoint which naturally incorporates the tree into the arithmetic surface approach of Sect. 4.3. That is, besides of the reductions $\mathscr{C}^{(p)}$ of the curves $\mathscr{C}$ one can investigate the curves $\mathscr{C}_p$ over the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ at the places p. They have a Schottky uniformization in place of the split reduction (see Sect. 4). Given the arithmetic surface $\mathfrak{C}$, one can using σ-processes "blow up" all these points, after blowing up the points produced before, etc. This process repeated infinitely many times leads to A-surface $\hat{\mathfrak{C}}$ (a "foam space" [41]). In particular, if $\mathscr{C}=P^1$,

$\bar{\mathscr{C}}^{(p)} = P^1(\mathbb{F}_p)$ has exactly $p+1$ points over the finite residue field $\mathbb{F}_p$, then one can blow up each point to the whole branch of the Bruhat-Tits tree [i.e. paste $P^1(\mathbb{F}_p)$ at each point of $P^1(\mathbb{F}_p)$ and then repeat this process infinitely many times]. One eventually obtains the Bruhat-Tits tree. Just the same, if the curve $\mathscr{C}$ has a split reduction at p^*, then $\bar{\mathscr{C}}^{(p^*)}$ is described by the reduced graph, and the points of this graph can be blown up to produce the branches of the factorized Bruhat-Tits tree (see Fig. 11). In the case of $\mathscr{C} = P^1$, the corresponding A-surface $\hat{\mathfrak{P}}^1$ is a union of the Bruhat-Tits trees growing over all places p together with the Poincaré disc over the

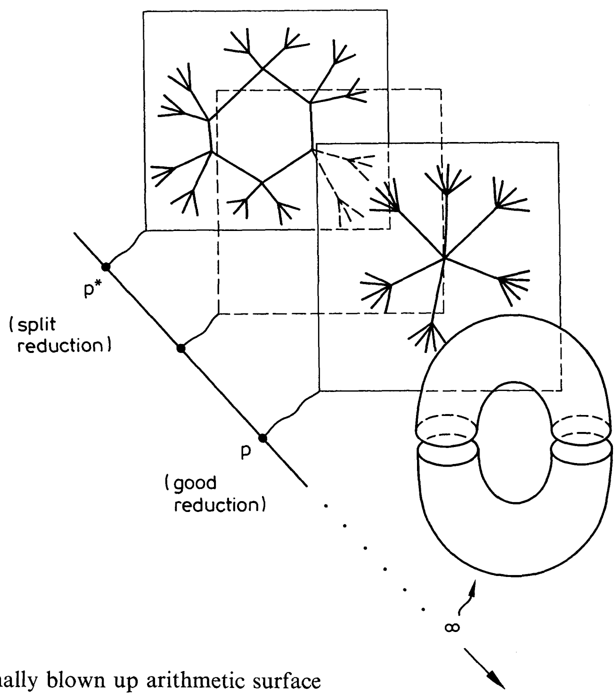

Fig. 11. Maximally blown up arithmetic surface

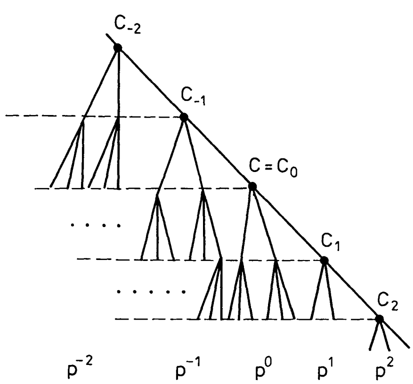

Fig. 12. The correspondence between rays and numbers

archimedean place ∞. So this Poincaré disc is analogous to the "maximally blown up" fibre over a finite place of A-surface. The general $\hat{\mathfrak{C}}$ is a union of trees over places of good reduction (Fig. 11) and a finite number of "singular fibres". Roughly speaking, the whole family $\hat{\mathfrak{C}}$ can be given by its singular fibres.

This "arithmetic catastrophe theory" viewpoint seems to be rather natural. We hope that such an approach can be more suitable for the further development in non-archimedean physics.

Appendix A

In this Appendix the basic results on the coordinatization of ∂T are briefly reviewed. We follow [10].

One can introduce a coordinate function on $P^1(\mathbb{Q}_p) \cong \partial T$, i.e. to identify the boundary of the tree T with the field of p-adic numbers. In terms of the tree this amounts to choosing three rays leading to the points of the boundary which are to be identified with $0, 1$ and ∞. Any three distinct points on ∂T (i.e. three rays) uniquely define a point inside T, namely, it is the common starting point of the corresponding three rays, the rays being chosen to have no common edges. We shall denote such a point for the rays $0, 1, \infty$ as C.

In order to clarify the rules of coordinatization, it is useful to interpret the tree in a somewhat different way (Fig. 12). Let us write down the number $x \in \mathbb{Q}_p$ in the form:

$$x = p^n \cdot u \ , \qquad u = a_0 + a_1 p + a_2 p^2 + \dots \ , \tag{A.1}$$

where the coefficients a_i take values in the residue field $\mathbb{F}_p$ and $a_0 \neq 0$. Then the ray $C \to x$ corresponding to (A.1) coincides with the path $\infty \to 0$ until the vertex C_n is encountered and, further, goes within the corresponding branch. The direction to be chosen at the i^{th} step when moving inside this branch is determined by the coefficient a_{i-1} in (A.1). So we have an identification (non-canonical) $\partial T \cong \hat{\mathbb{Q}}_p$, where $\hat{\mathbb{Q}}_p = \mathbb{Q}_p \cup \{\infty\}$.

The p-adic norm has a nice interpretation in terms of the "coordinatized" tree. For x_1, x_2, y_1 and $y_2 \in \partial T$ let $\langle x_1 \to x_2, y_1 \to y_2 \rangle$ be defined by Eq. (3.1.1). (In the notation of [10] this quantity was denoted as $\delta(x_1 \to x_2, y_1 \to y_2)$; it is the length of the common part of the oriented paths $x_1 \to x_2$, $y_1 \to y_2$ with the negative sign when the orientations are opposite.) Then the p-adic norm of the cross-ratio of the four points is

$$\left| \frac{(x_1 - y_1)(x_2 - y_2)}{(x_1 - y_2)(x_2 - y_1)} \right|_p = p^{-\langle x_1 \to x_2, y_1 \to y_2 \rangle} \ . \tag{A.2}$$

In particular, we have

$$|1 - x/y|_p = p^{-\langle x \to 0, y \to \infty \rangle} \ . \tag{A.3}$$

Appendix B

In this Appendix we present analytic constructions over $\mathbb{Q}_p$ used in the main body of the paper. They are well-defined only over $\bar{\Omega}$ but we shall restrict them to $\mathbb{Q}_p$.

At first, let us consider a formal Laurent series over $\bar{\Omega}: f(z)=\sum_{-\infty}^{+\infty} a_n z^n$, which converges in a point $z_0 \in \bar{\Omega}$ if and only if $\lim_{n\to\infty} |a_n z_0^n|_p = 0$. The point t is called critical for f if $\exists i, j (i \neq j): |a_i|_p^i t = |a_j|_p^j t = \max_{k\in\mathbb{Z}} |a_k|_p t^k$. Then an arbitrary function f (given on a finite segment) has the following properties [16]:

1. The set of critical points $\{t_i\}$ is finite, $t_i \in |\bar{\Omega}|_p$ and $|f(z)|_p$ may vanish only when $|z|_p = t_i$.

2. If $|z|_p \notin \{t_i\}$, $|f(z)|_p$ is the function of $|z|_p$ only. Then $\mathrm{ord}_p f(z)$ is a piecewise linear function of $\mathrm{ord}_p z$ with the slope growing with $\mathrm{ord}_p z$.

Now we define the p-adic one-dimensional θ-function which gives an example of converging Laurent series with the critical points $t_i = |q|_p^i$, $i \in \mathbb{Z}$, $|q|_p < 1$ [16, 17]:

$$\theta_p(z,q) = \prod_{n>0} (1-q^n z) \prod_{n\geq 0} (1-q^n z^{-1}) \ . \tag{B.1}$$

It is a literal counterpart of the usual θ-function over $\mathbb{C}$ [38]:

$$\theta_{11}(z,q) = c(q) z^{1/2} \prod_{n>0} (1-q^n z) \prod_{n\geq 0} (1-q^n z^{-1}) \ , \tag{B.2}$$

where we redenote $e^{2\pi i\tau} \to q$ and $e^{2\pi i z} \to z$; $c(q)$ does not depend on z. The only difference between (B.1) and (B.2) is the factor $z^{1/2}$. It can be easily included as $z^{1/2} = p^{\frac{1}{2}\log_p z} \to p^{-\frac{1}{2}\mathrm{ord}_p z}$. One can see [16] that:

$$\begin{cases} \mathrm{ord}_p \theta_p = 0 & \text{for} \quad -\mathrm{ord}_p q < \mathrm{ord}_p z < 0 \\ \mathrm{ord}_p \theta_p = -\mathrm{ord}_p z & \text{for} \quad 0 < \mathrm{ord}_p z < \mathrm{ord}_p q \\ \mathrm{ord}_p \theta_p = -\mathrm{ord}_p q - 2\,\mathrm{ord}_p z & \text{for} \quad \mathrm{ord}_p q < \mathrm{ord}_p z < 2\,\mathrm{ord}_p q \ , \end{cases} \tag{B.3}$$

and so on. We can rewrite (B.3) as

$$\mathrm{ord}_p \theta_p = -\frac{(\mathrm{ord}_p z)^2}{2\,\mathrm{ord}_p q} - \frac{\mathrm{ord}_p z}{2} + \psi \ , \tag{B.4}$$

where ψ is a periodic function with the period $\mathrm{ord}_p q$, which can be obtained from the condition $\mathrm{ord}_p \theta_p = 0$ for $-\mathrm{ord}_p q < \mathrm{ord}_p z < 0$.

Now we shall restrict all formulas to $\mathbb{Q}_p$. It is convenient to work in a special fundamental domain: $-\mathrm{ord}_p q < \mathrm{ord}_p z \leqq 0$. Then $\mathrm{ord}_p \theta_p$ is zero in non-critical points ($\mathrm{ord}_p z \neq 0$).

This technique we apply to θ_p-functions of many variables. That is, define a formal Laurent series $f(z_1, \ldots, z_n)$ like θ_p-function as in [16]. Then the above statement is valid with the only change that the critical points should be replaced by critical subspaces of unit codimension. They form a complex, which in fact is the Voronoj decomposition, and it results in the skeleton which is a visualization of the period lattice. Consider the simplest nontrivial example of $n=2$. Then the possible Voronoj decompositions together with the corresponding reduced graphs are depicted in Fig. 13. Moving along the critical line corresponds to moving along a cycle in the reduced graph. When two cycles do not overlap, the Voronoj lattice

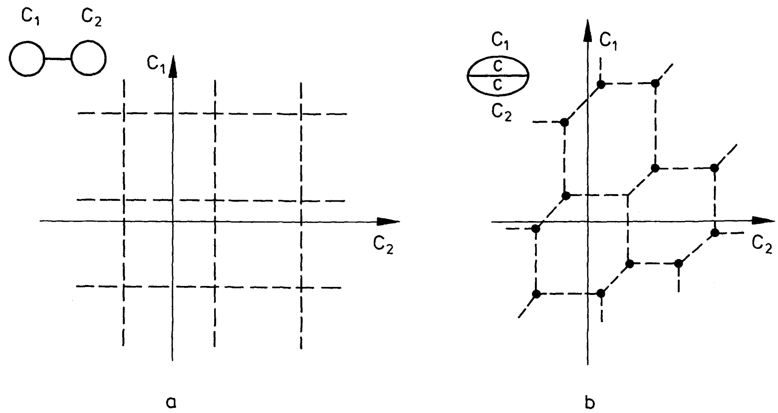

Fig. 13. The Voronoj decomposition in two dimensions

evidently is square (two axes giving this lattice are independent). The elementary cell corresponds to a fixed choice of the fundamental domain. We may again choose such a fundamental domain that $\mathrm{ord}_p\theta_p$ is zero outside the critical points. (It can be done due to the above theorem, see the proof below.) One can write the product formula for θ_p-functions analogous to (B.1) [39], but we omit it here, as we need a slightly different object, namely, the Prime form $E(x,y)$. In the archimedean case it is defined to be

$$E(x,y)=\frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int\limits_y^x \omega_i\right)}{h(x)h(y)}\ . \tag{B.4}$$

Here $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is any odd θ-characteristic, $\{\omega_i\}$ is a basis of holomorphic 1-differentials, $h(z)$ is the holomorphic 1/2-form:

$$h^2(z)=\sum_{i=1}^{g}\omega_i(z)\partial_{,i}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)\ .$$

Though the abelian differentials and other objects in (B.4) are rather complicated in the Schottky group terms [39, 40], the infinite product expansion for $E(x,y)$ has a simple form [27]:

$$E(x,y)=(x-y)\prod_{\gamma\in\Gamma}{}'\ \frac{[x-\gamma(y)]\,[y-\gamma(x)]}{[x-\gamma(x)]\,[y-\gamma(y)]}\ . \tag{B.5}$$

The product goes over all elements of the Schottky group (except the unit element) with γ and γ^{-1} elements counted only once. The p-adic formulas should be obtained simply by replacing the variables $x,y\in\mathbb{C}$ by $x,y\in\mathbb{Q}_p$.

Now we demonstrate that for any Schottky group Γ_g it is possible to choose such a fundamental domain $\partial F(\Gamma_g)\subset\hat{\mathbb{Q}}_p$ that $|E(x,y)|_p=|x-y|_p$ for $x,y\in\partial F(\Gamma_g)$, i.e. the infinite product term in (B.5) gives no contribution. Let us consider the Schottky tree for Γ_g. It is in fact the universal covering space for the reduced graph F_g^R. (The example of such a covering for Γ_2 is depicted in Fig. 14a). Given the graph F_g^R

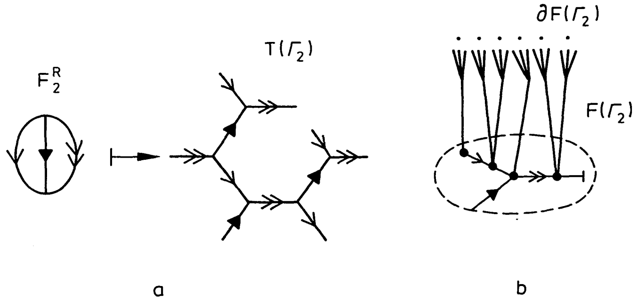

Fig. 14. An example of the Schottky tree $T(\Gamma_2)$. The choice of the connected fundamental domain $F(\Gamma_2)$ leads to trivialization of the Prime form (B.5)

containing R segments $\mathcal{S}_i$, $i=1,\ldots,R$. Any segment $\mathcal{S}_k$ is replicated infinitely many times in the Schottky tree. We denote these copies as $\mathcal{S}_k^{\{\alpha\}}$. Any appropriate fundamental domain $F(\Gamma_g)\subset T$ may be determined by the set of R arbitrary copies of these R segments $\mathcal{S}_1^{\{\alpha_1\}}\cup\mathcal{S}_2^{\{\alpha_2\}}\cup\ldots\cup\mathcal{S}_R^{\{\alpha_R\}}$. The fundamental domain $\partial F(\Gamma_g)$ is the collection of points at infinity of all branches growing from the vertices belonging to this union. The "good" choices of such domains are subtrees $F(\Gamma_g)$ which are *connected* in the tree T. It is easy to show that such domains do exist for an arbitrary Schottky tree. (They can be obtained if one cuts all $\mathfrak{g}$ loops of F_g^R by exactly $\mathfrak{g}$ cuts, the obtained graph being connected without loops and any of its connected replica in the Schottky tree gives us a connected graph $F(\Gamma_g)$). It means that for any two points $x,y\in\partial F(\Gamma_g)$ the path $x\to y\subset F(\Gamma_g)$. (The example of such a choice for Γ_2 is depicted in Fig. 14b.)

Consider now the action of an element γ on the subdomain $F(\Gamma_g)$. It follows that $\gamma(F(\Gamma_g))\cap F(\Gamma_g)=\emptyset$ and the path $\gamma(x)\to\gamma(y)\subset\gamma F(\Gamma_g))$. Let us calculate the cross-ratio (A.2),

$$\left|\frac{(x-\gamma(y))\,(y-\gamma(x))}{(x-\gamma(x))\,(y-\gamma(y))}\right|_p=p^{-\langle x\to y,\,\gamma(x)\to\gamma(y)\rangle}\;.$$

It is clear from the above that for connected $F(\Gamma_g)$ and any $\gamma\in\Gamma_g$ the intersection $\langle x\to y,\gamma(x)\to\gamma(y)\rangle=0$. Thus for this fundamental domain the infinite product in (B.5) gives no contribution.

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