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The Large-Scale Limit of Dyson's Hierarchical Vector-Valued Model at Low Temperatures. The marginal case $c=\sqrt{2}$

P. M. Bleher[1] and P. Major[2]

[1] Keldysh Institute for Applied Mathematics of the Academy of Sciences of the USSR, Moscow A-47 Miusskaya Square 4, USSR
[2] Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127, H-1364 Budapest, Hungary

Dedicated to Roland Dobrushin

Abstract. In this paper we construct the equilibrium states of Dyson's vector-valued hierarchical model with parameter $c=\sqrt{2}$ at low temperatures and describe their large-scale limit. The analogous problems for $\sqrt{2}<c<2$ and $1<c<\sqrt{2}$ were solved in our papers [1] and [2]. In the present case the large-scale limit is similar to the case $\sqrt{2}<c<2$, i.e. it is a Gaussian self-similar field with long-range dependence in the direction orthogonal to and a field consisting of independent Gaussian random variables in the direction parallel with the magnetization. The main difference between the two cases is that now the normalizing factor in the direction of the magnetization contains, beside the square-root of the volume, a logarithmic term too.

1. Introduction

First we briefly describe the model we are investigating. Dyson's hierarchical model is a one-dimensional classical spin model on the lattice $\mathbf{Z}=\{1,2,\dots\}$. Its Hamiltonian function depends on a parameter a, $1<a<2$, and is defined as

$$\mathscr{H}(\sigma)=-\sum_{i\in\mathbf{Z}}\sum_{\substack{j\in\mathbf{Z}\\ j>i}} d(i,j)^{-a}\sigma(i)\sigma(j), \tag{1.1}$$

where $d(i,j)=2^{n(i,j)-1}$, and

$$n(i,j)=\min\{n, \text{ there exists some } k \text{ such that } (k-1)2^n<i,j\leqq k2^n\}.$$

We are dealing with vector-valued models, where $\sigma(j)\in R^p$ with some $p\geqq 2$. If $x\in R^p$ and $y\in R^p$ then xy denotes scalar product. We consider models with the free measure ν,

$$\frac{d\nu}{dx}(x)=p_0(x)=p_0(x,t)=C(t)\exp\left\{-\frac{x^2}{2}-\frac{t}{4}|x|^4\right\},\qquad x\in R^p, \tag{1.2}$$

where $t>0$ is a sufficiently small number, and $C(t)$ is an appropriate norming constant which turns $p_0(x)$ into a density function. For the sake of convenience we shall work in the sequel with the number $c=2^{2-a}$ instead of the parameter a.

We investigate the following problem: First we construct an equilibrium state $\mu=\mu(T)$ at low temperatures with magnetization in the direction of the first coordinate $x^{(1)}$ and then we want to describe its large-scale limit. In more detail, let

$$\sigma=\{\sigma(j)=(\sigma^{(1)}(j),\dots,\sigma^{(p)}(j))\in R^p, j\in\mathbf{Z}\}$$

be a random field with the distribution of the equilibrium state $\mu=\mu(T)$, and define for all $n=1,2,\dots$ the random field

$$\mathscr{R}_n\sigma=\{(\mathscr{R}_n\sigma^{(1)}(j),\dots,\mathscr{R}_n\sigma^{(p)}(j))\in R^p, j\in\mathbf{Z}\},\tag{1.3}$$

$$\mathscr{R}_n\sigma^{(1)}(j)=\frac{1}{A_n}\sum_{k=(j-1)2^n+1}^{j2^n}[\sigma^{(1)}(k)-E\sigma^{(1)}(k)],\quad j\in\mathbf{Z},\tag{1.4}$$

$$\mathscr{R}_n\sigma^{(s)}(j)=\frac{1}{B_n}\sum_{k=(j-1)2^n+1}^{j2^n}\sigma^{(s)}(k),\quad j\in\mathbf{Z},\quad s=2,\dots,p,\tag{1.5}$$

where A_n and B_n are appropriate norming constants. We want to choose them in such a way that the finite dimensional distributions of the fields $\mathscr{R}_n\sigma$ converge as $n\to\infty$, and also want to describe the limit field. Here A_n is the norming constant in the direction of the magnetization and B_n in the direction orthogonal to it.

We have solved this problem for $\sqrt{2}<c<2$ in our paper [1] and for $1<c<\sqrt{2}$ in [2]. In both cases we have to choose a "critical" normalization $B_n=2^nc^{-n/2}$ in the direction orthogonal to the magnetization, and the limit is a self-similar Gaussian field with long-range correlation. On the other hand, in the direction of the magnetization we have a different situation in the two cases. For $\sqrt{2}<c<2$ we have to choose $A_n=2^{n/2}$ and get a field of independent Gaussian variables for the limit. For $1<c<\sqrt{2}$ the right choice in (1.4) is $A_n=2^nc^{-n}$, and the limit is a non-Gaussian field which we have described explicitly in [2]. Our aim in this paper is to solve this problem for $c=\sqrt{2}$. The answer is very similar to the case $\sqrt{2}<c<2$. Namely, we have to choose $B_n=2^nc^{-n/2}=2^{3n/4}$ and get a dependent Gaussian field in the direction orthogonal to the direction of the magnetization. In the direction of the magnetization we have to choose $A_n=2^{n/2}\sqrt{n}$, and the limit is a field consisting of independent Gaussian random variables. The main difference between the cases $\sqrt{2}<c<2$ and $c=\sqrt{2}$ is the appearance of multiplying term $\sqrt{n}$ in the normalizing factor A_n in the latter case. It is expected that translation invariant models with short-range interaction in the cases $d<4$, $d=4$ and $d>4$ show a behaviour similar to Dyson's model in the cases $1<c<\sqrt{2}$, $c=\sqrt{2}$ and $\sqrt{2}<c<2$. Thus Dyson's model with $c=\sqrt{2}$ corresponds to four-dimensional translation invariant models.

Let us formulate our results in more detail. In Theorem 1 formulated below we construct the equilibrium state whose large-scale limit will be investigated.

Given some $h\in R^1$, $h\geqq 0$, and a positive integer n let us define the Gibbs measure $\mu_n^h=\mu_n^h(T,t)$ on $(R^p)^{2^n}$ with the density function

$$p_n^h(x_1,\dots,x_{2^n})=p_n^h(x_1,\dots,x_{2^n},t,T),\quad x_j=(x_j^{(1)},\dots,x_j^{(p)})\in R^p,\quad j=1,\dots,2^n,$$

given by the formula

$$p_n^h(x_1, \ldots, x_{2^n})$$
$$= Z_n^{-1}(T, t, h) \exp\left\{-\frac{1}{T}\left(-\sum_{i=1}^{2^n-1} \sum_{j=i+1}^{2^n} d(i,j)^{-3/2} x_i x_j - h \sum_{j=1}^{2^n} x_j^{(1)}\right)\right\} \prod_{j=1}^{2^n} p_0(x_j, t), \tag{1.6}$$

where

$$Z_n = \int \exp\left\{-\frac{1}{T}\left(-\sum_{i=1}^{2^n-1} \sum_{j=i+1}^{2^n} d(i,j)^{-3/2} x_i x_j - h \sum_{j=1}^{2^n} x_j^{(1)}\right)\right\} \prod_{j=1}^{2^n} p_0(x_j, t) dx_j$$

is the grand partition function, and $p_0(x, t)$ is defined in (1.2). Let $p_n^h(x) = p_n^h(x, T)$ denote the density function of the average $2^{-n} \sum_{j=1}^{2^n} \sigma(j)$ of the μ_n^h distributed random vector $(\sigma(1), \ldots, \sigma(2^n))$. Put $\mu_n = \mu_n^h$, $p_n(x_1, \ldots, x_{2^n}) = p_n^h(x_1, \ldots, x_{2^n})$ and $p_n(x) = p_n^h(x)$ in the case $h = 0$.

Let us introduce the functions

$$q_n(x) = q_n(x, T) = K_n \exp\left\{\frac{a_0}{2a_1} 2^{n/2} x^2\right\} p_n\left(\sqrt{\frac{T}{a_1}}\, x\right) \tag{1.7}$$

with $a_0 = \dfrac{2}{2-\sqrt{2}}$, $a_1 = a_0 + 1$ and the above defined functions $p_n(x)$, where the norming constant K_n will be appropriately chosen. The function $q_n(x, T)$ is rotation invariant, i.e. the function $\bar{q}_n(z, T)$, $z \in R^1$, defined by the formula $\bar{q}_n(z, T) = q_n((z, 0), T)$, $z \in R^1$, $0 = (0, \ldots, 0) \in R^{p-1}$ satisfies the relation $q_n(x, T) = \bar{q}_n(|x|, T)$. Choose the constant K_n in (1.7) in such a way that

$$\int_0^\infty \bar{q}_n(x, T) dx = 1,$$

and define the numbers

$$M_n = \int_0^\infty x \bar{q}_n(x, T) dx. \tag{1.8}$$

Now we formulate the following

Theorem 1. *There are some thresholds $T_0 > 0$ and $t_0 > 0$ such that if $0 < T < T_0$ and $0 < t < t_0$ then the limit $M = \lim_{n\to\infty} M_n > 0$ exists, and $M^2 = \dfrac{a_1(a_0 - T)}{tT^2} + O(1)$ with $a_0 = \dfrac{2}{2-\sqrt{2}}$ and $a_1 = a_0 + 1$. Moreover, the following relation holds: Put*

$$\bar{M} = \sqrt{\frac{T}{a_1}}\, M, \tag{1.9}$$

and consider an arbitrary sequence of real numbers h_n, $n = 0, 1, 2, \ldots$ such that

$$\frac{2\bar{M}}{2-\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n \leq h_n \leq D \left(\frac{1}{\sqrt{2}}\right)^n \tag{1.10}$$

with some $\infty > D > \dfrac{2\bar{M}}{2-\sqrt{2}}$. *Then the measures* $\mu_n^{h_n}$ *tend to a probability measure* $\bar{\mu}=\bar{\mu}(t,T)$ *on* $(R^p)^{\mathbf{Z}}$. *More precisely, for all* $k \geqq 0$ *the measures* $\mu_{k,n}^{h_n}$, *the projections of the measures* μ_n^h *to* $(R^p)^{2^k}$, *converge to the projection of* $\bar{\mu}$ *to the first* 2^k *coordinates in variational metric as* $n \to \infty$. *The measure* $\bar{\mu}$ *does not depend on the choice of sequences* h_n.

The main result of this paper is the following

Theorem 2. *Let* $\sigma=\{\sigma(n)=(\sigma^{(1)}(n),\dots,\sigma^{(p)}(n))\in R^p,\ n\in\mathbf{Z}\}$ *be a* $\bar{\mu}$ *distributed random field with the distribution* $\bar{\mu}$ *defined in Theorem 1. Then the finite dimensional distributions of the random fields* $R_n\sigma$ *defined in* (1.3), (1.4), (1.5) *tend, with the choice* $A_n=2^{n/2}\sqrt{n}$ *and* $B_n=2^{3n/4}$, *to those of a Gaussian random field* $Y=(Y(n)) =(Y^{(1)}(n),\dots,Y^{(p)}(n))\in R^p$, $n\in\mathbf{Z}$). *For all* $k\geqq 0$ *the density function* $h_k(x_1,\dots,x_{2^k})$, $x_j=(x_j^{(1)},\dots,x_j^{(p)})\in R^p$ *of the random vector* $(Y(1),\dots,Y(2^k))$ *is given by the formula*

$$\begin{aligned} &h_k(x_1,\dots,x_{2^k})\\ &=C(k)\exp\left\{-\frac{1}{T}\left[\sum_{s=2}^{p}\left(\frac{2+\sqrt{2}}{2}\sum_{j=1}^{2^k}x_j^{(s)2}-(\sqrt{2}-1)\left(\frac{\sqrt{2}}{4}\right)^k\left(\sum_{j=1}^{2^k}x_j^{(s)}\right)^2\right.\right.\right.\\ &\quad\left.\left.\left.-\sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k}d(i,j)^{-3/2}x_i^{(s)}x_j^{(s)}\right)+(6+2\sqrt{2})M^2\sum_{j=1}^{2^k}x_j^{(1)2}\right]\right\}. \end{aligned} \tag{1.11}$$

It follows from the result in Appendix E of [2] that the measure constructed in Theorem 1 is an equilibrium state. We restricted ourselves to the construction of equilibrium states for low temperatures where we are interested in their large-scale limit. The proofs of Theorems 1 and 2 are based, similar to the papers [1] and [2], on two analytic problems, where the action of an integral operator must be investigated. We formulate these problems in the next section.

2. The Basic Steps of the Proof

In this section we discuss two analytical problems which play a central role in the proof of Theorems 1 and 2. The first one is connected with the asymptotic behaviour of the density function $p_n(x)$ of the average of a μ_n distributed vector defined after formula (1.6). It is proved (see e.g. Appendix A in [2]) that $p_n(x)$ satisfies the recursive relation

$$p_{n+1}(x)=C_n(T)\int\exp\left\{\frac{2^{n/2}}{T}(x^2-u^2)\right\}p_n(x-u)p_n(x+u)du \tag{2.1}$$

with the starting function $p_0(x)$ defined in formula (1.2). For us it is more convenient to work with the functions $q_n(x)$ defined in (1.7) instead of the functions $p_n(x)$. Simple calculation shows that relations (2.1) and (1.7) imply the recursive relations

$$q_{n+1}(x)=K_n\int\exp\{-2^{n/2}u^2\}q_n(x-u)q_n(x+u)du \tag{2.2}$$

with the starting function

$$q_0(x)=q_0(x,T,t)=K_0\exp\left\{\frac{a_0-T}{2a_1}x^2-\frac{tT^2}{4a_1^2}|x|^4\right\}, \tag{2.2'}$$

where $a_0 = \dfrac{2}{2-\sqrt{2}}$, $a_1 = a_0 + 1$, and K_n is an appropriate norming constant. (The numbers a_0 and a_1 will denote these numbers in the whole paper.)

In Theorem A formulated below we describe the asymptotic behaviour of the function $q_n(x)$. We recall that we have introduced the functions $\bar{q}_n(z) = \bar{q}_n(z, T)$, $z \in R^1$ in Sect. 1, and they satisfy the relation $q_n(x, T) = \bar{q}_n(|x|, T)$.

Theorem A. *There are some thresholds t_0 and T_0 such that for $0 < t < t_0$ and $0 < T < T_0$ the functions $q_n(x)$ defined by formulas (2.2) and (2.2′) satisfy the following relations:*

There are some $M = M(T, t) > 0$ and $n_0 = n_0(T, t) > 0$ such that for $n > n_0$,

$$2^{-n/2}\sqrt{n}q_n(x, T) = 2^{-n/2}\sqrt{n}\bar{q}_n(|x|, T) = \frac{\sqrt{2}M}{\sqrt{\pi}} \exp\left\{-\frac{2^{n+1}M^2}{n}(|x| - M)^2\right\} + r_n(x) \tag{2.3}$$

with

$$|r_n(x)| \leqq \frac{K}{\sqrt{n}} \tag{2.3′}$$

and

$$\left|M^2 - \frac{a_1(a_0 - T)}{tT^2}\right| \leqq K \tag{2.4}$$

with some $K > 0$. Also the estimate

$$2^{-n/2}\sqrt{n}\bar{q}_n(x, T) \leqq K \exp\left\{-\frac{2^{n/2}\mu}{\sqrt{n}}|x - M|\right\} \quad \textit{for all} \quad x > 0 \tag{2.5}$$

holds with some $K > 0$ and $\mu > 0$ depending on T and t.

For $|x - M| \gg 2^{-n/2}/\sqrt{n}$ we need a better bound on $\bar{q}_n(x, T)$ than that given in (2.5). This is given in the following

Proposition A. *Under the conditions of Theorem A*

$$2^{-n/2}\sqrt{n}\bar{q}_n(x, T) \leqq K \exp\left\{-\beta \frac{2^n}{n}(x - M)^2\right\} \quad \textit{for} \quad x > M \tag{2.6}$$

with some $\beta > 0$ and $K > 0$ depending on t and T.

If $0 < x < M$ then for all $\varepsilon > 0$ there are some thresholds $t_0 = t_0(\varepsilon)$, $T_0 = T_0(\varepsilon)$ and a real number r_n, $C_1 n 2^{-n/2} < M - r_n < C_2 n 2^{-n/2}$ with $C_2 > C_1 > 0$ such that if $0 < t < t_0$ and $< T < T_0$ then

$$2^{-n/2}\sqrt{n}\bar{q}_n(x, T) \leqq K \exp\left\{-\beta \frac{2^n}{n}(x - M)^2\right\} \quad \textit{for} \quad r_n < x < M \tag{2.7}$$

and

$$2^{-n/2}\sqrt{n}\bar{q}_n(x, T) \leqq K \exp\left\{-(1-\varepsilon)2^{n/2}(r_n^2 - x^2) - \beta \frac{2^n}{n}(r_n - M)^2\right\} \quad \textit{for} \quad 0 < x < r_n. \tag{2.7′}$$

(With some extra-work it can be shown that the number r_n can be chosen in the form $r_n = M - Cn2^{-n/2}$ with some $C>0$. We do not prove it, because the slightly weaker statement formulated above is sufficient for our purposes. Also the dependence of the thresholds t_0 and T_0 on ε can be dropped with the help of some additional investigation. We do not do it, because we do not need Proposition A with very small ε. What we need is that it holds with some $\varepsilon>0$ such that $1-\varepsilon > \frac{a_0}{2a_1}$.)

Now we formulate the second problem we are interested in. Given some integers $0 \leqq n \leqq N$ and positive real number $h>0$ consider the measure μ_N^h with density function $p_N^h(x_1, \ldots, x_{2^N})$ defined in (1.6) (we replace the number n by N in it), and define its projection $\mu_{n,N}^h$ to the first 2^n coordinates $x_1, \ldots, x_{2^n}$. We want to give a good asymptotic formula for the Radon-Nikodym derivative $\frac{d\mu_{n,N}^h}{d\mu_n}(x_1, \ldots, x_{2^n})$, where μ_n is μ_n^h with $h=0$. It can be expressed explicitly by the following recursive integral formula: (See e.g. Appendix C in [2].)

Formula for the Radon-Nikodym Derivative

$$\frac{d\mu_{n,N}^h}{d\mu_n}(x_1, \ldots, x_{2^n}) = f_{n,N}^h\left(2^{-n} \sum_{j=1}^{2^n} x_j\right), \qquad n \leqq N, \tag{2.8}$$

$$f_{N,N}^h(x) = K(N,h) \exp\left(\frac{2^N h x^{(1)}}{T}\right), \tag{2.9}$$

$$f_{n,N}^h(x) = K(n,N,h) \mathbf{S}_n f_{n+1,N}^h(x), \tag{2.10}$$

with

$$\mathbf{S}_n f(x) = \int_{R^p} \exp\left(\frac{2^{n/2}}{T} xy\right) f\left(\frac{x+y}{2}\right) p_n(y) dy, \tag{2.10'}$$

where $K(n,N,h)$ are appropriate norming factors and $p_n(x)$ is the density function of a μ_n distributed random vector.

In Theorem B formulated below we give an asymptotic formula for the function $f_{n,N}^h(x)$ if $h = h_N$ satisfies the relation

$$\frac{2\bar{M}}{2-\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^N \leqq h_N \leqq D\left(\frac{1}{\sqrt{2}}\right)^N \tag{2.11}$$

with some $\infty > D > \frac{2\bar{M}}{2-\sqrt{2}}$. To formulate it we introduce the sequences g_n, A_n, $n = 1, 2, \ldots, N$ defined by the recursive relations

$$g_N = g_N(N, h_N) = \frac{2^N h_N}{T}, \tag{2.12}$$

$$g_n = g_n(N, h_N) = \frac{g_{n+1}}{2} + \frac{2^{n/2}}{T}\bar{M} \quad \text{for} \quad n < N, \tag{2.12'}$$

$$A_N = A_N(N, h_N) = 0, \tag{2.13}$$

$$A_n = A_n(N, h_N) = \frac{A_{n+1}}{4} + \frac{\left(\frac{2^{n/2}}{T} + \frac{A_{n+1}}{2}\right)^2}{\frac{2^{(n+2)/2}}{T} + \frac{g_{n+1}}{\bar{M}} - A_{n+1}} \quad \text{for} \quad n < N, \tag{2.13'}$$

where $\bar{M}$ is defined in (1.9), and M and T are the same as in Theorem A.

For the sake of simpler notations let us restrict ourselves to the case $R^p = R^2$. Let us define the domains

$$\Omega_n^1 = \{x \in R^2, | \, |x| - \bar{M}| < 2^{-0.2n}, |x^{(2)}| < 2^{-0.2n}, x^{(1)} > 0\}, \tag{2.14}$$

$$\Omega_n^2 = \{x \in R^2 \, | \, |x| - \bar{M}| < 2^{-0.2n}\} \setminus \Omega_n^1, \tag{2.14'}$$

$$\Omega_n^3 = \{x \in R^2, | \, |x| - \bar{M}| \geqq 2^{-0.2n}\}. \tag{2.14''}$$

Clearly $\Omega_n^1 \cup \Omega_n^2 \cup \Omega_n^3 = R^2$. Now we formulate the following

Theorem B. *For all q, $2^{-0.1} < q < 1$, there is some $n_0 = n_0(T, \bar{M}, D, q)$ such that if* (2.11) *holds, and $N \geqq n \geqq n_0$ then the Radon-Nikodym derivative $f_n(x) = f_{n,N}^{h_N}(x)$ defined by formulas* (2.9)–(2.10′) *satisfies the following relations:*

a) *In the domain Ω_n^1*

$$f_n(x) = L_n \exp\{g_n(x^{(1)} - \bar{M}) + A_n x^{(2)2} + \varepsilon_n(x)\} \tag{2.15}$$

with

$$\sup_{x \in \Omega^1} |\varepsilon_n(x)| \leqq q^n .$$

b) *In the domain Ω_n^2*

$$0 \leqq f_n(x) \leqq L_n \exp\left\{g_n(|x| - \bar{M}) - \left(\frac{g_n}{2\bar{M}} - A_n\right) 2^{-0.4n} + q^n\right\}. \tag{2.16}$$

c) *In the domain Ω_n^3*

$$0 \leqq f_n(x) \leqq L_n \exp\left\{\frac{g_n}{2\bar{M}} (|x|^2 - \bar{M}^2)\right\}, \tag{2.17}$$

where the numbers A_n and g_n are defined in (2.12)–(2.13′), *and $L_n = L_n(N, h_N)$ is an appropriate norming constant.*

We also need the following result which describes the asymptotic behaviour of the sequences g_n and A_n defined by (2.12)–(2.13′).

Proposition B. *Let us choose some integer N and real number $h_N > 0$. Define the sequences g_n and A_n, $0 \leqq n \leqq N$, by formulas* (2.12)–(2.13′) *and put $\bar{g}_n = 2^{-n/2} g_n$, $\bar{A}_n = 2^{-n/2} A_n$. If h_N satisfies relation* (2.11) *then $\bar{g}_N \geqq \bar{g}_{N-1} \geqq \dots \geqq \bar{g}_0 \geqq \bar{g}$ and $0 = \bar{A}_N \leqq \bar{A}_{N-1} \leqq \dots \leqq \bar{A}_0 \leqq \bar{A}$ with $g = \frac{2}{2 - \sqrt{2}} \frac{\bar{M}}{T}$, and $\bar{A} = \frac{\sqrt{2} - 1}{T}$. If the relations $N > N_0$ and $N > n^B$ also hold with some appropriate $N_0 = N_0(\bar{M}, T, D)$ and $B = B(\bar{M}, T, D)$ then $|\bar{g}_n - \bar{g}| < 4^{-n}$ and $|\bar{A}_n - \bar{A}| < 4^{-n}$.*

The above results enable us to carry out a limiting procedure analogous to that in Sects. 6 and 7 in Part II of [2], which leads to the proof of Theorems 1 and 2. The main step of this limiting procedure is to give a good estimate for the expression

$$p_n\left(2^{-n}\sum_{j=1}^{2^n} x_j\right) f_{n,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2^n} x_j\right). \tag{2.18}$$

Since we can express the function $p_n(x)$ through $q_n(x)$ Theorems A and B together with Proposition B enable us to give a good asymptotic formula for this expression in a typical domain around the point $(\bar{M},0)\in R^2$. Then Theorem B together with Proposition A guarantee that the region outside this typical domain has a negligible effect.

3. On Theorem A. The Method of the Proof

The proof both of Theorem A and B is based on the ideas of [2]. Most proofs can be carried out in almost the same way, only the number c must be replaced by $\sqrt{2}$. The proofs of such parts will be omitted, we only refer to the corresponding result in [2]. From now on the letters C, C_1, K etc. will denote absolute constants. The same letter in different formulas may denote different constants if it is not stated otherwise.

Let us introduce, similarly to Part I in [2], the numbers M_n defined in (1.8) and the functions

$$f_n(x)=f_n(x,T)=2^{-n/2}\bar{q}_n(M_n+2^{-n/2}x,T), \tag{3.1}$$

where the function $\bar{q}_n(x)$ was defined after formula (1.7). We shall deduce Theorem A from the following

Theorem A'. *Under the conditions of Theorem A the limit* $\lim\limits_{n\to\infty} M_n=M>0$ *exists, and*

$$M^2=\frac{a_1(a_0-T)}{tT^2}+R(T,t) \tag{3.2}$$

with some $|R(T,t)|<\text{const}$. *Moreover, there is some* $n_0=n_0(t,T)$ *such that for* $n>n_0$

$$M_n=M+\frac{2+\sqrt{2}}{4M}2^{-n/2}+\delta(n)2^{-n/2}, \qquad |\delta(n)|<K2^{-n/2} \tag{3.3}$$

with some $K>0$. *The function* f_n *satisfies the relations*

$$\left|f_n(x,T)-\frac{\sqrt{2}M}{\sqrt{n\pi}}\exp\left\{-\frac{2M^2}{n}x^2\right\}\right|<\frac{K}{n} \quad \textit{for} \quad x>-2^{n/2}M_n \tag{3.4}$$

and

$$f_n(x,T)\leqq\frac{KM}{\sqrt{n}}\exp\{-\mu|x|\} \quad \textit{for} \quad x>-2^{n/2}M_n \tag{3.5}$$

for $n>n_0$ *with some* $\mu>0$ *and* $K>0$.

To prove Theorem A′ let us introduce, similarly to [2], the operator $\bar{\mathbf{Q}}_{n,M}$,

$$\bar{\mathbf{Q}}_{n,M}f(x)=\int \exp\{-2^{-n/2}u^2-v^2\}$$
$$f(2^{n/2}(\sqrt{(M+2^{-(n+1)/2}x+2^{-n/2}u)^2+2^{-n/2}v^2}-M))$$
$$f(2^{n/2}(\sqrt{(M+2^{-(n+1)/2}x-2^{-n/2}u)^2+2^{-n/2}v^2}-M))du\,dv\,, \tag{3.6}$$

its standardization defined by the formula

$$\mathbf{Q}_{n,M}f(x)=\frac{\bar{\mathbf{Q}}_{n,M}f(x+m_n)}{\int\limits_{-2^{(n+1)/2}M}^{\infty}\bar{\mathbf{Q}}_{n,M}f(x)dx} \tag{3.7}$$

with

$$m_n=\frac{\int\limits_{-2^{(n+1)/2}M}^{\infty}x\bar{\mathbf{Q}}_{n,M}f(x)dx}{\int\limits_{-2^{(n+1)/2}M}^{\infty}\bar{\mathbf{Q}}_{n,M}f(x)dx} \tag{3.7′}$$

together with their approximations $\mathbf{T}_M$ and $\bar{\mathbf{T}}_M$ given by the formulas

$$\bar{\mathbf{T}}_M f(x)=\int e^{-v^2}f\left(\frac{x}{\sqrt{2}}+u+\frac{v^2}{2M}\right)f\left(\frac{x}{\sqrt{2}}-u+\frac{v^2}{2M}\right)du\,dv \tag{3.8}$$

and

$$\mathbf{T}_M f(x)=\sqrt{\frac{2}{\pi}}\,\bar{\mathbf{T}}_M f\left(x-\frac{\sqrt{2}}{4M}\right). \tag{3.8′}$$

The same calculation as that in (2.20) of [2] yields that the Fourier transforms of the operators $\mathbf{T}_M$ and $\bar{\mathbf{T}}_M$ defined by the formulas $\tilde{\bar{\mathbf{T}}}_M\tilde{f}=(\bar{\mathbf{T}}_M f)^{\sim}$ and $\tilde{\mathbf{T}}_M\tilde{f}=(\mathbf{T}_M f)^{\sim}$ satisfy the relation

$$\tilde{\bar{\mathbf{T}}}_M\tilde{f}(\xi)=\sqrt{\frac{\pi}{2}}\frac{\tilde{f}\left(\frac{\xi}{\sqrt{2}}\right)^2}{\sqrt{1+\frac{i\xi}{\sqrt{2}M}}} \tag{3.9}$$

and

$$\tilde{\mathbf{T}}_M\tilde{f}(\xi)=\frac{\exp\left(\frac{i\sqrt{2}\xi}{4M}\right)}{\sqrt{1+\frac{i\xi}{\sqrt{2}M}}}\tilde{f}\left(\frac{\xi}{\sqrt{2}}\right)^2. \tag{3.9′}$$

The relation

$$(f_{n+1}(x),M_{n+1})=(\mathbf{Q}_{n,M_n}f_n(x),M_n+2^{-(n+1)/2}m_n) \tag{3.10}$$

holds with the starting pair $(f_0(x),M_0)$ defined by the relations

$$M_0=\int\limits_0^{\infty}x\bar{q}_0(x)dx\qquad f_0(x)=\bar{q}_0(x-M_0), \tag{3.10′}$$

where the function $\bar{q}_0(x)$ was defined after formula (1.7) (with $n=0$).

We have

$$\mathbf{Q}_{n,M} f(x) = \mathbf{T}_M f(x) + \varepsilon_n(x), \tag{3.11}$$

where $\varepsilon_n(x)$ is a small error term. We get a heuristic explanation of Theorem A′ by investigating the expression $\mathbf{T}^n_M f(x)$ for large n with a function $f(x)$ satisfying the relations $\int f(x)dx = 1$ and $\int x f(x) dx = 0$. Put

$$\varphi_k(\xi) = \log \tilde{\mathbf{T}}^k_M \tilde{f}(\xi) = \sum_{j=2}^{\infty} d_{j,k} \xi^k .$$

It follows from (3.9) that

$$d_{j,k+1} = 2^{(2-j)/2} d_{j,k} + \frac{(-i)^j}{2j(\sqrt{2}M)^j}, \quad j \geqq 2 .$$

Hence

$$\lim_{n\to\infty} d_{j,n} = \frac{(-i)^j}{2j(\sqrt{2}M)^j(1-2^{(2-j)/2})} \quad \text{for} \quad j \geqq 3,$$

and

$$d_{2,n} = -\frac{n}{8M^2} + d_{2,0} .$$

The above relations imply that

$$\lim_{n\to\infty} \varphi_n\left(\frac{\xi}{\sqrt{n}}\right) = -\frac{1}{8M^2}\xi^2 .$$

Since $f_n(x)$ behaves similarly to $\mathbf{T}^n_M f_0(x)$, the above calculation suggests that $\sqrt{n} f_n(\sqrt{n} x)$ is asymptotically Gaussian with variance $\frac{1}{8M^2}$. We justify this heuristic argument similarly to the method of [2]. First we show that if t and T are sufficiently small then for all not too large n $f_n(x)$ is asymptotically normal with variance $\sigma = \frac{a_1}{2(a_0 - T)}$. More precisely, we prove the following

Proposition 1. *For all positive integers $N \geqq 1$ there are some thresholds t_0 and T_0 such that if $0 < T < T_0$ and $0 < t < t_0$ then for all $n \leqq N$*

$$\left|\frac{d^j}{dx^j}[f_n(x) - \phi(x,\sigma)]\right| \leqq \frac{B(n)}{\sqrt{M_n}} \exp\{-2^{(n+2)/2}|x|\}$$
$$\text{if} \quad |x| < \log M_n,\ j = 0,1,2, \tag{3.12}$$

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq B(n) \exp\left\{-2^{n/2}\left|2x + \frac{2^{-n/2}x^2}{M_n}\right|\right\}$$
$$\text{if} \quad x > -2^{-n/2} M_n,\ j = 0,1,2, \tag{3.13}$$

and

$$|M_n - \hat{M}_0| \leqq B(n) t^{1/2} T, \tag{3.14}$$

where $\hat{M}_0^2 = \frac{a_1(a_0-T)}{Tt^2}$, $\sigma^2 = \frac{a_1}{2(a_0-T)}$, $\phi(x,\sigma)$ *denotes the normal density function with expectation zero and variance* σ, *and* $B(n)$ *is some appropriate multiplying factor depending on* n, *but not on* t *and* T.

If t_0 and T_0 are sufficiently small then $\hat{M}_0$ is very large, therefore (3.14) states that for fixed n (depending on t and T) M_n is very close to $\hat{M}_0$. Then (3.12) gives a good Gaussian approximation of $f_n(x)$ and (3.13) a good bound on its tail behaviour.

The proof of Proposition 1 is based on the observation that M_0 almost agrees with the positive maximum $\hat{M}_0$ of the function $\bar{q}_0(x)$, $f_0(x)$ is almost Gaussian, and we commit a small error by substituting the operator $\bar{\mathbf{Q}}_{n,M}$ for small n by the operator $\hat{T}_n$,

$$\begin{aligned}\hat{T}_n f_n(x) &= C\int \exp\{-v^2 - 2^{-n/2}u^2\} f_n(x+u) f_n(x-u)\,du\,dv \\ &= C\sqrt{\pi}\int \exp\{-2^{-n/2}u^2\} f_n(x+u) f_n(x-u)\,du\,.\end{aligned}$$

Since the proof is almost the same as the proof of the corresponding result for $1<c<\sqrt{2}$ given in Sect. 4 of Part I in [2] we omit it. By the same reason we omit the proof of its Corollary formulated below. To formulate this result first we have to introduce the following notion:

Definition of the Regularization of a Function. *Let us choose some fixed function* $\varphi(x) \in C_0^\infty(R^1)$ *such that* $1 \geqq \varphi(x) \geqq 0$ *for all* $x \in R^1$, $\varphi(x)=1$ *for* $|x|<1$, *and* $\varphi(x)=0$ *for* $|x| \geqq 2$. *Put* $\varphi_n(x) = \varphi\left(\frac{1}{100} 2^{-n/2} x\right)$. *Given some function* $f(x)$, $f(x) \geqq 0$, $\int f(x)dx < \infty$ *we define its n-th regularization* $\varphi_n(f)$ *as* $\varphi_n(f)(x) = \frac{1}{A_n}\varphi_n(x+B_n) f(x+B_n)$ *with* $A_n = \int \varphi_n(x) f(x)dx$ *and* $B_n = \frac{1}{A_n}\int x\varphi_n(x) f(x)dx$, *provided that the above formula is meaningful, i.e.* $A_n > 0$.

Now we formulate the following

Corollary of Proposition 1. *Under the conditions of Proposition 1 we have for all* $n \leqq N$

$$|\tilde{\varphi}_n(f_n)(t+is)| \leqq \frac{\exp s^2}{1+\frac{t^2}{200}} \quad \text{for} \quad |s|<2,\ t \in R^1\,,$$

and

$$\left|\frac{d^j}{dx^j} f_n(x)\right| \leqq 10^5 \exp\left\{-\left|2x + \frac{2^{-n/2}x^2}{M_n}\right|\right\} \quad \text{for} \quad x > -2^{n/2}M_n,\ j=0,1,2\,.$$

Let us fix some positive integer N, and define the sequences α_n, β_n, $n=N$, $N+1, \ldots$, as

$$\alpha_N = \frac{1}{200}, \tag{3.15}$$

$$\alpha_{n+1} = (1-2^{-n/4})\alpha_n + \frac{10^{-12}}{M_n^2} \quad \text{for} \quad n \geqq N \tag{3.15'}$$

and

$$\beta_N = 1, \tag{3.16}$$

$$\beta_{n+1} = (1 + 2^{-n/4})\beta_n + \frac{10}{M_n^2} \quad \text{for} \quad n \geqq N, \tag{3.16'}$$

where M_n is defined in (1.8).

Now we define the following Properties $I(n)$ and $J(n)$.

Property $I(n)$. *Let $n \geqq N$. The function $f(x)$ satisfies Property $I(n)$ (with the starting index N and parameter C) if*

$$\left|\frac{d^j}{dx^j} f(x)\right| \leqq \frac{C}{\beta_n^{(j+1)/2}} \exp\left\{-\frac{1}{\sqrt{\beta_n}}\left|2x + \frac{2^{-n/2}x^2}{M_n}\right|\right\} \quad \textit{for} \quad x > -2^{n/2}M_n, j = 0, 1, 2$$

with the above defined sequence β_n and the number M_n defined in (1.8).

Property $J(n)$. *Let $n \geqq N$. The function $f(x)$ satisfies Property $J(n)$ (with starting index N) if*

$$|\tilde{\varphi}_n(f)(t+is)| \leqq \frac{\exp\{\beta_n s^2\}}{1 + \alpha_n t^2} \quad \textit{for} \quad |s| < \frac{2}{\sqrt{\beta_n}}, \ t \in R^1,$$

with the above defined sequences α_n and β_n.

Now we formulate

Proposition 2. *The multiplying factor C and the starting index N can be chosen in Properties $I(n)$ and $J(n)$ in such a way that under the additional conditions $M_n > K$ with some universal constant K, $|M_n - M_{n-1}| < 1$, $100n > \beta_n > \max(9M_n^{-2}, 4^{-n})$, Properties $I(n)$ and $J(n)$ for the function $f_n(x)$ imply Properties $I(n+1)$ and $J(n+1)$ for the function $f_{n+1}(x)$ (with the same parameters N and C). Also the following relations hold true:*

$$M_{n+1} = M_n + 2^{-(n+1)/2} m_n, \qquad m_n = -\frac{\sqrt{2}}{4M_n} + \gamma(n)$$
$$\textit{with} \quad \gamma(n) < C_1 2^{-n/2}\sqrt{\beta_n}, \tag{3.17}$$

$$\left|\frac{d^j}{dx^j} f_{n+1}(x) - \mathbf{T}_{M_n}\varphi_n(f_n)(x)\right| \leqq \frac{C_1 C^4}{\beta_{n+1}^{(j+1)/2}} 2^{-n/2}$$
$$\times \left[\exp\left\{-\frac{1}{\sqrt{\beta_{n+1}}}\left|2x + \frac{2^{-(n+1)/2}x^2}{M_{n+1}}\right|\right\} + \exp\left\{-\frac{2|x|}{\sqrt{\beta_{n+1}}}\right\}\right]$$
$$\textit{for} \quad x > -2^{(n+1)/2}M_{n+1}, \quad j = 0, 1, 2, \tag{3.18}$$

and

$$\left|\frac{d^j}{dx^j} \mathbf{T}_{M_n}\varphi_n(f_n)(x)\right| \leqq \frac{C_1 C^2}{\beta_{n+1}^{(j+1)/2}} \exp\left\{-\frac{2|x|}{\sqrt{\beta_{n+1}}}\right\}, \qquad x \in R^1, j = 0, 1, 2, 3, 4 \tag{3.19}$$

with some absolute constant C_1. As a consequence, if $0 < T < T_0$ and $0 < t < t_0$ with some sufficiently small $t_0 > 0$ and $T_0 > 0$ then Properties $I(n)$ and $J(n)$ hold for the functions $f_n(x)$ with some appropriate parameters C and N, and these functions satisfy relations (3.17)–(3.19). *Also the relation $\beta_n < 100n$ holds.*

Proposition 2 is proved similarly to the analogous result for $1<c<\sqrt{2}$ in Sects. 5 and 6 in Part I of [2], only the number c must be replaced by $\sqrt{2}$ everywhere. The expressions $\mathbf{Q}_{n,M}f(x)$, $\mathbf{T}_M\varphi_n(f)(x)$, and $\mathbf{Q}_{n,M}f(x)-\mathbf{T}_M\varphi_n(f)$ can be bounded with the help of Property $I(n)$, as it is formulated in Proposition 3 and proved in Sect. 5 in Part I of [2]. This enables us to reduce the problem to the investigation of $\mathbf{T}_{M_n}\varphi_n(f_n)(x)$, which can be done with the help of Property $J(n)$ and formula (3.9′).

The only difference between the cases $1<c<\sqrt{2}$ and $c=\sqrt{2}$ is that for $c=\sqrt{2}$ the condition $\beta<100$ must be replaced by the condition $\beta<100n$ when the operator $\mathbf{Q}_{n,M}$ is investigated. This is so, because we apply our estimates with $\beta=\beta_n$, and the sequence β_n defined in (3.16), (3.16′) is of order const$n$. (In the case $1<c<\sqrt{2}$ it was bounded by a constant.) Nevertheless, this difference causes no problem and the estimate $\beta<100n$ is sufficient for our purposes.

Proposition 2 enables us to bound the error term $\varepsilon_n(x)$ in (3.11) when the operator $\mathbf{Q}_{n,M_n}$ is applied for $f_n(x)$. With the help of this estimate we can turn the heuristic argument after formula (3.11) into a rigorous proof.

4. The Proof of Theorem A

We prove Theorem A by estimating the Fourier transforms $\tilde{\varphi}_n(f_n)(t)$. Let us fix some constants N and C in such a way that Propositions 1 and 2 hold with this choice of the parameters. Let us introduce the functions $\psi_n(t)=\log\tilde{\varphi}_n(f_n)(t)$ and the numbers $\bar{\beta}_n=-\frac{d^2}{dt^2}\psi_n(t)\Big|_{t=0}$, provided that these quantities are well-defined, i.e. we can take logarithm in these expressions. We shall prove the following

Lemma 1. *If* $0<t<t_0$, $0<T<T_0$ *with some sufficiently small* $t_0>0$ *and* $T_0>0$ *then*

a)
$$\bar{\beta}_N=\frac{a_1}{2(a_0-T)}+\delta(N), \qquad |\delta(N)|\leqq 4^{-N}, \tag{4.1}$$

$$\bar{\beta}_{n+1}=\bar{\beta}_n+\frac{1}{4M_n^2}+\delta(n), \qquad |\delta(n)|\leqq 2^{-n/4} \quad \text{for} \quad n\geqq N. \tag{4.2}$$

b) *For* $|t|<\left(\frac{n}{\bar{\beta}_n}\right)^{1/3}$ *and* $n\geqq N$ $\psi_n(t)$ *is well-defined, and*

$$\left|\frac{d^3}{dt^3}\psi_n(t)\right|\leqq\frac{2}{M_n^3}+2^{-n/4} \quad \text{for} \quad |t|\leqq\left(\frac{n}{\bar{\beta}_n}\right)^{1/3} \quad \text{and} \quad n\geqq N. \tag{4.3}$$

Proof of Lemma 1. Because of Proposition 1 $\tilde{\varphi}_N(f_N)(t)$ is very close to the Fourier transform of the normal density function $\phi(x,\sigma)$ with $\sigma^2=\frac{a_1}{2(a_0-T)}$, and the analogous result also holds for its derivatives. This implies (4.2) and (4.3) for $n=N$, since if $\tilde{\varphi}_N(f_N)(t)$ were exactly normal then we would have $\bar{\beta}_N=\frac{a_1}{2(a_0-T)}$ and $\frac{d^3}{dt^3}\psi_N(t)=0$. We prove (4.2) and (4.3) in the general case by induction from n to $n+1$.

Let us introduce the operator $\hat{\mathbf{T}}_n$ by the formula $\hat{\mathbf{T}}_n\psi(t)=\log\tilde{\mathbf{T}}_{M_n}\exp\psi(t)$. It follows from (3.9′) that

$$-\frac{d^2}{dt^2}\hat{\mathbf{T}}_n\psi_n(t)\bigg|_{t=0}=\frac{1}{4M_n^2}+\bar{\beta}_n, \tag{4.4}$$

and

$$\frac{d^3}{dt^3}\hat{\mathbf{T}}_n\psi_n(t)=\frac{1}{\sqrt{2}}\psi_n\left(\frac{t}{\sqrt{2}}\right)+\frac{\sqrt{2}i}{16M_n^3\left(1+\frac{it}{\sqrt{2}M_n}\right)^3}. \tag{4.4′}$$

Since M_n is very large, (4.4′) together with our inductive hypothesis imply that

$$\left|\frac{d^3}{dt^3}\hat{\mathbf{T}}_n\psi_n(t)\right|\leqq\frac{2}{M_{n+1}^3}+\frac{1}{\sqrt{2}}2^{-n/4}\quad\text{if}\quad |t|<\left(\frac{n+1}{\bar{\beta}_{n+1}}\right)^{1/3}. \tag{4.5}$$

Because of the identities $\hat{\mathbf{T}}_n\psi_n(0)=\frac{d}{dt}\hat{\mathbf{T}}_n\psi_N(t)\Big|_{t=0}=0$ it follows from (4.4) and (4.5) that

$$\Re\hat{\mathbf{T}}_n\psi_n(t)\geqq-\left(\bar{\beta}_n+\frac{1}{4M_n^2}\right)\frac{t^2}{2}-\left(\frac{2}{M_{n+1}^3}+\frac{2^{-n/4}}{\sqrt{2}}\right)\frac{|t|^3}{6}\geqq-\frac{n}{10}$$
$$\text{for}\quad |t|\leqq\left(\frac{n+1}{\bar{\beta}_{n+1}}\right)^{1/3},$$

where $\Re$ denotes real part. (Observe that $1<\bar{\beta}_n<n/10$.) This relation implies that

$$|\tilde{\mathbf{T}}_{M_n}\tilde{\varphi}_n(f_n)(t)|\geqq e^{-n/10}\quad\text{for}\quad |t|\leqq\left(\frac{n+1}{\bar{\beta}_{n+1}}\right)^{1/3}. \tag{4.6}$$

We get similarly, by expressing the derivatives of $\tilde{\mathbf{T}}_{M_n}\tilde{\varphi}_n(f_n)(t)$ through $\psi_n(t)$ and its derivatives, that

$$\left|\frac{d^j}{dt^j}\tilde{\mathbf{T}}_{M_n}\tilde{\varphi}_n(f_n)(t)\right|\leqq e^{n/10}\quad\text{for}\quad |t|\leqq\left(\frac{n+1}{\bar{\beta}_{n+1}}\right)^{1/3},\quad j=1,2,3. \tag{4.6′}$$

On the other hand, some calculation with the help of (3.18) yields that

$$\left|\frac{d^j}{dt^j}\tilde{\varphi}_{n+1}(f_{n+1})(t)-\frac{d^j}{dt^j}\tilde{\mathbf{T}}_{M_n}\tilde{\varphi}_n(f_n)(t)\right|\leqq K2^{-n/2}$$
$$\text{for}\quad |t|\in R^1\quad\text{and}\quad j=0,1,2,3. \tag{4.7}$$

By expressing $\frac{d^3}{dt^3}\psi_{n+1}(t)$ and $\frac{d^3}{dt^3}\hat{\mathbf{T}}_n\psi_n(t)$ by the corresponding Fourier transforms we get that relations (4.6), (4.6′), and (4.7) imply that

$$\left|\frac{d^3}{dt^3}\psi_{n+1}(t)-\frac{d^3}{dt^3}\hat{\mathbf{T}}_n\psi_n(t)\right|\leqq\frac{1}{100}2^{-n/4}.$$

The last relation together with (4.5) imply (4.3) for $n+1$.

It can be proved similarly that

$$\left| \frac{d^2}{dt^2} \psi_{n+1}(t) \Big|_{t=0} - \frac{d^2}{dt^2} \hat{\mathbf{T}}_n \psi_n(t) \Big|_{t=0} \right| \leqq 2^{-n/4},$$

which together with (4.4) imply (4.2) for $n+1$. Lemma 1 is proved.

Proof of Theorem A'. It follows from Lemma A that

$$\tilde{\varphi}_n(f_n)(t) = \exp\left\{-\frac{\bar{\beta}_n}{2} t^2 + R_n(t) t^3\right\} \quad \text{with} \quad |R_n(t)| < \frac{2}{M_n^3} + 2^{-n/4}$$

$$\text{if} \quad t < \left(\frac{n}{\bar{\beta}_n}\right)^{1/3}.$$

Hence

$$\left| \int\limits_{|t| < \left(\frac{n}{\bar{\beta}_n}\right)^{1/3}} e^{-itx} \left[e^{-\frac{\bar{\beta}_n t^2}{2}} - \tilde{\varphi}_n(f_n)(t) \right] dt \right| \leqq 2 \left(\frac{2}{M_n^3} + 2^{-n/4} \right) \frac{1}{\bar{\beta}_n^2}. \tag{4.8}$$

On the other hand

$$\left| \int\limits_{|t| > \left(\frac{n}{\bar{\beta}_n}\right)^{1/3}} e^{-itx} e^{-\frac{\bar{\beta}_n t^2}{2}} dt \right| \leqq \exp\left\{ -\frac{n^{2/3}}{2} \bar{\beta}_n^{1/3} \right\}, \tag{4.8$'$}$$

and by Property $J(n)$ and the relation $\alpha_n > 10^{-14} \bar{\beta}_n$

$$\left| \int\limits_{|t| > \left(\frac{n}{\bar{\beta}_n}\right)^{1/3}} e^{-itx} \tilde{\varphi}_n(f_n)(t) dt \right| \leqq 10^{14} n^{-1/3} \bar{\beta}_n^{-2/3}. \tag{4.8$''$}$$

Relations (4.8), (4.8'), and (4.8'') imply that

$$\begin{aligned}
\left| \varphi_n(f_n)(x) - \frac{1}{\sqrt{2\pi \bar{\beta}_n}} \exp\left\{ -\frac{x^2}{2\bar{\beta}_n} \right\} \right| &= |\textstyle\int e^{-itx} [\tilde{\varphi}_n(f_n)(t) - \exp\{-\bar{\beta}_n t^2\} dt]| \\
&\leqq \frac{2}{\bar{\beta}_n^2} \left(\frac{2}{M_n^3} + 2^{-n/4} \right) + 10^{14} n^{-1/3} \bar{\beta}_n^{-2/3} + \exp\left\{ -\frac{n^{2/3}}{2} \bar{\beta}_n^{1/3} \right\} \\
&\leqq \frac{1}{\bar{\beta}_n^2} \left(\frac{4}{M_n^2} + 2^{-n/4} \right) + 2 \cdot 10^{14} n^{-1/3} \bar{\beta}_n^{-2/3}.
\end{aligned} \tag{4.9}$$

In relation (4.9) $\varphi_n(f)(x)$ can be replaced by $f_n(x)$, since for $|x| < 2^{n/2}$ they are very close to each other by (3.8), and for $|x| > 2^{n/2}$ both terms at the left-hand side of (4.9) are negligibly small. (The norming constants A_n and B_n appearing in the regularization are almost 0 and 1.)

Hence (4.9) implies that

$$\left| f_n(x) - \frac{1}{\sqrt{2\pi \bar{\beta}_n}} \exp\left\{ -\frac{x^2}{2\bar{\beta}_n} \right\} \right| \leqq \frac{1}{\bar{\beta}_n} \left(\frac{4}{M_n^2} + 2^{-n/5} + 10^{15} \left(\frac{\bar{\beta}_n}{n} \right)^{1/3} \right)$$

$$\text{for} \quad n \geqq N. \tag{4.10}$$

Since $\left|\bar\beta_n-\frac{n}{4M^2}\right|<10$, hence

$$\left|\frac{1}{\sqrt{2\pi\bar\beta_n}}\exp\left\{-\frac{1}{2\bar\beta_n}x^2\right\}-\frac{\sqrt{2}M}{\sqrt{\pi n}}\exp\left\{-\frac{2M^2}{n}x^2\right\}\right|\leqq\frac{\text{const}}{n}. \tag{4.10'}$$

For large n the term $\frac{1}{\bar\beta_n}$ can be replaced by $\frac{5M^2}{n}$ in (4.10), hence (4.10) and (4.10′) imply (3.4). Relation (3.5) holds because of Property $I(n)$, and relations (3.2) and (3.3) can be deduced from Proposition 2 in the same way as the analogous result in [2] in Lemma 10 of Part I. Theorem A′ is proved.

Proof of Theorem A. By Theorem A′ and (3.1)

$$2^{-n/2}\sqrt{n}\bar q_n(x,T)=\frac{\sqrt{2}M}{\sqrt{n}}\exp\left\{-\frac{2^{n+1}M^2}{n}(x-M_n)^2\right\}+r_n(x) \tag{4.11}$$

with

$$|r_n(x)|<\frac{K}{\sqrt{n}}.$$

We have to check that an error of order $O\left(\frac{1}{\sqrt{n}}\right)$ is committed if M_n is replaced by M in (4.11). We have

$$\begin{aligned}&\left|\exp\left\{-\frac{2^{n+1}M^2}{n}(x-M)^2\right\}-\exp\left\{-\frac{2^{n+1}M^2}{n}(x-M_n)^2\right\}\right|\\&\leqq\exp\left\{-\frac{2^{n+1}M^2}{n}(x-M)^2\right\}\frac{2^{n+1}}{n}M^2|(x-M)^2-(x-M_n)^2|\\&\leqq\frac{2^{n+1}M^2}{n}|M-M_n|(2|x-M|+2|M-M_n|)\exp\left\{-\frac{2^{n+1}}{n}M^2(x-M)^2\right\}\\&\leqq\frac{K}{\sqrt{n}},\end{aligned}$$

since $|M-M_n|<CM^{-1}2^{-n/2}$. This estimate together with (4.11) imply (2.3). The remaining statements of Theorem A also follow from Theorem A′.

Theorem A gives a good Gaussian approximation only for large n. On the other hand, the error term in (4.10) is small for all $n\geqq N$. Beside this, Proposition 1 yields a good Gaussian approximation for all $n\leqq N$ if $\hat M_0$ is very large. These observations imply the following

Corollary of Theorem A′. *Define the sequence $\bar\beta_n$ by* (4.1) *and* (4.2) *for $n\geqq N$ and $\bar\beta_n=\bar\beta_N$ for $n\leqq N$. For all $\delta>0$ some positive integer N and thresholds $t_0>0$ and $T_0>0$ can be chosen in such a way that*

$$\left|f_n(x)-\frac{1}{2\sqrt{\pi\bar\beta_n}}\exp\left\{-\frac{1}{2\bar\beta_n}x^2\right\}\right|<\delta\quad\textit{for all } n\geqq0 \textit{ and } x\in R^1 \tag{4.12}$$

if $0<t<t_0$ *and* $0<T<T_0$. *As a consequence, for arbitrary* $L>0$ *the inequality*

$$f_n(x)<\frac{10}{\sqrt{\hat{\beta}_n}}\exp\left\{-\frac{1}{2\hat{\beta}_n}x^2\right\} \quad \textit{for} \quad |x|<L\sqrt{\hat{\beta}_n},\ n=0,1,2,\ldots \tag{4.13}$$

holds with the sequence

$$\hat{\beta}_n=10+\frac{n}{M_n^{1/2}} \quad \textit{for} \quad 0\leqq n\leqq N, \tag{4.14}$$

$$\hat{\beta}_{n+1}=\hat{\beta}_n(1+2^{-n/4})+\frac{1}{8M_n^2} \quad \textit{for} \quad n\geqq N, \tag{4.14'}$$

if the conditions (4.12) *hold with a sufficiently small* $\delta=\delta(L)$.

5. The Proof of Proposition A

First we prove formula (2.6). Choose an appropriately small $\varepsilon>0$ and a large $L=L(\varepsilon)>0$. We are going to show that if $0<t<t_0$ and $0<T<T_0$ with some $t_0=t_0(\varepsilon,L)$, and $T_0=T_0(\varepsilon,L)$ then

$$f_n(x)\leqq\frac{10}{\sqrt{\hat{\beta}_n}}\exp\left\{-\frac{1}{2\hat{\beta}_n}x^2\right\} \quad \text{for} \quad |x|<L\sqrt{\hat{\beta}_n},\ n=0,1,2,\ldots, \tag{5.1}$$

and

$$f_n(x)\leqq\frac{\varepsilon}{\sqrt{\hat{\beta}_n}}\exp\left\{-\frac{1}{4\hat{\beta}_n}x^2\right\} \quad \text{for} \quad |x|>L\sqrt{\hat{\beta}_n},\ n=0,1,2,\ldots. \tag{5.1'}$$

Since $\lim\limits_{n\to\infty}\frac{\hat{\beta}_n}{n}=\frac{1}{8M^2}$, relations (5.1) and (5.1′) imply (2.6). Because of the corollary of Theorem A′ we may assume that relation (5.1) and relation (5.1′) for $L\sqrt{\hat{\beta}_n}<|x|<3L\sqrt{\hat{\beta}_n}$ hold. It is enough to apply this corollary for $3L$, and to choose L in such a way that $\exp\left\{-\frac{L^2}{4}\right\}<\frac{\varepsilon}{10}$. Moreover, it can be seen from the form of $f_0(x)$ that for $n=0$ (5.1′) holds for all $x>L\sqrt{\hat{\beta}_0}$. Hence it is enough to prove (5.1′) for $x>2L\sqrt{\hat{\beta}_n}$ by induction from n to $n+1$. We shall do it with the help of the following

Lemma 2. *If* $\varepsilon>0$ *and* $L>L(\varepsilon)>0$ *are appropriately chosen (in dependence of the number* C *appearing in the conditions of this lemma),* n *is some non-negative integer,* $M>K>0$ *with an appropriate* $K>0$ *and*

$$f(x)\leqq\frac{10}{\sqrt{\beta}}\exp\left\{-\frac{1}{\beta}x^2\right\} \quad \textit{for} \quad |x|<L\sqrt{\beta}, \tag{5.2}$$

$$f(x)\leqq\frac{\varepsilon}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}x^2\right\} \quad \textit{for} \quad |x|>L\sqrt{\beta}, \tag{5.2'}$$

$$f(x)\leqq\frac{C}{\sqrt{\beta}} \quad \textit{for all } x\in R^1, \tag{5.2''}$$

then

$$\bar{Q}_{n,M}f(x) \leqq \frac{\varepsilon^{3/2}}{\sqrt{\beta}} \exp\left\{-\frac{1}{2\beta}x^2\right\} \quad \textit{for} \quad x > 2L\sqrt{\beta}.$$

Proof of Lemma 2. The proof applies the same ideas as that of Lemma 19 in Part I of [2]. Let us introduce the functions

$$\begin{gathered} l^{\pm}_{n,M}(x,u,v) = 2^{n/2}\left(\sqrt{(M + 2^{-(n+1)/2}x \pm 2^{-n/2}u)^2 + 2^{-n/2}v^2} - M\right), \\ P(x,u) = \int \exp\{-v^2\} f(l^{+}_{n,M}(x,u,v)) f(l^{-}_{n,M}(x,u,v)) dv \end{gathered} \tag{5.3}$$

and

$$P(x) = P(x,0). \tag{5.3'}$$

Then

$$\bar{\mathbf{Q}}_{n,M}f(x) = 2\int_0^\infty \exp\{-2^{-n/2}u^2\} P(x,u) du, \tag{5.4}$$

and by the Schwarz inequality

$$P(x,u) \leqq [P(x+\sqrt{2}u) P(x-\sqrt{2}u)]^{1/2}. \tag{5.5}$$

Let us estimate $P(x)$. It follows from (5.2)–(5.2″) and the inequality $l^{\pm}_{n,M}(x,0,v) \geqq l^{\pm}_{n,M}(x,0,0)$ that

$$P(x) \leqq \frac{\varepsilon^2}{\beta}\sqrt{\pi}\exp\left\{-\frac{1}{2\beta}x^2\right\} \quad \text{for} \quad x > \sqrt{2\beta}L, \tag{5.6}$$

$$P(x) \leqq \frac{100}{\beta}\sqrt{\pi}\exp\left\{-\frac{1}{\beta}x^2\right\} \quad \text{for} \quad |x| < \sqrt{2\beta}L, \tag{5.6'}$$

$$P(x) \leqq \frac{C^2\sqrt{\pi}}{\beta} \quad \text{for all} \quad x \in R^1. \tag{5.6''}$$

These estimates together with (5.4) and (5.5) imply that for $x \geqq 2L\sqrt{\beta}$

$$\begin{aligned} \bar{\mathbf{Q}}_{n,M}f(x) \leqq\ & 2\int_0^{\frac{x}{\sqrt{2}} - L\sqrt{\beta}} \frac{\varepsilon^2\sqrt{\pi}}{\beta}\exp\left\{-\frac{x^2}{2\beta} - \frac{u^2}{\beta}\right\} du \\ & + 2\int_{\frac{x}{\sqrt{2}} - L\sqrt{\beta}}^{\frac{x}{\sqrt{2}} + L\sqrt{\beta}} \frac{10\varepsilon}{\beta}\sqrt{\pi}\exp\left\{-\frac{(x-\sqrt{2}u)^2}{\beta} - \frac{(x+\sqrt{2}u)^2}{2\beta}\right\} du \\ & + 2\int_{\frac{x}{\sqrt{2}} + L\sqrt{\beta}}^{\infty} \frac{C\varepsilon}{\beta}\sqrt{\pi}\exp\left\{-\frac{(x+\sqrt{2}u)^2}{2\beta}\right\} du \leqq \frac{\varepsilon^{3/2}}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}x^2\right\} \end{aligned}$$

if $L = L(\varepsilon)$ is sufficiently large. Lemma 2 is proved.

Let us apply Lemma 2 with $f(x) = f_n(x)$, $\beta = 2\hat{\beta}_n$ and $M = M_n$. Since

$$f_{n+1}(x) = \mathbf{Q}_{n,M_n} f_n(x) \leqq C_1 \bar{\mathbf{Q}}_{n,M_n} f_n(x + m_n)$$

with some $C_1>0$ hence in order to carry out our inductive procedure it is enough to show that

$$C_1\sqrt{\varepsilon}\exp\left\{-\frac{1}{4\hat{\beta}_n}(x+m_n)^2\right\}\leqq\exp\left\{-\frac{1}{4\hat{\beta}_{n+1}}\right\}.$$

This can be deduced from the inequality

$$\hat{\beta}_{n+1}(x+m_n)^2+K\hat{\beta}_n\hat{\beta}_{n+1}\geqq\hat{\beta}_n x^2 \tag{5.7}$$

with sufficiently large $K>0$ if $\varepsilon>0$ is chosen sufficiently small. Since $|m_n|<CM_n^{-1}$ for $n\leqq N$, $|m_n|<\frac{1}{2M}+C_1 2^{-n/2}$ for $n>N$ and $\hat{\beta}_n>10$ one gets formula (5.7) with the help of simple calculation from (4.13) and (4.13′).

The proof of formulas (2.7) and (2.7′) is based on the following

Lemma 3. *Let the function $f(x)$ satisfy the conditions of Lemma 2. Let some numbers $r>0$, $\beta>0$, and $\alpha>0$ be given in such a way that $r>\beta>10$, $\beta<\frac{9}{10}Mn$ and $\frac{1}{100}<\alpha<1-\varepsilon^{1/8}$. Let us assume that the function $f(x)$ satisfies, beside the conditions of Lemma 2, the estimates*

$$f(x)\leqq\frac{\varepsilon}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}x^2\right\}\quad for\quad -r<x<-L\sqrt{\beta}\,, \tag{5.8}$$

$$f(x)\leqq\frac{\varepsilon}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}r^2-\alpha 2^{(n-1)/2}[(M-2^{-n/2}r)^2-(M+2^{-n/2}x)^2]\right\}$$
$$for\quad -2^{-n/2}M<x<-r. \tag{5.8′}$$

Put $\bar{\alpha}=\min((1+\varepsilon)\alpha,1-\varepsilon^{1/8})$, $\bar{\bar{\alpha}}=(1+\varepsilon^{1/8})\bar{\alpha}$ and

$$\bar{r}=\frac{\sqrt{2\bar{\bar{\alpha}}\beta M}}{1+\bar{\bar{\alpha}}\beta 2^{-n/2}}. \tag{5.8″}$$

If $\bar{r}<\sqrt{2}r$ then

$$\bar{\mathbf{Q}}_{n,M}f(x)\leqq\frac{\varepsilon^{3/2}}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}x^2\right\}\quad for\quad -\bar{r}<x<-2L\sqrt{\beta}\,, \tag{5.9}$$

$$\bar{\mathbf{Q}}_{n,M}f(x)\leqq\frac{\varepsilon^{3/2}}{\sqrt{\beta}}\exp\left\{-\frac{1}{2\beta}\bar{r}^2-\bar{\alpha}2^{n/2}[(M-2^{-(n+1)/2}\bar{r})^2\right.$$
$$\left.-(M+2^{-(n+1)/2}x)^2]\right\}$$
$$for\quad -2^{(n+1)/2}M<x<-\bar{r}. \tag{5.9′}$$

The proof of Lemma 3 is similar to that of Lemma 2. The main difference is that in Lemma 2, i.e. when $x>0$, the main contribution to the integral $\bar{\mathbf{Q}}_{n,M}f(x)$ is given in a small neighbourhood of the point $(u,v)=(0,0)$. For $x<0$ this statement remains valid only for $x>-\bar{r}$. For $x<-\bar{r}$ the main contribution to this integral is given in a small neighbourhood of the points

$$(u,v)=(0,\pm v^*)\quad\text{with}\quad v^{*2}=2^{n/2}\{(M-2^{-(n+1)/2}\bar{r})^2-(M+2^{-(n+1)}x)^2\}.$$

Proof of Lemma 3. Define the function

$$K(x)=\begin{cases}\dfrac{10}{\sqrt{\beta}}\exp\left\{-\dfrac{1}{\beta}x^2\right\} & \text{for } \ |x|<-\sqrt{2\beta}L\\ \dfrac{\varepsilon}{\sqrt{\beta}}\exp\left\{-\dfrac{1}{2\beta}x^2\right\} & \text{for } \ x>L\sqrt{\beta} \ \text{ or } \ -r<x<-\sqrt{2\beta}L\\ \dfrac{\varepsilon}{\sqrt{\beta}}\exp\left\{-\dfrac{1}{2\beta}r^2-\alpha 2^{(n-1)/2}[(M-2^{-n/2}r)^2-(M+2^{-n/2}x)^2]\right\} & \\ \quad\text{for } \ -2^{n/2}M<x<-r. & \end{cases}$$

Some calculation shows that for fixed x the function

$$\bar{K}(x,v)=\exp\{-\bar{\alpha}v^2\}K^2(l^{\pm}_{n,M}(x,0,v))$$

takes its maximum in the point $v=0$ for $x>-\dfrac{\bar{r}}{\sqrt{2}}$ and in the points $\pm v^*$ satisfying the equation $l^{\pm}_{n,M}(x,0,v^*)=-\dfrac{\bar{r}}{\sqrt{2}}$ for $x<-\dfrac{\bar{r}}{\sqrt{2}}$. (At this point we need the condition $\bar{r}<\sqrt{2}r$ which guarantees that the estimate (5.8) holds in the point $\dfrac{\bar{r}}{\sqrt{2}}$.) The function $P(x)$ defined in formula (5.3) can be estimated in the following way:

$$P(x)\leqq\int\exp\{-\varepsilon^{1/8}v^2\}\bar{K}(x,v)dv\leqq\varepsilon^{-1/4}\sqrt{\pi}\sup_v\bar{K}(x,v).$$

Hence we obtain that

$$P(x)\leqq\frac{\varepsilon^{7/4}\sqrt{\pi}}{\beta}\exp\left\{-\frac{1}{2\beta}x^2\right\}\quad\text{for}\quad -\bar{r}<x<-\sqrt{2\beta}L, \tag{5.10}$$

$$P(x)\leqq\frac{\varepsilon^{7/4}\pi}{\beta}\exp\left\{-\frac{1}{2\beta}\bar{r}^2-\bar{\alpha}2^{n/2}[(M-2^{-(n+1)/2}\bar{r})^2-(M+2^{-(n+1)/2}x)^2]\right\}$$
$$\text{for}\quad -2^{(n+1)/2}M<x<\bar{r}. \tag{5.10'}$$

We estimate the integral in (5.4) with the help of (5.5), (5.6), (5.6′), (5.10), and (5.10′). Let us first consider the case $-2^{(n+1)/2}M<x<-\bar{r}$ and integrate in the domain $\{u>0,\ x+\sqrt{2}u<-\bar{r}\}$. This integral can be estimated in the following way:

$$\begin{aligned}\int_{\{u>0,x+\sqrt{2}u<-\bar{r}\}}P(x,u)du&\leqq\frac{\varepsilon^{7/4}\sqrt{\pi}}{\beta}\exp\left\{-\frac{1}{2\beta}\bar{r}^2-\bar{\alpha}2^{n/2}[(M-2^{-(n+1)/2}\bar{r})^2\right.\\&\quad\left.-(M+2^{-(n+1)/2}x)^2]\right\}\\&\quad\times\int_{\{u>0,x+\sqrt{2}u<-\bar{r}\}}\exp\{-2^{-n/2}(1-\bar{\alpha})u^2\}du.\end{aligned}\tag{5.11}$$

We give an upper bound on the right-hand side of (5.11) by replacing $\bar{\alpha}$ with α in it and multiplying the expression by $\exp\{-(\bar{\alpha}-\alpha)|\bar{r}+x|\}$. The integral in this expression can be estimated by the rather rough bound $|\bar{r}+x|$. These estimates show that the right-hand side of (5.11) is much less than the expression at the right-hand side of (5.9′). To estimate the integral $\int P(x,u)du$ in the case $-2^{(n+1)/2}M<x<-\bar{r}$ in the domain $\{x+\sqrt{2}u>-\bar{r}\}$ observe that some calculation yields that

$$\exp\left\{-\frac{1}{2\beta}x^2\right\}=\exp\left\{-\frac{1}{2\beta}\bar{r}^2-\bar{\alpha}2^{n/2}[(M-2^{-(n+1)/2}\bar{r})^2-(M+2^{-(n+1)/2}x)^2]\right\}$$
$$\times\exp\left\{-\left(\frac{1}{2\beta}+2^{-(n+2)/2}\bar{\bar{\alpha}}\right)(x+\bar{r})^2\right\}, \tag{5.12}$$

because of the definition of $\bar{r}$.

Because of this identity the estimates (5.10) and (5.10′) enable us to estimate the integral $\int P(x,u)du$ in this case similarly to the estimation of (5.11), only in this case the last term in (5.12) helps us to bound the pre-exponential term. Similar calculations enable us to bound the integral (5.4) for $x>-\bar{r}$ and to deduce the estimates (5.9) and (5.9′). Lemma 3 is proved.

Formulas (5.8) and (5.8′) hold for $f(x)=f_0(x)$ with $\beta=2\hat{\beta}_0=20$, $\alpha=\frac{1}{100}$, $M=M_0$, and $r=\sqrt{2\alpha\beta M}$. If the conditions of Lemma 3 are satisfied for $f_n(x)$ with $M=M_n$, $\beta=2\hat{\beta}_n$ and some α_n and r_n, then Lemma 3 gives an estimate on $\bar{\mathbf{Q}}_{n,M_n}f_n(x)$. An argument similar to that given after Lemma 2 gives an estimate when the operator $\bar{\mathbf{Q}}_{n,M_n}$ is replaced by $\mathbf{Q}_{n,M_n}$. In such a way we get by induction the estimates (5.8) and (5.8′) for $f_n(x)$ with $\beta=2\hat{\beta}_n$, an increasing sequence α_n which tends to $1-\varepsilon^{1/8}$ and a number r_n which is a small perturbation of the expression given in (5.8″). Since $\frac{\hat{\beta}_n}{n}$ has a positive limit $n\to\infty$, the number $r=r_n$ which appears in the estimates (5.8) and (5.8′) for $f_n(x)$ during this induction has the order n. By rewriting these estimates for $\bar{q}_n(x)$ with the help of (3.1) we obtain the estimates (2.7) and (2.7′) (with $\varepsilon^{1/8}$ instead of ε).

6. The Proof of Theorem B

The proof of Proposition B is the same as that of Lemma 1 in Part II of [2], hence we omit it. The proof of Theorem B is also very similar to the method of Part II in [2], only the number c must be replaced by $\sqrt{2}$ and M by the constant $\bar{M}$ defined in (1.9) everywhere. The main difference is that now we have a weaker control about the tail behaviour of the density function of the average spin $p_n(x)$. As a consequence, we can prove some estimates only in a weaker form. Nevertheless, they are sufficient for our purposes.

Let us discuss this question in more detail. Introduce the functions $\bar{p}_n(x)$ and $g_n(x)$, $x\in R^1$ as

$$\bar{p}_n(x)=K_n\exp\left\{\frac{a_0}{2a_1}2^{n/2}M^2\right\}p_n(\tilde{x}),\qquad \tilde{x}=(x,0)\in R^2, \tag{6.1}$$

$$g_n(x)=2^{-n/2}\bar{p}_n(\bar{M}+2^{-n/2}x), \tag{6.2}$$

where $p_n(x)$ is defined after formula (1.6), the number $\bar{M}$ in (1.9), and K_n is the same norming constant as in (1.7). By formula (1.7)

$$g_n(x)=2^{-n/2}\exp\left\{-\frac{a_0}{2T}x(2\bar{M}+2^{-n/2}x)\right\}\bar{q}_n\left(M+2^{-n/2}\sqrt{\frac{a_1}{T}}\right),$$

hence Theorem A yields that

$$g_n(x)=\frac{1}{\sqrt{n}}\exp\left\{-\frac{a_0}{2T}x(\bar{M}+2^{-n/2}x)\right\}\left[\frac{\sqrt{2}M}{\sqrt{\pi}}\exp\left\{-\frac{2\bar{M}^2}{n}x^2\right\}+R_n(x)\right] \tag{6.3}$$

with

$$|R_n(x)|\leqq\frac{K}{\sqrt{n}}. \tag{6.3'}$$

On the other hand, we get by rewriting Proposition A for $g_n(x)$ that there are some numbers $B>0$, $D>0$ and R_n, $-C_1n<R_n<-C_2n$ with some $C_1>C_2>0$ such that

$$g_n(x)\leqq\frac{K}{\sqrt{n}}\exp\left\{-\frac{a_0}{2T}(2\bar{M}+2^{-n/2}x)-\frac{B}{n}x^2\right\}\quad\text{for}\quad x>R_n, \tag{6.4}$$

and

$$g_n(x)\leqq\frac{K}{\sqrt{n}}\exp\left\{-\frac{a_0}{2T}R_n(2\bar{M}+2^{-n/2}R_n)-\frac{B}{n}R_n^2\right.$$
$$\left.-D(R_n-x)(2\bar{M}+2^{-n/2}(R_n+x))\right\}\quad\text{for}\quad -2^{-n/2}\bar{M}<x<R_n. \tag{6.4'}$$

[We have to choose $B=\frac{\beta a_1}{T}$, $R_n=2^{n/2}\sqrt{\frac{T}{a_1}}(r_n-M)$, and $D=\left(1-\varepsilon-\frac{a_0}{2a_1}\right)\frac{a_1}{T}$ in Proposition A. We may assume that $D>0$ by choosing ε in (2.7) sufficiently small.]

The estimates (6.4) and (6.4′) are the natural counterparts of the estimates (4.11′) and (4.11″) in Part II of [2]. The function $f_n(x)$ defined by formula (4.11) of that work is the analogue of our function $g_n(x)$.

The bound given on $g_n(x)$ decreases at infinity slower than its counterpart in [2] because of the multiplying term $1/n$ in formula (6.4). Another, and even more important difference between the two cases is that in the points $x\sim-\text{const}\,n$ relations (6.4) and (6.4′) give no better bound on the function $g_n(x)$ than $\exp\{Cn\}$ with some positive $C>0$. As a consequence, in several estimates we have to multiply the upper bound by an exponential term instead of a constant, as is the case in [2]. But these estimates suffice for us, because in the final estimates we have a double exponential term which compensates this effect.

Applying the same argument as in [2] we get that Theorem B follows from an analogue of Proposition 1′ in Part II of [2] which is obtained if c is replaced by $\sqrt{2}$ and M by $\bar{M}$ in this result. For the sake of convenience we also make the following modification. From now on we shall work with the function $K_n\exp\left\{\frac{a_0}{2a_1}2^{n/2}M^2\right\}p_n(x)$ instead of the original function $p_n(x)$ and we denote it in the same way. This modification influences only the norming constant L_n in the Radon-Nikodym derivative.

The proof of this modified version of Proposition 1′ of [2] is very similar to the original one. We have to estimate certain integral expressions in the domains Ω_n^i, $i=1,2,3$, defined in (2.14)–(2.14″). We rewrite these integrals in a polar coordinate system and first estimate the integrals on a circle of fixed radius r. This can be done in the same way as in [2]. Then the integrals with respect to r can be estimated with the help of formulas (6.4) and (6.4′) instead of formulas (4.11′) and (4.11″) in [2]. We get in such a way slightly weaker estimates than those in [2], but they suffice for our purposes. Lemmas 2 and 3 of Part II of [2] remain valid after the replacement of c and M by $\sqrt{2}$ and $\bar{M}$ in the following weaker form: In Lemma 3 the multiplying term K and in Part a) of Lemma 2 the multiplying term c^n before the exponent must be replaced by K^n, where K is some appropriate constant depending on t and T. Also the estimates of Sect. 5 of Part II of [2] remain valid. The only place where the argument of the proof must be slightly changed is Part a) where $\mathrm{S}_n^1 f(x)$ is estimated for $x\in\Omega_2^1$. The argument of [2] works if we show that the expressions $J_{n,\bar{\varepsilon}}(x_1)$ defined by the formula

$$J_{n,\bar{\varepsilon}}(x^{(1)})=\int_{|t|<\bar{\varepsilon}2^{0.3n}} \exp\left\{\frac{tx^{(1)}}{T}+\sqrt{2}\frac{\bar{g}_{n+1}}{2}t\right\} g_n(t)dt \tag{6.5}$$

with some sufficiently small $\bar{\varepsilon}>0$ satisfy the following relations:

$$J_{n,\bar{\varepsilon}}(x^{(1)})=(1+O(2^{-0.1n}))J_{n,\bar{\varepsilon}}(M) \quad \text{if} \quad x\in\Omega_n^1, \tag{6.6}$$

and

$$J_{n,\bar{\varepsilon}}(\bar{M})>K_1>0. \tag{6.7}$$

Relation (6.7) simply follows from (6.3) if we restrict the domain of integration in (6.5) to the domain $|t|<\dfrac{1}{3M}\sqrt{n\log n}$. [The corresponding estimate (5.11) in Part II of [2] also contained an upper bound on $J_{n,\bar{\varepsilon}}(\bar{M})$, but we do not need this bound.] Then relation (6.5) follows from the following observations: The ratio of the integrands in the expressions $J_{n,\bar{\varepsilon}}(x^{(1)})$ and $J_{n,\bar{\varepsilon}}(\bar{M})$ are closer to 1 than const $2^{0.05n}$ if $|t|<2^{0.05n}$ and $x^{(1)}\in\Omega_n^1$ and therefore $|x^{(1)}-\bar{M}|<2^{-0.2n}$, and the contribution of the domain $|t|>2^{0.05n}$ to these integrals is less than $\exp\{-\text{const}\,2^{0.05n}\}$. The remaining part of the proof works with some natural modification of the proof given in [2], hence we omit it.

7. The Proof of Theorems 1 and 2

To prove Theorem 1 first we show that for all q, $2^{-0.1}<q<1$ there are some thresholds n_0 and $N_0(n,q)$ such that if $n\geq n_0$ and $N\geq N_0(n,q)$ then

$$\frac{d\mu_{n,N}^{h_N}}{d\mu_n}(x_1,\dots,x_{2^n})=f_{n,N}^{h_N}\left(2^{-n}\sum_{j=1}^{2n}x_j\right) \tag{7.1}$$

with

$$f_{n,N}^{h_N}(x)=L_n\exp\{\bar{g}2^{n/2}(x^{(1)}-\bar{M})+\bar{A}2^{n/2}x^{(2)2}+\varepsilon_n(x)\} \quad \text{for} \quad x\in\Omega_n^1, \tag{7.2}$$

where
$$\sup_{x\in\Omega_n^1} |\varepsilon_n(x)| \leqq q^n, \tag{7.2'}$$

$$f_{n,N}^{h_N}(x) \leqq L_n \exp\left\{\bar{g}2^{n/2}(|x|-\bar{M}) - \left(\frac{\bar{g}}{2\bar{M}} - \bar{A}\right)2^{0.1n} + q^n\right\} \quad \text{for} \quad x\in\Omega_n^2, \tag{7.3}$$

$$f_{n,N}^{h_N}(x) \leqq L_n \exp\left\{\frac{\bar{g}2^{n/2}}{\bar{M}}(|x|^2 - \bar{M}^2)\right\} \quad \text{if} \quad x > \bar{M} + 2^{-0.2n}, \tag{7.4}$$

$$f_{n,N}^{h_N}(x) \leqq L_n \exp\left\{\frac{\bar{g}2^{n/2}}{2\bar{M}}(|x|^2 - \bar{M}^2)\right\} \quad \text{if} \quad 0 < x < \bar{M} - 2^{-0.2n}, \tag{7.4'}$$

with the numbers $\bar{g}$ and $\bar{A}$ appearing in Proposition B and some appropriate norming constant L_n which satisfies the relation

$$C_1 < 2^{-n/4}\bar{L}_n < C_2 \quad \text{with some} \quad 0 < C_1 < C_2 < \infty, \tag{7.5}$$

where $\bar{L}_n = L_n K_n \exp\left\{\frac{a_0}{2T}2^{n/2}\bar{M}^2\right\}$ with the same norming constant K_n as in (1.9).

In the proof of this statement we argue just as in Sect. 6 of Part II of [2]. Because of Theorem B and Proposition B the constants g_n and A_n can be replaced by $\bar{g}2^{n/2}$ and $\bar{A}2^{n/2}$ in (2.15)–(2.17) by slightly changing the error terms. To show that $L_n = L_n(N, h_N)$ can be chosen independently of N and h_N we observe that a calculation analogous to that in Sect. 6 of [2] yields that

$$\mu_{n,N}^{h_N}(\Omega_n^2 \cup \Omega_n^3) \leqq L_n(N, h_N)\exp\{-K2^{0.3n}\}, \tag{7.6}$$

and for

$$T_n = \int_{\Omega_n^1} \exp\{\bar{g}2^{n/2}(x^{(1)} - \bar{M}) + \bar{A}2^{n/2}x^{(2)2}\}p_n(x)dx, \tag{7.7}$$

the relation

$$\mu_{n,N}^{h_n}(\Omega_1^n) = L_n(N, h_N)T_n(1 + O(q^n)), \qquad 2^{-0.1} < q < 1, \tag{7.8}$$

holds. The estimate

$$C_1 2^{-n/4} < K_n \exp\left\{\frac{a_0}{2T}2^{n/2}\bar{M}^2\right\} T_n < C_2 2^{-n/4} \quad \text{with some} \quad 0 < C_1 < C_2 < \infty. \tag{7.9}$$

also holds true.

The proof of (7.9) is similar to that of (6.9) in [2], only one has to observe that $\bar{g}$ equals $\frac{a_0\bar{M}}{T}$, i.e. -1 times the coefficient of x in (6.3)–(6.4′), and this causes some cancellation if we express $p_n(x)$ through $g_n(x)$ in the integral (7.7). Since $\mu_{n,N}^{h_n}(R^2) = 1$, relations (7.6) and (7.8) imply that L_n can be chosen as T_n^{-1}, and then (7.9) implies (7.5). Theorem 1 can be proved with the help of this information in the following way:

Fix some integer $k \geqq 0$, and define for all $n \geqq k$ and measurable sets $A \subset (R^2)^{2^k}$ the cylindrical set $A(n) = A \times (R^2)^{2^n - 2^k} \subset (R^2)^{2^n}$. Put

$$\tilde{\mu}_n(A) = L_n \int_{\tilde{A}(n)} \exp\left\{\bar{g}2^{-n/2}\sum_{j=1}^{2^n}(x_j^{(1)} - \bar{M}) + \bar{A}2^{-3n/2}\left(\sum_{j=1}^{2^n} x_j^{(2)}\right)^2\right\}\prod_{j=1}^{2^n} p(x_j)dx_j$$

with $\tilde{A}(n)=A(n)\cap\{(x_1,\dots,x_{2^n}),\ 2^{-n}\sum_{j=1}^{2^n} x_j\in\Omega_n^1\}$. We prove similarly to [2] that if $n>n_0$ and $N>N_0(n,q)$, then

$$|\bar{\mu}_n(A(n))-\mu_{n,N}^{h_N}(A(n))|\leqq Kq^n$$

with some $K>0$ independent of the set A.

Theorem 1 can be proved with the help of the above relation similarly to [2]. Moreover, this argument also yields the following

Corollary of Theorem 1. *Let $\bar{\mu}_n$ denote the projection of the measure $\bar{\mu}$ constructed in Theorem 1 to $(R^2)^{2^n}$. There is some function $\bar{f}_n(x)$ such that*

$$\frac{d\bar{\mu}_n}{d\mu_n}(x_1,\dots,x_{2^n})=\bar{f}_n\left(2^{-n}\sum_{j=1}^{2^n} x_j\right).$$

Let $n>n_0$ with some threshold $n_0>0$. Then relations (7.1)–(7.5) remain valid if $f_{n,N}^{h_N}$ is replaced by $\bar{f}_n(x)$ in them.

Now we turn to the proof of Theorem 2. Let us introduce the Hamiltonian $\mathscr{H}_k$ in the volume $(R^2)^{2^k}$ by the formula

$$\mathscr{H}_k(x_1,\dots,x_{2^k})=-\sum_{i=1}^{2^k-1}\ \sum_{j=i+1}^{2^k} d(i,j)^{-3/2}x_ix_j.$$

Let $\sigma=\{\sigma(j)=(\sigma_1(j),\sigma_2(j)), j\in\mathbf{Z}\}$ be a $\bar{\mu}$ distributed vector and consider the random vector $\{(\mathscr{R}_n\sigma^{(1)}(j),\mathscr{R}_n\sigma^{(2)}(j)),\ 1\leqq j\leqq 2^k\}$ defined by formulas (1.3)–(1.5) with $A_n=2^{n/2}\sqrt{n}$ and $B_n=2^{3n/4}$. The argument at the beginning of Sect. 7 in Part II of [2] also shows that the density function $h_{n,k}(x_1,\dots,x_{2^k})$ of this vector can be expressed in the following way:

$$\begin{aligned}h_{n,k}(x_1,\dots,x_{2^k})&=L_{n,k}\bar{f}_{n+k}\left(2^{-k}\sum_{j=1}^{2^k}\tilde{x}_j\right)\\&\quad\times\exp\left\{-\frac{1}{T}\mathscr{H}_k(2^{n/4}\tilde{x}_1,\dots,2^{n/4}\tilde{x}_{2^k})\right\}\prod_{j=1}^{2^k}p_n(\tilde{x}_j)\end{aligned}\tag{7.10}$$

with

$$\tilde{x}=\tilde{x}(x)=(\bar{M}+2^{-n/2}\sqrt{n}x^{(1)},2^{-n/4}x^{(2)})\quad\text{for}\quad x=(x^{(1)},x^{(2)}).\tag{7.10'}$$

Let us define the sets $W_n\subset R^2$ and $\bar{W}_n\subset R^2$ by the formulas

$$\begin{aligned}\bar{W}_n=\Bigg\{&(x^{(1)},x^{(2)}),\ \bar{M}-\frac{\sqrt{T}}{8\bar{M}}2^{-n/2}\sqrt{n\log n}<|x|<\bar{M}+\frac{\sqrt{T}}{8\bar{M}}2^{-n/2}\sqrt{n\log n},\\&|x^{(2)}|<2^{-n/4}n^{1/5},\ x^{(1)}>0\Bigg\},\end{aligned}$$

$$W_n=\{(x^{(1)},x^{(2)}),\ \tilde{x}(x)\in\bar{W}\}.$$

We claim that for all $j=1,2,\dots,2^k$

$$P((\mathscr{R}_n\sigma^{(1)}(j),\mathscr{R}_n\sigma^{(2)}(j))\notin W_n)\leqq n^{-1/100}\quad\text{if}\quad n\geqq n_0.\tag{7.11}$$

and

$$h_{n,k}(x_1,\dots,x_{2^k})=h_k(x_1,\dots,x_{2^k})(1+O(n^{-1/9}))$$
$$\text{if} \quad x_j\in W_n \quad \text{for all} \quad j=1,2,\dots,2^k, \tag{7.12}$$

where $h_k(x_1,\dots,x_{2^k})$ is the function defined in (1.11) (with $s=2$).

Relations (7.11) and (7.12) together imply Theorem 2. Relation (7.12) can be proved with the help of the following estimates:

$$\begin{aligned} p_n(x)&=K_n\left[\exp\left\{-\frac{2^{n+1}M^2a_1}{nT}(|x|-\bar{M})^2\right\}+O\left(\frac{1}{\sqrt{n}}\right)\right]\exp\left\{-\frac{a_0}{2T}2^{n/2}x^2\right\}\\ &=K_n\exp\left\{-\frac{2^{n+1}M^2a_1}{nT}(|x|-\bar{M})^2-\frac{a_0}{2T}2^{n/2}(x^{(1)2}+x^{(2)2})+O(n^{-1/9})\right\}\\ &=\bar{K}_n\exp\left\{-\frac{2^{n+1}M^2a_1}{nT}(x^{(1)}-\bar{M})^2\right.\\ &\quad\left.-\frac{a_0}{2T}2^{n/2}(2\bar{M}(x^{(1)}-\bar{M})+x^{(2)2})+O(n^{-1/9})\right\} \quad \text{if} \quad x\in\bar{W}_n, \end{aligned} \tag{7.13}$$

since in this case we can put the $O(\cdot)$ term into the exponent by appropriately decreasing the power of n in it,

$$\frac{2^n}{n}(|x|-\bar{M})^2=\frac{2^n}{n}(x^{(1)}-\bar{M})^2+O(n^{-3/10}\sqrt{\log n}) \quad \text{for} \quad x\in\bar{W}_n, \tag{7.14}$$

and

$$2^{n/2}x^{(1)2}=2^{n/2}\bar{M}^2+2\cdot 2^{n/2}\bar{M}(x^{(1)}-\bar{M})+O(2^{-n/2}n\log n) \quad \text{for} \quad x\in\bar{W}_n. \tag{7.15}$$

We also have

$$\begin{aligned} \mathscr{H}_k(2^{n/4}x_1,\dots,2^{n/4}x_{2^k})&=C_{k,n}-2^{n/2}\left[\sum_{i=1}^{2^k-1}\sum_{j=i+1}^{2^k}d(i,j)^{-3/2}x_i^{(2)}x_j^{(2)}\right.\\ &\quad\left.+a_0\bar{M}\sum_{i=1}^{2^k}(1-2^{-k/2})(x_i^{(1)}-\bar{M})\right]+O(2^{-n/2}n\log n)\\ &\quad \text{if} \quad x_i\in\bar{W}_n,\ j=1,\dots,2^k, \end{aligned} \tag{7.16}$$

since $\sum_{j=1}^{2^k} d(i,j)^{-3/2}=a_0(1-2^{-k/2})$ for all $1\leqq i\leqq 2^k$, and

$$2^{n/2}(x_i^{(1)}-\bar{M})(x_j^{(1)}-\bar{M})=O(2^{-n/2}n\log n)$$

in this case.

Because of the corollary of Theorem 1 and the relation $\bar{g}=\dfrac{a_0\bar{M}}{T}$,

$$\begin{aligned} \bar{f}_{n+k}\left(2^{-k}\sum_{j=1}^{2^k}x_j\right)&=C_{n,k}\exp\left\{2^{n/2}\left(\frac{a_0\bar{M}}{T}2^{-k/2}\sum_{j=1}^{2^k}(x^{(1)}-\bar{M})\right.\right.\\ &\quad\left.\left.+\bar{A}2^{-3k/2}\left(\sum_{j=1}^{2^k}x_j^{(2)}\right)^2\right)+O(q^n)\right\} \quad \text{if} \quad x_j\in\bar{W}_n,\ j=1,\dots,2^k. \end{aligned} \tag{7.17}$$

Relation (7.12) follows from (7.10), (7.13), (7.16), and (7.17). Relation (7.11) can be proved in the same way as it is done in Sect. 7 of Part II of [2], only the relations $||x|-M|<c^{-0.4n}$ and $|x^{(2)}|<c^{-0.45n}$ must be replaced by

$$||x|-\bar{M}|<\frac{\sqrt{T}}{8\bar{M}}2^{-n/2}\sqrt{n\log n} \text{ and } |x^{(2)}|<2^{-n/4}n^{1/5} \quad \text{and} \quad |x^{(2)}|<2^{-n/4}n^{1/5}$$

in the definition of the set $\tilde{\Omega}_n^1$. We get a weaker bound in (7.11) than the corresponding estimate in [2], but it is sufficient for our purposes. Theorem 2 is proved.

References

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