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# Super-Extensions of Energy Dependent Schrödinger Operators

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**Abstract.** We consider the energy dependent super Schrödinger linear problem $\sum_{i=0}^{N} \lambda^i[(\varepsilon_i\partial^2+u_i)\psi+\eta_i\phi]=0$, $\sum_{i=0}^{N} \lambda^i(\varepsilon_i\partial\phi+\eta_i\psi)=0$ which is a direct generalization of the purely even, energy dependent Schrödinger equation discussed in [1]. We show that the isospectral flows of that problem possess $(N+1)$ compatible Hamiltonian structures. We also extend a generalised factorisation approach of [2] to this case and derive a sequence of $N$ modifications for the $2N$ component systems. The $n^{\text{th}}$ such modification possesses $(N-n+1)$ compatible Hamiltonian structures.

## 1. Introduction

In this paper we generalise the results of [1, 2] to a super extension of the Schrödinger linear problem. The present paper follows very closely the format of [2], so we omit much of the motivation, concentrating on the features which distinguish the super and purely even cases.

We start with Kupershmidt's spectral problem:

$$(\partial^2+u)\psi+\eta\varphi=\lambda\psi, \qquad \partial\phi+\eta\psi=0, \tag{1.1}$$

for the following bi-super Hamiltonian sKdV equation [3]:

$$\begin{aligned} u_t &= \tfrac{1}{4}(u_{xxx}+6uu_x+12\eta\eta_{xx}), \\ \eta_t &= \tfrac{1}{4}(4\eta_{xxx}+6u\eta_x+3u_x\eta). \end{aligned} \tag{1.2}$$

In Sect. 2 we generalise (1.1) to its "energy dependent" counterpart (2.1). The corresponding $2N$ component isospectral flows can be written in Hamiltonian form with respect to $(N+1)$ compatible, locally defined Hamiltonian structures.

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The general class of equations we present contains, as special cases, a number of interesting examples, both known and new. Among the previously known examples [apart from (1.2)] are the super Harry Dym [4] and super dispersive water waves (sDWW) [5] equations, respectively bi- and tri-Hamiltonian. We remark that Kupershmidt only presents 2 Hamiltonian structures for the sDWW equations. One of the new equations is a super-Ito equation, which is tri-Hamiltonian.

In Sect. 3 we present the Miura maps associated with the equations of Sect. 2. For the purely even case it is possible to construct these through the factorisation of the Lax operator $\mathbb{L}$ [2], but this is not easily carried out for the operator (1.1) and its generalisation (2.1). However starting with Kupershmidt's Miura map [3], $M$, for (1.2):

$$u=-v_x-v^2-\vartheta\vartheta_x, \qquad \eta=-\vartheta_x-v\vartheta, \tag{1.3}$$

we generalise the corresponding factorisation of the second Hamiltonian structure of (1.2):

$$\mathbf{B}_1=m(-\mathbf{D})m^{s\dagger}, \tag{1.4}$$

where $m=M^0$ is the (even) Fréchet derivative of $M$ [see (1.5) below], $m^{s\dagger}$ is its superadjoint and $\mathbf{D}=\begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix}$ is the first (and only) Hamiltonian structure of the corresponding sMKdV equation. The Fréchet derivatives of the generalised Miura maps are constructed out of copies of the $2\times 2$ blocks $m$ and $m^{s\dagger}$ of (1.4). In this way Kupershmidt's Miura map for the sDWW equations [5] is constructed in a natural way out of his Miura map for (1.2). Kupershmidt's modification is, in fact, only the first. We also present the second modifications for these equations, thus giving a natural super generalisation of the well known results for the purely even case [2, 6]. The existence of a 2 step modification is a natural consequence of the existence of 3 Hamiltonian structures for the sDWW equations.

Some of our $2N$ component system are shown to have a sequence of $N$ modifications. This is a direct generalisation of results presented in [2].

The basic facts concerning (even) Hamiltonian structures can be found in [2]. The super Hamiltonian formalism is slightly more complicated because of the presence of odd variables. For basic super-linear algebra and super matrix theory we refer to [7]. For basic super Hamiltonian theory we refer to [4].

In this paper, however, we only need the notions of the Fréchet and variational derivatives. Let $\mathbf{U}=F[\mathbf{V}]$ be a change of variables (invertible or otherwise). The Fréchet derivative of $F$ is defined in the usual way (with $\tau$ an even parameter):

$$F^0[\mathbf{V}]\mathbf{W}=\frac{d}{d\tau}F[\mathbf{V}+\tau\mathbf{W}]|_{\tau=0}, \tag{1.5}$$

where $\mathbf{V}$ and $\mathbf{W}$ are even vectors. The notation $F^0$ is that used by Kupershmidt [4], who calls this operator the "even" Fréchet derivative. Hamiltonian structures are then transformed through

$$\mathbf{B}\to\tilde{\mathbf{B}}=F^0\mathbf{B}(F^0)^{s\dagger}, \tag{1.6}$$

where $s\dagger$ denotes the superadjoint, defined in the same way as the usual adjoint, but with the transpose of matrices being replaced by supertranspose [7]. With this definition, the skew symmetry of the Poisson bracket (related to **B**) is *not* equivalent to the skew-superadjointness of the operator **B** (see Leites' discussion of quadratic forms [7]). The conditions implied upon the elements of **B**, by the skew symmetry of the Poisson bracket, are precisely those used by Kupershmidt to define his superadjoint [4].

The variational derivative is defined in the usual way, but remembering that all derivatives are assumed to act from the left.

## 2. Evolution Equations and their Hamiltonian Structures

We consider here the isospectral flows of the energy dependent super-Schrödinger equation:

$$(\varepsilon\partial^2+u)\psi+\eta\varphi=0\,,\qquad \varepsilon\partial\phi+\eta\psi=0\,, \tag{2.1}$$

where, $\varepsilon$, $u$, $\psi$ are even and $\eta$, $\varphi$ are odd variables. We specify the $\lambda$-dependence of $\varepsilon$, $u$, and $\eta$ to be

$$\varepsilon=\sum_0^N \varepsilon_i\lambda^i\,,\qquad u=\sum_0^N u_i\lambda^i\,,\qquad \eta=\sum_0^N \eta_i\lambda^i\,, \tag{2.2}$$

with $\varepsilon_i$ being constants and $u_i$, $\eta_i$ functions of $x$ and with spectral parameter $\lambda$ being even. We postulate the linear time evolution of the wave functions $\psi$, $\varphi$ of the form

$$\psi_t=2p\psi_x+q\psi+\gamma\varphi\,,\qquad \varphi_t=\varrho\psi_x+\sigma\psi+r\varphi\,, \tag{2.3}$$

where the even $p$, $q$, $r$, as well as the odd $\varrho$, $\sigma$, $\gamma$ variables are functions of the potentials $u_i$, $\eta_i$ and of $\lambda$. If we time-differentiate (2.1) (assuming $\lambda_t=0$) and use (2.3), (2.1) to remove all time and higher order $x$-derivatives of $\psi$ and $\varphi$, we are left with

$$\begin{aligned}&(\eta_t+\eta q-\eta r+\varepsilon\sigma_x-u\varrho)\psi+(2p\eta+\varepsilon\varrho_x+\varepsilon\sigma)\psi_x+(\eta\gamma+\eta\varrho+\varepsilon r_x)\varphi=0\,,\\ &(u_t-4up_x-2u_xp+\varepsilon q_{xx}+\eta\sigma+2\eta\gamma_x+\eta_x\gamma)\psi+(2\varepsilon p_{xx}+2\varepsilon q_x+\eta\varrho+\eta\gamma)\psi_x\\ &\qquad+(\eta_t+\eta r-\eta q-4\eta p_x-2\eta_xp+\varepsilon\gamma_{xx}+u\gamma)\phi=0\,.\end{aligned} \tag{2.4}$$

Since (2.4) must be satisfied identically, all coefficients of $\psi$, $\psi_x$, and $\varphi$ must vanish. Thus we obtain

$$\gamma=-\varrho\,,\qquad \varepsilon\sigma=-2\eta p-\varepsilon\varrho_x\,,\qquad q=-p_x\,,\qquad r=0\,, \tag{2.5}$$

where we put inessential constants of integration equal to zero, and

$$\begin{aligned}u_t&=\varepsilon p_{xxx}+4up_x+2u_xp+3\eta\varrho_x+\eta_x\varrho\,,\\ \eta_t&=3\eta p_x+2\eta_xp+\varepsilon\varrho_{xx}+u\varrho\,.\end{aligned} \tag{2.6}$$

Introducing the operator $J=\sum_0^N J_k\lambda^k$ with:

$$J_k=\begin{pmatrix}\varepsilon_k\partial^3+2u_k\partial+2\partial u_k & 2\eta_k\partial+\partial\eta_k\\ 2\partial\eta_k+\eta_k\partial & \varepsilon_k\partial^2+u_k\end{pmatrix}, \tag{2.7}$$

we can rewrite (2.6) as

$$U_t=JP \tag{2.8}$$

with $P=(p,\varrho)^T$ and $U=(u,\eta)^T$. If we choose $P$ to be a polynomial in $\lambda$:

$$P=\sum_{i=0}^{m} P_{m-i}\lambda^i=\begin{pmatrix}\sum_0^m p_{m-i}\lambda^i\\ \sum_0^m \varrho_{m-i}\lambda^i\end{pmatrix} \tag{2.9}$$

with $p_i$ and $\varrho_i$ being functions of $u_j$ and $\eta_j$ only, then (2.8) decomposes into

$$J_0P_{k-N}+J_1P_{k-N+1}+\ldots+J_NP_k=0\,,\qquad k=0,1,\ldots,m-1\,, \tag{2.10a}$$

and

$$\begin{aligned}&U_{0t}=J_0P_m\,,\\&U_{1t}=J_0P_{m-1}+J_1P_m\,,\\&\quad\vdots\\&U_{Nt}=J_0P_{m-N}+\ldots+J_NP_m\,.\end{aligned} \tag{2.10b}$$

Comparing (2.10a) and (2.10b) we see that the recursion relation (2.10a) permits us (in principle) to determine $P_0,\ldots,P_{m-1}$ while $P_m$ is arbitrary. This corresponds to the freedom of gauge in our linear problem (2.1). There are two distinct cases:

i) $u_N=\text{constant}=-1$, $\eta_N=(\text{odd})\text{ constant}=-\gamma$.
The last equation of (2.10b) is then added to (2.10a), enabling us to determine $P_m$. The remaining equations of (2.10b) form the equations of motion for $U_0,\ldots,U_{N-1}$. We refer to this choice as the (generalised) sKdV case.

ii) $u_0=\text{constant}$, $\eta_0=(\text{odd})\text{ constant}$.
This requires $P_m=0$ and is referred to as the (generalised) super Harry Dym case.

Since both cases are identical from the point of view of underlying algebraic structure we concentrate, in what follows, on the (generalised) sKdV choice.

Further development is almost the same as in the purely even case [2]. The only difference is that now $U_i\equiv(u_i,\eta_i)^T$ and $P_i\equiv(p_i,\varrho_i)^T$ are 2 component vectors rather than scalar functions and $J_k$ are matrix (not scalar) differential operators.

To prove that the recursion relation (2.10a) has a solution of the form (2.9) for any $m\geqq 0$ we consider the formal power series solution:

$$\mathscr{P}=\sum_0^\infty P_n\lambda^{-n} \tag{2.11}$$

of the equation $J\mathscr{P}=0$. Then

$$P=(\lambda^m\mathscr{P})_+\,, \tag{2.12}$$

where $(\ )_+$ means only terms with non-negative powers of $\lambda$, is of the form (2.9) and satisfies (2.10a). Explicitly, the equation $J\mathscr{P}=0$ is

$$\varepsilon p_{xxx}+4up_x+2u_xp+3\eta\varrho_x+\eta_x\varrho=0\,, \tag{2.13a}$$

$$3\eta p_x+2\eta_xp+\varepsilon\varrho_{xx}+u\varrho=0\,. \tag{2.13b}$$

Multiplying (2.13a) by $p$ and (2.13b) by $\varrho$ (from the right), adding the resulting equations together and integrating once we get

$$2up^2+3p\eta\varrho+\varepsilon(pp_{xx}-\tfrac{1}{2}p_x^2-\varrho\varrho_x)=C(\lambda), \tag{2.14a}$$

$$u\varrho+3\eta p_x+2\eta_x p+\varepsilon\varrho_{xx}=0, \tag{2.14b}$$

where $C(\lambda)$ is a constant of integration which we choose to be $C\lambda^N$. The recursion relation resulting from (2.14) [when substituting (2.11)] is solvable whenever $\varepsilon_N=0$ since the leading term coefficients of $\lambda^{N-n}$ are then $4p_0u_N^{-1}(u_N^2-3\eta_N\eta_{Nx})p_n$ and $u_N\varrho_n$. The series starts with

$$\begin{aligned} p_0&=(Cu_N)^{1/2}(2u_N^2-6\eta_N\eta_{Nx})^{-1/2}, \\ \varrho_0&=-3u_N^{-1}p_{0x}\eta_N-2u_N^{-1}p_0\eta_{Nx}. \end{aligned} \tag{2.14c}$$

Thus we have proven the existence of an infinite power series solution (2.11) of Eq. (2.13) with $p_i$, $\varrho_i$ being differential functions of $u_i$, $\eta_i$, provided $\varepsilon_N=0$. We note that the above proof holds for the KdV as well as Harry Dym case. Equations (2.10b) and (2.12) give us an infinite series of flows isospectral to (2.1):

$$\mathbf{U}_{t_m}=\mathbf{B}_N\mathbf{P}^{(m)}, \qquad m=0,1,\dots, \tag{2.15}$$

where

$$\mathbf{B}_N=\begin{pmatrix} 0 & & J_0 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ J_0 & \cdots & J_{N-1} \end{pmatrix}, \tag{2.16}$$

$\mathbf{U}=(U_0,\dots,U_{N-1})^T\equiv(u_0,\eta_0,\dots,u_{N-1},\eta_{N-1})^T$ and $\mathbf{P}^{(m)}=(P_{m-N+1},\dots,P_m)^T$.

To prove that (2.15) gives us an infinite hierarchy of *Hamiltonian* flows we need to know that $\mathbf{B}_N$ is a Hamiltonian operator and that $\mathbf{P}^{(m)}$ are variational derivatives of some functionals $\mathscr{H}_m$, $\mathbf{P}^{(m)}=\delta\mathscr{H}_m$, where $\delta$ stands for $(\delta_{u_0},\delta_{\eta_0},\dots,\delta_{u_{N-1}},\delta_{\eta_{N-1}})^T$. It is possible to check both properties directly, but we prefer the more instructive methods employed below.

The hierarchy (2.15) is, in fact, not just Hamiltonian but actually $(N+1)$-Hamiltonian. The recursion relation (2.10a) can be written as a bi-Hamiltonian ladder

$$\mathbf{B}_n\mathbf{P}^{(k)}=\mathbf{B}_{n-1}\mathbf{P}^{(k+1)} \tag{2.17}$$

in $N$ different ways corresponding to $n=1,\dots,N$ in (2.17). The Hamiltonian operators $\mathbf{B}_n$ are given by

$$\mathbf{B}_n=\left[\begin{array}{ccc|ccc} 0 & & J_0 & & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & & 0 & \\ J_0 & \cdots & J_{n-1} & & & \\ \hline & & & -J_{n+1} & \cdots & -J_N \\ & 0 & & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & & -J_N & & 0 \end{array}\right] \tag{2.18}$$

and satisfy $\mathbf{B}_n = \mathbf{R}\mathbf{B}_{n-1}$, with

$$\mathbf{R} = \left( \begin{array}{ccc|c} 0 & \cdots & 0 & -J_0 J_N^{-1} \\ \hline 1 & & 0 & -J_n J_N^{-1} \\ & \ddots & & \vdots \\ 0 & & 1 & -J_{N-1} J_N^{-1} \end{array} \right) \tag{2.19}$$

being the recursion operator ($J_N^{-1}$ is the formal inverse of $J_N$, well defined if $\varepsilon_N = 0$). From (2.17) it follows that (2.15) is the $(N+1)$-Hamiltonian system

$$\mathbf{U}_{t_m} = \mathbf{B}_{N-l} \delta \mathscr{H}_{m+l}, \qquad l = 0, \ldots, N. \tag{2.20}$$

A simple proof that all the differential operators $\mathbf{B}_n$ of (2.18) are Hamiltonian will be given in the next section. Now we will show that the $N$-tuples $\mathbf{P}^{(m)}$ obtained from (2.12), (2.14) are indeed variational derivatives of some functionals $\mathscr{H}_m$. Our proof will be constructive. Let us consider the Riccati equation

$$u + \varepsilon(y_x + y^2 + \xi\xi_x) = 0, \qquad \eta + \varepsilon(\xi_x + y\xi) = 0, \tag{2.21a}$$

satisfied by $y = \dfrac{\psi_x}{\psi}$, $\xi = \dfrac{\phi}{\psi}$ as a consequence of the linear problem (2.1). The formal power series solution of (2.21a):

$$y = \zeta^s \sum_{i=0}^{-\infty} y_i \zeta^i, \tag{2.22a}$$

$$\xi = \zeta^s \sum_{i=0}^{-\infty} \xi_i \zeta^i, \tag{2.22b}$$

where $\zeta^2 = \lambda$ and $s = N - r$ ($r$ is the largest $n$ such that $\varepsilon_n \neq 0$), gives us an infinite series of conserved quantities $y_i$. Half of these (those with odd $i$) are trivial while the others (with $i = 2n$) can be chosen as our Hamiltonians $\mathscr{H}_n$. To prove that Hamiltonians defined this way are compatible with the multi-Hamiltonian ladder (2.17) we consider (2.21a) as a change of variables

$$\begin{aligned} u &= F_1[y, \xi] = -\varepsilon(y_x + y^2 + \xi\xi_x), \\ \eta &= F_2[y, \xi] = -\varepsilon(\xi_x + y\xi). \end{aligned} \tag{2.21b}$$

Then according to the well known rule (compare with [4]) $\delta_Y = (F^0)^{s\dagger} \delta_U$ we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} \dfrac{\delta y}{\delta y} \\ \dfrac{\delta y}{\delta \xi} \end{bmatrix} = -\varepsilon \begin{pmatrix} -\partial + 2y & \xi \\ \partial\xi + \xi_x & -\partial + y \end{pmatrix} \begin{bmatrix} \dfrac{\delta y}{\delta u} \\ \dfrac{\delta y}{\delta \eta} \end{bmatrix}. \tag{2.23}$$

Introducing $p = \dfrac{\delta y}{\delta u}$, $\varrho = \dfrac{\delta y}{\delta \eta}$ we rewrite (2.23) as

$$\varepsilon(-p_x + 2yp + \xi\varrho) = 1, \tag{2.24a}$$

$$\varepsilon(p_x\xi + 2p\xi_x - \varrho_x + y\varrho) = 0. \tag{2.24b}$$

Differentiating (2.24a) twice and (2.24b) once and making use of (2.21) to eliminate $y$ and $\xi$ we end up with Eqs. (2.13). Since $p$ and $\varrho$ defined above differ from the variational derivatives $\frac{\delta y}{\delta u_k}, \frac{\delta y}{\delta \eta_k}$ only by a factor $\lambda^k$, we see that the variational derivatives of the Hamiltonians $\mathscr{H}_n$ satisfy (2.13) and thus are compatible with the multi-Hamiltonian ladder (2.17). In fact, comparing the initial conditions for the relevant recursion relations, one can show that the variational derivatives of the formal power series (2.22a) coincide with the formal power series solution of (2.13) constructed previously.

*Remark.* The possibility of expanding solutions of (2.13) in the powers of $\lambda$ rather than $\zeta$ is explained by the fact that the odd $y_i$ are trivial (i.e. their variational derivatives vanish) and thus do not contribute to the series for $\mathscr{P} = \left(\frac{\delta y}{\delta u}, \frac{\delta y}{\delta \eta}\right)^T$.

## 3. Miura Maps

In this section we generalise our construction [2] of Miura maps associated with the energy dependent Schrödinger spectral problem. The factorisation of that operator, presented in [2], does not survive the super extension to (2.1), but we can still factorise our basic operator $J = \sum_0^N J_k \lambda^k$ of (2.7) as:

$$J = (m_0, \dots, m_N)(-D)\Lambda \begin{pmatrix} m_0^{s\dagger} \\ \vdots \\ m_N^{s\dagger} \end{pmatrix}, \tag{3.1}$$

where $\Lambda$ is a symmetric ($\lambda$-dependent) $(N+1)\times(N+1)$ matrix,

$$D = \begin{pmatrix} \partial & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

and

$$m_k = \begin{pmatrix} -\alpha_k\partial - 2v_k & -\vartheta_k\partial + \vartheta_{kx} \\ -\vartheta_k & -\alpha_k\partial - v_k \end{pmatrix}, \tag{3.3a}$$

$$m_k^{s\dagger} = \begin{pmatrix} \alpha_k\partial - 2v_k & -\vartheta_k \\ -\partial\vartheta_k - \vartheta_{kx} & \alpha_k\partial - v_k \end{pmatrix}, \tag{3.3b}$$

are copies of the Fréchet derivative of the elementary Miura map

$$u = -\alpha v_x - v^2 - \vartheta\vartheta_x, \qquad \eta = -\alpha\vartheta_x - v\vartheta. \tag{3.4}$$

*Remark.* This elementary Miura map is a slight modification of Kupershmidt's map (1.3) so is easily seen to be non-degenerate.

Choosing $\Lambda$ to be

$$\Lambda=\Lambda_r=\left[\begin{array}{ccc|ccc} 1 & \cdots & \lambda^{r-1} & & & \\ \vdots & ⋰ & & & 0 & \\ \lambda^{r-1} & & 0 & & & \\ \hline & & & 0 & & \lambda^r \\ & 0 & & & ⋰ & \vdots \\ & & & \lambda^r & \cdots & \lambda^N \end{array}\right], \tag{3.5}$$

we obtain the following maps $\mathbf{V}\to\mathbf{U}$,

$$\varepsilon_k=\sum_0^k \alpha_i\alpha_{k-i}, \tag{3.6a}$$

$$u_k=\tfrac{1}{2}\sum_0^k \mathscr{V}_{i,k-i}, \tag{3.6b}$$

$$\eta_k=\tfrac{1}{2}\sum_0^k \Theta_{i,k-i}, \tag{3.6c}$$

for $k=0,\ldots,r-1$, and

$$\varepsilon_k=\sum_0^{N-k} \alpha_{k+i}\alpha_{N-i}, \tag{3.7a}$$

$$u_k=\tfrac{1}{2}\sum_0^{N-k} \mathscr{V}_{k+i,N-i}, \tag{3.7b}$$

$$\eta_k=\tfrac{1}{2}\sum_0^{N-k} \Theta_{k+i,N-i}, \tag{3.7c}$$

for $k=r,\ldots,N$, where

$$\begin{aligned}\mathscr{V}_{kn}&=-\alpha_k v_{nx}-\alpha_n v_{kx}-2v_k v_n-\vartheta_k\vartheta_{nx}-\vartheta_n\vartheta_{kx},\\ \Theta_{kn}&=-\alpha_k\vartheta_{nx}-\alpha_n\vartheta_{kx}-v_k\vartheta_n-v_n\vartheta_k.\end{aligned} \tag{3.8}$$

Equations (3.6) and (3.7) result from the formula

$$m_k(-D)m_n^{\mathrm{st}\dagger}+m_n(-D)m_k^{\mathrm{st}\dagger}=\begin{pmatrix}2\alpha_k\alpha_n\partial^3+2\mathscr{V}_{kn}\partial+2\partial\mathscr{V}_{kn} & 2\Theta_{kn}\partial+\partial\Theta_{kn}\\ 2\partial\Theta_{kn}+\Theta_{kn}\partial & 2\alpha_k\alpha_n\partial^2+\mathscr{V}_{kn}\end{pmatrix}. \tag{3.9}$$

The analogy between the above construction and that for the purely even case [2], is so complete that we need not repeat the details of the calculations here, but concentrate on the results only. The essential difference between the two cases is that $J$, $m_k$, $D$ are now matrices rather than scalars. As in [2], we explicitly present the (generalised) sKdV case ($u_N=-1$, $\eta_N=-\gamma$) only, leaving details concerning the super Harry Dym case to the reader. The sKdV choice is compatible with the factorisation (3.1) with $\Lambda=\Lambda_r$ for $r=0,\ldots,N$ (in the super Harry Dym case we must choose $r=1,\ldots,N+1$). Assuming $v_N=-1$, $\vartheta_N=-\gamma$ we rewrite (3.7) as:

$$\varepsilon_k=\sum_0^{N-k} \alpha_{k+i}\alpha_{N-i}, \qquad k=r,\ldots,N, \tag{3.10a}$$

and

$$u_k = -\alpha_N v_{kx} + 2v_k + \gamma \vartheta_{kx} + \tfrac{1}{2} \sum_{1}^{N-k-1} \mathscr{V}_{k+i,N-i}, \tag{3.10b}$$

$$\eta_k = -\alpha_N \vartheta_{kx} + \vartheta_k + v_k \gamma + \tfrac{1}{2} \sum_{1}^{N-k-1} \Theta_{k+i,N-i}, \tag{3.10c}$$

for $k=r,\ldots,N-1$. We denote by $\mathbf{V}^{(r)}$ modified variables corresponding to the Miura map $\mathbf{U}=M_r[\mathbf{V}^{(r)}]$ defined by (3.6), (3.10).

**Proposition 1.** *If the factorisation* (3.1) *with* $\Lambda=\Lambda_r$ *is consistent (meaning that* $\varepsilon$ *is such that Eqs.* (3.6a), (3.10a) *can be solved for* $\alpha_k$, $k=0,\ldots,N$*) then the operator* $\mathbf{B}_r$ *is* $M_r$*-related to a constant coefficient Hamiltonian operator* $\tilde{\mathbf{B}}_r$,

$$\mathbf{B}_r = (M_r^0)\tilde{\mathbf{B}}_r(M_r^0)^{st\dagger}|_{U=M_r[V^{(r)}]}, \tag{3.11}$$

*where*

$$\tilde{\mathbf{B}}_r = \left[\begin{array}{ccc|ccc} 0 & & -D & & & \\ & \cdot^{\cdot^{\cdot}} & & & 0 & \\ -D & & 0 & & & \\ \hline & & & 0 & & D \\ & 0 & & & \cdot^{\cdot^{\cdot}} & \\ & & & D & & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} r \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} N-r \end{array}. \tag{3.12}$$

*Proof.* The proposition follows from the formula (3.9) and the fact that the Fréchet derivative of the transformation $M_r$ takes the simple form:

$$M_r^0 = \left[\begin{array}{ccc|ccc} m_0 & & 0 & & & \\ \vdots & \ddots & & & 0 & \\ m_{r-1} & \cdots & m_0 & & & \\ \hline & & & m_N & \cdots & m_{r+1} \\ & 0 & & & \ddots & \vdots \\ & & & 0 & & m_N \end{array}\right]. \tag{3.13}$$

**Corollary.** *For any choice of* $\varepsilon_0,\ldots,\varepsilon_N$ *all the operators* $\mathbf{B}_n$ *of* (2.18) *are Hamiltonian.*

*Proof.* If we assume that $\varepsilon_0\varepsilon_N \neq 0$, then Eqs. (3.6a), (3.10a) can always be solved for $\alpha_n$. Thus any $\mathbf{B}_r$ with $\varepsilon_0\varepsilon_N \neq 0$ is Miura related by (3.11) to a constant coefficient Hamiltonian operator $\tilde{\mathbf{B}}_r$ of (3.12) and thus (since $M_r$ is nondegenerate) is itself a Hamiltonian operator. Taking the limit $\varepsilon_0 \to 0$ and/or $\varepsilon_N \to 0$ we see that all the $\mathbf{B}_n$ are Hamiltonian.

*Remark.* The non-degeneracy of this Miura map follows from the simple structure of its Fréchet derivative (3.13) (see the discussion following (3.10b) of [2]), together with the non-degeneracy of the map (1.3). This is also true of the other Miura maps discussed in this paper.

The form of $M_r^0$ shows that $M_r$ is a non-invertible transformation (and thus a genuine Miura map) only when $\alpha_0$ or $\alpha_N$ are different from zero. Since the very existence of the (generalised) sKdV hierarchy requires $\varepsilon_N=0$ and thus $\alpha_N=0$ the only Miura maps correspond to $\alpha_0\neq 0$, say $\alpha_0=1$, and consequently $\varepsilon_0=1$.

In what follows we limit ourselves, for the sake of simplicity to the case $\varepsilon_0=1$, $\varepsilon_k=0$ for $k>0$. This guarantees the solvability of (3.6a), (3.10a) for any $r=1,\dots,N$ (with the solution $\alpha_0=1$, $\alpha_k=0$ for $k>0$) and thus Proposition 1 gives us $N$ distinct Hamiltonian Miura maps $M_r$ relating $\mathbf{B}_r$ to the constant coefficient operator $\tilde{\mathbf{B}}_r$, $r=1,\dots,N$. The "ultimate" Miura map $M_N$ is the most interesting one.

**Proposition 2.** i) *The Miura map $M_N$ decomposes into $N$ consecutive Miura maps $M_N^n$, $n=1,\dots,N$: $M_N=M_N^1\circ\dots\circ M_N^N$, each $M_N^{n+1}$ being a map $\mathbf{U}^{(n+1)}\to\mathbf{U}^{(n)}$ given by*

$$\begin{aligned} &u_n^{(n)}=\tfrac{1}{2}\sum_0^n \mathcal{U}_{i,n-i}^{(n+1)}, \qquad \eta_n^{(n)}=\tfrac{1}{2}\sum_0^n H_{i,n-i}^{(n+1)},\\ &u_k^{(n)}=u_k^{(n+1)}, \qquad \eta_k^{(n)}=\eta_k^{(n+1)} \quad \textit{for} \quad k\neq n, \end{aligned} \tag{3.14}$$

*where $\mathcal{U}_{kl}^{(n+1)}$, $H_{kl}^{(n+1)}$ are defined by (3.8) with $u_i^{(n+1)}$, $\eta_i^{(n+1)}$ instead of $v_i$, $\vartheta_i$. In particular we have $\mathbf{U}^{(0)}=\mathbf{U}$, $\mathbf{U}^{(N)}=\mathbf{V}^{(N)}$.*

ii) *Each Miura map $M_N^n$ is Hamiltonian and the $n^{\text{th}}$ modification $\mathbf{U}^{(n)}$ possesses $N+1-n$ compatible Hamiltonian structures:*

$$\mathbf{B}_k^n=(M_N^{n+1})^0 B_k^{n+1}((M_N^{n+1})^0)^{s\dagger}\big|_{\mathbf{U}^{(n)}=M_N^{n+1}[\mathbf{U}^{(n+1)}]}, \tag{3.15}$$

*where $k=n+1,\dots,N$ (the first Hamiltonian structure $\mathbf{B}_n^n$ of the $n^{th}$ modification is not the image of any local Hamiltonian operator under the map $M_N^{n+1}$). This is illustrated by the Fig. 1.*

$$\begin{array}{l} \mathbf{U}=\mathbf{U}^{(0)}\xleftarrow{M_N^1}\mathbf{U}^{(1)}\xleftarrow{M_N^2}\dots\xleftarrow{M_N^n}\mathbf{U}^{(n)}\xleftarrow{M_N^{n+1}}\dots\xleftarrow{M_N^N}\mathbf{U}^{(N)}\equiv\mathbf{V}^{(N)}\\ \mathbf{B}_N\equiv\mathbf{B}_N^0\longleftarrow\mathbf{B}_N^1\,\dots\dots\dots\longleftarrow\mathbf{B}_N^n\longleftarrow\dots\longleftarrow\mathbf{B}_N^N\\ \quad\vdots\\ \mathbf{B}_n\equiv\mathbf{B}_n^0\longleftarrow\mathbf{B}_n^1\,\dots\dots\dots\longleftarrow\mathbf{B}_n^n\\ \quad\vdots\\ \mathbf{B}_0\equiv\mathbf{B}_0^0 \end{array}$$

**Fig. 1**

iii) The chain of modified systems is given by

$$\mathbf{U}_{t_m}^{(n)}=B_{N-l}^{(n)}\delta\mathcal{H}_{m+l}^n, \qquad l=0,\dots,N-n, \qquad m=0,1,\dots, \tag{3.16}$$

where $\mathcal{H}_i^n=\mathcal{H}_i\circ M_N^1\circ\dots\circ M_N^n$ are the modified Hamiltonians.

*Remark.* The only difference between the sKdV and super Harry Dym cases is that in the latter the map $M_r$ is derived from the factorisation (3.1) with $\Lambda=\Lambda_{r+1}$ rather than $\Lambda_r$.

## 4. Examples

We present, in this section, a few of the simplest super extensions of well known classical integrable systems corresponding to $N=1$ and $N=2$ in our general scheme. We limit ourselves to coupled KdV systems leaving their Harry Dym analogues to the reader. Thus throughout this section we assume that $u_N=-1$, $\eta_N=-\gamma$ and $\varepsilon_N=0$.

*The 2-Component System*

If $N=1$ our construction reduces to a one (odd) parameter extension of the super Schrödinger linear problem of Kupershmidt [3]. The isospectral flows of that problem are bi-Hamiltonian with Hamiltonian operators given by $\mathbf{B}_0=-J_1$ and $\mathbf{B}_1=J_0$:

$$\mathbf{B}_0=\begin{pmatrix} 4\partial & 3\gamma\partial \\ 4\gamma\partial & 1 \end{pmatrix}, \quad \mathbf{B}_1=\begin{pmatrix} \varepsilon\partial^2+2u\partial+2\partial u & 2\eta\partial+\partial\eta \\ 2\partial\eta+\eta\partial & \partial^2+u \end{pmatrix}, \tag{4.1}$$

where we write $\varepsilon$, $u$, $\eta$ instead of $\varepsilon_0$, $u_0$, $\eta_0$. If $\varepsilon=1$ the corresponding hierarchy is a one (odd) parameter extension of Kupershmidt's sKdV [3] while $\varepsilon=0$ gives its dispersionless limit. We concentrate on the case $\varepsilon=1$ here. The first two Hamiltonians are

$$\mathscr{H}_0=\tfrac{1}{2}u, \quad \mathscr{H}_1=\tfrac{1}{8}u^2+\tfrac{1}{2}\eta\eta_x-\tfrac{3}{4}u\gamma\eta_x, \tag{4.2}$$

and the first nontrivial flow is given by

$$\begin{aligned} u_t&=\tfrac{1}{4}(u_{xx}-3\gamma\eta_{xxx}+3u^2+12\eta\eta_x+9u_x\gamma\eta-12u\gamma\eta_x)_x, \\ \eta_t&=\tfrac{1}{4}(4\eta_{xxx}-3u_{xxx}\gamma+3u_x\eta+6u\eta_x-3uu_x\gamma+9\gamma\eta\eta_{xx}). \end{aligned} \tag{4.3}$$

When $\gamma=0$ it reduces to the sKdV of (1.2). Kupershmidt's Miura map (1.3) is unchanged:

$$u=-v_x-v^2-\vartheta\vartheta_x, \quad \eta=-\vartheta_x-v\vartheta, \tag{4.4}$$

and relates $\mathbf{B}_1$ to the (only) Hamiltonian structure of the sMKdV:

$$\tilde{\mathbf{B}}_1=-D=-\begin{pmatrix} \partial & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

The modified Hamiltonians are

$$\begin{aligned} \tilde{\mathscr{H}}_0&=-\tfrac{1}{2}v^2-\tfrac{1}{2}\vartheta\vartheta_x, \\ \tilde{\mathscr{H}}_1&=\tfrac{1}{8}v_x^2+\tfrac{1}{8}v^4+\tfrac{3}{4}(v^2-v_x)\vartheta\vartheta_x+\tfrac{1}{2}\vartheta_x\vartheta_{xx}+\tfrac{3}{4}\vartheta\vartheta_x\vartheta_{xx}\gamma+\tfrac{3}{4}(v^2+v_x)(\vartheta_{xx}+v_x\vartheta+v\vartheta_x)\gamma. \end{aligned} \tag{4.6}$$

The sMKdV itself is given (when $\gamma=0$) by

$$\begin{aligned} v_t&=\tfrac{1}{4}(v_{xx}-2v^3-6v\vartheta\vartheta_x-3\vartheta\vartheta_{xx})_x, \\ \vartheta_t&=\tfrac{1}{4}(\vartheta_{xxx}+6(v_x-v^2)\vartheta_x+(3v_{xx}-6vv_x)\vartheta). \end{aligned} \tag{4.7}$$

*The 4-Component Systems*

These are tri-Hamiltonian with Hamiltonian operators given by

$$\mathbf{B}_0=\begin{pmatrix} -J_1 & -J_2 \\ -J_2 & 0 \end{pmatrix}, \quad \mathbf{B}_1=\begin{pmatrix} J_0 & 0 \\ 0 & J_2 \end{pmatrix}, \quad \mathbf{B}_2=\begin{pmatrix} 0 & J_0 \\ J_0 & J_1 \end{pmatrix}, \tag{4.8}$$

where $J_2$ is now given by

$$J_2=\begin{pmatrix} -4\partial & -3\gamma\partial \\ -3\gamma\partial & -1 \end{pmatrix}.$$

The first few Hamiltonians (with $\gamma=0$) are

$$\begin{aligned} &\mathscr{H}_0=\tfrac{1}{2}u_1, \qquad \mathscr{H}_1=\tfrac{1}{2}u_0+\tfrac{1}{8}u_1^2+\tfrac{1}{2}\eta_1\eta_{1x}, \\ &\mathscr{H}_2=\tfrac{1}{4}u_0u_1+\tfrac{1}{16}u_1^3-\frac{\varepsilon_1}{32}u_{1x}^2+\tfrac{3}{4}u_1\eta_1\eta_{1x}+\eta_0\eta_{1x}-\frac{\varepsilon_1}{2}\eta_{1x}\eta_{1xx}. \end{aligned} \tag{4.9}$$

We will concentrate on two special cases: $\varepsilon_0=1$, $\varepsilon_1=0$ (dispersive water waves) and $\varepsilon_0=0$, $\varepsilon_1=1$ (Ito's equation).

*Super* DWW

Specializing our general formulae by the choice $\varepsilon_0=1$, $\varepsilon_1=0$ we obtain a tri-Hamiltonian hierarchy with 2 even and 2 odd variables and first nontrivial flow given by

$$\begin{aligned} u_{0t}&=\tfrac{1}{4}u_{1xxx}+u_0u_{1x}+\tfrac{1}{2}u_{0x}u_1+3\eta_0\eta_{1xx}+\eta_{0x}\eta_{1x}, \\ \eta_{0t}&=\eta_{1xxx}+u_0\eta_{1x}+\tfrac{1}{2}u_1\eta_{0x}+\tfrac{3}{4}u_{1x}\eta_0, \\ u_{1t}&=u_{0x}+\tfrac{3}{2}u_1u_{1x}+3\eta_1\eta_{1xx}, \\ \eta_{1t}&=\eta_{0x}+\tfrac{3}{2}u_1\eta_{1x}+\tfrac{3}{4}u_{1x}\eta_1. \end{aligned} \tag{4.10}$$

An invertible change of variables

$$\begin{aligned} q&=\tfrac{1}{2}u_0+\tfrac{1}{8}u_1^2-\tfrac{1}{4}u_{1x}+\tfrac{1}{2}\eta_1\eta_{1x}, \\ \mu&=\eta_0+\tfrac{1}{2}u_1\eta_1-\eta_{1x}, \\ r&=\tfrac{1}{2}u_1, \\ v&=\eta_1, \end{aligned} \tag{4.11}$$

transforms (4.10) into the "super long waves" equation of Kupershmidt [5]. The Hamiltonians and first Hamiltonian structure of sDWW hierarchy take a very simple form when written in $(q,\mu,r,v)$ coordinates:

$$\tilde{\mathbf{B}}_0=\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \tag{4.12a}$$

$$\tilde{\mathscr{H}}_0=r, \qquad \tilde{\mathscr{H}}_1=q, \qquad \tilde{\mathscr{H}}_2=qr+\mu v_x. \tag{4.12b}$$

The sDWW hierarchy possesses two consecutive Hamiltonian modifications, Propositions 1 and 2 giving us two Miura maps:

$$\begin{aligned} u_0&=-w_{0x}-w_0^2-\kappa_0\kappa_{0x}, \\ \eta_0&=-\kappa_{0x}-w_0\kappa_0, \\ u_1&=w_1, \\ \eta_1&=\kappa_1, \end{aligned} \tag{4.13a}$$

and

$$
\begin{aligned}
w_0 &= v_0\,, \\
\kappa_0 &= \vartheta_0\,, \\
w_1 &= -v_{1x} - 2v_0 v_1 - \vartheta_0 \vartheta_{1x} - \vartheta_1 \vartheta_{0x}\,, \\
\kappa_1 &= -\vartheta_{1x} - v_0 \vartheta_1 - v_1 \vartheta_0\,,
\end{aligned} \tag{4.13b}
$$

where we write $(w_0, \kappa_0, w_1, \kappa_1)$ instead of $(u_0^{(1)}, \eta_0^{(1)}, u_1^{(1)}, \eta_1^{(1)})$. The transformation (4.13a) is equivalent to Kupershmidt's single Miura map [5].

*Super Ito Equation*

The choice $\varepsilon_0 = 0$, $\varepsilon_1 = 1$ leads to a super extension of Ito's hierarchy. The first nontrivial flow is

$$
\begin{aligned}
u_{0t} &= u_0 u_{1x} + \tfrac{1}{2} u_{0x} u_1 + 3\eta_0 \eta_{1xx} + \eta_{0x} \eta_{1x}\,, \\
\eta_{0t} &= u_0 \eta_{1x} + \tfrac{1}{2} u_1 \eta_{0x} + \tfrac{3}{4} u_{1x} \eta_0\,, \\
u_{1t} &= \tfrac{1}{4} u_{1xxx} + u_{0x} + \tfrac{3}{2} u_1 u_{1x} + 3\eta_1 \eta_{1xx}\,, \\
\eta_{1t} &= \eta_{1xxx} + \eta_{0x} + \tfrac{3}{2} u_1 \eta_{1x} + \tfrac{3}{4} u_{1x} \eta_1\,.
\end{aligned} \tag{4.14a}
$$

The invertible map $\mathbf{U} = M_1[\mathbf{V}]$ (of Sect. 3) given by

$$
\begin{aligned}
u_0 &= -r^2 - v v_x\,, \\
\eta_0 &= -r v\,, \\
u_1 &= 2q\,, \\
\eta_1 &= \mu\,,
\end{aligned} \tag{4.15}
$$

where we write $(r, v, q, \mu)$ instead of $(v_0, \vartheta_0, v_1, \vartheta_1)$, transforms (4.14a) into

$$
\begin{aligned}
r_t &= (rq - \mu_x v)_x\,, \\
v_t &= \tfrac{1}{2} q_x v + q v_x + r\mu_x\,, \\
q_t &= \tfrac{1}{4}(q_{xx} + 6q^2 - 2r^2 - 2v v_x + 6\mu\mu_x)_x\,, \\
\mu_t &= \mu_{xxx} + \tfrac{3}{2} q_x \mu + 3q\mu_x - (rv)_x\,.
\end{aligned} \tag{4.14b}
$$

The system (4.14b) reduces to Ito's equation when $v = \mu = 0$ and to the sKdV when $r = 0$, $v = 0$.

The second Hamiltonian structure of the Ito hierarchy takes a particularly simple form when written in $(r, v, q, \mu)$ coordinates. According to Proposition 1 it is given by

$$
\tilde{\mathbf{B}}_1 = \begin{pmatrix} -D & 0 \\ 0 & D \end{pmatrix}. \tag{4.16}
$$

The Hamiltonians $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_2$ are now given by

$$
\begin{gathered}
\tilde{\mathscr{H}}_0 = q\,, \qquad \tilde{\mathscr{H}}_1 = \tfrac{1}{2} q^2 - \tfrac{1}{2} r^2 + \tfrac{1}{2}\mu\mu_x - \tfrac{1}{2} v v_x\,, \\
\tilde{\mathscr{H}}_2 = -\tfrac{1}{8} q_x^2 + \tfrac{1}{2} q^3 - \tfrac{1}{2} q r^2 - \tfrac{1}{2} q v v_x + \tfrac{3}{2} q\mu\mu_x + r\mu_x v - \tfrac{1}{2}\mu_x \mu_{xx}\,.
\end{gathered} \tag{4.17}
$$

## 5. Conclusions

In this paper we have generalised all the results of [2] from the purely even to the super case. First, we have used the energy dependent super Schrödinger operator (2.1a) to construct hierarchies of super integrable equations with $2N$ components and $(N+1)$ compatible Hamiltonian structures. As a general class of equation, this is completely new. However, for $N=1$ and $N=2$ this class includes a number of known, *but disparate*, examples, such as Kupershmidt's sKdV, sDWW and super Harry Dym equations. One of the new examples in our general class is a super Ito's equation.

As well as the isospectral flows of the spectral problem (2.1a) we also presented a sequence of modifications. These are obtained through the factorisations (using a sequence of quadratic forms) of the operator $J$ of (2.8). Specifically, to the $2N$ component hierarchy associated with (2.1), $\varepsilon_0=1$, there corresponds a chain of $N$ modifications, the $n^{\text{th}}$ modification possessing $(N-n+1)$ compatible Hamiltonian structures. This should be contrasted with Kupershmidt's expectation [5] that "... super extensions ... usually destroy the Miura maps." Perhaps this is true in general, but we believe that our construction has wide applications, both to purely even and super integrable equations.

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