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The Inverse Backscattering Problem in Three Dimensions

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Abstract. This article is a study of the mapping from a potential $q(x)$ on $\mathbf{R}^3$ to the backscattering amplitude associated with the Hamiltonian $-\Delta + q(x)$. The backscattering amplitude is the restriction of the scattering amplitude $a(\theta, \omega, k), (\theta, \omega, k) \in S^2 \times S^2 \times \mathbb{R}_+$, to $a(\theta, -\theta, k)$. We show that in suitable (complex) Banach spaces the map from $q(x)$ to $a(x/|x|, -x/|x|, |x|)$ is usually a local diffeomorphism. Hence in contrast to the overdetermined problem of recovering q from the full scattering amplitude the inverse backscattering problem is well posed.

This article is a study of the mapping from a potential on $\mathbf{R}^3$ to its quantum mechanical scattering amplitude. The scattering amplitude associated with a potential $q(x)$ can be described as follows. One assumes that for each $k > 0$ and each $\omega \in S^2$,

$$(-\Delta + q - k^2)u = 0$$

has a unique solution of the form $\exp(ik\omega \cdot x) + v(x, \omega, k)$ such that $v = \lim_{\varepsilon \downarrow 0} v_\varepsilon$, where v_ε is the square-integrable solution of

$$-\Delta v_\varepsilon + q v_\varepsilon - (k + i\varepsilon)^2 v_\varepsilon = -e^{ik\omega \cdot x} q. \tag{I.1}$$

Much work has been devoted to showing that, under general hypotheses on $q, v(x, \omega, k)$ exists and is unique (see Agmon [1], and the references given there). When $q \in C_0^\infty(\mathbf{R}^3)$ and hence $\Delta v + k^2 v \in C_0^\infty(\mathbf{R}^3)$, it is an elementary consequence of (I.1) that

$$v(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{ik|x-y|}}{|x-y|} (\Delta + k^2) v(y) dy, \quad \text{and hence}$$

$$v(x) = \left(\frac{e^{ik|x|}}{4\pi |x|} \right) (a(x/|x|, \omega, k) + O(|x|^{-1})) \tag{I.2}$$

as $|x| \to \infty$. The function $a(\theta, \omega, k)$ on $S^2 \times S^2 \times \mathbf{R}_+$ is known as the scattering amplitude. If we replace functions in (I.2) by their Fourier transforms, we have

$$v(x,\omega,k)=\lim_{\varepsilon\downarrow 0}(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}g(\xi,\omega,k)}{|\xi|^2-(k+i\varepsilon)^2}d\xi$$

$$\equiv(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}g(\xi,\omega,k)}{|\xi|^2-(k+i0)^2}d\xi, \tag{I.3}$$

where g is the Fourier transform of $-(\Delta+k^2)v$. Evaluating (I.3) in spherical coordinates and using stationary phase in the angular integration to derive asymptotics as $|x|\to\infty$, we find that

$$a(\theta,\omega,k)=g(k\theta,\omega,k). \tag{I.4}$$

Given $q\in C_0^\infty(\mathbf{R}^3)$, taking the Fourier transform of (I.1) and the limit $\varepsilon\downarrow 0$, one arrives at

$$g(\xi,\omega,k)+(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{\hat{q}(\xi-\eta)g(\eta,\omega,k)}{|\eta|^2-(k+i0)^2}d\eta=-\hat{q}(\xi-k\omega). \tag{I.5}$$

In this article we will take (I.4) and (I.5) as the definition of the scattering amplitude, i.e., when the integral equation (I.5) has a unique solution g for $(\omega,k)\in S^2\times\mathbf{R}_+$, the scattering amplitude is defined by (I.4).

Since we are dealing with a singular integral equation involving the Fourier transform of the potential q, we will assume $\hat{q}$ belongs to one of the weighted Hölder spaces $H_{\alpha,N}$ with $0<\alpha<1$ and $N>1$. Spaces of this type have been used in scattering theory by L. D. Faddeev in [3] and K. O. Friedrichs in [5]. The norm in $H_{\alpha,N}$ is $\|f\|_{\alpha,N}=\|(1+|\xi|^2)^{N/2}f\|_\alpha$, where

$$\|f\|_\alpha=\sup_{\substack{|\Delta|\leq 1\\ \xi\in\mathbf{R}^3}}(|f(\xi)|+|\Delta|^{-\alpha}|f(\xi+\Delta)-f(\xi)|),$$

and $H_{\alpha,N}$ is defined as the closure of $C_0^\infty(\mathbf{R}^3)$ in this norm. We do not assume that q is real-valued, though our main interest is in potentials with small imaginary parts.

As our title implies we are interested in the inverse problem of determining the potential given the scattering amplitude. This problem is quite overdetermined and there has been considerable work devoted to characterizing which scattering amplitudes actually arise for given classes of potentials, beginning with L. D. Faddeev [4] and more recently Newton [11], Beals–Coifman [2], Nachman–Ablowitz [9], Melin [7] and Novikov–Khenkin [6]. We are concerned here with the inverse *back*scattering problem, i.e. determining q from $a(\omega,-\omega,k)$. In dimensions $n>1$ the only work that we know of is the numerical study of Bayliss, Lin and Morawetz [8] using wave equation methods, and the formal solution of the three-dimensional problem for small potentials by Prosser [13].

For technical reasons we will replace (I.5) by

$$h(\xi,\zeta,k)+(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{\hat{q}(\xi-\eta)h(\eta,\zeta,k)}{|\eta|^2-(k+i0)^2}d\eta=-\hat{q}(\xi-\zeta), \tag{I.6}$$

where now (ξ,ζ,k) ranges over $\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+$, i.e. $k=0$ is now included. Thus (I.4) becomes

$$a(\theta,\omega,k)=h(k\theta,k\omega,k). \tag{I.7}$$

Let $H^r_{\alpha,N}$ denote the (real) subspace of $H_{\alpha,N}$ consisting of Fourier transforms of real-valued potentials, i.e. the set of $\hat{q}\in H_{\alpha,N}$ such that $\hat{q}(\xi)=\overline{\hat{q}(-\xi)}$. The backscattering map is well behaved on $H_{\alpha,N}$ and we have the following result which is proven in Corollary 3.5 and Remark 4 after Theorem 3.1 in the text:

Theorem A. *The backscattering map*

$$S:\hat{q}\to h(\xi,-\xi,|\xi|)$$

is a continuously Frechet differentiable function from an open, dense set $\mathcal{O}$ in $H_{\alpha,N}$ into $H_{\alpha,N}$. Moreover, $\mathcal{O}\cap H^r_{\alpha,N}$ is dense in $H^r_{\alpha,N}$.

Since continuously differentiable functions on complex Banach spaces are analytic, S is analytic. The set $\mathcal{O}$ is the set of $\hat{q}$ such that $I+A(\hat{q},k)$ is injective on $H_{\alpha,N}$ for $k\geqq 0$, where

$$[A(\hat{q},k)f](\xi)=(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{\hat{q}(\xi-\eta)f(\eta)d\eta}{|\eta|^2-(k+i0)^2}.$$

The proof that $h(\xi,-\xi,|\xi|)$ belongs to precisely the same space $H_{\alpha,N}$ as $\hat{q}(\xi)$ for $\hat{q}\in\mathcal{O}$, i.e. the proof of Theorem A, is quite technical and takes up about half of this paper.

Next we prove that the Frechet derivative of S is a Fredholm operator of index zero for $\hat{q}\in\mathcal{O}$ (Theorem 4.3) and that $\mathcal{O}\cap H^r_{\alpha,N}$ is contained in a connected component $\mathcal{O}_1$ of $\mathcal{O}$ (Proposition 5.3). This leads to the following theorem (Theorem 5.4):

Theorem B. *The Frechlet derivative of S at $\hat{q}$ is an isomorphism of $H_{\alpha,N}$ for $\hat{q}$ in an open, dense subset $\mathcal{O}_2$ of $\mathcal{O}_1$. Moreover, $\mathcal{O}_2\cap H^r_{\alpha,N}$ is an open, dense subset of $H^r_{\alpha,N}$.*

The implicit function theorem then implies:

Corollary C. *S is a local analytic homeomorphism in a neighbourhood of each $\hat{q}\in\mathcal{O}_2$.*

This is the main result of this paper. Corollary C implies that (locally) recovering $\hat{q}$ from backscattering data is a well-posed problem, since small changes in $h(\xi,-\xi,|\xi|)$ will lead to small changes in $\hat{q}(\xi)$ in $H_{\alpha,N}$ norm. Note also that the results in Theorem B and Corollary C do not depend on the number of negative eigenvalues of $-\Delta+q$. This follows from the fact that $\mathcal{O}_2$ is a subset of the *connected* set $\mathcal{O}_1$.

Even the backscattering problem is overdetermined when we restrict the domain of our mapping to real-valued potentials. Therefore in the final section we consider a restricted backscattering problem for the case of real-valued potentials. Let S_r denote the mapping

$$S_r:\hat{q}\to\frac{h(\xi,-\xi,|\xi|)+\overline{h(-\xi,\xi,|\xi|)}}{2}.$$

Note that $\mathcal{F}^{-1}S_r$ is the real part of $\mathcal{F}^{-1}S$. This map is well-behaved on $H^r_{\alpha,N}$: S_r is real-analytic with a Frechet derivative which is Fredholm and index zero for $\hat{q}\in H^r_{\alpha,N}\cap\mathcal{O}$, (Theorem 6.1). However, we only know that its Frechet derivative is an isomorphism on an open dense set $\mathcal{O}^r_2$ of the component $\mathcal{O}^r_1$ of $H^r_{\alpha,N}\cap\mathcal{O}$ containing the zero potential (Theorem 6.2). The component $\mathcal{O}^r_1$ does contain all $\hat{q}$ such that $q\in C^\infty_0(\mathbf{R}^3)$ and $-\Delta+q$ has no bound states with energies $E\leqq 0$ or half-bound states at $E=0$ (Proposition 6.3). We plan to study other approaches to the formulation of the restricted backscattering problem in the future.

Section 1. Preliminaries

We will use the weight function $\Lambda(\xi)=(1+|\xi|^2)^{1/2}$ and the Lipschitz norms

$$\|f\|_\alpha=\sup\left(|f(\xi)|+\frac{|f(\xi+\Delta)-f(\xi)|}{|\Delta|^\alpha}\right),$$

where $0<\alpha\leqq 1$ and the supremum is taken over $\{\xi\in\mathbf{R}^3, \Delta\in\mathbf{R}^3: 0<|\Delta|\leqq 1\}$. The Banach space of all functions f on $\mathbf{R}^3$ with $\|f\|_\alpha<\infty$ will be denoted by $C^\alpha(\mathbf{R}^3)$. We also use $\|f\|_0$ to denote the supremum of $|f(\xi)|$ over $\mathbf{R}^3$. The principal Banach spaces in this paper are $H_{\alpha,N}, 0<\alpha<1, N>1$, the closures of $C_0^\infty(\mathbf{R}^3)$ in the norms

$$\|f\|_{\alpha,N}=\|\Lambda^N f\|_\alpha.$$

While $H_{\alpha,N}$ does not contain all functions f on $\mathbf{R}^3$ with $\|f\|_{\alpha,N}<\infty$, one does have the following.

Lemma 1.1. *$H_{\alpha,N}$ contains all functions f on $\mathbf{R}^3$ such that $\|f\|_{\alpha',N'}<\infty$ for some $\alpha'>\alpha$ and $N'>N$.*

Proof. Let j_ε be the standard mollifier and choose $\varphi\in C_0^\infty(\mathbf{R}^3)$ with $\varphi(\xi)=1$ for $|\xi|<1$. Then for $R\geqq 1$

$$\|(1-\varphi(\cdot/R))f\|_{\alpha,N}\leqq CR^{N-N'}\|f\|_{\alpha,N'}$$

and for fixed R, setting $g(\xi)=\varphi(\xi/R)f(\xi)$,

$$\|g-j_\varepsilon * g\|_{\alpha,N}$$
$$\leqq C\left(\sup|g(\xi+\eta)-g(\xi)|+\sup\frac{|g(\xi+\eta+\Delta)-g(\xi+\Delta)-g(\xi+\eta)+g(\xi)|}{|\Delta|^\alpha}\right),$$

where the suprema are taken over $\{\xi,\eta,\Delta:|\eta|\leqq\varepsilon,|\Delta|\leqq 1\}$. Thus

$$\|g-j_\varepsilon * g\|_{\alpha,N}\leqq C\left(\varepsilon^{\alpha'}\|g\|_{\alpha'}+\sup_{|\Delta|\leqq 1}\left(\frac{\varepsilon^{\alpha'-\alpha}}{|\Delta|^\alpha}\|g(\cdot+\Delta)-g(\cdot)\|_{\alpha'-\alpha}\right)\right).$$

Hence, since $|\Delta|^{-\alpha}\|g(\cdot+\Delta)-g(\cdot)\|_{\alpha'-\alpha}\leqq 3\|g\|_{\alpha'}$,

$$\|g-j_\varepsilon * g\|_{\alpha,N}\leqq C\varepsilon^{\alpha'-\alpha}\|g\|_{\alpha'}. \quad \blacksquare$$

We will also deal with functions defined on $\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+$. For functions on $\bar{\mathbf{R}}_+$, we define

$$\|f\|_\alpha=\sup\left(|f(k)|+\frac{|f(k+\Delta)-f(k)|}{\Delta^\alpha}\right),$$

where the supremum is taken over $\{k\in\bar{\mathbf{R}}_+, 0<\Delta\leqq 1\}$. Note that, since we take the supremum in k and Δ, $\|f\|_\alpha<\infty$ does imply $f\in C^\alpha[0,\infty)$. For $0<\alpha<1$, we define a C^α-norm on functions on $\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+$ by

$$\|f\|_{\underline{\alpha}} = \sup(\|f(\cdot,\zeta,k)\|_\alpha + \|f(\xi,\cdot,k)\|_\alpha + \|f(\xi,\zeta,\cdot)\|_\alpha) \tag{1.1}$$

with the supremum taken over $(\xi,\zeta,k)\in\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+$.

Translations of functions will often be denoted by subscripts, i.e. $f_\zeta(\xi)=f(\xi-\zeta)$. In particular we will often use $\Lambda_\zeta(\xi)$ for $\Lambda(\xi-\zeta)$.

Section 2. Estimates of the Operator $A(\hat{q},k)$

We define for $\hat{q}\in H_{\alpha,N}$ and $f\in C_0^\infty(\mathbf{R}^3)$,

$$[A(\hat{q},k)f](\xi)=(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{\hat{q}(\xi-\eta)f(\eta)d\eta}{|\eta|^2-(k+i0)^2}.$$

Theorem 2.1. *The operator $A(\hat{q},k)$ satisfies the following estimate for $\zeta\in\mathbf{R}^3$ and $k\geqq 0$,*

$$\|\Lambda_\zeta^N A(\hat{q},k)\Lambda_\zeta^{-N}\Lambda^\delta f\|_\alpha \leqq \frac{C}{(1+k)^\gamma}\|\hat{q}\|_{\alpha,N}\|f\|_{\alpha-\varepsilon},$$

where $0<\alpha<1$, $N>1$, $0\leqq\varepsilon<\alpha$, $0\leqq\delta<\min\{1,N-1\}$, and $\gamma<\min\{1-\delta, N-1-\delta\}$. The constant C is independent of $k,\zeta,\hat{q}$ and f.

Theorem 2.1 is the principal estimate in this article. To prove it we need to know the asymptotic behavior of integrals of the weight functions.

Lemma 2.2. *Define for $k>0, N>0$ and $(\xi,\zeta)\in\mathbf{R}^6$,*

$$I(k,\xi,\zeta)=\int_{|\omega|=1}\frac{(1+|\xi-\zeta|^2)^{N/2}}{(1+|\xi-k\omega|^2)^{N/2}(1+|k\omega-\zeta|^2)^{N/2}}d\omega.$$

Then

$$I(k,\xi,\zeta)\leqq C_N\max\{(1+k)^{-2}\log(1+k),(1+k)^{-N}\}.$$

Proof of Lemma 2.2.

$$\begin{aligned} I(k) &\leqq C_N\int_{|\omega|=1}[(1+|\xi-k\omega|^2)^{-N/2}+(1+|k\omega-\zeta|^2)^{-N/2}]d\omega \\ &\leqq 2C_N\sup_\xi\int_{|\omega|=1}(1+|\xi-k\omega|^2)^{-N/2}d\omega. \end{aligned}$$

Introducing spherical coordinates with the z-axis in direction ξ,

$$\begin{aligned} \int_{|\omega|=1}(1+|\xi-k\omega|^2)^{-N/2}dw &= 2\pi\int_0^\pi(1+|\xi|^2-2|\xi|k\cos\theta+k^2)^{-N/2}\sin\theta d\theta \\ &= 2\pi\int_{-1}^{1}(1+|\xi|^2-2|\xi|k\tau+k^2)^{-N/2}d\tau. \end{aligned}$$

Letting $u=|\xi|^2-2|\xi|k\tau+k^2$, we have

$$\int_{|\omega|=1}(1+|\xi-k\omega|^2)^{-N/2}d\omega=\frac{\pi}{|\xi|k}\int_{(|\xi|-k)^2}^{(|\xi|+k)^2}(1+u)^{-N/2}du$$

$$=\frac{\pi}{|\xi|k}\begin{cases}\ln\left(\dfrac{(|\xi|+k)^2+1}{(|\xi|-k)^2+1}\right) & \text{if } N=2\\ \dfrac{2}{N-2}((1+(k-|\xi|)^2)^{(2-N)/2}-(1+(k+|\xi|)^2)^{(2-N)/2}) & \text{if } N\neq 2.\end{cases} \tag{2.1}$$

If $||\xi|-k|>\frac{1}{2}k$, we have

$$\int_{|\omega|=1}(1+|\xi-k\omega|^2)^{-N/2}d\omega \leqq 4\pi(1+\tfrac{1}{4}k^2)^{-N/2}$$

and, if $||\xi|-k|<\frac{1}{2}k$, formula (2.1) shows

$$\int_{|\omega|=1}(1+|\xi-k\omega|^2)^{-N/2}d\omega \leqq \frac{2\pi}{k^2}\begin{cases}\dfrac{2}{N-2} & \text{if } N>2\\ \ln\left(1+\dfrac{25k^2}{4}\right) & \text{if } N=2\\ \dfrac{2}{2-N}\left(1+\dfrac{25k^2}{4}\right)^{(2-N)/2} & \text{if } N<2.\end{cases}$$

Thus we have the desired estimate (note that for $0\leqq k\leqq 1$ the estimate is trivial). ∎

An immediate corollary of Lemma 2.2 is the following.

Lemma 2.3. *For* $0<\delta<\min\{1,N-1\}$ *let*

$$J(k,\xi,\zeta)=\int_{||\eta|-k|>1}\frac{(1+|\xi-\zeta|^2)^{N/2}(1+|\eta|^2)^{\delta/2}}{(1+|\xi-\eta|^2)^{N/2}(|\eta|^2-k^2)(1+|\eta-\zeta|^2)^{N/2}}d\eta.$$

Then for $N>1+\delta$ *and* $\gamma<\min\{N-1-\delta,1-\delta\}$ *we have* $J(k,\xi,\zeta)\leqq C_{\gamma,N,\delta}(1+k)^{-\gamma}$.

Proof of Lemma 2.3. By applying Lemma 2.2 with $|\eta|$ playing the role of k and $\omega=\eta/|\eta|$, we see

$$J\leqq C\int_{||\eta|-k|\geqq 1}\frac{(1+|\eta|^2)^{\delta/2}|\eta|^2d|\eta|}{||\eta|^2-k^2|(1+|\eta|^2)^{\beta/2}},\tag{2.2}$$

where $\beta=2$ if $N>2$, and $\beta=N-\varepsilon$, $\varepsilon>0$, for $N\leqq 2$. Substituting $k\eta'=\eta$, we have

$$J\leqq Ck\int_{||\eta'|-1|>k^{-1}}\frac{(1+k^2|\eta'|^2)^{(\delta-\beta)/2}|\eta'|^2d|\eta'|}{||\eta'|^2-1|}.$$

For $k>1/2$ we have

$$J\leqq Ckk^{\delta-\beta}\int_{||\eta'|-1|>1/k}\frac{|\eta'|^{2+\delta-\beta}}{||\eta'|-1|(|\eta'|+1)}d|\eta'|$$

and, hence for $\beta-\delta>1$, we have

$$J\leqq Ck^{1+\delta-\beta}(1+\ln k).\tag{2.3}$$

For $k<1/2$, we have immediately from (2.2)

$$J\leqq C\int_1^\infty(1+|\eta|^2)^{(\delta-\beta)/2}d|\eta|.\tag{2.4}$$

From (2.3) and (2.4) we conclude

$$J\leqq C(1+k)^{-\gamma}$$

for any $\gamma < \beta - \delta - 1$ when $\beta - \delta > 1$, which is the desired result for $N > 2$. Choosing ε so that $N - \varepsilon > 1 + \delta$, when $N \leqq 2$, completes the proof. ∎

Proof of Theorem 2.1. We begin by reducing the theorem to the case $\varepsilon = 0$. For this let $\Delta(\mu)$ denote the operator $(\Delta(\mu)f)(\xi) = f(\xi + \mu) - f(\xi)$. Then $\Delta(\mu)A(\hat{q}, k)f = A(\Delta(\mu)\hat{q}, k)f$, and assuming Theorem 2.1 in the case $\varepsilon = 0$, we have

$$|\mu|^{-\varepsilon} \| \Lambda_\zeta^N \Delta(\mu) A(\hat{q}, k) \Lambda^\delta \Lambda_\zeta^{-N} f \|_{\alpha - \varepsilon} \leqq \frac{C}{(1+k)^\gamma} |\mu|^{-\varepsilon} \| \Delta(\mu)\hat{q} \|_{\alpha - \varepsilon, N} \| f \|_{\alpha - \varepsilon}.$$

For $p \in \mathbf{R}$ the mean value theorem implies

$$|\Delta(\mu)\Lambda^p(\xi)| \leqq |\mu| \, |\partial_\xi \Lambda^p(\xi')|,$$

where $|\xi' - \xi| < |\mu|$. Since $|\partial_\xi \Lambda(\xi)| \leqq 1$, and hence $\Lambda(\xi')/\Lambda(\xi)$ and $\Lambda(\xi)/\Lambda(\xi')$ are bounded for $|\xi' - \xi| \leqq 1$, we have for $|\mu| \leqq 1$,

$$|\Delta(\mu)\Lambda_\zeta^p(\xi)| \leqq C|\mu|\Lambda_\zeta^{p-1}(\xi). \tag{2.5}$$

As in the proof of Lemma 1.1, we have

$$\sup_{|\mu| \leqq 1} |\mu|^{-\varepsilon} \| \Delta(\mu)\hat{q} \|_{\alpha - \varepsilon, N} \leqq 3 \| \hat{q} \|_{\alpha, N}.$$

Moreover, it is also true (see Proposition 8, Sect. 4, Chap. V in Stein [14]) that $(\| f \|_{\alpha'} + \sup |\mu|^{-\alpha} \| \Delta(\mu) f \|_{\alpha'}) \geqq 1/C \| f \|_{\alpha + \alpha'}$. Thus, using (2.5) we have,

$$\begin{aligned} \| \Lambda_\zeta^N A(\hat{q}, k) \Lambda^\delta \Lambda_\zeta^{-N} f \|_\alpha \leqq C(&\| \Lambda_\zeta^N A(\hat{q}, k) \Lambda^\delta \Lambda_\zeta^{-N} f \|_{\alpha - \varepsilon} \\ &+ \sup_{|\mu| \leqq 1} |\mu|^{-\varepsilon} \| \Lambda_\zeta^N \Delta(\mu) A(\hat{q}, k) \Lambda^\delta \Lambda_\zeta^{-N} f \|_{\alpha - \varepsilon}). \end{aligned}$$

Thus we only need to consider Theorem 2.1 in the case $\varepsilon = 0$.

To prove Theorem 2.1, we begin by defining $h(\xi, \eta) = \Lambda_\xi^N(\eta)\hat{q}(\xi - \eta) f(\eta)$. Then, using (2.5) we conclude

$$\| h(\cdot, \eta) \|_\alpha + \| h(\xi, \cdot) \|_\alpha \leqq C \| \hat{q} \|_{\alpha, N} \| f \|_\alpha$$

uniformly for $(\xi, \eta) \in \mathbf{R}^6$.

Next we decompose $(2\pi)^3 \Lambda_\zeta^N A(\hat{q}, k) \Lambda_\zeta^{-N} \Lambda^\delta f$ into three terms:

$$\begin{aligned} &\int_{||\eta| - k| > 1} \frac{\Lambda_\zeta^N(\xi)\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)\Lambda_\xi^N(\eta)} \frac{h(\xi, \eta)}{|\eta|^2 - k^2} d\eta \\ &+ \int_{||\eta| - k| < 1} \frac{\Lambda_\zeta^N(\xi)}{|\eta|^2 - k^2} \left[\frac{\Lambda^\delta(\eta) h(\xi, \eta)}{\Lambda_\zeta^N(\eta)\Lambda_\xi^N(\eta)} - \frac{\Lambda^\delta(k\omega) h(\xi, k\omega)}{\Lambda_\zeta^N(k\omega)\Lambda_\xi^N(k\omega)} \right] d\eta \\ &+ \int_{||\eta| - k| < 1} \frac{\Lambda_\zeta^N(\xi)\Lambda^\delta(k\omega) h(\xi, k\omega)}{(|\eta|^2 - (k + i0)^2)\Lambda_\zeta^N(k\omega)\Lambda_\xi^N(k\omega)} d\eta \\ &\equiv I_1 + I_2 + I_3, \end{aligned}$$

where $\omega = \eta/|\eta|$ in I_2 and I_3. In I_3 we introduce polar coordinates and compute

$$I_3 = \int_{S^2} \frac{\Lambda_\zeta^N(\xi) h(\xi, k\omega)}{\Lambda_\zeta^N(k\omega)\Lambda_\xi^N(k\omega)} d\omega \int_{||\eta| - k| < 1, |\eta| > 0} \frac{(1 + k^2)^{\delta/2} |\eta|^2 d|\eta|}{|\eta|^2 - (k + i0)^2}.$$

Moreover

$$\lim_{\varepsilon\to 0}\int_{||\eta|-k|<a,|\eta|>0}\frac{|\eta|^2}{|\eta|^2-(k+i\varepsilon)^2}d|\eta| = \begin{cases} 2a-\dfrac{k}{2}\left(\ln\left(\dfrac{2k+a}{2k-a}\right)-\pi i\right) & \text{if } k>a \\ a+k-\dfrac{k}{2}\left(\ln\left(1+\dfrac{2k}{a}\right)-\pi i\right) & \text{if } k\leqq a. \end{cases} \tag{2.6}$$

Hence

$$\int_{||\eta|-k|<1,|\eta|>0}\frac{|\eta|^2}{|\eta|^2-(k+i0)^2}d|\eta| = \begin{cases} \dfrac{k\pi i}{2}+O(1) & \text{as } k\to\infty \\ 1+O(k) & \text{as } k\to 0. \end{cases}$$

Applying Lemma 2.2 and (2.5) we have $|I_3|\leqq(1+k)^{\delta-\beta-1}\sup_{(\xi,\eta)}|h(\xi,\eta)|$ and for $|\mu|\leqq 1$, $|\mu|^{-\alpha}|\Delta(\mu)I_3|\leqq(1+k)^{\delta-\beta+1}\sup_{\eta}\|h(\cdot,\eta)\|_\alpha$, where as in the proof of Lemma 2.3 $\beta=2$ for $N>2$ and $\beta=N-\varepsilon,\varepsilon>0$, for $N\leqq 2$. Taking ε small enough that $N-1-\delta-\varepsilon>\gamma$, if $N\leqq 2$, this shows that I_3 satisfies the estimate of the theorem. Hence we need only consider I_1 and I_2.

The estimates of I_1 follow immediately from Lemma 2.3 and (2.5). We have $|I_1|\leqq C(1+k)^{-\gamma}\sup_{(\xi,\eta)}|h(\xi,\eta)|$ and for $|\mu|\leqq 1$, $|\mu|^{-\alpha}|\Delta(\mu)I_1|\leqq C(1+k)^{-\gamma}\sup_{\eta}\|h(\cdot,\eta)\|_\alpha$, which again is the estimate of the theorem.

The estimate of $|I_2|$ is also easy. Once again (2.5) implies for $||\eta|-k|<1$,

$$\left|\frac{\Lambda^\delta(\eta)h(\xi,\eta)}{\Lambda_\zeta(\eta)\Lambda_\xi^N(\eta)}-\frac{\Lambda^\delta\left(k\dfrac{\eta}{|\eta|}\right)h\left(\xi,k\dfrac{\eta}{|\eta|}\right)}{\Lambda_\zeta^N\left(k\dfrac{\eta}{|\eta|}\right)\Lambda_\xi^N\left(k\dfrac{\eta}{|\eta|}\right)}\right|\leqq C||\eta|-k|^\alpha\frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)\Lambda_\xi^N(\eta)}\sup_\xi\|h(\xi,\cdot)\|_\alpha.$$

Hence by Lemma 2.2

$$|I_2|\leqq C\int_{||\eta|-k|<1}\frac{(1+|\eta|^2)^{(\delta-\beta)/2}|\eta|^2}{(|\eta|+k)|\,|\eta|-k|^{1-\alpha}}d|\eta|\left(\sup_\xi\|h(\xi,\eta)\|_\alpha\right),$$

where β is as before. This gives

$$|I_2|\leqq C(1+k)^{\delta-\beta+1}\|\hat{q}\|_{\alpha,N}\|f\|_\alpha$$

as desired.

It is the estimate of $|\Delta(\mu)I_2|$ that presents some problems. For this we need first to split the domain of integration in the integral into $\{||\eta|-k|<2|\mu|\}$, getting J_1, and $\{2|\mu|<||\eta|-k|<1\}$, getting J_2. To estimate $\Delta(\mu)J_1=J_1(\xi+\mu)-J_1(\xi)$, we use $|\Delta(\mu)J_1|\leqq|J_1(\xi+\mu)|+|J_1(\xi)|$. Since the procedure used to estimate $|I_2|$, shows that for $|\mu|\leqq 1$,

$$|J_1(\xi+\mu)|+|J_1(\xi)|\leqq C\int_{||\eta|-k|<2|\mu|}||\eta|-k|^{-1+\alpha}d|\eta|(1+k)^{\delta-\beta+1}\|\hat{q}\|_{\alpha,N}\|f\|_\alpha,$$

and

$$\int_{||\eta|-k|<2|\mu|} ||\eta|-k|^{-1+\alpha} d|\eta| \leqq C|\mu|^{\alpha},$$

we have the estimate required for $|\Delta(\mu)J_1|$.

To estimate $\Delta(\mu)J_2$ we must use the special form of $h(\xi,\eta)$, i.e. $h(\xi,\eta) = \Lambda_\xi^N(\eta)\hat{q}(\xi-\eta)f(\eta)$. We have

$$\begin{aligned}
\Delta(\mu)J_2 = & \int_{2|\mu|<||\eta|-k|<1} \frac{(\Delta(\mu)\Lambda_\zeta^N(\xi))}{|\eta|^2-k^2}\left[\frac{\Lambda^\delta(\eta)h(\xi+\mu,\eta)}{\Lambda_\zeta^N(\eta)\Lambda_{\xi+\mu}^N(\eta)} - \frac{\Lambda^\delta(k\omega)h(\xi+\mu,k\omega)}{\Lambda_\zeta^N(k\omega)\Lambda_{\xi+\mu}^N(k\omega)}\right]d\eta \\
& + \int_{2|\mu|<||\eta|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{|\eta|^2-k^2}\left[\frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)}\hat{q}(\xi+\mu-\eta) - \frac{\Lambda^\delta(k\omega)}{\Lambda_\zeta^N(k\omega)}\hat{q}(\xi+\mu-k\omega)\right]f(\eta)d\eta \\
& - \int_{2|\mu|<||\mu|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{|\eta|^2-k^2}\left[\frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)}\hat{q}(\xi-\eta) - \frac{\Lambda^\delta(k\omega)}{\Lambda_\zeta^N(k\omega)}\hat{q}(\xi-k\omega)\right]f(\eta)d\eta \\
& + \int_{2|\mu|<||\eta|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{|\eta|^2-k^2}\frac{\Lambda^\delta(k\omega)}{\Lambda_\zeta^N(k\omega)}(\hat{q}(\xi+\mu-k\omega)-\hat{q}(\xi-k\omega))(f(\eta)-f(k\omega))d\eta \\
& \equiv K_1+K_2-K_3+K_4.
\end{aligned}$$

We can estimate $|K_1|$ exactly as $|I_2|$ was estimated and, using Lemma 2.2 and (2.5), one can easily verify that for $|\mu| \leqq 1$

$$|K_4| \leqq C|\mu|^{\alpha}(1+k)^{\delta-\beta+1}\|\hat{q}\|_{\alpha,N}\|f\|_{\alpha} \int_{||\eta|-k|<1} ||\eta|-k|^{-1+\alpha}d|\eta|.$$

Hence K_4 also satisfies the required estimate.

In estimating K_2-K_3 we need to make the cancellation between $\hat{q}(\xi+\mu-\eta)$ and $\hat{q}(\xi-\eta)$ as good as possible. For this we replace η in K_2 by $\eta+\mu$. This gives

$$\begin{aligned}
K_2-K_3 = & \int(\chi_+-\chi_-)\frac{\Lambda_\zeta^N(\xi)}{|\eta+\mu|^2-k^2}\left[\frac{\Lambda^\delta(\eta+\mu)}{\Lambda_\zeta^N(\eta+\mu)}\hat{q}(\xi+\mu-\eta-\mu)\right. \\
& \left. - \frac{\Lambda^\delta(k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\hat{q}(\xi+\mu-k\tilde{\omega})\right]f(\eta+\mu)d\eta + \int_{2|\mu|<||\eta|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{|\eta+\mu|^2-k^2} \\
& \cdot\left[\frac{\Lambda^\delta(\eta+\mu)}{\Lambda_\zeta^N(\eta+\mu)}\hat{q}(\xi-\eta) - \frac{\Lambda^\delta(k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\tilde{q}(\xi+\mu-k\tilde{\omega})\right](f(\eta+\mu)-f(\eta))d\eta \\
& + \int_{2|\mu|<||\eta|-k|<1} \Lambda_\zeta^N(\xi)\left[\frac{1}{|\eta+\mu|^2-k^2} - \frac{1}{|\eta|^2-k^2}\right] \\
& \cdot\left[\frac{\Lambda^\delta(\eta+\mu)}{\Lambda_\zeta^N(\eta+\mu)}\hat{q}(\xi+\mu-\eta-\mu) - \frac{\Lambda^\delta(k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\hat{q}(\xi+\mu-k\tilde{\omega})\right]f(\eta)d\eta \\
& + \int_{2|\mu|<||\eta|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{|\eta|^2-k^2}\left[\frac{\Lambda^\delta(\eta+\mu)}{\Lambda_\zeta^N(\eta+\mu)} - \frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)}\right]\hat{q}(\xi-\eta)f(\eta)d\eta \\
& + \int_{2|\mu|<||\eta|-k|<1} \frac{\Lambda_\zeta^N(\xi)}{(|\eta|^2-k^2)}(1+k^2)^{\delta/2}
\end{aligned}$$

$$\cdot\left[\frac{\hat{q}(\xi - k\omega)}{\Lambda_\zeta^N(k\omega)} - \frac{\hat{q}(\xi + \mu - k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\right] f(\eta)d\eta$$

$$\equiv L_1 + L_2 + L_3 + L_4 + L_5.$$

Here $\tilde{\omega} = (\eta + \mu)/|\eta + \mu|$ and χ_+ is the characteristic function of $\{\eta : 2|\mu| < ||\eta + \mu| - k| < 1$ and $||\eta| - k| > 1$ or $||\eta| - k| < 2|\mu|\}$ and χ_- is the characteristic function of $\{\eta : 2|\mu| < ||\eta| - k| < 1$ and $||\eta + \mu| - k| > 1$ or $||\eta + \mu| - k| < 2|\mu|\}$.

The first two terms in the expansion of $K_2 - K_3$ are like terms we have already considered. The integral L_1 can be estimated as J_1 was, and L_2 is another term like K_4. The remaining three terms require further explanation. Since

$$||\eta + \mu| \pm k| > \tfrac{1}{2}||\eta| \pm k| \quad \text{when} \quad ||\eta| - k| > 2|\mu|,$$

we have

$$|L_3| \leqq C \int_{2|\mu| < ||\eta| - k| < 1} \frac{|\mu|(|\eta| + 1)|\eta|^2}{(|\eta| + k)^2} ||\eta| - k|^{\alpha - 2} d|\eta| (1 + k)^{\delta - \beta} \|\hat{q}\|_{\alpha,N} \|f\|_0$$

$$\leqq C(1 + k)^{\delta - \beta + 1} \|\hat{q}\|_{\alpha,N} \|f\|_0 \int_{2|\mu| < ||\eta| - k| < 1} \frac{|\mu| d|\eta|}{||\eta| - k|^{2 - \alpha}}$$

$$\leqq C(1 + k)^{\delta - \beta + 1} |\mu|^\alpha \|\hat{q}\|_{\alpha,N} \|f\|_0.$$

By (2.5) we have

$$\left|\frac{\Lambda^\delta(\eta + \mu)}{\Lambda_\zeta^N(\eta + \mu)} - \frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)}\right| \leqq C|\mu| \frac{\Lambda^\delta(\eta)}{\Lambda_\zeta^N(\eta)}.$$

Thus, we can estimate L_4 by

$$|L_4| \leqq C|\mu|(1 + k)^{\delta - \beta} \|\hat{q}\|_{0,N} \|f\|_0 \int_{2|\mu| < ||\eta| - k| < 1} \frac{|\eta|^2}{||\eta|^2 - k^2|} d|\eta|$$

$$\leqq C|\mu|(|\ln|\mu|| + 1)(1 + k)^{\delta - \beta + 1} \|\hat{q}\|_{0,N} \|f\|_0.$$

Since $\alpha < 1$, this is stronger than the estimate we need.

The term L_5 must be decomposed again (but this is the last decomposition we will use):

$$L_5 = (1 + k^2)^{\delta/2} \Lambda_\zeta^N(\xi) \left[\int_{2|\mu| < ||\eta| - k| < 1} \left[\frac{\hat{q}(\xi - k\omega)}{\Lambda_\zeta^N(k\omega)} - \frac{\hat{q}(\xi + \mu - k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\right]\right]$$

$$\cdot\left(\frac{f(\eta) - f(k\omega)}{|\eta|^2 - k^2}\right) d\eta + \int_{2|\mu| < ||\eta| - k| < 1} \left[\frac{\hat{q}(\xi - k\omega)}{\Lambda_\zeta^N(k\omega)} - \frac{\hat{q}(\xi + \mu - k\beta)}{\Lambda_\xi^N(k\beta)}\right] \frac{f(k\omega)}{|\eta|^2 - k^2} d\eta$$

$$+ \int_{2|\mu| < ||\eta| - k| < 1} \left[\frac{\hat{q}(\xi + \mu - k\beta)}{\Lambda_\zeta^N(k\beta)} - \frac{\hat{q}(\xi + \mu - k\tilde{\omega})}{\Lambda_\zeta^N(k\tilde{\omega})}\right] \frac{f(k\omega)}{|\eta|^2 - k^2} d\eta$$

$$\equiv M_1 + M_2 + M_3.$$

Here

$$\beta = \frac{k\omega + \mu}{|k\omega + \mu|} = \tilde{\omega}|_{|\eta| = k}.$$

The point of this decomposition is that the mean of the integrand in M_2 over spheres $|\eta| = c$ is independent of $|\eta|$, and hence we can estimate the integral in $|\eta|$ accurately. On the other hand $\tilde{\omega} - \beta$ is so small that we can control M_3.

We claim that

$$|k(\tilde{\omega} - \beta)| \leqq C|\mu|\,||\eta| - k|(|\eta| + |\mu|)^{-1} + C|\mu|^2((k + |\mu|)^{-1} + (|\eta| + |\mu|)^{-1}) \quad (2.7)$$

for all η, k and μ. One can arrive at this estimate in the following way. If $k < 2|\mu|$, we have

$$|k(\tilde{\omega} - \beta)| \leqq 2k \leqq 4|\mu| \leqq \frac{12|\mu|^2}{k + |\mu|}.$$

Similarly, if $|\eta| < 2|\mu|$, we have

$$|k(\tilde{\omega} - \beta)| \leqq 2k \leqq 2|\eta| + 2||\eta| - k| \leqq \frac{12|\mu|^2}{|\eta| + |\mu|} + \frac{6|\mu|\,||\eta| - k|}{|\eta| + |\mu|}.$$

When $|\eta| > 2|\mu|$ and $k > 2|\mu|$, we use Taylor series in μ. Thus

$$\begin{aligned}\tilde{\omega} &= \frac{\eta + \mu}{|\eta + \mu|} = \frac{\eta + \mu}{|\eta|}\left(1 + \frac{2\mu\cdot\eta}{|\eta|^2} + \frac{|\mu|^2}{|\eta|^2}\right)^{-1/2} \\ &= \omega + \frac{\mu}{|\eta|} - \frac{(\mu\cdot\omega)\omega}{|\eta|} + O\left(\frac{|\mu|^2}{|\eta|^2}\right), \end{aligned} \quad (2.8)$$

and, since $\beta = \tilde{\omega}|_{|\eta| = k}$,

$$\beta = \omega + \frac{\mu}{k} - \frac{(\mu\cdot\omega)\omega}{k} + O\left(\frac{|\mu|^2}{k^2}\right). \quad (2.9)$$

Thus

$$\tilde{\omega} - \beta = \mu\left(\frac{k - |\eta|}{k|\eta|}\right) + (\mu\cdot\omega)\omega\left(\frac{|\eta| - k}{k|\eta|}\right) + O\left(\frac{|\mu|^2}{k^2} + \frac{|\mu|^2}{|\eta|^2}\right),$$

and,

$$|k(\tilde{\omega} - \beta)| \leqq \frac{2|\mu|\,|k - |\eta||}{|\eta|} + C|\mu|^2\left(\frac{1}{k} + \frac{1}{|\eta|} + \frac{||\eta| - k|}{|\eta|^2}\right).$$

Thus, since $2|\mu| < k$ and $2|\mu| < |\eta|$, wee see that (2.7) holds. However, since $|\mu|^2(k + |\mu|)^{-1} \leqq |\mu|^2(|\eta| + |\mu|)^{-1} + |\mu|\,||\eta| - k|(|\eta| + |\mu|)^{-1}$, we actually have

$$|k(\beta - \tilde{\omega})| \leqq C\frac{|\mu|\,||\eta| - k|| + |\mu|^2}{|\eta| + |\mu|}. \quad (2.10)$$

From (2.8) we have for $|\eta| > 2|\mu|$

$$\tilde{\omega} - \omega = \frac{\mu}{|\eta|} - \frac{(\mu\cdot\omega)\omega}{|\eta|} + O\left(\frac{|\mu|^2}{|\eta|^2}\right).$$

Thus for $|\eta| > 2|\mu|$,

$$|k(\tilde{\omega} - \omega)| \leqq Ck\frac{|\mu|}{|\eta| + |\mu|}, \quad (2.11)$$

and this estimate also holds (with $C=6$) for $|\eta|<2|\mu|$. Finally, from (2.9) we see for $k>2|\mu|$,

$$|k(\omega-\beta)| \leqq 2|\mu| + C\frac{|\mu|^2}{k} \leqq C|\mu|, \tag{2.12}$$

and again this estimate also holds (with $C=4$) for $k<2|\mu|$. We will use (2.11) to estimate M_1, (2.12) to estimate M_2 and (2.10) to estimate M_3. We have by Lemma 2.2 and (2.11),

$$\begin{aligned}|M_1| &\leqq C(1+k)^{\delta-\beta}|\mu|^\alpha \|\hat{q}\|_{\alpha,N}\|f\|_\alpha \int_{||\eta|-k|<1}\left(1+\frac{k^\alpha}{(|\eta|+|\mu|)^\alpha}\right)\frac{|\eta|^2 d|\eta|}{(|\eta|+k)||\eta|-k|^{1-\alpha}} \\ &\leqq C(1+k)^{\delta-\beta+1}|\mu|^\alpha \|\hat{q}\|_{\alpha,N}\|f_\alpha\end{aligned}$$

as desired.

The integral M_2 is given by

$$M_2 = (1+k^2)^{\delta/2} P(k,\mu,\xi,\zeta) \int_{2|\mu|<||\eta|-k|<1} \frac{|\eta|^2}{|\eta|^2-k^2} d|\eta|,$$

where

$$P = \Lambda_\zeta^N(\xi) \int_{S^2}\left(\frac{\hat{q}(\xi-k\omega)}{\Lambda_\zeta^N(k\omega)} - \frac{\hat{q}(\xi+\mu-k\beta)}{\Lambda_\zeta^N(k\beta)}\right) f(k\omega)\, d\omega.$$

Lemma 2.2 and (2.12) show

$$|P| \leqq C(1+k)^{-\beta}|\mu|^\alpha \|\hat{q}\|_{\alpha,N}\|f\|_0. \tag{2.13}$$

We have

$$\int_{1>||\eta|-k|>2|\mu|} \frac{|\eta|^2}{|\eta|^2-k^2} d|\eta| = \frac{k}{2}\int_{1>||\eta|-k|>2|\mu|}\frac{d|\eta|}{|\eta|-k} + \int_{1>||\eta|-k|>2|\mu|}\frac{2|\eta|+k}{2(|\eta|+k)} d|\eta|.$$

The second integral is bounded by 2, and

$$\int_{1>||\eta|-k|>2|\mu|}\frac{d|\eta|}{|\eta|-k} = \begin{cases} 0, & k>1, \\ -\ln k, & 2|\mu|<k<1 \\ -\ln 2|\mu|, & k<2|\mu|. \end{cases}$$

Since $k \ln k$ is bounded for $k<1$, we conclude from (2.13),

$$|M_2| \leqq C(1+k)^{\delta-\beta}|\mu|^\alpha \|\hat{q}\|_{\alpha,N}\|f\|_0,$$

which suffices.

By Lemma 2.2 and (2.10) we have

$$\begin{aligned}|M_3| &\leqq C(1+k)^{\delta-\beta}\|\hat{q}\|_{\alpha,N}\|f\|_0 \\ &\quad \int_{2|\mu|<||\eta|-k|<1}\frac{|\eta|^2}{||\eta|^2-k^2|}\left(\frac{|\mu|^\alpha||\eta|-k|^\alpha+|\mu|^{2\alpha}}{(|\eta|+|\mu|)^\alpha}\right)d|\eta| \\ &\leqq C(1+k)^{\delta-\beta+1-\alpha}\|\hat{q}\|_{\alpha,N}\|f\|_0(|\mu|^\alpha+|\mu|^{2\alpha}(1+|\ln|\mu||))\end{aligned}$$

which suffices. ∎

In addition to the estimate in Theorem 2.1 we also need control of Lipschitz norms in the variable k. This is provided by the following theorem.

Theorem 2.2. *Let $\Delta(s), 0 < s < 1$ denote operator $(\Delta(s)f)(k) = f(k+s) - f(k)$. Then one has the estimate*

$$\sup_{\xi,\zeta} \Lambda_\zeta^N(\xi) \left| \int_{\mathbf{R}^3} f(\eta,\xi,\zeta,k)\left(\Delta(s)\left(\frac{1}{|\eta|^2-(k+i0)^2}\right)\right)d\eta\right|$$
$$\leqq \frac{Cs^\alpha}{(1+k)^\gamma} \sup_{\xi,\zeta} \|\Lambda^{-\delta}(\cdot)\Lambda_\xi^N(\cdot)\Lambda_\zeta^N(\cdot)f(\cdot,\xi,\zeta,k)\|_\alpha$$

with C independent of k for α, N, δ and γ in the set given in Theorem 2.1.

Proof of Theorem 2.2. Here we will write

$$\Lambda_\zeta^N(\xi)\int_{\mathbf{R}^3} f(\eta)\left(\Delta(s)\left(\frac{1}{|\eta|^2-(k+i0)^2}\right)\right)d\eta$$
$$= \Lambda_\zeta^N(\xi)\int_{||\eta|-k|>1} f(\eta)\left(\Delta(s)\left(\frac{1}{|\eta|^2-k^2}\right)\right)d\eta$$
$$+\Lambda_\zeta^N(\xi)\int_{||\eta|-k|<1} \Delta(s)\left((f(\eta)-f(k\omega))\left(\frac{1}{|\eta|^2-k^2}\right)\right)d\eta$$
$$+\Lambda_\zeta^N(\xi)\int_{||\eta|-k|<1} \Delta(s)\left(f(k\omega)\left(\frac{1}{|\eta|^2-(k+i0)^2}\right)\right)d\eta$$
$$\equiv I_1+I_2+I_3,$$

where $f(\eta)=f(\eta,\xi,\zeta,k)$ and $\Delta(s)f(k\omega)=f((k+s)\omega,\xi,\zeta,k)-f(k\omega,\xi,\zeta,k)$.

By Lemma 2.2 for some $\beta>1+\delta$, setting $h=\Lambda^{-\delta}(\eta)\Lambda_\zeta^N(\eta)\Lambda_\xi^N(\eta)f(\eta,\xi,\zeta,k)$, we have

$$|I_1| \leqq C\sup_{\xi,\zeta} h| \int_{||\eta|-k|>1} \frac{(ks+s^2)(1+|\eta|)^{-\beta+\delta}}{||\eta|^2-k^2||\,|\eta|^2-(k+s)^2|}d\eta$$
$$\leqq C\sup_{\xi,\zeta}|h|(k+1)s\int_{||\eta|-k|>1/2}\frac{(1+|\eta|)^{-\beta+\delta}}{||\eta|-k|^2}d|\eta|$$
$$\leqq C\sup_{\xi,\zeta}|h|(k+1)s\left[\int_{1/2<||\eta|-k|<k/2}\frac{(1+|\eta|)^{-\beta+\delta}}{||\eta|-k|^2}d|\eta|\right.$$
$$\left.+\int_{||\eta|-k|>\max\{1/2,k/2\}}\frac{(1+|\eta|)^{-\beta+\delta}}{||\eta|-k|^2}d|\eta|\right] \leqq C\sup_{\xi,\zeta}|h|(k+1)s$$
$$\cdot\left((1+k)^{-\beta+\delta}\int_{||\eta|-k|>1/2}||\eta|-k|^{-2}d|\eta|+(1+k)^{-2}\int_{|\eta|>0}(1+|\eta|)^{-\beta+\delta}d|\eta|\right)$$
$$\leqq C\left(\sup_{\xi,\zeta}|h|\right)s(1+k)^{-\beta+\delta+1}.$$

The last term I_3 is also easy to estimate,

$$I_3 = \Lambda_\zeta^N(\xi)\left((\Delta(s)g) \int_{S^2} f((k+s)\omega)d\omega + g \int \Delta(s)(f(k\omega))d\omega \right),$$

where (see (2.6))

$$g(k) = \begin{cases} 2 - \dfrac{k}{2}\left(\ln\left(\dfrac{2k+1}{2k-1}\right) - \pi i \right) & \text{if} \quad k > 1 \\ 1 + k - \dfrac{k}{2}(\ln(2k+1) - \pi i) & \text{if} \quad k < 1. \end{cases}$$

Since g has Lipschitz constant bounded on $\mathbf{R}_+$,

$$|I_3| \leqq \frac{Cs^\alpha}{(1+k)^{\beta-\delta-1}} \sup_{\xi,\zeta} \| \Lambda^{-\delta}(\cdot)\Lambda_\xi^N(\cdot)\Lambda_\zeta^N(\cdot) f(\cdot,\xi,\zeta,k)\|_\alpha.$$

The term I_2 here we decompose to

$$\begin{aligned} I_2 &= \Lambda_\zeta^N(\xi) \int_{||\eta|-k|<2s} (f(\eta) - f(k+s)\omega) \frac{d\eta}{|\eta|^2 - (k+s)^2} \\ &\quad - \Lambda_\zeta^N(\xi) \int_{||\eta|-k|<2s} (f(\eta) - f(k\omega)) \frac{d\eta}{|\eta|^2 - k^2} \\ &\quad + \Lambda_\zeta^N(\xi) \int_{2s<||\eta|-k|<1} (f(\eta) - f(k\omega)) \left(\frac{1}{|\eta|^2 - (k+s)^2} - \frac{1}{|\eta|^2 - k^2} \right) d\eta \\ &\quad + \Lambda_\zeta^N(\xi) \int_{2s<||\eta|-k|<1} (f(k\omega) - f((k+s)\omega)) \left(\frac{1}{|\eta|^2 - (k+s)^2} \right) d\eta \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Here $|J_1|$ and $|J_2|$ can be estimated in the same way that $|\Delta(\mu)J_1|$ was estimated in the proof of Theorem 2.1 with s in place of $|\mu|$. Likewise $|J_3|$ can be estimated as L_3 was estimated. Finally J_4 is like M_2 in the proof of Theorem 2.1. Carrying out the integration in $|\eta|$, we have

$$J_4 = \left(\Lambda_\zeta^N(\xi) \int_{S^2} (f(k\omega) - f((k+s)\omega))d\omega \right) P(k,s),$$

where

$$\begin{aligned} P(k,s) &= \int_{2s<||\eta|-k|<1} \frac{|\eta|^2}{|\eta|^2 - (k+s)^2} d|\eta| \\ &= \frac{k+s}{2} \int_{2s<||\eta|-k|<1} \frac{d|\eta|}{|\eta| - (k+s)} + \int_{2s<||\eta|-k|<1} \frac{2|\eta| + k + s}{2(|\eta| + k + s)} d|\eta|. \end{aligned}$$

As in the proof of Theorem 2.1, this suffices. ■

Section 3. Existence and Regularity of $h(\xi, \zeta, k)$

The function $h(\xi, \zeta, k)$ on $\mathbf{R}^3 \times \mathbf{R}^3 \times \bar{\mathbf{R}}_+$ is defined to be the solution of

$$h(\xi, \zeta, k) + (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{\hat{q}(\xi - \eta) h(\eta, \zeta, k)}{|\eta|^2 - (k + i0)^2} d\eta = -\hat{q}(\xi - \zeta). \tag{3.1}$$

We will assume that $\hat{q} \in H_{\alpha,N}$ for some α and N. We will not assume that $\hat{q}$ is the Fourier transform of a real-valued function. In this situation one has the following existence theorem, considering ζ and k as parameters.

Theorem 3.1. *Given* (α, N), $0 < \alpha < 1$, $N > 1$, *for all* $\zeta \in \mathbf{R}^3$ *and* $k \geqq 0$, *(3.1) has a unique solution* $h(\xi, \zeta, k)$ *such that* $\Lambda_\zeta^N(\cdot) h(\cdot, \zeta, k) \in C^\alpha(\mathbf{R}^3)$, *when* $\hat{q}$ *belongs to an open set* $\mathcal{O}$ *in* $H_{\alpha,N}$. *Moreover, the intersection of* $\mathcal{O}$ *with* $H^r_{\alpha,N} = \{\hat{q} \in H_{\alpha,N} : \hat{q}(-\xi) = \overline{\hat{q}(\xi)}\}$ *is dense in* $H^r_{\alpha,N}$.

Remark 1. Note that $H^r_{\alpha,N}$ is simply the subspace of $H_{\alpha,N}$ (considered as vector space with real scalars) consisting of Fourier transforms of real-valued functions. The set $\mathcal{O}$ in this theorem is actually dense in $H_{\alpha,N}$ (see Remark 4 following the proof), but it is the stated density of $\mathcal{O} \cap H^r_{\alpha,N}$ in $H^r_{\alpha,N}$ that is important for our main results here.

Remark 2. One does not have existence for *all* real-valued $q \in C_0^\infty(\mathbf{R}^3)$, as the following family of examples shows. Let $u(x)$ be any positive function in $C^\infty(\mathbf{R}^3)$ such that $u(x) = |x|^{-1}$ for $|x| > R$, and define $q = \Delta u / u \in C_0^\infty(\mathbf{R}^3)$. Then $-\Delta u + qu = 0$. Since $|D^\alpha u(x)| \leqq C_\alpha (1 + |x|)^{-1-|\alpha|}$, for all $\alpha, |\hat{u}(\xi)| \leqq C_k |\xi|^{-k}$ for $|\xi| > 1$ for all k. Moreover, since $u = |x|^{-1} + g, g$ supported in $|x| \leqq R, \hat{u}(\xi) = -4\pi |\xi|^{-2} + \hat{g}$, and $\hat{g}$ is entire. We have

$$|\eta|^2 \hat{u}(\eta) + (2\pi)^{-3} \int_{\mathbf{R}^3} \hat{q}(\eta - \xi) \hat{u}(\xi) d\xi = 0. \tag{3.2}$$

Assuming that (3.1) has a solution $h(\xi, 0, 0) \in H_{\alpha,N}$ for $\zeta = 0, k = 0$ and taking the inner product with $\hat{u}(\xi)$ we conclude from (3.2) (note $\hat{q}(\xi - \eta) = \bar{\hat{q}}(\eta - \xi)$)

$$0 = \int_{\mathbf{R}^3} \bar{\hat{u}}(\xi) \hat{q}(\xi) d\xi.$$

However, by Plancherel's theorem

$$(2\pi)^{-3} \int_{\mathbf{R}^3} \bar{\hat{u}}(\xi) \hat{q}(\xi) d\xi = \int_{\mathbf{R}^3} u(x) q(x) dx \equiv \int_{\mathbf{R}^3} \Delta u dx = -4\pi. \quad \blacksquare$$

Throughout this section we will work with the modified operators and functions,

$$\tilde{A}(\hat{q}, \zeta, k) = \Lambda_\zeta^N A(\hat{q}, k) \Lambda_\zeta^{-N}, \tilde{h}(\xi, \zeta, k) = \Lambda_\zeta^N(\xi) h(\xi, \zeta, k),$$
$$\tilde{q}(\xi) = \Lambda^N(\xi) \hat{q}(\xi) \quad \text{and} \quad \tilde{q}_\zeta(\xi) = \Lambda_\zeta^N(\xi) \hat{q}(\xi - \zeta).$$

We will also frequently suppress some or all of the variables $\hat{q}, \xi, \zeta, k$ in $\tilde{A}$ and $\tilde{h}$. In this notation (3.1) becomes

$$\tilde{h}(\xi, k, \zeta) + [\tilde{A}(\tilde{q}, \zeta, k) \tilde{h}(\cdot, \zeta, k)](\xi) = -\tilde{q}_\zeta(\xi) \tag{3.3}$$

or, more compactly

$$\tilde{h} + \tilde{A}\tilde{h} = -\tilde{q}_\zeta.$$

Proof of Theorem 3.1. Theorem 2.1 implies that for $\hat{q}\in H_{\alpha,N}$, $0<\alpha'\leqq\alpha$ and $0\leqq\delta<\min\{1,N-1\}$,

$$\|\tilde{A}\Lambda^{\delta}f\|_{\alpha}\leqq C\|f\|_{\alpha'}.$$

Thus, since for $\alpha'<\alpha$ and $\delta>0$ $\{g:\|g\|_{\alpha}\leqq 1\}$ has compact closure in $C^{\alpha',\delta}=\{g:\|\Lambda^{-\delta}g\|_{\alpha'}<\infty\}$, we see that $\tilde{A}$ is a compact operator on $C^{\alpha}(\mathbf{R}^3)$. Hence, since $\tilde{q}_{\zeta}$ is in $C^{\alpha}(\mathbf{R}^3)$ by hypothesis, (3.3) is a Fredholm equation in $C^{\alpha}(\mathbf{R}^3)$ for $\tilde{h}$. We will prove the first part of Theorem 3.1 by showing that the set $\mathcal{O}$ of $\hat{q}$ such that $I+\tilde{A}(\hat{q},\zeta,k)$ has trivial kernel in $C^{\alpha}(\mathbf{R}^3)$ for all $(\zeta,k)\in\mathbf{R}^3\times\bar{\mathbf{R}}_+$ is open. For $\hat{q}\in\mathcal{O}$ (3.3) has a unique solution $\tilde{h}$ in $C^{\alpha}(\mathbf{R}^3)$. Since $C(\zeta)^{-1}\leqq\Lambda^N\Lambda_{\zeta}^{-N}\leqq C(\zeta)$, one sees that $\Lambda_{\zeta}^{-N}\tilde{h}$ is the unique solution to (3.1) with $\Lambda^N h\in C^{\alpha}(\mathbf{R}^3)$.

Theorem 2.1 implies that given $\hat{q}_0\in H_{\alpha,N}$ the operator norm on $C^{\alpha}(\mathbf{R}^3)$, $\|\tilde{A}(\hat{q},\zeta,k)\|_{\alpha}$ will be less than 1, for $k>k_0$ and $\|\hat{q}-\hat{q}_0\|_{\alpha,N}\leqq 1$. Thus $I+\tilde{A}(\hat{q},\zeta,k)$ is injective for $k>k_0$ and $\|\hat{q}-\hat{q}_0\|_{\alpha,N}\leqq 1$. Since $(C(\zeta))^{-1}\leqq\Lambda_{\zeta}^{-N}\Lambda^N\leqq C(\zeta)$, if $I+\tilde{A}(\hat{q},0,k)$ is injective on $C^{\beta}(\mathbf{R}^3)$, then $I+\tilde{A}(\hat{q},\zeta,k)$ is injective on $C^{\beta}(\mathbf{R}^3)$ for all $\zeta\in\mathbf{R}^3$. Applying Theorem 2.2, we have

$$|\Delta(s)\Delta(\mu)\tilde{A}(\hat{q},0,k)f|\leqq Cs^{\alpha'}\|\Delta(\mu)\hat{q}\|_{\alpha',N}\|f\|_{\alpha'},$$

where $\Delta(s)$ and $\Delta(\mu)$ are the difference operators in k and ξ, respectively. Hence arguing as in the initial reduction in the proof of Theorem 2.1, we see for $\alpha'=\alpha/2$,

$$\|\Delta(s)\tilde{A}(\hat{q},0,k)f\|_{\alpha/2}\leqq Cs^{\alpha/2}\|\hat{q}\|_{\alpha,N}\|f\|_{\alpha/2},\tag{3.4}$$

uniformly for $k\geqq 0$. Thus, as an operator acting on $C^{\alpha/2}(\mathbf{R}^3)$, $\tilde{A}(\hat{q},0,k)$ is norm continuous in $(\hat{q},k)$ with the topology of $H_{\alpha,N}\times\bar{\mathbf{R}}_+$.

Now suppose $I+\tilde{A}(\hat{q}_0,\zeta,k)$ has no nullspace in $C^{\alpha}(\mathbf{R}^3)$ for $(\zeta,k)\in\mathbf{R}^3\times\bar{\mathbf{R}}_+$. If $f+\tilde{A}(\hat{q}_0,\zeta,k)f=0$ for some $f\in C^{\alpha/2}(\mathbf{R}^3)$, then Theorem 2.1, implies $f\in C^{\alpha}(\mathbf{R}^3)$. Hence $I+\tilde{A}(\hat{q}_0,\zeta,k)$ has no nullspace in $C^{\alpha/2}(\mathbf{R}^3)$ for $(\zeta,k)\in\mathbf{R}^3\times\bar{\mathbf{R}}_+$. Thus by the remarks in the preceding paragraph $I+\tilde{A}(\hat{q},\zeta,k)$ is injective on $C^{\alpha/2}(\mathbf{R}^3)$ for $k\geqq k_0$ when $\|\hat{q}-\hat{q}_0\|_{\alpha,N}<1$ and injective on $C^{\alpha/2}(\mathbf{R}^3)$ for $0\leqq k\leqq k_0$ when $\|q-\hat{q}_0\|_{\alpha,N}<\varepsilon$ for some $\varepsilon>0$. Thus, the set of $\hat{q}$ for which $I+\tilde{A}(\hat{q},\zeta,k)$ is injective on $C^{\alpha}(\mathbf{R}^3)$ is open in $H_{\alpha,N}$.

To verify the density assertion in Theorem 3.1 we consider real-valued $q\in C_0^{\infty}(\mathbf{R}^3)$. The Fourier transforms of these q are easily seen to be dense in $H^r_{\alpha,N}$. If for $k>0$, $\hat{f}+A(\hat{q},k)\hat{f}=0$ has a nontrivial solution with $\Lambda^N\hat{f}\in C^{\alpha}(\mathbf{R}^3)$, we set for $\varepsilon\geqq 0$,

$$u_{\varepsilon}=\int_{\mathbf{R}^3}\frac{e^{ix\cdot\xi}\hat{f}(\xi)d\xi}{|\xi|^2-(k+i\varepsilon)^2}.$$

Note $u_{\varepsilon}\in L^2(\mathbf{R}^3)$ for $\varepsilon>0$, and, taking the inverse Fourier transform of $\hat{f}+A(\hat{q},k)\hat{f}=0$, $-\Delta u_0+qu_0=k^2u_0$ which implies $qu_0\in C_0^{\infty}(\mathbf{R}^3)$. We also have $(-\Delta-(k+i\varepsilon)^2)u_{\varepsilon}+qu_0=0$, which implies

$$u_{\varepsilon}(x)=\frac{-1}{4\pi}\int_{\mathbf{R}^3}\frac{e^{i(k+i\varepsilon)|x-y|}}{|x-y|}q(y)u_0(y)dy,$$

and hence

$$u_0(x) = \frac{-1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{ik|x-y|}}{|x-y|} q(y) u_0(y) dy.$$

Now standard arguments show $u_0 \in L^2(\mathbf{R}^3)$ and hence $u_0 \equiv 0$. Thus $I + A(\hat{q}, k)$ is injective for $k > 0$, and $I + \tilde{A}(\hat{q}, 0, k)$ is invertible on $C^\alpha(\mathbf{R}^3)$ for $k > 0$.

Suppose $I + \tilde{A}(\hat{q}, 0, 0)$ has nontrivial nullspace for $\hat{q} \in S = \{ \| \hat{q} - \hat{q}_0 \|_{\alpha,N} < \delta \} \cap H^r_{\alpha,N}$. Let $m = \dim \mathrm{Null}\{I + \tilde{A}(\hat{q}_1, 0, 0)\}$ be minimal for $\hat{q} \in S$. Then $\dim \mathrm{Null}\{I + \tilde{A}(\hat{q}, 0, 0)\} = m$, for all $\hat{q}$ with $\| \hat{q} - \hat{q}_1 \|_{\alpha,N} < \delta'$, for some $\delta' > 0$. This follows from the continuity of the projection

$$P(\hat{q}) = \oint_{|z-1|=\varepsilon} (zI + \tilde{A}(\hat{q}, 0, 0))^{-1} dz \tag{3.5}$$

in $\hat{q}$ on a neighborhood of $\hat{q}_1$ for ε sufficiently small. Moreover, for all $\hat{f} \in C^\alpha(\mathbf{R}^3)$,

$$(I + \tilde{A}(\hat{q}, 0, 0)) P(\hat{q}) \hat{f} = 0$$

for $\| \hat{q} - \hat{q}_1 \|_{\alpha,N} < \delta''$. Let $\hat{q}(t) = \hat{q}_1 + t\hat{q}, \hat{q} \in H^r_{\alpha,N}$. For t sufficiently small, one sees by substituting the power series for $(zI + \tilde{A}(q_1, 0, 0) + t\tilde{A}(\hat{q}, 0, 0))^{-1}$ into (3.5) that $P(\hat{q}(t))$ is analytic in t. Differentiating

$$(I + \tilde{A}(\hat{q}(t), 0, 0) P(\hat{q}(t)) \hat{f} = 0$$

with respect to t at $t = 0$, we have

$$(I + \tilde{A}(\hat{q}_1, 0, 0)) \dot{V} = -\tilde{A}(\hat{q}, 0, 0) V,$$

where $\dot{V} = d/dt P(\hat{q}(t)) \hat{f}|_{t=0}$ and $V = P(\hat{q}_1)\hat{f}$. As in Remark 2, taking the inner product with $V(\xi)|\xi|^{-2} \Lambda^{-N}(\xi) = w(\xi)|\xi|^{-2}$,

$$0 = \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\hat{q}(\xi - \eta) w(\eta)}{|\eta|^2} \frac{\bar{w}(\xi)}{|\xi|^2} d\xi d\eta = (2\pi)^3 \int_{\mathbf{R}^3} q(x) |h|^2 dx,$$

where h is the inverse Fourier transform of $w(\xi)|\xi|^{-2}$. Since we can choose $\hat{f}$ so that $w \neq 0$ and q is arbitrary, this is a contradiction.

Finally we note that, since the Fourier transform $\hat{R}$ of the set R of real-valued $q \in C_0^\infty(\mathbf{R}^3)$ is dense in $H^r_{\alpha,N}$, if $I + \hat{A}(\hat{q}, 0, 0)$ has a nontrivial nullspace for all $\hat{q} \in \hat{R} \cap S$ it must have a nontrivial nullspace for all $\hat{q} \in S$. This follows from the compactness of $\tilde{A}(\hat{q}, 0, 0)$ for $\hat{q} \in H_{\alpha,N}$ and its continuity in $\hat{q}$. Thus the preceding contradiction shows that given $\hat{q}_1 \in \hat{R}$ there is no δ such that $I + \hat{A}(\hat{q}, 0, 0)$ has a nontrivial nullspace for $\hat{q} \in \hat{R} \cap \{ \| \hat{q} - \hat{q}_1 \|_{\alpha,N} < \delta \}$. Thus we conclude $I + \tilde{A}(\hat{q}, \zeta, k)$ is injective for $\hat{q}$ in a dense subset of $\hat{R}$. ■

Remark 3. The computations following (3.5) are much more transparent in x-space. In x-space the equation $(I + A(\hat{q}, 0))\hat{f} = 0$, becomes

$$(I + qE_0) f = 0,$$

where E_0 is the operator

$$[E_0 g](x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{g(y)}{|x-y|} dy.$$

Setting $q=q(t)$ and $f=f(t)$ and differentiating in t, we have

$$(I+qE_0)\dot{f}=-\dot{q}E_0f.$$

Since q is real,

$$\int_{\mathbf{R}^3}\overline{E_0f}(I+qE_0)g\,dx=\int_{\mathbf{R}^3}\overline{(f+qE_0f)}(E_0g)dx=0$$

for all g. Thus

$$0=\int_{\mathbf{R}^3}\dot{q}|E_0f|^2dx.$$

We work in ξ-space in the proof of Theorem 3.1 and elsewhere because we have no simple characterization of the inverse Fourier transform of $H_{\alpha,N}$.

Remark 4. Though our interest here is primarily in potentials with small imaginary parts, it is not at all difficult to extend the arguments used to prove Theorem 3.1 to show that the set of complex potentials q in $C_0^\infty(\mathbf{R}^3)$ such that $\hat{q}\in\mathcal{O}$ is large enough that $\mathcal{O}\cap H_{\alpha,N}$ is a dense, open subset of $H_{\alpha,N}$. A sketch of one way to do this follows.

Given $q\in C_0^\infty(|x|<R)$, if $\hat{f}\in H_{\alpha,N}$ and $\hat{f}+A(\hat{q},k)\hat{f}=0$, then $f\in C_0^\infty(|x|<R)$ and $k\leqq k_0(\|\hat{q}\|_{\alpha,N})$ (by Theorem 2.1). Thus, taking s large enough that $\|\hat{q}\|_{\alpha,N}\leqq C\|q\|_s$, where $\|\ \|_s$ is the norm on the Sobolev space $\mathring{H}_s(|x|<R)$, to show the injectivity of $I+A(\hat{q},k)$ on $H_{\alpha,N}$ when $k\geqq 0$ for a dense set of $\hat{q}$ in $H_{\alpha,N}$, it will suffice to show that for any $R, I+qE_z$ is injective on $L^2(|x|<R)$ for $z\geqq 0$ for a dense set of q in $\mathring{H}_s(|x|<R)$, where

$$E_zf=\frac{1}{4\pi}\int_{|x|<R}\frac{e^{iz|x-y|}}{|x-y|}f(y)dy.$$

Given $q_0\in C_0^\infty(|x|<R)$, since q_0E_z is both compact and entire in z as an operator on $L^2(|x|<R)$ and $I+q_0E_z$ is injective for $z\gg 0$, $(I+q_0E_z)^{-1}$ is meromorphic with only a finite number of poles $k_1,\dots,k_M$ on $k\geqq 0$. Using contour integrals to define projections on the nullspaces of $I-q_0E_{k_j}$ as in the proof of Theorem 3.1, one can get $\varepsilon>0$ and functions $\lambda_j(q,z)$ analytic on $D_j=\{\|q-q_0\|_s<\varepsilon,|z-k_j|<\varepsilon\}$, $j=1,\dots,M$, such that, for $\|q-q_0\|_s<\varepsilon$ and $|z-k_j|<\varepsilon$, $I+qE_z$ fails to be injective if and only if $\lambda_j(q,z)=0$.

For each j an argument similar to the one given in the proof of Theorem 3.1 shows $\lambda_j(q,k_j)\not\equiv 0$. Thus one can choose $h\in C_0^\infty(|x|<R)$ such that for $j=1,\dots,M$,

$$d_j(w,z)=\lambda_j(q_0+wh,z)$$

is an analytic function on $\{|w|<\varepsilon',|z-k_j|<\varepsilon'\}$ such that $(\partial^pd_j/\partial w^p)(0,k_j)=0$ for $p<N_j$ and

$$\frac{\partial^{N_j}d_j}{\partial w^{N_j}}(0,k_j)\neq 0$$

for some $N_j>0$. By the Weierstrass preparation theorem, for each j

$$d_j(w,z)=(w^{N_j}+a_1(z)w^{N_j-1}+\cdots+a_{N_j}(z))r(w,z),$$

where $r(0, k_j) \neq 0$. Thus the zero set of d_j in $\{|w| < \varepsilon' < \varepsilon, |z - k_j| < \varepsilon' < \varepsilon\} \cap \{\text{real } z\}$ is the union of a finite set of curves $(w_l(k), k)$ where either $w_l \equiv 0$ or

$$w_l(k) = a_l(k - k_j)^{r_l} + o((k - k_j)^{r_l}$$

with $a_l \neq 0$ and r_l rational. Thus we can choose $w_n \to 0$ such that $d_j(w_n, k) \neq 0$ for $|k - k_j| < \varepsilon', j = 1, \ldots, M$ for all n. Since $I + q_0 E_k$ is injective for $k \neq k_j$ and $w_n \to 0$, we see that $I + (q_0 + w_n h)E_k$ is injective for all $k \geqq 0$ for $n > n_0$. ■

Our estimates on the regularity and growth of $h(\xi, \zeta, k)$ are primarily directed toward showing that the backscattering amplitude $h(\xi, -\xi, |\xi|)$ belongs to $H_{\alpha,N}$ when $\hat{q}$ is in the set $\mathcal{O} \subset H_{\alpha,N}$ of Theorem 3.1. However, the expression we use for the Frechet derivative of the backscattering map $\hat{q}(\xi) \to h(\xi, -\xi, |\xi|)$ involves $h(\xi, \zeta, |\zeta|)$, and it is actually easier to treat ξ, ζ, k as independent variables. Thus our estimate takes the following form.

Theorem 3.2. *Let $\mathcal{O}$ be the open subset of $H_{\alpha,N}$ in Theorem 3.1,* i.e. *let $\mathcal{O}$ be the set of $\hat{q} \in H_{\alpha,N}$ such that $I + \tilde{A}(\hat{q}, \zeta, k)$ in injective on $C^\alpha(\mathbf{R}^3)$ for all $(\zeta, k) \in \mathbf{R}^3 \times \bar{\mathbf{R}}_+$. Then, for $\hat{q} \in \mathcal{O}$, $h(\xi, \zeta, k)$ satisfies*

$$\| \Lambda_\zeta^N h \|_{\underline{\alpha}} < \infty.$$

Here $\| \ \|_{\underline{\alpha}}$ is the norm on functions on $\mathbf{R}^3 \times \mathbf{R}^3 \times \bar{\mathbf{R}}_+$ introduced in (1.1).

Proof. From Theorem 3.1 we know that $(I + \tilde{A}(\hat{q}, \zeta, k))^{-1}$, and hence $\tilde{h}$ exist for $\hat{q} \in \mathcal{O}$. However, here we want to show that $\sup_{\zeta,k} \| \tilde{h}(\cdot, \zeta, k) \|_\alpha < \infty$. For this we will show that

$$\sup_{\zeta,k} \| (I + \Lambda^{-\delta/2} \tilde{A}(\hat{q}, \zeta, k) \Lambda^{\delta/2})^{-1} \|_{\alpha/2} < \infty. \tag{3.6}$$

Note that Theorem 2.1 implies that if $f + \Lambda^{-\delta/2} \tilde{A} \Lambda^{\delta/2} f = 0$ and $f \in C^{\alpha/2}(\mathbf{R}^3)$, then $\Lambda^{\delta/2} f \in C^\alpha(\mathbf{R}^3)$. Hence $I + \Lambda^{-\delta/2} \tilde{A} \Lambda^{\delta/2}$ is injective on $C^{\alpha/2}(\mathbf{R}^3)$ for $\hat{q} \in \mathcal{O}$. Moreover, $\Lambda^{-\delta/2} \tilde{A} \Lambda^{\delta/2}$ is compact on $C^{\alpha/2}(\mathbf{R}^3)$ by the argument used in the proof of Theorem 3.1. Thus $I + \Lambda^{-\delta/2} \tilde{A}(\hat{q}, \zeta, k) \Lambda^{\delta/2}$ is invertible on $C^{\alpha/2}(\mathbf{R}^3)$ for $(\zeta, k) \in \mathbf{R}^3 \times \bar{\mathbf{R}}_+$ for $\hat{q} \in \mathcal{O}$. Using Theorems 2.1 and 2.2 as in the proof of (3.4), one has uniformly for $(\zeta, k) \in \mathbf{R}^3 \times \bar{\mathbf{R}}_+$,

$$\| \Lambda^{-\delta/2} (\tilde{A}(\hat{q}, \zeta, k + s) - \tilde{A}(\hat{q}, \zeta, k)) \Lambda^{\delta/2} \|_{\alpha/2} \leqq C s^{\alpha/2}. \tag{3.7}$$

Moreover, simply by using (2.5) we can extend (3.7) to

$$\| \Lambda^{-\delta/2} (\tilde{A}(\hat{q}, \zeta + \mu, k + s) - \tilde{A}(\hat{q}, \zeta, k)) \Lambda^{\delta/2} \|_{\alpha/2} \leqq C(s^{\alpha/2} + |\mu|), \tag{3.8}$$

where C is independent of ζ and k.

Since Theorem 2.1 implies that $\| \Lambda^{-\delta/2} \tilde{A}(\hat{q}, \zeta, k) \Lambda^{\delta/2} \|_{\alpha/2} \leqq 1/2$ for $k > k(\hat{q})$ for all $\zeta \in \mathbf{R}^3$, we can use the Neumann series representation of $(I + \Lambda^{-\delta/2} \tilde{A}(\hat{q}, \zeta, k) \Lambda^{\delta/2})^{-1}$ to conclude that

$$\| (I + \Lambda^{-\delta/2} \tilde{A}(\hat{q}, \zeta, k) \Lambda^{\delta/2})^{-1} \|_{\alpha/2} \leqq C \tag{3.9}$$

for $\zeta \in \mathbf{R}^3, k \geqq k(\hat{q})$. Since for any invertible operators $I + B$ and $I + B_0$,

$$(I + B)^{-1} - (I + B_0)^{-1} = (I + B)^{-1}(B - B_0)(I + B_0)^{-1},$$

the estimate (3.8) implies

$$\|(I+\Lambda^{-\delta/2}\tilde{A}(\zeta,k)\Lambda^{\delta/2})^{-1}\|_{\alpha/2} \leqq 2\|(I+\Lambda^{-\delta/2}\tilde{A}(\zeta_0,k_0)\Lambda^{\delta/2})^{-1}\|_{\alpha/2}$$

for $|k-k_0|+|\zeta-\zeta_0|<\varepsilon_0$. Thus, for any $R<\infty$,

$$\|(I+\Lambda^{-\delta/2}\tilde{A}(\hat{q},\zeta,k)\Lambda^{\delta/2})^{-1}\|_{\alpha/2} \leqq C_R \tag{3.10}$$

for $0\leqq k\leqq k(\hat{q})$, $|\zeta|\leqq R$.

To bound $\|(I+\Lambda^{-\delta/2}\tilde{A}(\hat{q},\zeta,k)\Lambda^{\delta/2})^{-1}\|_{\alpha/2}$ as $|\zeta|\to\infty$, we will begin by showing that

$$\|\Lambda^{-\delta/2}(\tilde{A}(\hat{q},\zeta,k)-A(\hat{q},k))\Lambda^{\delta/2}\|_{\alpha/2}\to 0 \tag{3.11}$$

as $|\zeta|\to\infty$, uniformly in k and $\hat{q}$ on bounded sets of $\hat{q}$.

Given $\varphi\in C_0^\infty(\mathbf{R}^3)$ with $\varphi(\xi)=1$ for $|\xi|\leqq 1$, one sees easily that the operator norm of multiplication by $(1-\varphi(\xi/R))\Lambda^{-\delta/2}(\xi)$ on $C^{\alpha/2}(\mathbf{R}^3)$ tends to zero as $R\to\infty$. Since Theorem 2.1 implies $\|\tilde{A}(\hat{q},\zeta,k)\Lambda^\delta\|_{\alpha/2}\leqq C\|\hat{q}\|_{\alpha,N}$ for $(\zeta,k)\in\mathbf{R}^3\times\bar{\mathbf{R}}_+$, we see, letting $\varphi_R(\xi)=\varphi(\xi/R)$,

$$\|(1-\varphi_R)\Lambda^{-\delta/2}\tilde{A}(\hat{q},\zeta,k)\Lambda^{\delta/2}\|_{\alpha/2}\to 0$$

and

$$\|\Lambda^{-\delta/2}\tilde{A}(\hat{q},\zeta,k)\Lambda^{\delta/2}(1-\varphi_R)\|_{\alpha/2}\to 0$$

as $R\to\infty$ uniformly in (ζ,k) on bounded sets of $\hat{q}$.

To obtain the estimate $\|A(\hat{q},k)\Lambda^\delta\|_{\alpha/2}\leqq C\|\hat{q}\|_{\alpha,N}$, we must repeat the derivation of the bounds on $|I_1|,|I_2|$ and $|I_3|$ without the weight factor $\Lambda_\zeta^N(\xi)/\Lambda_\zeta^N(\eta)$. We have

$$|I_1|\leqq C\|f\|_0\|\hat{q}\|_{0,N}\int_{||\eta|-k|>1}\frac{(1+|\eta|^2)^{\delta/2}}{(1+|\xi-\eta|^2)^{N/2}||\eta|^2-k^2|}d\eta,$$

and as in the proof of (2.2), this implies

$$|I_1|\leqq C\|f\|_0\|\hat{q}\|_{0,N}\int_{||\eta|-k|>1}\frac{(1+|\eta|^2)^{(\delta-\beta)/2}}{||\eta|^2-k^2|}|\eta|^2d|\eta|$$

with $\beta>1+\delta$. Hence $|I_1|\leqq C\|f\|_0\|\hat{q}\|_{0,N}$. For I_2 we have

$$|I_2|\leqq C\|f\|_{\alpha/2}\|\hat{q}\|_{\alpha/2,N}\int_{||\eta|-k|\leqq 1}\frac{(1+|\eta|^2)^{\delta/2}(1+|\xi-\eta|^2)^{-N/2}}{(|\eta|+k)||\eta|-k|^{1-\alpha/2}}d\eta$$

so that $|I_2|\leqq C\|f\|_{\alpha/2}\|\hat{q}\|_{\alpha/2,N}$. Likewise,

$$|I_3|\leqq C\|f\|_0\|\hat{q}\|_{0,N}.$$

Thus we conclude $\sup_\xi|[A(\hat{q},k)\Lambda^\delta f](\xi)|\leqq C\|f\|_{\alpha/2}\|\hat{q}\|_{\alpha/2,N}$. Then, since $\Delta(\mu)(A(\hat{q},k)f)=A(\Delta(\mu)\hat{q},k)f$, we have

$$\|A(\hat{q},k)\Lambda^\delta f\|_{\alpha/2}\leqq C\|\hat{q}\|_{\alpha,N}\|f\|_{\alpha/2},$$

uniformly in k as desired. Thus

$$\|(1-\varphi_R)\Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}\|_{\alpha/2}\to 0$$

and

$$\|\Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}(1-\varphi_R)\|_{\alpha/2}\to 0$$

as $R \to \infty$ uniformly in k on bounded sets of $\hat{q}$.

Next we consider

$$\varphi_R \Lambda^{-\delta/2}(\tilde{A}(\hat{q},\zeta,k) - A(\hat{q},k))\Lambda^{\delta/2}\varphi_R.$$

We view this as a modification of the operator $\tilde{A}(q,\zeta,k)$ in which the weight factor $\omega(\xi,\eta,\zeta) = \Lambda_\zeta^N(\xi)\Lambda_\zeta^{-N}(\eta)$ has been replaced by

$$\omega_R(\xi,\eta,\xi) = \varphi_R(\xi)\Lambda^{-\delta/2}(\xi)(1 - \Lambda_\zeta^N(\eta)\Lambda_\zeta^{-N}(\xi))\Lambda^{\delta/2}(\eta)\varphi_R(\eta)\omega(\xi,\eta,\zeta).$$

Since for any $M, \Lambda_\zeta^N(\eta)\Lambda_\zeta^{-N}(\xi) \to 1$ uniformly on $\{|\xi| < M, |\eta| < M\}$ as $|\zeta| \to \infty$, given ε, we have $\omega_R(\xi,\eta,\zeta) \leqq \varepsilon\omega(\xi,\eta,\zeta)$ for $|\zeta| > C(R)$. Likewise, letting $\Delta(\mu)$ denote the difference operator in ξ or $\eta, (1/|\mu|)\Delta(\mu)(\Lambda_\zeta^N(\eta)\Lambda_\zeta^{-N}(\xi)) \to 0$ uniformly on $\{|\xi| < M, |\eta| < M\}$ as $|\zeta| \to \infty$, and we have $|\Delta(\mu)\omega_R(\xi,\eta,\zeta)| < \varepsilon|\mu|\omega(\xi,\eta,\zeta)$ for $|\zeta| > C(R)$. In the proof of Theorem 2.1 we only used

$$\omega(\xi,\eta,\zeta) \leqq \Lambda_\zeta^N(\xi)\Lambda_\zeta^{-N}(\eta)$$

and

$$|\Delta(\mu)\omega(\xi,\eta,\zeta)| \leqq C|\mu|\Lambda_\zeta^N(\xi)\Lambda_\zeta^{-N}(\eta).$$

Thus for $|\zeta| > C(R)$

$$\|\varphi_R\Lambda^{-\delta/2}(\tilde{A}(\hat{q},\zeta,k) - A(\hat{q},k))\Lambda^{-\delta/2}\varphi_R\|_{\alpha/2} \leqq \varepsilon\|\hat{q}\|_{\alpha,N}. \tag{3.12}$$

Combining (3.12), with the previous estimates on terms with factors of $(1-\varphi_R)$ yields (3.11).

From (3.11) we conclude that $\Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}$ is a compact operator-valued function on $C^{\alpha/2}(\mathbf{R}^3)$ which is norm continuous in $(k,\hat{q})$. Thus to conclude that $\|(I + \Lambda^{-\delta/2}\tilde{A}(\hat{q},\zeta,k)\Lambda^{\delta/2})^{-1}\|_{\alpha/2}$ is uniformly bounded for $0 \leqq k \leqq k(\hat{q})$ and $|\zeta| > R, R$ sufficiently large, we only need to show that $I + \Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}$ is injective on $C^{\alpha/2}(\mathbf{R}^3)$ for $0 \leqq k \leqq k(\hat{q})$. Note that $f + \Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}f = 0$ implies $\tilde{f} + \Lambda^{-\delta/2}\tilde{A}(\hat{q},0,k)\Lambda^{\delta/2}\tilde{f} = 0$, where $\tilde{f} = \Lambda^N f$. Hence, since $\hat{q} \in \mathcal{O}$, to complete the proof of (3.6) we only need the following.

Lemma 3.3. *Assume* $f + \Lambda^{-\delta/2}A(\hat{q},k)\Lambda^{\delta/2}f = 0$, $f \in C^{\alpha/2}(\mathbf{R}^3)$ *and* $\hat{q} \in H_{\alpha,N}$. *Then* $\Lambda^{N+\delta/2}f$ *is in* $C^{\alpha/2}(\mathbf{R}^3)$. *Here* $0 \leqq \delta < \min\{1, N-1\}$ *as before.*

Proof of Lemma 3.3. We only need consider $f(\xi)$ when $|\xi| > k+1$. Then we have

$$\begin{aligned}
|f(\xi)| &= |\Lambda^{-\delta/2}(\xi)[A(\hat{q},k)\Lambda^{\delta/2}f](\xi)| \\
&= \left| \int_{|\eta|<k+1} \frac{\Lambda^{-\delta/2}(\xi)\hat{q}(\xi-\eta)\Lambda^{\delta/2}(\eta)f(\eta)}{|\eta|^2 - (k+i0)^2} d\eta \right| + C\|\hat{q}\|_{0,N} \\
&\quad \int_{k+1<|\eta|<\infty} (1 + |\eta-\xi|^2)^{-N/2}(1+|\eta|^2)^{-1+\delta/4}(1+|\xi|^2)^{-\delta/4}|f(\eta)|d\eta \\
&\equiv I_1 + I_2.
\end{aligned}$$

We have $|I_1| \leqq C(1+|\xi|^2)^{-N/2-\delta/4}$, since $\|\Lambda^N\hat{q}\|_\alpha < \infty$, and the proof will proceed by repeated application of the inequality

$$|f(\xi)| \leqq I_2 + C(1+|\xi|^2)^{-N/2-\delta/4}. \tag{3.13}$$

Assume that we have shown $|f(\xi)| \leqq C(1+|\xi|^2)^{-r/2}$ for some $r \geqq 0$. Then

$$I_2 \leqq C \int_{k+1<|\eta|<\infty} (1+|\eta-\xi|^2)^{-N/2}(1+|\eta|^2)^{\delta/4-r/2-1}(1+|\xi|^2)^{-\delta/4}\,d\eta.$$

We divide the region of integration into

$$k+1<|\eta|<\tfrac{1}{2}|\xi| \quad \text{and} \quad \tfrac{1}{2}|\xi|<|\eta|<\infty,$$

getting J_1, and J_2. We have

$$J_1 \leqq \begin{cases} C(1+|\xi|^2)^{-N/2}(1+|\xi|^2)^{(1-r)/2}\ln|\xi|, & r \leqq 1+\delta/2, \\ C(1+|\xi|^2)^{-N/2-\delta/4}, & r>1+\delta/2. \end{cases}$$

Since

$$\int_{S^2} (1+||\eta|\omega-\xi|^2)^{-N/2}\,d\omega \leqq \begin{cases} C|\eta|^{-2}, & N>2, \\ C|\eta|^{-2}\ln|\eta|, & N=2, \\ C|\eta|^{-N}, & 1<N<2, \end{cases}$$

(see proof Lemma 2.2), and $0<\delta<\min\{1, N-1\}$, we have

$$J_2 \leqq (1+|\xi|^2)^{1/2-(r+\beta)/2},$$

where β is defined as in the proof of Lemma 2.3. Thus repeated use (3.13) gives

$$|f(\xi) \leqq C(1+|\xi|^2)^{-N/2-\delta/4}.$$

To show that

$$|\Delta(\mu)f(\xi)| \leqq C|\mu|^{\alpha}(1+|\xi|^2)^{-N/2-\delta/4},$$

one merely notes that

$$|\Delta(\mu)\hat{q}(\xi-\eta)| \leqq C|\mu|^{\alpha}\|\hat{q}\|_{\alpha,N}(1+|\xi-\eta|^2)^{-N/2}$$

and uses the preceding estimates with $r=N+\delta/2$. ■

Continuation of the Proof of Theorem 3.2. Since from (3.3) one has

$$\Lambda^{-\delta/2}\tilde{h} = -(I+\Lambda^{-\delta/2}\tilde{A}\Lambda^{\delta/2})^{-1}\Lambda^{-\delta/2}\tilde{q}_\zeta,$$

(3.6) implies

$$\|\Lambda^{-\delta/2}(\cdot)\tilde{h}(\cdot,\zeta,k)\|_{\alpha/2} \leqq C\|\Lambda^{-\delta/2}\tilde{q}_\zeta\|_{\alpha/2} \leqq C\|\tilde{q}_\zeta\|_\alpha,$$

where C is independent of $(\zeta,k)\in\mathbf{R}^3\times\bar{\mathbf{R}}_+$. Now, writing $\tilde{h} = -\tilde{q}_\zeta - \tilde{A}\Lambda^{\delta/2}\Lambda^{-\delta/2}\tilde{h}$, and using Theorem 2.1, we have

$$\sup_{\zeta,k}\|\tilde{h}(\cdot,\zeta,k)\|_\alpha \leqq C\|\tilde{q}_\zeta\|_\alpha = C\|\hat{q}\|_{\alpha,N}. \tag{3.14}$$

Since $\tilde{h}=\Lambda_\zeta^N h$, $\sup_{\zeta,k}\|\tilde{h}(\cdot,\zeta,k)\|_\alpha$ is the first of the three norms in (1.1) whose sum is $\|\Lambda_\zeta^N h\|_\alpha$. Note that, if we replace $\tilde{q}_\zeta$ by an arbitrary element of $C^\alpha(\mathbf{R}^3)$, (3.14) shows

$$\sup_{(\zeta,k)}\|I+\tilde{A}(\hat{q},\zeta,k))^{-1}\|_\alpha<\infty, \quad \text{for} \quad \hat{q}\in\mathcal{O}. \tag{3.14'}$$

Since Theorem 2.1 fails for $\alpha=0$, we cannot obtain estimates on $\|\tilde{h}(\xi,\cdot,k)\|_\alpha$

and $\|\tilde{h}(\xi,\zeta,\cdot)\|_\alpha$ by applying difference operators to (3.3). Instead we use the following procedure. Since $\hat{q}$ can be approximated in $\|\ \|_{\alpha,N}$ by $\hat{q}_\infty \in C_0^\infty(\mathbf{R}^3)$, we have

$$\tilde{h} + \tilde{A}(\hat{q}_1,\zeta,k)\tilde{h} + \tilde{A}(\hat{q}_\infty,\zeta,k)\tilde{h} = -\tilde{q}_\zeta \tag{3.15}$$

with $\|\hat{q}_1\|_{\alpha,N} < \varepsilon_0, \varepsilon_0$ to be chosen small enough that the Neumann series for $(I + \tilde{A}(\hat{q}_1,\zeta,k))^{-1}$ converges. Then we set

$$\tilde{q}_\zeta + \tilde{h} + \tilde{A}(\hat{q}_1,\zeta,k)\tilde{h} = \tilde{g},$$

so that (3.15) becomes

$$\tilde{g} + \tilde{A}(\hat{q}_\infty,\zeta,k)(I + \tilde{A}(\hat{q}_1,\zeta,k))^{-1}(\tilde{g} - \tilde{q}_\zeta) = 0. \tag{3.16}$$

The extra regularity of $\hat{q}_\infty$ and the explicit representation of $(I + \tilde{A}(\hat{q}_1,\zeta,k))^{-1}$ via Neumann series will permit us to get regularity results for $\tilde{g}$ by applying difference operators to (3.16), and then pass to $\tilde{h}$ via

$$\tilde{h} = (I + \tilde{A}(\hat{q}_1,\zeta,k))^{-1}(\tilde{g} - \tilde{q}_\zeta). \tag{3.17}$$

The Neumann series expansion of $(I + A(\hat{q},k))^{-1}f$ is given by

$$(I + A(\hat{q},k))^{-1}f = f + \sum_{n=1}^{\infty} (-1)^n A^n(q,k)f,$$

where

$$[A^n f](\xi,k) = \int_{R^{3n}} \frac{\hat{q}(\xi-\eta_1)\hat{q}(\eta_1-\eta_2)\cdots\hat{q}(\eta_{n-1}-\eta_n)f(\eta_n)}{\prod_{j=1}^{n}(|\eta_j|^2-(k+i0)^2)}\,d\eta_1\cdots d\eta_n.$$

Expanding $\Delta(s)A^n f = [A^n f](\xi,k+s) - [A^n f](\xi,k)$ by Leibnitz' formula, we have

$$\Delta(s)A^n f = \sum_{p=1}^{n}\int_{\mathbf{R}^3} \Delta(s)\left(\frac{1}{|\eta_p|^2-(k+i0)^2}\right) Q_p(\xi,\eta_p)R_p(\eta_p)\,d\eta_p, \tag{3.18}$$

where for $p > 1$,

$$Q_p(\xi,\eta_p) = \int_{R^{3(p-1)}} \frac{\hat{q}(\xi-\eta_1)\cdots\hat{q}(\eta_{p-1}-\eta_p)}{\prod_{j=1}^{p-1}(|\eta_j|^2-(k+s+i0)^2)}\,d\eta_1,\ldots,d\eta_{p-1},$$

and for $p < n$

$$R_p(\eta_p) = \int_{R^{3(n-p)}} \frac{\hat{q}(\eta_p-\eta_{p+1})\cdots\hat{q}(\eta_{n-1}-\eta_n)f(\eta_n)}{\prod_{j=p+1}^{n}(|\eta_j|^2-(k+i0)^2)}\,d\eta_{p+1}\cdots d\eta_n$$

with $Q_1 = \hat{q}(\xi-\eta_1)$ and $R_n = f(\eta_n)$. Applying Theorem 2.2 with $\delta = 0$ and then Theorem 2.1 with $\delta = 0$, we have

$$\sup_{0<s<1} |\Lambda_\zeta^N(\xi)s^{-\alpha}\Delta(s)A^n f| \le C\sum_{p=1}^{n} \|\Lambda_\zeta^N Q_p\|_\alpha \|\Lambda_\zeta^N R_p\|_\alpha \le C(n\tilde{C}^n)\|\Lambda^N\hat{q}\|_\alpha^n \|\Lambda_\zeta^N f\|_\alpha, \tag{3.19}$$

where $\tilde{C}$ is the constant from Theorem 2.1. Combining (3.19) with the direct estimate from Theorem 2.1,

$$\| \Lambda_\zeta^N(\cdot)[A^n f](\cdot) \|_\alpha \leqq \tilde{C}^n \| \Lambda^N \hat{q} \|_\alpha^n \| \Lambda_\zeta^N f \|_\alpha,$$

we conclude that, given $g \in C^\alpha(\mathbf{R}^3)$, the Neumann series expansion of

$$\Lambda_\zeta^N (I + A(\hat{q}, k))^{-1} \Lambda_\zeta^{-N} g = (I + \tilde{A}(\hat{q}, \zeta, k))^{-1} g$$

converges in $C^\alpha(\mathbf{R}^3)$ to a function which is C^α in k when $\| \hat{q} \|_{\alpha,N} < \varepsilon_0$.

Now, given $\hat{q} \in H_{\alpha,N}$, we choose $\hat{q}_\infty \in C_0^\infty(\mathbf{R}^3)$ so that $\hat{q}_1 = \hat{q} - \hat{q}_\infty$ satisfies $\| \hat{q}_1 \|_{\alpha,N} < \varepsilon_0$. Thus we have Eq. (3.16) for $\tilde{g} = \tilde{h} + \tilde{A}(\hat{q}_1, \zeta, k)\tilde{h} + \tilde{q}_\zeta$. Our next objective is to show that

$$\sup_{\zeta,k} \| \tilde{g}(\cdot, \zeta + v, k + s) - \tilde{g}(\cdot, \zeta, k) \|_\alpha \leqq C(|v|^\alpha + s^\alpha)$$

for $|v| < 1, 0 < s < 1$.

Since

$$(I + \tilde{A}(\hat{q}_\infty, \zeta, k)(I + \tilde{A}(\hat{q}_1, \zeta, k))^{-1})^{-1} = (I + \tilde{A}(\hat{q}_1, \zeta, k))(I + \tilde{A}(\hat{q}, \zeta, k))^{-1},$$

it follows from the uniform boundedness of $(I + \tilde{A}(\hat{q}, \zeta, k))^{-1}$ and Theorem 2.1 applied to $\tilde{A}(\hat{q}_1, \zeta, k)$ that

$$\sup_{\zeta,k} \| (I + \tilde{A}(\hat{q}_\infty, \zeta, k)(I + \tilde{A}(\hat{q}_1, \zeta, k))^{-1})^{-1} \|_\alpha < \infty.$$

Thus (3.16) shows that $\sup_{\zeta,k} \| \tilde{g}(\cdot, \zeta, k) \|_\alpha < \infty$. Applying the difference operator in $k, \Delta(s)$, to (3.16) we have

$$\begin{aligned} &\Delta(s)\tilde{g} + \tilde{A}(\hat{q}_\infty, \zeta, k)(I + \tilde{A}(\hat{q}_1, \zeta, k))^{-1} \Delta(s)\tilde{g} \\ &\quad = -[\Delta(s)(\tilde{A}(\hat{q}_\infty, \zeta, k)(I + \tilde{A}(\hat{q}_1, \zeta, k))^{-1}](\tilde{g}(\cdot, \zeta, k + s) - \tilde{q}_\zeta) \equiv r(\xi, \zeta, k, s). \end{aligned} \tag{3.20}$$

Viewing (3.20) as a linear equation for $\Delta(s)\tilde{g}$, we need to show that the inhomogeneous term in this equation, $r(\xi)$, is bounded in $C^\alpha(\mathbf{R}^3)$ by a multiple of s^α uniformly in (ζ, k). To do this we will substitute the Neumann series for $(I + \tilde{A}(\hat{q}_1, \zeta, k))^{-1}$ into (3.20) and consider $\partial_\xi^\beta r$. The terms in the resulting expansion for $\partial_\xi^\beta r$ are precisely those in (3.18) with $\hat{q}(\xi - \eta_1)$ in each Q_p replaced by $\partial_\xi^\beta(\Lambda_\zeta^N(\xi)\hat{q}_\infty(\xi - \eta_1))$, all other $\hat{q}$'s in Q_p and R_p replaced by $\hat{q}_1$'s and $f(\eta_n) = \Lambda_\zeta^{-N}(\eta)$ $(\tilde{g}(\eta_n, \zeta, k + \zeta) - q_\zeta(\eta_n))$. Thus (3.19) implies

$$\sup_{(\xi,\zeta,k)} |\partial_\xi^\beta r(\xi, \zeta, k, s)| \leqq C s^\alpha. \tag{3.21}$$

Hence, using $\sup_\xi \left(|r(\xi)| + \sum_{|\beta|=1} |\partial_\xi^\beta r(\xi)| \right)$ to bound $\| r(\cdot, \zeta, k, s) \|_\alpha$, we conclude

$$\sup_{\zeta,k} \| \tilde{g}(\cdot, \zeta, k + s) - \tilde{g}(\cdot, \zeta, k) \|_\alpha \leqq C s^\alpha$$

for $0 \leqq s \leqq 1$.

To get the analogous result in ζ we let $\Delta(v)f = f(\zeta + v) - f(\zeta)$ for functions depending on ζ. The analogue of (3.20) is

$$\begin{aligned}&\Delta(v)\tilde{g}+\tilde{A}(\hat{q}_\infty,\zeta,k)(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1}\Delta(v)\tilde{g}\\&\quad=-[\Delta(v)(\tilde{A}(\hat{q}_\infty,\zeta,k)(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1})](\tilde{g}(\cdot,\zeta+v,k)-\tilde{q}_{\zeta+v})\\&\qquad-\tilde{A}(\hat{q}_\infty,\zeta,k)(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1}(\Delta(v)\tilde{q}_\zeta)\equiv r_1+r_2.\end{aligned}\tag{3.22}$$

Since

$$\tilde{A}(\hat{q}_\infty,\zeta,k)(I+\tilde{A}(\hat{q}_1,\zeta,k)^{-1}=\Lambda_\zeta^N A(\hat{q}_\infty,k)(I+A(\hat{q}_1,k))^{-1}\Lambda_\zeta^{-N},$$

it follows directly from (2.5) and our bound on $\|\tilde{g}(\cdot,\zeta,k)\|_\alpha$ that

$$\sup_{\zeta,k}\|r_1(\cdot,\zeta,k,v)\|_\alpha\leqq C|v|.$$

To estimate r_2 we again substitute the Neumann series for $(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1}$ and consider $\partial_\xi^\beta r_2$. The n^{th} term in the resulting series for $\partial_\xi^\beta r_2$ is

$$I_n(\xi,\zeta,k,v)=\int_{\mathbf{R}^{3n}}\Lambda_\zeta^{-N}(\xi)$$

$$\cdot\frac{\partial_\xi^\beta(\Lambda_\zeta^N(\xi)\hat{q}_\infty(\xi-\eta_1))\hat{q}_1(\eta_1-\eta_2)\cdots\hat{q}_1(\eta_{n-1}-\eta_n)(\Lambda_\zeta^{-N}(\eta_n)(\Lambda_\zeta^N(\xi)\Delta(v)\hat{q}_\zeta(\eta_n))}{(2\pi)^{3n}\prod_{j=1}^{n}(|\eta_j|^2-(k+i0)^2)}$$

$$\cdot d\eta_1\cdots d\eta_n.$$

We have

$$\begin{aligned}\Lambda_\zeta^{-N}(\eta_n)(\Delta(v)\tilde{q}_\zeta(\eta_n))&=\hat{q}(\eta_n-\zeta-v)-\hat{q}(\eta-\zeta)\\&\quad+\Lambda_\zeta^{-N}(\eta_n)(\Lambda_{\zeta+v}^N(\eta_n)-\Lambda_\zeta^N(\eta_n))\hat{q}(\eta_n-\zeta-v).\end{aligned}$$

If we think of ζ as the variable for which we expect functions to be C^α and ξ as a parameter (note that $\Lambda_\zeta(\xi)=\Lambda_\xi(\zeta)$), Theorem 2.1 and (2.5) imply

$$|I_n(\xi,\zeta,k,v)|\leqq C|v|^\alpha\tilde{C}^n\|\Lambda^N\hat{q}\|_\alpha\|\Lambda^N\hat{q}_1\|_\alpha^{n-1}(\|\Lambda_\xi^N(\cdot)\Lambda_\zeta^{-N}(\xi)\partial_\xi^\beta(\Lambda_\zeta^N(\xi)\hat{q}_\infty(\xi-\cdot))\|)_\alpha.$$

Thus

$$\sup_{(\xi,\zeta,k)}|\partial_\xi^\beta r_2(\xi,\zeta,k,v)|\leqq C|v|^\alpha\tag{3.23}$$

and, using this to bound $\|r_2(\cdot,\zeta,k,v)\|_\alpha$ as before, we conclude

$$\sup_{\zeta,k}\|\tilde{g}(\cdot,\zeta+v,k)-\tilde{g}(\cdot,\zeta,k)\|_\alpha\leqq C|v|^\alpha$$

for $|v|\leqq 1$.

Now we are ready to go back to $\tilde{h}$ via the relation

$$\tilde{h}=(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1}\tilde{g}-(I+\tilde{A}(\hat{q}_1,\zeta,k))^{-1}\tilde{q}_\zeta\equiv\tilde{h}_1-\tilde{h}_2.$$

That the C^α-norm in (ξ,ζ,k), $\|\tilde{h}_2\|_\alpha$, is finite follows by substitution of the Neumann series exactly as in the derivation of (3.21) and (3.23). That $\|\tilde{h}_1\|_\alpha<\infty$ also follows by substitution of the Neumann series but first we separate terms:

$$\Delta(\rho)(I+\tilde{A})^{-1}\tilde{g}=(\Delta(\rho)(I+\tilde{A})^{-1})\tilde{\tilde{g}}+(I+\tilde{A})^{-1}\Delta(\rho)\tilde{g},\tag{3.24}$$

where $\rho=s$ or v and $\tilde{\tilde{g}}$ is $\tilde{g}$ with ρ added to the appropriate variable. The first term on the right of (3.24) is estimated by (3.19) with $f=\tilde{\tilde{g}}$ when $\rho=s$ and is estimated trivially when $\rho = \nu$. The second term can be estimated directly by (3.14′) since $\|\Delta(\rho)\tilde{g}(\cdot,\zeta,k)\|_\alpha \leqq C|\rho|^\alpha$. ■

From the proof of Theorem 3.2 one can see that the mapping

$$\Psi : \hat{q} \to \tilde{h}$$

is analytic from $\mathcal{O}$ to $C^\alpha(\mathbf{R}^3 \times \mathbf{R}^3 \times \bar{\mathbf{R}}_+)$. To do this, given $\hat{q}_0 \in \mathcal{O}$, we consider for $\|\hat{q}\|_{\alpha,N} \leqq 1$ and $|z| \leqq \delta$,

$$\tilde{h}(\xi,\zeta,k,z) = -(I + \tilde{A}(\hat{q}_0 + z\hat{q},\zeta,k))^{-1}(\tilde{q}_0 + z\tilde{q})_\zeta.$$

Writing

$$\begin{aligned}&(I + \tilde{A}(\hat{q}_0 + z\hat{q},\zeta,k))^{-1}(\tilde{q}_0 + z\tilde{q})_\zeta \\ &\quad = (I + z(I + \tilde{A}(\hat{q}_0,\zeta,k))^{-1}\tilde{A}(\hat{q},\zeta,k))^{-1}(I + \tilde{A}(\hat{q}_0,\zeta,k))^{-1}(\tilde{q}_0 + z\tilde{q})_\zeta,\end{aligned}$$

Theorem 2.1 and (3.14′) imply that for δ sufficiently small we can expand the first factor on the right of (3.25) in a Neumann series which converges in $C^\alpha(\mathbf{R}^3)$ for all $(\zeta,k) \in \mathbf{R}^3 \times \bar{\mathbf{R}}_+$. Thus for some δ independent of (ξ,ζ,k) we have for $\|\hat{q}\|_{\alpha,N} \leqq 1$ and $|z| < \delta$,

$$\tilde{h}(\xi,\zeta,k,z) = \frac{1}{2\pi i} \oint_{|w|=\delta} \frac{\tilde{h}(\xi,\zeta,k,w)}{w - z}\, dw,$$

and hence for all $k \geqq 0$,

$$\frac{\partial^p \tilde{h}}{\partial z^p}(\xi,\zeta,k,0) = \frac{p!}{2\pi i} \int_{|w|=\delta} \frac{\tilde{h}(\xi,\zeta,k,w)\,dw}{w^{p+1}}.$$

Since $\Psi(\hat{q}_0 + z\hat{q}) = \tilde{h}(\xi,\zeta,k,z)$, to conclude that $\Psi(\hat{q}_0 + z\hat{q})$ can be expanded for $|z| < \delta$ in a power series in z convergent in $C^\alpha(\mathbf{R}^3 \times \mathbf{R}^3 \times \bar{\mathbf{R}}_+)$ uniformly on $\|\hat{q}\|_{\alpha,N} \leqq 1$, we only need to show that

$$\|\tilde{h}(\cdot,\cdot,\cdot,w)\|_{\underline{\alpha}} \leqq C$$

for $|w| = \delta$. However, this is just the statement that the estimate in Theorem 3.2 is locally uniform in $\hat{q}$. This uniformity is clear from the proof. Thus we have shown that Ψ satisfies one of the definitions of analyticity (see Pöschel–Trubowitz [12], Appendix A, or Nachbin [11]) and have

Corollary 3.4. *The mapping $\Psi : \hat{q} \to \tilde{h}$ considered as a function from $\mathcal{O}$ to $C^\alpha(\mathbf{R}^3 \times \mathbf{R}^3 \times \bar{\mathbf{R}}_+)$ is analytic in $\hat{q}$.*

Analyticity in the sense above is equivalent to the fact that Ψ has a continuous Frechet derivative with respect to $\hat{q}$ (see references above), as one can easily verify. In what follows we will often make use of the continuous differentiability of Ψ. If we restrict to the backscattering map on $\mathcal{O}$

$$S : \hat{q} \to h(\xi, -\xi, |\xi|) = \Lambda_{-\xi}^{-N}(\xi)\tilde{h}(\xi, -\xi, |\xi|),$$

Theorem 3.2 implies $\|S(\hat{q})\|_{\alpha,N} < \infty$. Moreover, choosing $\hat{q}_n \in \mathcal{O} \cap C_0^\infty(\mathbf{R}^3)$ converging to $\hat{q}$ in $\|\ \|_{\alpha,N}$, it follows from the analyticity and hence continuity of Ψ that $\|\Psi(\hat{q}_n) - \Psi(\hat{q})\|_{\underline{\alpha}} \to 0$. Thus $\|S(\hat{q}_n) - S(\hat{q})\|_{\alpha,N} \to 0$. Since $\|S(\hat{q}_n)\|_{\alpha' N'} < \infty$ for all $N' > 1, 0 < \alpha' < 1$, it follows by Lemma 1.1 that $S(\hat{q}_n) \in H_{\alpha,N}$ and hence $S(\hat{q}) \in H_{\alpha,N}$. This gives:

Corollary 3.5. *The backscattering map $S: \hat{q} \to h(\xi, -\xi, |\xi|)$ is an analytic function from $\mathcal{O}$ to $H_{\alpha,N}$.*

Section 4. The Derivative of the Backscattering Map

Since by Corollary 3.4 $\Psi: \hat{q} \to \tilde{h}$ is a continuously Frechet differentiable function on $\mathcal{O}$, we may compute its derivative. To do this we will differentiate Eq. (3.3) with respect to $\hat{q}$. Note that Theorem 2.1 implies $\tilde{A}(\hat{q})\tilde{h}(\hat{q})$ is the composition of a bounded operator valued function linear in $\hat{q}$ with a continuously differentiable function, and is hence continuously differentiable. We have for $v \in H^{\alpha,N}, \hat{q} \in \mathcal{O}$,

$$d\tilde{h}(v) + \tilde{A}(\hat{q}) d\tilde{h}(v) = -\tilde{v}_\zeta - \tilde{A}(v)\tilde{h},$$

and hence

$$d\tilde{h}(v) = (I + \tilde{A}(\hat{q}))^{-1}(-\tilde{v}_\zeta - \tilde{A}(v)\tilde{h}). \tag{4.1}$$

Lemma 4.1. *The operator $(I + \tilde{A}(\hat{q}))^{-1}, \hat{q} \in \mathcal{O}$, has the following form:*

$$[(I + \tilde{A}(\hat{q}, \zeta, k))^{-1} f](\xi) = f(\xi) + (2\pi)^{-3} \int \frac{\Lambda_\zeta^N(\xi) h(\xi, \eta, k) \Lambda_\zeta^{-N}(\eta) f(\eta)}{|\eta|^2 - (k + i0)^2} d\eta. \tag{4.2}$$

Proof. Let $f + \tilde{D} f$ denote the right-hand side of (4.2). Then we have from (3.3)

$$(I + \tilde{A}(\hat{q}))(I + \tilde{D}) f = f + \tilde{A}(\hat{q}) f + \tilde{D} f + \tilde{A}(\hat{q}) \tilde{D} f = f + \tilde{A}(\tilde{q}) f + \tilde{D} f - \tilde{D} f - \tilde{A}(q) f = f.$$

Thus $I + \tilde{D}$ is a right inverse for $I + \tilde{A}(\hat{q})$. Since $I + \tilde{A}(\hat{q})$ is invertible, it follows that $I + \tilde{D} = (I + \tilde{A}(\hat{q}))^{-1}$. ■

Substituting (4.2) into (4.1) we have

$$\begin{aligned} d\tilde{h}(v) = &-\tilde{v}_\zeta - (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{\Lambda_\zeta^N(\xi) h(\xi, \eta, k) v_\zeta(\eta) d\eta}{|\eta|^2 - (k + i0)^2} \\ &- (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{h(\eta, \zeta, k) \Lambda_\zeta^N(\xi) v(\xi - \eta) d\eta}{|\eta|^2 - (k + i0)^2} \\ &- (2\pi)^{-6} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{\Lambda_\zeta^N(\xi) h(\xi, t, k) v(t - \eta) h(\eta, \zeta, k) d\eta\, dt}{(|t|^2 - (k + i0)^2)(|\eta|^2 - (k + i0)^2)}. \end{aligned} \tag{4.3}$$

Changing variables so as to get integrals of $v(\eta)$ in all integrals in (4.3) and cancelling $\Lambda_\zeta^N(\xi)$ in all terms, one arrives at

$$\begin{aligned} dh(\xi, \zeta, k)(v) = &-v(\xi - \zeta) - (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{h(\xi, \eta + \zeta, k) v(\eta)}{|\eta + \zeta|^2 - (k + i0)^2} d\eta \\ &- (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{h(\xi - \eta, \zeta, k) v(\eta)}{|\eta - \xi|^2 - (k + i0)^2} d\eta \\ &- (2\pi)^{-6} \int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} \frac{h(\xi, t, k) h(t - \eta, \zeta, k) v(\eta) d\eta}{(|t|^2 - (k + i0)^2)(|t - \eta|^2 - (k + i0)^2)} \right) dt. \end{aligned} \tag{4.4}$$

Since $I+\tilde{D}$ is the left inverse of $I+\tilde{A}(\hat{q})$, we have

$$0=\hat{q}(\xi-\zeta)+h(\xi,\zeta,k)+(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{h(\xi,\eta,k)\hat{q}(\eta-\zeta)}{|\eta|^2-(k+i0)^2}d\eta.$$

Sending $\xi\to-\zeta,\zeta\to-\xi$ and $\eta\to-\eta$, we have

$$0=\hat{q}(\xi-\zeta)+h(-\zeta,-\xi,k)+(2\pi)^{-3}\int_{\mathbf{R}^3}\frac{\hat{q}(\xi-\eta)h(-\zeta,-\eta,k)}{|\eta|^2-(k+i0)^2}d\eta. \tag{4.5}$$

Comparing (4.5) and (3.1) one sees that for $\hat{q}\in\mathcal{O}$,

$$h(-\zeta,-\xi,k)=h(\xi,\zeta,k) \tag{4.6}$$

for $(\xi,\zeta,k)\in\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+$. Hence, setting $(\xi,\zeta,k)=(\xi,-\xi,|\xi|)$ in (4.4), sending $\eta\to2\eta$ and using (4.6), we have

$$\begin{aligned}dh(\xi,-\xi,|\xi|)(v)=&-v(2\xi)-2\pi^{-3}\int_{\mathbf{R}^3}\frac{h(\xi-2\eta,-\xi,|\xi|)v(2\eta)}{|\xi-2\eta|^2-(|\xi|+i0)^2}d\eta\\&-2^{-3}\pi^{-6}\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\frac{h(\xi,t,|\xi|)h(t-2\eta,-\xi,|\xi|)v(2\eta)d\eta}{(|t|^2-(|\xi|+i0)^2)(|2\eta-t|^2-(|\xi|+i0)^2)}\right)dt.\end{aligned} \tag{4.7}$$

From (4.7) one sees that the Frechet derivative of the backscattering map S is given by

$$dS=(I+B+F)T,$$

$$[Tf](\xi)=-f(2\xi),[Bf](\xi)=2\pi^{-3}\int_{\mathbf{R}^3}\frac{h(\xi-2\eta,-\xi,|\xi|)f(\eta)}{|\xi-2\eta|^2-(|\xi|+i0)^2}d\eta$$

and

$$[Ff](\xi)=2^{-3}\pi^{-6}\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\frac{h(\xi,t,|\xi|)h(t-2\eta,-\xi,|\xi|)f(\eta)d\eta}{(|t|^2-(|\xi|+i0)^2)(|2\eta-t|^2-(|\xi|+i0)^2)}\right)dt.$$

Since S is an analytic function, dS is continuous on $\mathcal{O}$ as a function with values in $\mathcal{L}(H_{\alpha,N})$, the space of bounded linear operators from $H_{\alpha,N}$ to itself. Since we need to know that B and F are individually continuous functions from $\mathcal{O}$ to $\mathcal{L}(H_{\alpha,N})$, we prove the following.

Lemma 4.2. *$B(\hat{q})$ is an analytic function from $\mathcal{O}$ to $\mathcal{L}(H_{\alpha,N})$.*

Proof. As in the proof of Corollary 3.4, the analyticity will follow from the local boundedness of the operator norm $\||B(\hat{q})\||_{\alpha,N}$ on $\mathcal{O}$.

From (3.17) we have the representation

$$h(\xi-2\eta,-\xi,|\xi|)=[(I+A(\hat{q}_1,|\xi|))^{-1}(\Lambda_{-\xi}^{-N}(\cdot)g(\cdot,-\xi,|\xi|)-\hat{q}(\cdot+\xi))](\xi-2\eta).$$

As in the proof of Theorem 3.2 we will substitute the Neumann series for $(I+A(\hat{q}_1,|\xi|))^{-1}$ in h and hence in $B(\hat{q})$. This gives

$$[B(\hat{q})f](\xi)=2\pi^{-3}\sum_{n=0}^{\infty}\left(\frac{-1}{(2\pi)^3}\right)^n I_n(\xi),$$

where

$$I_n(\xi) = \int_{\mathbf{R}^{3(n+1)}} \frac{f(\eta)\hat{q}_1(\xi-2\eta-\eta_1)\hat{q}_1(\eta_1-\eta_2)\cdots\hat{q}_1(\eta_{n-1}-\eta_n)(\Lambda_{-\xi}^{-N}(\eta_n)g(\eta_n,-\xi,|\xi|)-\hat{q}(\eta_n+\xi))}{(|\xi-2\eta|^2-(|\xi|+i0)^2)\prod_{i=1}^{n}(|\eta_i|^2-(|\xi|+i0)^2)} \cdot d\eta\, d\eta_1\cdots d\eta_n.$$

Setting $\eta_0 = \xi - 2\eta$, we have

$$I_n(\xi) = \frac{1}{8}\int_{\mathbf{R}^{3(n+1)}} \frac{f\left(\dfrac{\xi-\eta_0}{2}\right)\hat{q}_1(\eta_0-\eta_1)\hat{q}_1(\eta_1-\eta_2)\cdots\hat{q}_1(\eta_{n-1}-\eta_n)(\Lambda_{-\xi}^{-N}(\eta_n)g(\eta_n,-\xi,|\xi|)-\hat{q}(\eta_n+\xi))}{\prod_{i=0}^{n}(|\eta_i|^2-(|\xi|+i0)^2)} \cdot d\eta_0\cdots d\eta_n.$$

These are precisely the terms which arose at the end of the proof of Theorem 3.2 with one factor of $\hat{q}$, replaced by $f, \zeta = -\xi$ and $k = |\xi|$. Hence, the argument given there shows that

$$\|\Lambda^N B(\hat{q})f\|_\alpha \leqq C\|f\|_{\alpha,N} \tag{4.8}$$

for $f\in H_{\alpha,N}$, where C is locally uniform in $\hat{q}$ on $\mathcal{O}$.

To prove analyticity we proceed as follows. Defining $h(\eta,\zeta,k,z)$ as in the proof of Corollary 3.4, we have

$$[B(\hat{q}_0+z\hat{q})f](\xi) = 2\pi^{-3}\int_{\mathbf{R}^3} \frac{f\left(\dfrac{\xi-\eta}{2}\right)h(\eta,-\xi,|\xi|,z)}{|\eta|^2-(|\xi|^2+i0)^2}\, d\eta,$$

and we know that $\Lambda_{-\xi}^N(\eta)h(\eta,-\xi,|\xi|,z)$ is an analytic function from $|z|\leqq\delta$ to $C^\alpha(\mathbf{R}^3)$ for each $\xi\in\mathbf{R}^3$ and $\hat{q}$ with $\|\hat{q}\|_{\alpha,N}\leqq 1$. Thus Theorem 2.1 implies that for each $\xi\in\mathbf{R}^3$ and $f\in H_{\alpha,N}$ we can represent $[B(\hat{q}_0+z\hat{q})f](\xi)$ as a Cauchy integral over $|z|=\delta/2$ with δ independent of ξ and $\hat{q}$, when $\|\hat{q}\|_{\alpha,N}\leqq 1$. Now analyticity follows from (4.8) just as in the proof of Corollary 3.4.

To see that the range of $B(\hat{q})$ on $H_{\alpha,N}$ is contained in $H_{\alpha,N}$, we approximate $\hat{q}$ by $\hat{q}_n\in C_0^\infty(\mathbf{R}^3)$ as in the proof of Corollary 3.5. ■

The main result of this section is that, for $\hat{q}\in\mathcal{O}$, the operator $dS(\hat{q})$ is Fredholm of index zero on $H_{\alpha,N}$. To prove this we will show that B^2 and F are compact on $H_{\alpha,N}$. To see that this is sufficient, note that for $0\leqq\varepsilon\leqq 1$,

$$T^{-1}(1-\varepsilon B)(I+\varepsilon B+F)T = I+K_1,$$
$$(I+\varepsilon B+F)T^{-1}(I-\varepsilon B)) = I+K_2,$$

where K_1 and K_2 are compact if B^2 and F are. Hence, $(I+\varepsilon B+F)T$ is Fredholm for $0 \leqq \varepsilon \leqq 1$, and for $\varepsilon = 0$ it is a compact perturbation of an invertible operator, and hence of index zero. Thus, to conclude that $dS(\hat{q})$ is a Fredholm operator of index zero on $H_{\alpha,N}$ for $\hat{q} \in \mathcal{O}$, we only need the following:

Theorem 4.3. *The operators $B^2(\hat{q})$ and $F(\hat{q})$ are compact on $H_{\alpha,N}$ for $\hat{q} \in \mathcal{O}$.*

To prove Theorem 4.3 we will first take advantage of the fact that operator norm limits of compact operators are compact to replace B^2 and F by the operators:

$$[T_1 f](\xi) = \int_{\mathbf{R}^6} \frac{g_1(\xi,\eta,t) f(t)}{(|\xi - 2\eta|^2 - (|\xi| + i0)^2)(|\eta - 2t|^2 - (|\eta| + i0)^2)} d\eta\, dt \tag{4.9}$$

and

$$[T_2 f](\xi) = \int_{\mathbf{R}^6} \frac{g_2(\xi,\eta,t) f(t)}{|\eta|^2 - (|\xi| + i0)^2)(|\eta - 2t|^2 - (|\xi| + i0)^2)} d\eta\, dt,$$

respectively, where $g_i, i = 1, 2$, satisfies

(i) $g_i \in C^\infty(\mathbf{R}^9)$ and all of its partial derivatives are bounded,
(ii) $g_i(\xi,\eta,t) = 0$ for $|\xi| + |\eta| + |t| < \delta$ for some $\delta > 0$, and
(iii) $g_i(\xi,\eta,t) = 0$, if $|\xi - \eta| > M$ or $|\eta - t| > M$ for some $M < \infty$.

Then the proof proceeds by analysis of the singularities of the kernels $t_1(\xi, t)$ and $t_2(\xi, t)$ of T_1 and T_2. For this we will use estimates modelled on the following simple lemma.

Lemma 4.4. *Assume that $g(\xi,\eta)$ is supported in $|\eta - \eta(\xi)| < M$ and that $\|g\|_\alpha < \infty$ for some $\alpha \in (0,1)$. Assume that $h(\xi)$ satisfies $|h(\xi + \mu) - h(\xi)| \leqq C|\mu|$ for $|\mu| \leqq 1$, $\xi \in \mathbf{R}^n$. Let*

$$s(\xi) = \int_{\mathbf{R}^n} \frac{g(\xi,\eta)}{\eta_1 - h(\xi) - i0} d\eta. \tag{4.10}$$

Then

$$\|s\|_{\alpha'} < C(M, \|g\|_\alpha, \|h\|_1, \alpha') \tag{4.11}$$

for any $\alpha' < \alpha$.

Proof. Changing variables we have

$$s(\xi) = \int_{\mathbf{R}^n} \frac{g(\xi, \eta + h(\xi)\hat{e}_1)}{\eta_1 - i0} d\eta.$$

Letting $f(\xi,\eta) = g(\xi, \eta + h(\xi)\hat{e}_1)$, we see f satisfies the same hypotheses g did. Expanding (4.10) we have, letting $\eta = (\eta_1, \eta')$,

$$s(\xi) = \int_{|\eta_1| > 1} \frac{f(\xi,\eta)}{\eta_1} d\eta + \int_{|\eta_1| < 1} \frac{f(\xi,\eta) - f(\xi,0,\eta')}{\eta_1} d\eta + \int_{|\eta_1| < 1} \frac{f(\xi,0,\eta')}{\eta_1 - i0} d\eta = I_1 + I_2 + I_3.$$

Carrying out the integration in η_1 in I_3,

$$I_3 = \pi i \int_{\mathbf{R}^{n-1}} f(\xi, 0, \eta') d\eta'.$$

Since $\|f(\cdot,\eta) - f(\cdot,0,\eta')\|_{\alpha'} \leqq 3|\eta_1|^{\alpha - \alpha'} \|f\|_\alpha$, the α'-norm of I_2 is easily estimated, and (4.11) follows directly from the representation of $s(\xi)$ as $I_1 + I_2 + I_3$. ■

The problem of obtaining (4.11) for singular integrals with more general denominators can be reduced to Lemma 4.4 by change of variables as long as the gradient in η of the denominator is bounded away from zero near the surface where the denominator vanishes. In what follows we will leave such reductions to the reader and simply refer to Lemma 4.4.

Proof of Theorem 4.3. Lemma 4.2 shows that $B(\hat{q})$ is analytic in $\hat{q}$, and hence, since $dS(\hat{q})$ is analytic, $F(\hat{q})$ is also an analytic function of $\hat{q}$. Thus, making a change of arbitrarily small norm in $B^2(\hat{q})$ and $F(\hat{q})$, we may assume $\hat{q}\in C_0^\infty(\mathbf{R}^3)\cap\mathcal{O}$, and hence by Theorem 3.2

$$\|\Lambda_\zeta^{N'}h\|_{\alpha'}<\infty \tag{4.12}$$

for all $N'>1$ and $\alpha'<1$.

The operators $B^2(\hat{q})$ and $F(\hat{q})$ are given by

$$[B^2f](\xi)=4\pi^{-6}\int_{\mathbf{R}^6}\frac{h(\xi-2\eta,-\xi,|\xi|)h(\eta-2t,-\eta,|\eta|)f(t)dt\,d\eta}{(|\xi-2\eta|^2-(|\xi|+i0)^2)(|\eta-2t|^2-(|\eta|+i0)^2)}$$

and

$$[Ff](\xi)=2^{-3}\pi^{-6}\int_{\mathbf{R}^6}\frac{h(-\eta,-\xi,|\xi|)h(\eta-2t,-\xi,|\xi|)f(t)}{(|\eta|^2-(|\xi|+i0)^2)(|\eta-2t|^2-(|\xi|+i0)^2)}dt\,d\eta.$$

By the argument used in the proof of Lemma 1.1, (4.12) implies that, given $\alpha_1,\alpha<\alpha_1<1$, we can choose $h_n(\xi,\zeta)\in C^\infty(\mathbf{R}^6)$ such that $h_n(\xi,\zeta)=0$ for $|\xi+\zeta|$ sufficiently large, $\partial^\beta_{\xi,\zeta}h_n$ is bounded for all β and

$$\Lambda^N(\xi+\zeta)(h_n(\xi,\zeta)-h(\xi,-\zeta,|\zeta|))$$

tends to zero in $C^{\alpha_1}(\mathbf{R}^6)$. Replacing the h's in B^2 and F by h_n's with the appropriate arguments, we get B_n^2 and F_n. We claim that $\|\,|B_n^2-B^2\|\,|_{\alpha,N}$ and $\|\,|F_n-F\|\,|_{\alpha,N}$ go to zero as $n\to\infty$. Expanding $B_n^2-B^2=B_n(B_n-B)+(B_n-B)B$ and making the analogous expansion of F_n-F, one sees by the estimates on $|I_1|,|I_2|$ and $|I_3|$ in the proof Theorem 2.1, that $\|\,|B_n^2-B^2\|\,|_{0,N}$ and $\|\,|F_n-F\|\,|_{0,N}$ go to zero. To estimate $\Delta(\mu)(\Lambda^N(B^2-B_n^2))$ and $\Delta(\mu)(\Lambda^N(F_n-F))$ we first change variables in η and t so that when ξ appears in the denominator of an integrand it is in a factor of the form $(|\eta|^2-(|\xi|+i0)^2)$ or $(|t|^2-(|\xi|+i0)^2)$. Then $[\Delta(\mu)(\Lambda^N(B^2-B_n^2)f](\xi)$ and $[\Delta(\mu)(\Lambda^N(F-F_n)f](\xi)$ can be expanded into sums of terms where the difference operator acts on $(|\beta|^2-(|\xi|+i0)^2)^{-1}$, $\beta=\eta$ or t, which we estimate by Theorem 2.2; terms where the operator acts on $f(l(\xi,\eta,t))$, l a linear function, which we estimate by Theorem 2.1 with f playing the role of $\hat{q}$, and terms where the operator acts on Λ^N,h,h_n or $h-h_n$, which we again expand as $I_1+I_2+I_3$ and then estimate $|I_1|,|I_2|$ and $|I_3|$ as the proof of Theorem 2.1. It is estimating terms of the last type that we use $\alpha_1>\alpha$ and this makes all estimates substantially easier. Thus, making a change of arbitrarily small operator norm, we can replace B^2 and F by operators T_1 and T_2 as in (4.9) with g_1 and g_2 satisfying (i) and (iii).

To see that we can make the integrands in B_n^2 and F_n vanish for $|\xi|+|\eta|+|t|<\delta_0$ so that g_1 and g_2 will satisfy (ii) in (4.9), we proceed as follows. Given any $\varphi\in C_0^\infty(|x|<1)$ and $\delta>0$, let

$$[R_n f](\delta\xi) = \int_{\mathbf{R}^6} \frac{h_n(\delta\xi - 2\eta, \delta\xi)\varphi^2\left(\frac{\eta}{\delta}\right) h_n(\eta - 2t, \eta)\varphi^2\left(\frac{t}{\delta}\right) f(t)}{(|\delta\xi - 2\eta|^2 - (|\delta\xi| + i0)^2)(|\eta - 2t|^2 - (|\eta| + i0)^2)} dt\, d\eta$$
$$= \delta^2 \int_{\mathbf{R}^6} \frac{h_n(\delta\xi - 2\delta\eta, \delta\xi)\varphi^2(\eta) h_n(\delta\eta - 2\delta t, \delta\eta)\varphi^2(t) f(\delta t)}{(|\xi - 2\eta|^2 - (|\xi| + i0)^2)(|\eta - 2t|^2 - (|\eta| + i0)^2)} dt\, d\eta.$$

Thus, by Theorems 2.1 and 2.2,

$$\| \varphi(\xi)[R_n f](\delta\xi) \|_{\alpha,N} \leqq C\delta^2 \| \varphi(t) f(\delta t) \|_{\alpha,N}.$$

Thus, since $\| g(\xi) \|_\alpha \leqq \delta^{-\alpha} \| g(\delta\xi) \|_\alpha$ for $\delta < 1$, we see that $\| |\varphi(\xi/\delta) R_n \| |_{\alpha,N} \leqq C\delta$. Thus, making an arbitrarily small norm change in B_n^2, we can assume that its integrand vanishes for $|\xi| + |\eta| + |t| < \delta_0$ for some $\delta_0 > 0$. This argument applies to F_n as well. Thus we may replace B_n^2 and F_n by the operators T_1 and T_2 in (4.9) with g_1 and g_2 satisfying (i), (ii) and (iii).

We will now study T_1. The analysis of T_2 is very similar and somewhat easier, and we will sketch it at the end of the proof.

In terms of η the integral defining T_1 is singular on the sphere (if $\xi \neq 0$)

$$\sum = \{\eta : |\xi - 2\eta| = |\xi|\}$$

and the plane (if $t \neq 0$)

$$\prod = \{\eta : |t|^2 - \eta \cdot t = 0\}.$$

We will see that the kernel $t_1(\xi, t)$ of T_1 is most singular at points (ξ, t) for which $\sum$ and $\prod$ are tangent. This happens when

$$\left\{\eta : \eta = \frac{\xi}{2} \pm \frac{|\xi|}{2} \frac{t}{|t|}\right\} \cap \{\eta : \eta \cdot t - |t|^2 = 0\} \neq \phi,$$

i.e. when $\xi \cdot t \pm |\xi||t| - 2|t|^2 = 0$. With these facts in mind we will break up the integration in η by summing over a partition of unity generated by $\rho_1 = \rho(|\xi - 2\eta| - |\xi|)$ and $\rho_2 = \rho(|t| - (\eta \cdot t)|t|^{-1})$, where $\rho \in C_0^\infty(\mathbf{R})$ satisfies $\rho(s) = 1$ for $|s| < \varepsilon_1$ and $\rho(s) = 0$ for $|s| > 2\varepsilon_1$. Since $g_1 = 0$ for $|\xi| + |\eta| + |t| < \delta$, choosing ε_1 sufficiently small, we can assume that $|\xi| > \delta/4$ on the support of $\rho_1 \rho_2 g_1$.

We will also need cutoffs in t near the most singular set,

$$\beta_1 = \beta\left(\frac{2|t| - (\xi \cdot t)|t|^{-1} - |\xi|}{|\xi|}\right) \quad \text{and} \quad \beta_2 = \beta\left(\frac{2|t| - (\xi \cdot t)|t|^{-1} + |\xi|}{|\xi|}\right),$$

where $\beta \in C_0^\infty(\mathbf{R})$ satisfies $\beta(s) = 1$ for $|s| < \varepsilon_2$ and $\beta(s) = 0$ for $|s| > 2\varepsilon_2$. The constants ε_1 and ε_2 are chosen small enough that on the support of $\rho_1 \rho_2 \beta_i g_1, i = 1, 2$, the component of $\xi - 2\eta$ orthogonal to t has length less than $1/2|\xi|$. Note that on support $\rho_1 \rho_2 \beta_i g_1, |\xi - 2\eta| < |\xi| + 2\varepsilon_1$,

$$\left| \frac{t}{|t|} \cdot (\xi - 2\eta) \right| > |\xi| - 2\varepsilon_2 |\xi| - 4\varepsilon_1, \quad \text{and} \quad |\xi| > \delta/4.$$

Now we replace g_1 in the definition of T_1 by $(1 - \rho_1) g_1$ to define S_1, by

$(1-\rho_2)\rho_1 g_1$ to define S_2, by $\rho_1\rho_2(1-\beta_1-\beta_2)g_1$ to define S_3 and by $\rho_1\rho_2\beta_i g_1$ to define S_{3+i}. Thus $T_1 f = \sum_{i=1}^{5} S_i f$.

Letting $s_1(\xi,t)$ denote the kernel of S_1, we have

$$s_1(\xi,t) = -\frac{1}{4}\int_{\mathbf{R}^3} \frac{h_1(\xi,\eta,t)}{\eta\cdot t - |t|^2 + i0}\,d\eta,$$

where $h_1 = (1-\rho_1)g_1(|\xi-2\eta|^2 - |\xi|^2)^{-1}$. Applying Lemma 4.4, we conclude $|t|\,\|s_1(\cdot,t)\|_{\alpha'}$ is bounded in t for all $\alpha'<1$. Since we also have $s_1(\xi,t)=0$ for $|t-\xi|>2M$, we conclude

$$\|\Lambda^{N+1}S_1 f\|_{\alpha'} \leqq C_N \|\Lambda^N f\|_0.$$

Thus S_1 is a compact operator on $H_{\alpha,N}$.

Letting $s_2(\xi,t)$ denote the kernel of S_2, we have

$$s_2(\xi,t) = \int_{\mathbf{R}^3} \frac{h_2(\xi,\eta,t)}{|\xi-2\eta|^2 - (|\xi|+i0)^2}\,d\eta,$$

where $h_2 = (1-\rho_2)\rho_1 g_1(-4\eta\cdot t + 4|t|^2)^{-1}$. Hence, changing variables

$$s_2(\xi,t) = \int_{\mathbf{R}^3} \frac{h_2\left(\xi,\dfrac{\eta+\xi}{2},t\right)}{|\eta|^2 - (|\xi|+i0)^2}\,d\eta.$$

Since $|t|h_2(\xi,(\eta+\xi)/2,t)$ is bounded, vanishes for $|\xi-\eta|>2M$, and has Lipschitz constant in (ξ,η) uniformly bounded in t, it follows that $|t|\,\|s_2(\cdot,t)\|_{\alpha'}$ is uniformly bounded in t for some $\alpha'>\alpha$. To verify this one can write

$$s_2(\xi,t) = \int_{\mathbf{R}^3} \frac{h_2\left(\xi,\dfrac{\eta+\xi}{2},t\right)\varphi(\eta)}{|\eta|^2 - (|\xi|+i0)^2}\,d\eta + \int_{\mathbf{R}^3} \frac{h_2\left(\xi,\dfrac{\eta+\xi}{2},t\right)(1-\varphi(\eta))}{|\eta|^2 - (|\xi|+i0)^2}\,d\eta \equiv s_{2,1} + s_{2,2},$$

where $\varphi\in C_0^\infty(\mathbf{R}^3)$ satisfying $\varphi(\eta)\equiv 1$ for $|\eta|<1$. Then $\|s_{2,1}(\cdot,t)\|_{\alpha'}$ can be estimated using Theorem 2.2 and the early steps in the proof of Theorem 2.1 and $\|s_{2,2}(\cdot,t)\|_{\alpha'}$ can be estimated directly by Lemma 4.4. Since $s_2(\xi,t)=0$ for $|\xi-t|>2M$, it follows that S_2 like S_1 is compact on $H_{\alpha,N}$.

Letting $s_3(\xi,t)$ denote the kernel of S_3, we have

$$s_3(\xi,t) = -\frac{1}{4}\int_{\mathbf{R}^3} \frac{h_3(\xi,\eta,t)}{|\xi-2\eta|^2 - (|\xi|+i0)^2)(\eta\cdot t - |t|^2 + i0)}\,d\eta,$$

where $h_3 = \rho_1\rho_2(1-\beta_1-\beta_2)g_1$. Since $|\xi|>\delta/4$ on support h_3, by taking ε_1 sufficiently small we can assume that $|\xi-2\eta|$ does not vanish on the support of h_3. Thus all partial derivatives of h_3 with respect to ξ and η are bounded on $\mathbf{R}^9$.

We want to use the coordinates $\mu_1 = |\eta-\xi/2|$ and $\mu_2 = (\eta-\xi/2)\cdot t/|t|$ on η-space to study s_3, since the singularities of the integrand are on level surfaces of μ_1 and μ_2. To see that these coordinates are independent on support h_3 and estimate derivatives with respect to μ_1 and μ_2, it is convenient to introduce cylindrical coordinates (ρ,θ,z) with origin $\xi/2$ and axis in the direction of t. Then we have $\rho=\sqrt{\mu_1^2-\mu_2^2}$ and $z=\mu_2$.

The factor $\rho_1\rho_2(1-\beta_1-\beta_2)$ in h_3 insures that

$$\mu_1-|\mu_2|=\left|\left(|\eta-\xi/2|-\frac{|\xi|}{2}\right)-\left(\left|\frac{\xi\cdot t}{2|t|}-\frac{\eta\cdot t}{|t|}\right|-\left|\frac{\xi\cdot t}{2|t|}-|t|\right|\right)\right.$$
$$\left.-\left(\left|\frac{\xi\cdot t}{2|t|}-|t|\right|-\frac{|\xi|}{2}\right)\right|>\varepsilon_2|\xi|-3\varepsilon_1$$

on support h_3. Thus, choosing ε_1 sufficiently small once ε_2 has been fixed, we have

$$\mu_1-|\mu_2|\geqq\varepsilon_3|\xi|$$

on support h_3. Setting

$$m_3(\rho,z,\xi,t)=\int_0^{2\pi}\frac{h_3\left(\xi,\frac{\xi}{2}+z\frac{t}{|t|}+\rho\cos\theta\hat{e}_1(t)+\rho\sin\theta\hat{e}_2(t),t\right)}{(z^2+\rho^2)^{1/2}+|\xi|}d\theta,$$

where $(t/|t|,\hat{e}_1(t),\hat{e}_2(t))$ is an orthonormal frame, we see that $|\xi|m_3$ is bounded together with its derivatives in ρ,z and ξ. We have

$$s_3(\xi,t)=-\frac{1}{4|t|}\int_{\mathbf{R}\times\mathbf{R}_+}\frac{m_3\rho d\rho dz}{(2\sqrt{\rho^2+z^2}-(|\xi|+i0))\left(z+\frac{\xi\cdot t}{2|t|}-|t|+i0\right)}$$

and, since $\partial(\mu_1,\mu_2)/\partial(\rho,z)=\rho/\mu_1$,

$$s_3(\xi,t)=-\frac{1}{4|t|}\int_{\mathbf{R}\times\mathbf{R}_+}\frac{\mu_1 m_3 d\mu_1 d\mu_2}{(2\mu_1-(|\xi|+i0))\left(\mu_2+\frac{\xi\cdot t}{2|t|}-|t|+i0\right)}.$$

Since m_3 is supported in

$$\{|2\mu_1-|\xi||<2\varepsilon_1\}\cap\left\{\left|\mu_2+\frac{\xi\cdot t}{2|t|}-|t|\right|<2\varepsilon_1\right\},$$

we have $\mu_1^2-\mu_2^2>\varepsilon_3|\xi|\mu_1>\varepsilon_3(2\mu_1-2\varepsilon_1)\mu_1$ on support m_3. Thus all partial derivatives of ρ with respect to μ_1 and μ_2 are bounded on support m_3, and $\mu_1 m_3$ and its derivatives in μ and ξ are bounded. Thus, applying Lemma 4.4 twice, one sees that $|t|\,\|s_3(\cdot,t)\|_{\alpha'}$ is bounded on $\mathbf{R}^3$ for $\alpha'<1$, and, since $s_3(\xi,t)=0$ for $|\xi-t|>2M$, it follows that S_3 is compact on $H_{\alpha,N}$.

The kernels s_4 and s_5 of S_4 and S_5 require a more detailed analysis. We have for $i=4,5$,

$$s_i(\xi,t)=\int_{\mathbf{R}^3}\frac{h_i(\xi,\eta,t)}{(|\xi-2\eta|^2-(|\xi|+i0)^2)(|\eta-2t|^2-(|\eta|+i0)^2)}d\eta,$$

where $h_i=\rho_1\rho_2\beta_i g_1$. Thus, as for h_3,h_i has bounded derivatives with respect to (ξ,η) of all orders. Moreover, writing t in spherical coordinates, one sees that $h_i, i=4,5$, as bounded derivatives of all orders as a function on $\mathbf{R}^3\times\mathbf{R}^3\times\bar{\mathbf{R}}_+\times S^2$.

We let μ_1 be the t-component of $2\eta-\xi$, i.e. $\mu_1=((2\eta-\xi)t)|t|^{-1}$, and μ_2 be the projection of $2\eta-\xi$ on the plane orthogonal to t, i.e.

$$\mu_2=2\eta-\xi-\frac{(2\eta-\xi)\cdot t}{|t|^2}t.$$

By our choices of ε_1 and $\varepsilon_2, |\mu_2|<1/2|\xi|$ on support h_i, and, since $h_i=0$ if $|\xi-\eta|>M$ or $|t-\eta|>M$, we also have $|\mu_2|<2M$ on support h_i.

We set $\mu_1(\xi,t)=2|t|-(\xi\cdot t)|t|^{-1}$, expand $s_i(\xi,t)$ as

$$\begin{aligned} s_i(\xi,t) &= \frac{1}{2}\int_{\mathbf{R}^3}\frac{h_i-h_i\restriction_{\mu_1=\mu_1(\xi,t)}}{(\mu_1^2+|\mu_2|^2-(|\xi|+i0)^2)(2|t|)(\mu_1(\xi,t)-\mu_1)}d\mu_1\,dm \\ &\quad+\frac{1}{2}\int_{\mathbf{R}^3}\frac{h_i\restriction_{\mu_1=\mu_1(\xi,t)}}{(\mu_1^2+|\mu_2|^2-(|\xi|+i0)^2)(2|t|)(\mu_1(\xi,t)-\mu_1-i0)}d\mu_1\,dm \\ &\equiv s_{i,1}+s_{i,2}, \end{aligned}$$

where dm is Lebesgue measure on $t\cdot\eta=0$. We consider $s_{i,1}$ as a function of the form (4.10) with

$$g=\frac{1}{4}\frac{h_i(\xi,\eta,t)-h_i\restriction_{\mu_1=\mu_1(\xi,t)}}{(|\eta|+|\xi|)(\mu_1(\xi,t)-\mu_1)}.$$

Although g is not supported in a bounded set, it has bounded support in μ_2 and the expansion used in Lemma 4.4 shows that (4.11) holds for $s=|t|s_{i,1}$. Thus, since $s_{i,1}(\xi,t)=0$ for $|\xi-t|>2M$, $s_{i,1}$ is the kernel of a compact integral operator on $H_{\alpha,N}$.

We evaluate $s_{i,2}$ by computing the integral in μ_1 by residues (there is a simple pole in $\operatorname{Im}\mu_1>0$ at $\mu_1=\sqrt{(|\xi|+i0)^2-|\mu_2|^2}$). This gives

$$|t|s_{i,2}(\xi,t)=\int_{\mathbf{R}^2}\frac{k_i(\mu_2,\xi,t)}{\sqrt{|\xi|^2-|\mu_2|^2}\left(2|t|-\dfrac{\xi\cdot t}{|t|}-\sqrt{|\xi|^2-|\mu_2|^2}-i0\right)}d\mu_2,$$

where

$$k_i=\frac{\pi i}{2}h_i\restriction_{\mu_1=2|t|-\xi\cdot t/|t|}.$$

On the support of $h_5, 2|t|-(\xi\cdot t)|t|^{-1}<-|\xi|+2\varepsilon_2|\xi|, |\xi|>\delta/4$ and $|\mu_2|<\min\{|\xi|/2, 2M\}$. Thus the integrand defining $s_{5,2}$ is smooth in ξ with bounded support in μ_2. Since one has $|t||\partial_\xi^\beta s_{5,2}(\xi,t)|$ bounded for all β and $s_{5,2}(\xi,t)=0$ for $|\xi-t|>2M$, the integral operator corresponding to $s_{5,2}$ is compact.

To simplify the study of $s_{4,2}$ we use polar coordinates in the plane $\eta\cdot t=0$ and set $m_4(|\mu_2|^2,\xi,t)=\int_0^{2\pi}k_4(|\mu_2|\cos\theta,|\mu_2|\sin\theta,\xi,t)d\theta$. It is important that m_4 is a smooth function of $|\mu_2|^2$ on $\bar{\mathbf{R}}_+$-note that only homogeneous functions of $(\cos\theta,\sin\theta)$ of even degree survive the integration. Thus $m_4(s,\xi,t)$ is smooth in (s,ξ) on $\bar{\mathbf{R}}_+\times\mathbf{R}^3$ and its partial derivatives with respect to s and ξ are bounded on $\bar{\mathbf{R}}_+\times\mathbf{R}^3\times\mathbf{R}^3$. It also remains true that, if we write t in spherical coordinates, m_4 is smooth on $\bar{\mathbf{R}}_+\times\mathbf{R}^3\times\bar{\mathbf{R}}_+\times S^2$.

We now have

$$s_{4,2}(\xi,t)=\lim_{\varepsilon\downarrow 0}\frac{1}{2|t|}\int_0^\infty \frac{m_4(s,\xi,t)}{\sqrt{|\xi|^2-s}\left(2|t|-\frac{\xi\cdot t}{|t|}-\sqrt{|\xi|^2-s}-i\varepsilon\right)}ds$$

and integrating by parts gives

$$s_{4,2}(\xi,t)=-\frac{m_4(0,\xi,t)\ln\left(2|t|-\frac{\xi\cdot t}{|t|}-|\xi|-i0\right)}{|t|}$$

$$-\frac{1}{|t|}\int_0^\infty\frac{\partial m_4}{\partial s}\ln\left(2|t|-\frac{\xi\cdot t}{|t|}-\sqrt{|\xi|^2-s}-i0\right)ds\equiv v_{4,1}+v_{4,2}.$$

Note that the integration in $v_{4,2}$ is over $0<s<(2M)^2$ and $|2|t|-(\xi\cdot t)/|t|-|\xi||<2\varepsilon_2|\xi|$ on support m_4. The kernel $v_{4,2}$ is superposition of the kernels

$$w_a(\xi,t)=\frac{m(a,\xi,t)\ln(2|t|-(\xi\cdot t)|t|^{-1}-\sqrt{|\xi|^2-a}-i0)}{|t|}$$

for $0\leqq a\leqq a_0$, where $m(a,\xi,r\omega)$ is smooth on $\bar{\mathbf{R}}_+\times\mathbf{R}^3\times\bar{\mathbf{R}}_+\times S^2$ with bounded derivatives in ξ and r and $m=0$ for $|\xi-t|>2M$ and for $|\xi|^2<\max\{4a,\delta^2/16\}$. The kernel $v_{4,1}$ is $w_0(\xi,t)$ with $m(0,\xi,t)=-m_4(0,\xi,t)$. Thus to complete the proof that T_1 is compact it will suffice to show for W_a with kernel w_a,

$$\|\Lambda^{N+1}W_af\|_1\leqq C_\alpha\|\Lambda^Nf\|_\alpha \tag{4.13}$$

for all $\alpha>0$ with C^α uniform on $0\leqq a\leqq a_0$.

We have

$$|\Lambda^{N+1}(\xi)W_af(\xi)|\leqq C\int_{|\xi-t|<2M}|\ln(2|t|-(\xi\cdot t)|t|^{-1}-\sqrt{|\xi|^2-a}-i0)|dt\,\|\Lambda^Nf\|_0$$

and, since the integral is bounded uniformly for $(a,\xi)\in[0,a_0]\times\mathbf{R}^3$, this gives

$$\sup|\Lambda^{N+1}(\xi)W_af(\xi)|\leqq C\|\Lambda^Nf\|_0. \tag{4.14}$$

To estimate the Lipschitz norm of $\Lambda^{N+1}(\xi)W_af(\xi)$, we use $\varphi(s)\in C^\infty(\mathbf{R})$, satisfying $\varphi(s)=1$ for $|s|<1$ and $\varphi(s)=0$ for $|s|>2$, to write

$$W_af(\xi)=\int_{\mathbf{R}^3}\varphi(|t|)w_a(\xi,t)f(t)dt+\int_{\mathbf{R}^3}(1-\varphi(|t|))w_a(\xi,t)f(t)dt\equiv I_1+I_2.$$

In I_1 we will use spherical coordinates, $t=r\omega$. Since $m(a,\xi,r\omega)$ is smooth in r uniformly in (a,ξ,ω), extending $rm(a,\xi,r\omega)$ to be zero for $r<0$ gives a Lipschitz function of r uniformly in ω, which we denote by $\tilde{m}(a,\xi,r,\omega)$. Thus

$$I_1=\int_{S^2}d\omega\int_{\mathbf{R}}\varphi(r)\tilde{m}\ln(2r-(\xi\cdot\omega)-\sqrt{|\xi|^2-a}-i0)f(r,\omega)dr,$$

where for $r\geqq 0$ $f(\pm r,\omega)=f(r\omega)$. Expanding I_1 as in the proof of Lemma 4.4, one sees that for $f\in C^\alpha$, $\alpha>0, I_1$ is differentiable in ξ with

$$\frac{\partial I_1}{\partial \xi_i} = \int_{S^2} d\omega \int_{\mathbf{R}} \varphi(r) \frac{\partial \tilde{m}}{\partial \xi_i} \ln(2r - (\xi \cdot \omega) - \sqrt{|\xi|^2 - a} - i0) f(r, \omega) dr$$

$$+ \int_{S^2} d\omega \int_{\mathbf{R}} - \frac{(\omega_i + \xi_i(|\xi|^2 - a)^{-1/2}) \varphi(r) \tilde{m} f(r, \omega)}{2r - (\xi \cdot \omega) - \sqrt{|\xi|^2 - a} - i0} dr \equiv J_1 + J_2.$$

One has

$$|J_1(\xi)| \leqq C \| f \|_0$$

by the reasoning that gave (4.14) and

$$|J_2(\xi)| \leqq C \| f \|_{\alpha'}$$

for any $\alpha' > 0$ by Lemma 4.4 applied to the integral over $\mathbf{R}$. Since $|\xi|$ is bounded on support $\varphi(|t|)m$, we have

$$|\Lambda^{N+1}(\xi) I_1(\xi, a)| \leqq C \| f \|_{\alpha'} \tag{4.15}$$

for any $\alpha' > 0$.

Since all functions are smooth in t for $|t| \geqq 1$, the expansion used in the proof of Lemma 4.4 can be used to show that I_2 is differentiable in ξ with

$$\frac{\partial I_2}{\partial \xi_i} = \int_{\mathbf{R}^3} - \frac{(t_i |t|^{-1} + \xi_i(|\xi|^2 - a)^{-1/2})(1 - \varphi(|t|)) m f(t)}{(2|t| - (\xi \cdot t)|t|^{-1} - \sqrt{|\xi|^2 - a} - i0)|t|} dt$$

and, since ξ and t have comparable magnitudes on support $(1 - \varphi(|t|)m)$, Lemma 4.4 shows

$$\sup \left| \Lambda^{N+1}(\xi) \frac{\partial I_2}{\partial \xi_i} \right| \leqq C \| \Lambda^N f \|_{\alpha'} \tag{4.16}$$

for any $\alpha' > 0$. Combining (4.14)–(4.16) gives (4.13), and completes the proof that T_1 is compact on $H_{\alpha,N}$.

In terms of η the integrand defining T_2 is singular on the spheres (for $\xi \neq 0$)

$$\Pi_1 = \{\eta : |\eta| = |\xi|\}$$

and

$$\Pi_2 = \{\eta : |\eta - 2t| = |\xi|\}.$$

At $t = 0$ these spheres coincide but the most singular part of the kernel $t_2(\xi, t)$ of T_2 is the set corresponding to tangency of Π_1 and Π_2, i.e.

$$|t| = |\xi|.$$

As in the proof of the compactness of T_1, we introduce a partition of unity adapted to these sets generated by

$$\rho_1 = \rho(|\eta| - |\xi|) \quad \text{and} \quad \rho_2 = \rho(|\eta - 2t| - |\xi|),$$

where $\rho(s) = 1$ for $|s| < \varepsilon_1$ and $\rho \in C_0^\infty(|s| < 2\varepsilon_1)$. Again for ε_1 sufficiently small one has $|\xi| > \delta/4$ on the support of $\rho_1 \rho_2 g_2$, and $\rho_1 \rho_2 g_2$ is smooth. The cutoffs corresponding to the more singular parts are

$$\beta_1=\beta\left(\frac{|t|}{|\xi|}\right) \quad \text{and} \quad \beta_2=\beta\left(\frac{|t|-|\xi|}{|\xi|}\right),$$

where $\beta(s)=1$ for $|s|<\varepsilon_2$ and $\beta\in C_0^\infty(|s|<2\varepsilon_2)$. Note that for ε_2 sufficiently small β_1 and β_2 are smooth on support $\rho_1\rho_2 g_2$.

Next we define $S_1,\dots,S_5$ precisely as in the proof for T_1. Thus the integrand of S_1 vanishes on a neighborhood of Π_1, the integrand of S_2 vanishes on a neighborhood of Π_2, and on the support of the integrand of S_3 we can introduce coordinates for which Π_1 and Π_2 are level sets. These three terms are treated exactly as before: in place of $|t|$ the weight factor in the denominator is $|\eta-2t|+|\xi|$.

For S_4 we introduce spherical coordinates in $\eta, \eta=r\omega, |\omega|=1, r>0$. Then

$$\begin{aligned}|\eta-2t|^2-|\xi|^2&=r^2-4r\omega\cdot t+4|t|^2-|\xi|^2\\&=(r-2t\cdot\omega-\sqrt{|\xi|^2+4(t\cdot\omega)^2-4|t|^2})\\&\quad(r-2t\cdot\omega+\sqrt{|\xi|^2+4(t\cdot\omega)^2-4|t|^2}),\end{aligned}$$

and for ε_2 sufficiently small $|\xi|^2-8|t|^2>1/2|\xi|^2$ on support h_4. Thus, the kernel $s_4(\xi,t)$ of S_4 given by

$$s_4(\xi,t)=\int_{S^2}d\omega\int_0^\infty\frac{k_4(\xi,r,\omega,t)}{(r-|\xi|-i0)(r-2t\cdot\omega-\sqrt{|\xi|^2+4(t\cdot\omega)^2-4|t|^2}-i0)}dr,$$

where

$$k_4=\frac{r^2h_4}{r+|\xi|}(r-2t\cdot\omega+\sqrt{|\xi|^2+4(t\cdot\omega)^2-4|t|^2})^{-1}$$

is a smooth function on $\mathbf{R}^3\times\bar{\mathbf{R}}_+\times S^2\times\mathbf{R}^3$.

Expanding in the usual manner, we have (with $A=|\xi|^2+4(t\cdot\omega)^2-4|t|^2$)

$$\begin{aligned}s_4(\xi,t)&=\int_{S^2}d\omega\int_0^\infty\frac{k_4(\xi,r,\omega,t)-k_4(\xi,|\xi|,\omega,t)}{(r-|\xi|)(r-2t\cdot\omega-\sqrt{A}-i0)}dr\\&\quad+\int_{S^2}d\omega\int_0^\infty\frac{k_4(\xi,|\xi|,\omega,t)}{(r-|\xi|-i0)(r-2t\cdot\omega-\sqrt{A}-i0)}dr\\&\equiv s_{4,1}(\xi,t)+s_{4,2}(\xi,t).\end{aligned}$$

The main point here is that, since the integral in $s_{4,2}$ is the limit as $\varepsilon\downarrow 0$ of the same integral with $i0$ replaced by $i\varepsilon$, we can deform the integration on $[0,\infty)$ to a contour in the upper half plane—for instance $z=r(1+i)$. Since $|\xi|>\delta/4$ and $|\xi-t|<2M$ on support k_4, this shows $s_{4,2}(\xi,t)\in C_0^\infty(\mathbf{R}^6)$. Since

$$\frac{k_4(\xi,r,\omega,t)-k_4(\xi,|\xi|,\omega,t)}{r-|\xi|}$$

is a smooth function supported on

$$\{|\xi|>\delta_0/4\}\cap\{|\xi-t|<2M\}\cap\{|t|<2\varepsilon_2|\xi|\}\cap\{|r-|\xi||<2\varepsilon_1\},$$

it follows that $s_{4,1}(\xi,t)$ also has compact support, and it has enough regularity in

ξ that the corresponding integral operator is compact on $H_{\alpha,N}$. Thus S_4 is compact.

For S_5 we introduce the t-component of η $\mu_1 = \eta \cdot t/|t|$, and set $\mu_2 = \eta - (\eta \cdot t/|t|^2)t$. Since $|t-\eta| < M$ on support h_5, we have $|\mu_2| < M$ on support h_5. Moreover, since we also have

$$||\xi| - |t|| < 2\varepsilon_2 |\xi|, \quad ||\eta| - |\xi|| < 2\varepsilon_1, \quad ||\eta - 2t| - |\xi|| < 2\varepsilon_1$$

and $|\xi| > \delta_0/4$ on support h_5, it also follows that, choosing ε_1 and ε_2 sufficiently small we can make $|\mu_2| < 1/2|\xi|$ on support h_5. The kernel of S_5 is given by

$$\begin{aligned} s_5(\xi,t) &= \int_{\mathbf{R}^3} \frac{h_5(\xi,\eta,t)\,d\eta}{(\mu_1^2 + |\mu_2|^2 - |\xi|^2 - i0)((\mu_1 - 2|t|)^2 + |\mu_2|^2 - |\xi|^2 - i0)} \\ &= \int_{\mathbf{R}^3} \frac{k_5(\xi,\eta,t)\,d\eta}{(\mu_1 - \sqrt{|\xi|^2 - |\mu_2|^2} - i0)(\mu_1 - 2|t| + \sqrt{|\xi|^2 - |\mu_2|^2} + i0)}, \end{aligned}$$

where

$$k_5 = \frac{h_5}{(\mu_1 + \sqrt{|\xi|^2 - |\mu_2|^2})(\mu_1 - 2|t| - \sqrt{|\xi|^2 - |\mu_2|^2})}.$$

Note that, since for ε_1 and ε_2 sufficiently small one has $|\mu_1 - |t|| < 1/4|\xi|$ and $|t| > 3/4|\xi|$ on support h_5, k_5 is a smooth function satisfying $(1+|\xi|)^2 |\partial^\beta_{\xi,\eta,t} k_5| \leqq C$ for all β.

Expanding $s_5(\xi,t)$, we have

$$\begin{aligned} s_5(\xi,t) &= \int_{\mathbf{R}^3} \frac{(k_5(\xi,\eta,t) - k_5)\restriction_{\mu_1 = \sqrt{|\xi|^2 - |\mu_2|^2}}}{(\mu_1 - \sqrt{|\xi|^2 - |\mu_2|^2})(\mu_1 - 2|t| + \sqrt{|\xi|^2 - |\mu_2|^2} + i0)}\,d\eta \\ &\quad + \int_{\mathbf{R}^2} d\mu_2 \int_{\mathbf{R}} \frac{k_5 \restriction_{\mu_1 = \sqrt{|\xi|^2 - |\mu_2|^2}}}{(\mu_1 - \sqrt{|\xi|^2 - |\mu_2|^2} - i0)(\mu_1 - 2|t| + \sqrt{|\xi|^2 - |\mu_2|^2} + i0)}\,d\mu_1 \\ &\equiv s_{5,1}(\xi,t) + s_{5,2}(\xi,t). \end{aligned}$$

From the restrictions on the support of h_5 one sees that $s_{5,1}$ is a smooth function supported in $|\xi - t| < 2M$, satisfying $\sup(1+|\xi|)|\partial^\beta_\xi s_{5,1}(\xi,t)| < \infty$, for all β. Thus the integral operator corresponding to $s_{5,1}$ is compact on $H_{\alpha,N}$.

Calculating the integral in μ_1 in $s_{5,2}$ by residues, we have

$$s_{5,2}(\xi,t) = \pi i \int_{\mathbf{R}^2} \frac{k_5 \restriction_{\mu_1 = \sqrt{|\xi|^2 - |\mu_2|^2}}}{\sqrt{|\xi|^2 - |\mu_2|^2} - |t| + i0}\,d\mu_2.$$

Multiplying numerator and denominator by $\sqrt{|\xi|^2 - |\mu_2|^2} + |t|$, which is smooth on support h_5, we have

$$s_{5,2} = \pi i \int_{\mathbf{R}^2} \frac{l_5(\xi,t,\mu_2)}{|\xi|^2 - |\mu_2|^2 - |t|^2 + i0}\,d\mu_2,$$

where $l_5 = (\sqrt{|\xi|^2 - |\mu_2|^2} + |t|)k_5 \restriction \mu_1 = \sqrt{|\xi|^2 - |\mu_2|^2}$. Note that $m_5 = \pi i/2 \int_0^{2\pi} l_5\,d\theta$ is a smooth function of (ξ, r^2, t) supported in $\{|t - \xi| < 2M\} \cap \{r < M\}$, satisfying

$\sup(1+|\xi|)|\partial^{\beta}_{\xi,r,t}m_5|<\infty$. Thus

$$s_{5,2}(\xi,t)=2\int_0^{\infty}\frac{m_5(\xi,r^2,t)}{|\xi|^2-r^2-|t|^2+i0}rdr=-\int_0^{\infty}\frac{m_5(\xi,s,t)}{s-|\xi|^2+|t|^2-i0}ds$$

$$=m_5(\xi,0,t)\ln(|t|^2-|\xi|^2-i0)+\int_0^{\infty}\frac{\partial m_5}{\partial s}(\xi,s,t)\ln(s+|t|^2-|\xi|^2-i0)ds$$

$$=m_5(\xi,0,t)\ln(|t|-|\xi|-i0)+\int_0^{\infty}\frac{\partial m_5}{\partial s}(\xi,s,t)\ln((s+|t|^2)^{1/2}-|\xi|-i0)ds$$

$$+m_5(\xi,0,t)\ln(|t|+|\xi|)+\int_0^{\infty}\frac{\partial m_5}{\partial s}(\xi,s,t)\ln((s+|t|^2)^{1/2}+|\xi|)ds$$

$$\equiv v_1+v_2+v_3+v_4,$$

The kernels v_3 and v_4 are supported in $|\xi-t|<2M$ and they satisfy $\|\Lambda^s(\cdot)v_i(\cdot,t)\|_1<C, t\in\mathbf{R}^3$, $i=3,4$, for $s+1$. Thus the corresponding integral operators, V_3 and V_4, are compact on $H_{\alpha,N}$.

The remaining terms in T_2f, V_1f and V_2f, are super positions of the operators

$$[V_af](\xi)=\int_{\mathbf{R}^3}m_5(\xi,a,t)\ln((a+|t|^2)^{1/2}-|\xi|-i0)f(t)dt$$

for $0\leqq a\leqq M$.

Since t and ξ are bounded away from zero on the support of m_5, the expansion used in the proof of Lemma 4.4 again shows that V_1f is differentiable and

$$\frac{\partial V_af}{\partial\xi_i}=\int_{\mathbf{R}^3}\frac{\partial m_5}{\partial\xi_i}\ln((a+|t|^2)^{1/2}-|\xi|-i0)f(t)dt+\int_{\mathbf{R}^3}m_5\left(-\frac{\xi_i|\xi|^{-1}f(t)}{(a+|t|^2)^{1/2}-|\xi|-i0}\right)dt.$$

Thus, since ξ and t have comparable magnitude on the support of m_5, V_a satisfies the estimate (4.13), i.e.

$$\|\Lambda^{N+1}(\xi)V_a\|_1\leqq C_{\alpha'}\|\Lambda^Nf\|_{\alpha'}$$

for any $\alpha'>0$. Thus V_a is compact on $H_{\alpha,N}$. ∎

As we showed earlier, Theorem 4.3 has the following corollary.

Corollary 4.5. *The Frechet derivative of the backscattering map, $dS(\hat{q})$, is a Fredholm operator on $H_{\alpha,N}$ of index zero for $\hat{q}\in\mathcal{O}$.*

Section 5. Local Invertibility of the Backscattering Map

In this section we present the consequences of the results of Sects. 3 and 4 for the inverse backscattering problem. The extent of the connected component of $\mathcal{O}$ containing the zero potential is of interest here. We can show that the intersection of $\mathcal{O}$ with $H^r_{\alpha,N}$ is contained in a connected component of $\mathcal{O}$. The proof of that fact requires the following pair of lemmas.

Lemma 5.1. *For some $\alpha', \alpha<\alpha'<1$, let $\hat{q}(t)$ be a curve in $H_{\alpha',N}$ continuous in the topology of $H_{\alpha',N}$, such that $q(t)$ is a real-valued function in $C_0^{\infty}(\mathbf{R}^3)$ for all t. Assume* *that $I + A(\hat{q}(0), 0)$ has a one-dimensional nullspace, and that $I + A(\hat{q}(t), 0)$ is invertible for $t \neq 0$. Then, given $\delta > 0$, there is a curve $q_1(t)$ in $C_0^\infty(\mathbf{R}^3)$ continuous in the topology of $H_{\alpha',N}$, such that*

(i) *$q_1(t) = q(t)$ for $|t| > \delta$, and*
(ii) *$I + A(\hat{q}_1(t), k)$ is invertible for all t and $k \geqq 0$,* i.e. *$\hat{q}_1(t) \in \mathcal{O}$ for all t.*

Proof. Let $\hat{f}(\hat{q}, k) \neq 0$ be an element of the range of the projection:

$$P(\hat{q}, k) = \frac{1}{2\pi i} \int_{|\omega+1|=c} (A(\hat{q}, k) - \omega I)^{-1} \, d\omega. \tag{5.1}$$

Since $\alpha' > \alpha$, for $\hat{q}$ in $H_{\alpha',N}$, $A(\hat{q}, k)$ is continuous in $(\hat{q}, k)$ in operator norm on $H_{\alpha,N}$ and compact on $H_{\alpha,N}$ by Theorems 2.1 and 2.2 (see 3.4). Thus it follows that for c sufficiently small P has 1-dimensional range and is continuous in $(\hat{q}, k)$ on $\{\|\hat{q} - \hat{q}(0)\|_{\alpha',N} < c_1, 0 \leqq k \leqq c_1\}$ for c_1 sufficiently small. Moreover, $P(\hat{q}, k)$ is differentiable in $\hat{q}$ and $\partial P / \partial \hat{q}$ is also continuous in $(\hat{q}, k)$.

We have

$$(I + A(\hat{q}, k))\hat{f}(\hat{q}, k) = \lambda(\hat{q}, k)\hat{f}(\hat{q}, k), \tag{5.2}$$

where $\lambda(\hat{q}, k) \in \mathbf{C}$ and λ has the regularity of P. Evaluating (5.2) at $(\hat{q}, k) = (z\hat{q}(0), 0)$, differentiating with respect to z, evaluating at $z = 1$, and taking the inner product with $[\hat{f}(\hat{q}(0), 0)](\xi)|\xi|^{-2}$, we have (see Remark after Theorem 3.1)

$$\left.\frac{\partial \lambda(z\hat{q}(0), 0)}{\partial z}\right|_{z=1} = -1.$$

We split $H_{\alpha',N}$ into the direct sum of span $\hat{q}(0)$ and

$$H' = \left\{ \hat{q} \in H_{\alpha',N} : \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \hat{q}(\xi - \eta) \frac{\hat{f}_0(\xi)\hat{f}_0(\eta)}{|\eta|^2 |\xi|^2} \, d\xi \, d\eta = 0 \right\},$$

where $\hat{f}_0 = \hat{f}(\hat{q}(0), 0)$.

Let $\lambda(\hat{q}', z, k) = \lambda(\hat{q}' + z\hat{q}(0), k)$. We consider λ as a function on $H' \times \{|z-1| < \delta\} \times \{0 \leqq k \leqq \delta\}$. By the implicit function theorem there is an $\varepsilon > 0$ such that for $\|\hat{q}'\|_{\alpha',N} < \varepsilon, 0 \leqq k \leqq \varepsilon$ the unique solution to $\lambda(\hat{q}', z, k) = 0$ in $|1 - z| < \varepsilon$ is given by $z = z(\hat{q}', k)$ and $z(\hat{q}', k)$ is continuous in $(\hat{q}', k)$. Note that $\lambda(\hat{q}', z, k) = 0$ means $\mathrm{Null}\,(I + A(z\hat{q}(0) + \hat{q}', k)) \neq \{0\}$.

Now suppose $\hat{q}(t_i) = \hat{q}'_i + z_i\hat{q}(0)$, $i = 1, 2$, with $t_1 < 0$, $t_2 > 0$, $\|\hat{q}'_i\|_{\alpha',N} < \varepsilon' < \varepsilon$ and $|z_i - 1| < \varepsilon'$. By hypothesis this will hold for $|t_i| < \delta$, δ sufficiently small. Also by hypothesis $z_i \in \mathbf{R}$ and $z(\hat{q}'_i, 0) \neq z_i$. Since $q(t)$ is real valued, q'_1 and q'_2 are real, and hence $z((1-s)\hat{q}'_1 + s\hat{q}'_2, k)$ does not intersect the real-axis for $(s, k) \in [0, 1] \times (0, \varepsilon]$. Thus we may choose $z(t)$ with $z(t_1) = z_1$ and $z(t_2) = z_2$ with $z(t)$ for $t \in (t_1, t_2)$ lying in the half-plane, $\{\mathrm{Im}\, z > 0\}$ or $\{\mathrm{Im}\, z < 0\}$, which does not intersect $\{z((1-s)\hat{q}'_1 + s\hat{q}'_2, k) : (s, k) \in [0, 1] \times [0, \varepsilon]\}$. Then we define

$$\hat{q}_1(t) = \left(1 - \frac{t - t_1}{t_2 - t_1}\right)\hat{q}'_1 + \frac{t - t_1}{t_2 - t_1}\hat{q}'_2 + z(t)\hat{q}(0)$$

for $t \in [t_1, t_2]$. Finally, we note that we may construct $\hat{q}_1(t)$ so that

$$\|\hat{q}_1(t) - \hat{q}(0)\|_{\alpha',N} \leqq (2 + \|\hat{q}(0)\|_{\alpha',N})\varepsilon'$$

for $[t_1, t_2]$. Thus, taking ε' sufficiently small, we will have $I + A(\hat{q}_1(t), k)$ invertible for $k \geqq \varepsilon, t \in [t_1, t_2]$. For $t \notin [t_1, t_2]$ we set $q_1(t) = q(t)$. ■

Lemma 5.2. *Suppose that $I + A(\hat{q}_0, 0)$ has a kernel of dimension $m > 0$, for some $\hat{q}_0 \in H^r_{\alpha,N}$. Then for some $\varepsilon > 0$, the set of $\hat{q}$ in $H^r_{\alpha,N} \cap \{\|\hat{q} - \hat{q}_0\|_{\alpha,N} < \varepsilon\}$ such that $I + A(\hat{q}, 0)$ has a kernel of dimension m is contained in a smooth surface of codimension m in $H^r_{\alpha,N}$. For all $\hat{q}$ on this surface $(I + A(\hat{q}, 0))$ has a kernel of dimension $\geqq 1$.*

Proof. Let $P(\hat{q}) = 1/2\pi i \int_{|\omega+1|=c} (A(\hat{q}, 0) - \omega I)^{-1} d\omega$, as in (5.1). Here P has an m-dimensional range for c sufficiently small, and is differentiable for $\|\hat{q} - \hat{q}_0\|_{\alpha,N} < c_1$ for c_1 sufficiently small. Since $A(\hat{q}, 0)$ leaves $H^r_{\alpha,N}$ invariant when $\hat{q} \in H^r_{\alpha,N}$, as one sees taking inverse Fourier transforms, $P(\hat{q})$ inherits this property. By construction $I + A(\hat{q}, 0)$ has an m-dimensional null space if and only if $(I + A(\hat{q}, 0))P(\hat{q}) = 0$. Let $\hat{f}_1, \ldots, \hat{f}_m$ be a basis for range $P(\hat{q}_0)$. Note that, since $\hat{q}_0 \in H^r_{\alpha,N}$, we may choose $\hat{f}_i \in H^r_{\alpha,N}$. Let

$$d_i(\hat{q}) = (2\pi)^{-3} \int_{\mathbf{R}^3} \frac{\bar{\hat{f}}_1(\xi)}{|\xi|^2} [(I + A(\hat{q}, 0))P(\hat{q})\hat{f}_i](\xi) d\xi$$

$$= \int_{\mathbf{R}^3} [E_0 f_1](x)[(I + qE_0)g_i](x) dx$$

by Plancherel's theorem, where g_i is the inverse Fourier transform of $P(\hat{q})\hat{f}_i$, see Remark 3 after Theorem 3.1. The set of $\hat{q}$ for which $I + A(\hat{q}, 0)$ has an m-dimensional nullspace intersected with $\|\hat{q} - \hat{q}_0\|_{\alpha,N} < c_1$ is contained in

$$\sum = \{\hat{q} : d_i(\hat{q}) = 0, i = 1, \ldots, m\}$$

and d_i is real-valued on $H^r_{\alpha,N}$. Taking Frechet derivatives at $\hat{q} = \hat{q}_0$,

$$d_i'(\hat{q}_0)\hat{r} \equiv \int_{\mathbf{R}^3} (E_0 f_1)(E_0 f_i) r \, dx,$$

since $g_i(\hat{q}_0) = f_i$. Since $-\Delta E_0 f_i + q_0 E_0 f_i = 0$, unique continuation implies no $E_0 f_i$ can vanish on an open set. The linear independence of $\{f_i\}_{i=1}^m$ implies the linear independence of $\{E_0 f_i\}_{i=1}^m$. Thus we conclude $\{(E_0 f_1)(E_0 f_i)\}_{i=1}^m$ is linearly independent as well. Thus we may choose real-valued $\varphi_j \in C_0^\infty(\mathbf{R}^3), j = 1, \ldots, m$ such that

$$\int_{\mathbf{R}^3} (E_0 f_1)(E_0 f_i)\varphi_j \, dx = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Now we restrict $d_1, \ldots, d_m$ to $H^r_{\alpha,N}$ and let H' be a closed complementary subspace to span $\{\varphi_j\}_{j=1}^m$ in $H^r_{\alpha,N}$. By the implicit function theorem the system of equations

$$d_i\left(\hat{q}' + \sum_{j=1}^m s_j \hat{\varphi}_j\right) = 0 \quad i = i, \ldots, m,$$

where $\hat{q} \in H'$ and $s = (s_1, \ldots, s_m) \in \mathbf{R}^m$ can be solved for $s(\hat{q}')$ when $\hat{q}' + \sum_{j=1}^m s_j \hat{\varphi}_j$ is near $\hat{q}_0$. Now we are ready to prove,

Proposition 5.3. *The set $\mathcal{O} \cap H^r_{\alpha,N}$ is contained in a connected component of $\mathcal{O}$.*

Proof. Since $\mathcal{O} \cap H^r_{\alpha,N}$ is an open dense set in $H^r_{\alpha,N}$ by Theorem 3.1 and the density of C_0^∞ in $H_{\alpha,N}$ implies the density of Fourier transforms of real-valued C_0^∞ in $H^r_{\alpha,N}$, it will suffice to show that we can connect any pair of functions $\hat{q}_1$ and $\hat{q}_2$ in $\mathcal{O}$, when q_1 and q_2 are real-valued functions in $C_0^\infty(\mathbf{R}^3)$. Given two such functions, let $q(t) = tq_1 + (1-t)q_2, t\in[0,1]$. Since $A(\hat{q}(t),0)$ is real-analytic as an operator-valued function of t on $H_{\alpha,N}$, and $I + A(\hat{q}(0),0)$ is injective, $I + A(\hat{q}(t),0)$ has a nontrivial nullspace for at most a finite set S of t in $[0,1]$.

Suppose that $t_0 \in S$ and

$$\dim \operatorname{Null}(I + A(\hat{q}(t_0),0)) = m$$

is maximal for $t\in[0,1]$. If $m > 1$, we choose ε_0 small enough that $\dim \operatorname{Null}(I + A(\hat{q}(t),0)) < m$ for $0 < |t - t_0| < \varepsilon_0$. Taking ε_0 smaller if necessary, we may assume $\| \hat{q}(t) - \hat{q}(t_0) \|_{\alpha,N} < \varepsilon_1$, where by Lemma 5.2 the set of $\hat{q}$ in $\| \hat{q} - \hat{q}(t_0) \|_{\alpha,N} < \varepsilon_1$ such that $\dim \operatorname{Null}(I + A(\hat{q},0)) = m$ is contained in the set of $\hat{q}' + \sum_{j=1}^{m} s_j(\hat{q}')\hat{\varphi}_j$ with $\hat{q}' \in H'$, a closed complement of span $\{\hat{\varphi}_j\}_{j=1}^m$ in $H^r_{\alpha,N}$. Then, $\hat{q}(t) = \hat{q}'(t) + \sum_{j=1}^{m} r_j(t)\hat{\varphi}_j$ for $|t - t_0| < \varepsilon_0$, where $\hat{q}'(t)$ and $r(t)$ are affine linear in t. Since $m > 1$, the set in $\mathbf{R}^{m+1}$,

$$\Sigma_{\delta,\varepsilon} = \{(u,s)\in\mathbf{R}^{m+1} : |s - s(\hat{q}'(t_0))| < \delta, |u - t_0| \leqq \varepsilon, s \neq s(\hat{q}'(u))\}$$

is connected for all δ and ε. For δ sufficiently small

$$\left\{\hat{q}' + \sum_{j=1}^{m} s_j\hat{\varphi}_j : |s - s(\hat{q}'(t_0))| < \delta \text{ and } \|\hat{q}' - \hat{q}'(t_0)\|_{\alpha,N} < \varepsilon_1/2\right\}$$

is contained in $\|\hat{q} - \hat{q}(t_0)\| < \varepsilon_1$. Likewise for ε sufficiently small $(t, r(t)) \in \sum_{\delta,\varepsilon}$ for $|t - t_0| = \varepsilon$. Hence, we can replace $(t, r(t))$ by a piecewise linear function $(a(t), \tilde{r}(t))$ for $|t - t_0| \leqq \varepsilon$ such that $\tilde{r}(t_0 \pm \varepsilon) = r(t_0 \pm \varepsilon)$, $a(t_0 \pm \varepsilon) = t_0 \pm \varepsilon$ and $(a(t), \tilde{r}(t)) \in \sum_{\delta,\varepsilon}$. Now we set

$$\hat{q}_1(t) = \begin{cases} \hat{q}(t) & \text{for} \quad |t - t_0| > \varepsilon \\ \hat{q}'(a(t)) + \sum_{j=1}^{m} \tilde{r}_j(t)\hat{\varphi}_j & \text{for} \quad |t - t_0| \leqq \varepsilon. \end{cases}$$

The function $\hat{q}_1(t)$ is piecewise linear, and, since $\mathcal{O} \cap H^r_{\alpha,N}$ is dense in $H^r_{\alpha,N}$, we may assume its corners are in $\mathcal{O}$.

Continuing in this way, we arrive at a piecewise linear function $\hat{q}_N(t)$ with corners in $\mathcal{O}$ such that $\dim \operatorname{Null}(I + A(\hat{q}_N(t),0)) < m$ for $t\in[0,1]$, $\hat{q}_N(0) = \hat{q}_1$, $\hat{q}_N(1) = \hat{q}_2$, and $q_N(t) \in C_0^\infty(\mathbf{R}^3)$ for $t\in[0,1]$. Since the set of t in $[0,1]$ such that $\dim \operatorname{Null}(I + A(\hat{q}_N(t),0)) > 0$ is again finite, we can repeat the preceding argument until we have a piecewise linear $\hat{q}_M(t)$ with corners in $\mathcal{O}$ such that $\hat{q}_M(0) = \hat{q}_1, \hat{q}_M(1) = \hat{q}_2, q_M(t)$ is a real-valued function in $C_0^\infty(\mathbf{R}^3)$ for $t\in[0,1]$, and $\dim \operatorname{Null}(I + A(\hat{q}_M(t),0)) < 2$ for $t\in[0,1]$. Since $(I + A(\hat{q}_M(t),0))$ can have a nontrivial nullspace for only a finite number of t in $[0,1]$, and $I + A(\hat{q}_M(t),k)$ does not have a nullspace for $k > 0$, we complete the proof with a finite number of applications of Lemma 5.1. ∎

We are now ready to prove the main result of this work. Let $\mathcal{O}_1$ be the connected component of $\mathcal{O}$ containing $\mathcal{O} \cap H^r_{\alpha,N}$. Recall that $\mathcal{O} \cap H^r_N$ is dense in $H^r_{\alpha,N}$.

Theorem 5.4. *The Frechet derivative of the backscattering map at $\hat{q}$ is an isomorphism of $H_{\alpha,N}$ when $\hat{q}$ belongs to an open, dense subset $\mathcal{O}_2$ of $\mathcal{O}_1$. Moreover, $\mathcal{O}_2 \cap H^r_{\alpha,N}$ is dense in $H^r_{\alpha,N}$. By the implicit function theorem the backscattering map is an analytic homeomorphism on a neighborhood in $H_{\alpha,N}$ of any $\hat{q} \in \mathcal{O}_2$.*

Proof. The zero potential belongs to $\mathcal{O}_1$. Moreover, the Frechet derivative of the backscattering map at the zero potential is $[Tf](\xi) = -f(2\xi)$ which is an isomorphism. Thus, letting $\mathcal{O}_2$ be the subset of $\mathcal{O}_1$ for which $dS(\hat{q})$ is an isomorphism, $\mathcal{O}_2$ is nonempty. Since $dS(\hat{q})$ is analytic in $\hat{q}$ and Fredholm, $\mathcal{O}_2$ is open. If $\mathcal{O}_2$ is not dense in $\mathcal{O}_1$, then, since $\mathcal{O}_1$ is open and connected, the boundary of the interior of $\mathcal{O}^c_2 \cap \mathcal{O}_1$ must be nonempty. Choose $\hat{q}_0$ in this set. Then any ball $B_\varepsilon = \{ \| \hat{q} - \hat{q}_0 \|_{\alpha,N} \leqq \varepsilon \}$ must contain points in the interior of $\mathcal{O}^c_2 \cap \mathcal{O}_1$ and in $\mathcal{O}_2$. Choose ε small enough that $B_\varepsilon \subset \mathcal{O}_1$, and pick $\hat{q}_1 \in \mathcal{O}_2 \cap B_\varepsilon$ and $\hat{q}_2 \in$(interior $\mathcal{O}^c_2 \cap \mathcal{O}_1) \cap B_\varepsilon$. Let

$$\hat{q}(t) = t\hat{q}_i + (1-t)\hat{q}_2 \quad t \in [0, 1].$$

Since $dS(\hat{q})$ is analytic in $\hat{q}$ on $\mathcal{O}$ and Fredholm of index $0, dS(\hat{q}(t))$ can fail to be an isomorphism for only a finite number of t in $[0, 1]$. This contradicts $\hat{q}_2 \in$ interior $\mathcal{O}^c_2 \cap \mathcal{O}_1$, and hence $\mathcal{O}_2$ is dense in $\mathcal{O}_1$.

Now suppose that we have $\hat{q}_0 \in H^r_{\alpha,N} \cap \mathcal{O}_1$ such that $dS(\hat{q})$ has a nontrivial kernel for $\hat{q} \in H^r_{\alpha,N}$ with $\| \hat{q} - \hat{q}_0 \|_{\alpha,N} < \delta$, for some $\delta > 0$. Introducing a finite rank operator K such that $dS(\hat{q}_0) + K$ is invertible and taking the determinant of $(dS(\bar{q}) + K)^{-1}(dS(\hat{q})) = I - (dS(\hat{q}) + K)^{-1}K$, we get a $\mathbf{C}$-valued analytic function $\lambda(\hat{q})$ on $H_{\alpha,N}$ such that for $\| \hat{q} - \hat{q}_0 \|_{\alpha,N} < \delta' < \delta, dS(\hat{q})$ has a nontrivial kernel if and only if $\lambda(\hat{q}) = 0$. As the Fourier transform of a space of real-valued functions, $H^r_{\alpha,N}$ is a real subspace of $H_{\alpha,N}$, i.e. given $f \in H_{\alpha,N}$, $f = f_1 + if_2$, f_1 and $f_2 \in H^r_{\alpha,N}$. It is a standard result that an analytic function vanishing on an open subset of a real subspace vanishes identically. One can see this by checking that complex Frechet derivatives of all orders must vanish on such a subset—as in the proof of this result for functions of one complex variable. Thus we conclude that $dS(\hat{q})$ has a nontrivial kernel for $\hat{q}$ in a neighborhood of $\hat{q}_0$ is $H_{\alpha,N}$. This contradicts the density of $\mathcal{O}_2$ in $\mathcal{O}_1$. ■

Section 6. Real Potential and the Restricted Backscattering Map

When we restrict the backscattering map to $H^r_{\alpha,N} \cap \mathcal{O}$, we cannot expect its range lie in $H^r_{\alpha,N}$. Since $H^r_{\alpha,N}$ is the Fourier transform of a space of real-valued functions, one natural way to proceed is to take the projection of backscattering which is the Fourier transform of taking the real part. Thus, we define the "restricted backscattering map":

$$S_r : \hat{q} \to \frac{h(\xi, -\xi, |\xi|) + \bar{h}(-\xi, \xi, |\xi|)}{2}.$$

Thus S_r maps all of $H_{\alpha,N} \cap \mathcal{O}$ into $H^r_{\alpha,N}$. When we restrict S_r to $H^r_{\alpha,N} \cap \mathcal{O}$, it is a real-analytic function with Frechet derivative given by (see (4.7)),

$$[dS_r(\hat{q})](v) = -\frac{v(2\xi)+\bar{v}(-2\xi)}{2}$$
$$-\pi^{-3}\int_{\mathbf{R}^3}\left[\frac{h(\xi-2n,-\xi,|\xi|)v(2n)}{|\xi-2\eta|^2-(|\xi|+i0)^2}+\frac{\bar{h}(-\xi-2\eta,\xi,|\xi|)\bar{v}(2\eta)}{|\xi+2\eta|^2-(|\xi|-i0)^2}\right]d\eta$$
$$-2^{-4}\pi^{-6}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left[\frac{h(\xi,\tau,|\xi|)h(\tau-2\eta,-\xi,|\xi|)v(2n)}{(|\tau|^2-(|\xi|+i0)^2)(|2\eta-\tau|^2-(|\xi|+i0)^2)}\right.$$
$$\left.+\frac{\bar{h}(-\xi,\tau,|\xi|)\bar{h}(\tau-2\eta,\xi,|\xi|)\bar{v}(2\eta)}{(|\tau|^2-(|\xi|-i0)^2)(|2\eta-\tau|^2-(|\xi|-i0))^2}\right]d\eta\,dt.$$

If we make use of the identity $v(-\xi)=\bar{v}(\xi)$ and change variables in the appropriate integrals, this becomes

$$[dS_r(\hat{q})](v) = -v(2\xi)$$
$$-\pi^{-3}\int_{\mathbf{R}^3}\left[\frac{h(\xi-2\eta,-\xi,|\xi|)}{|\xi-2\eta|^2-(|\xi|+i0)^2}+\frac{\bar{h}(-\xi+2\eta,\xi,|\xi|)}{|\xi-2\eta|^2-(|\xi|-i0)^2}\right]v(2\eta)d\eta$$
$$-2^{-4}\pi^{-6}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\left[\frac{h(\xi,\tau,|\xi|)h(\tau-2\eta,-\xi,|\xi|)}{(|\tau^2-(|\xi|+i0)^2)(|2\eta-\tau|^2-(|\xi|+i0)^2)}\right.$$
$$\left.+\frac{\bar{h}(-\xi,\tau,|\xi|)\bar{h}(\tau+2\eta,\xi,|\xi|)}{(|\tau|^2-(|\xi|-i0)^2)(|\tau+2\eta|^2-(|\xi|-i0)^2)}\right]v(2\eta)d\eta\,d\tau.$$

Thus $dS_r=(I+B+\underline{B}+C+\underline{C})T$. The proof of Theorem 4.3 can be repeated without change to show that $B^2, \underline{B}^2$, C and $\underline{C}$ are compact on $H_{\alpha,N}$. The proof applies to $B\underline{B}$ as well after one notes that changing $+i0$ to $-i0$ in one factor of the denominator of T_1 (see (4.9)) does not invalidate the proof: it merely interchanges the arguments for S_4 and S_5. Thus we conclude:

Theorem 6.1. *S_r is a real-analytic mapping of $H^r_{\alpha,N}\cap\mathcal{O}$ into $H^r_{\alpha,N}$ and its differential is a Fredholm operator of index zero.*

Analogue of Theorem 5.4 here is the following theorem. Its proof coincides with the first paragraph of the proof of Theorem 5.4.

Theorem 6.2. *Let $\mathcal{O}^r_1$ denote the component of $H^r_{\alpha,N}\cap\mathcal{O}$ containing the zero potential. Then the set $\mathcal{O}^r_2$ of $\hat{q}\in\mathcal{O}^r_1$ such that $dS_r(\hat{q})$ is an isomorphism of $H^r_{\alpha,N}$ is open and dense in $\mathcal{O}^r_1$. Hence, the implicit function theorem implies that S_r is a real analytic homeomorphism on a neighborhood of each $\hat{q}\in\mathcal{O}^r_2$.*

The set $\mathcal{O}^r_1$ is certainly not dense in $H^r_{\alpha,N}$. However, one does have the following.

Proposition 6.3. *The set $\mathcal{O}^r_1$ contains all $\hat{q}$ such that $I+A(\hat{q},0)$ injective, $q\in C^\infty_0(\mathbf{R}^3)$ and $-\Delta+q$ has no negative eigenvalues as an operator on $L^2(\mathbf{R}^3)$.*

Proof. It will suffice to show there is a curve $q(t)$ of real-valued functions in $C^\infty_0(\mathbf{R}^3)$ with $q(0)=q$ and $q(t_0)=0$ such that $I+A(\hat{q}(t),0)$ is injective for $t\in[0,t_0]$.

Let $E_0 f=(4\pi)^{-1}\int_{\mathbf{R}^3}|x-y|^{-1}f(y)dy$. If q is a real-valued function in $C^\infty_0(\mathbf{R}^3)$

and $-\Delta + q$ has no negative eigenvalues, we claim

$$\int_{\mathbf{R}^3} (\overline{E_0 f})(f + qE_0 f)\,dx \geqq 0 \tag{6.1}$$

for all $f \in C_0^\infty(\mathbf{R}^3)$. Let φ be a smooth function satisfying $\varphi(x) = 1$ for $|x| < 1$ and $\varphi(x) = 0$ for $|x| > 2$. Let $\varphi_R(x) = \varphi(x/R)$. Given $f \in C_0^\infty(\mathbf{R}^3)$, let $u = E_0 f$ and $u_R = \varphi_R u$. By assumption

$$\int_{\mathbf{R}^3} \bar{u}_R(-\Delta u_R + qu_R)\,dx \geqq 0.$$

Since $|u| = 0(|x|^{-1})$ and $|\nabla u| = 0(|x|^{-2})$ for $|x|$ large, one checks easily that

$$\lim_{R \to \infty} \int_{\mathbf{R}^3} \bar{u}_R(-\Delta u_R + qu_R)\,dx = \int_{\mathbf{R}^3} \bar{u}(-\Delta u + qu)\,dx,$$

which implies (6.1).

If we now assume that $I + A(\hat{q}, 0)$ is injective on $H_{\alpha,N}$, it follows that $I + qE_0$ is injective on $C_0^\infty(\mathbf{R}^3)$. Since (6.1) implies that

$$\left| \int_{\mathbf{R}^3} (\overline{E_0 g})(f + qE_0 f)\,dx \right|^2 \leqq \int_{\mathbf{R}^3} (\bar{E}_0 g)(g + qE_0 g)\,dx \int_{\mathbf{R}^3} (\overline{E_0 f})(f + qE_0 f)\,dx$$

for all $f, g \in C_0^\infty(\mathbf{R}^3)$, if $\int_{\mathbf{R}^3} (\overline{E_0 f})(f + qE_0 f)\,dx = 0$, then $\int_{\mathbf{R}^3} (\overline{E_0 g})(f + qE_0 f)\,dx = 0$ for all $g \in C_0^\infty(\mathbf{R}^3)$. Hence $f + qE_0 f = 0$, which contradicts the injectivity of $I + qE_0$, if $f \neq 0$. Thus

$$\int_{\mathbf{R}^3} (\overline{E_0 f})(f + qE_0 f)\,dx > 0 \tag{6.2}$$

for all nonzero $f \in C_0^\infty(\mathbf{R}^3)$.

Let $\chi \in C_0^\infty(\mathbf{R}^3)$ be a nonnegative function which is identically 1 on the support of q. We define

$$q(t) = \begin{cases} q + t\chi, & t \in [0, t_1] \\ (t_1 + 1 - t)(q + t_1\chi), & t \in [t_1, t_1 + 1] \end{cases}$$

where t_1 is chosen large enough that $q + t_1\chi$ is nonnegative. Now

$$\int_{\mathbf{R}^3} (\overline{E_0 f})(f + q(t)E_0 f)\,dx > 0 \tag{6.3}$$

for all nonzero $f \in C_0^\infty(\mathbf{R}^3)$. For $t \in [0, t_1]$ (6.3) follows from (6.2) and for $t \in [t_1, t_1 + 1]$ it follows from the strict positivity of $\int_{\mathbf{R}^3} f E_0 \bar{f}\,dx$. If $I + A(\hat{q}(t), 0)$ had a null vector $\hat{f} \in H_{\alpha,N}$ for some $t \in [0, t_1 + 1]$, then one would have $\partial_\xi^\beta \hat{f} \in H_{\alpha,N'}$ for all N' and β by Lemma 3.3. Thus $f + q(t)E_0 f = 0$ and $E_0 f \in C^\infty(\mathbf{R}^3)$. Hence $f \in C_0^\infty(\mathbf{R}^3)$ contradicting (6.3). ■

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