Commun. Math. Phys. 123, 425–452 (1989)

Communications in
Mathematical
Physics

The Sine-Gordon Field Theory Model at $\alpha^2 = 8\pi$, the Non-Superrenormalizable Theory

Francesco Nicolò[1] and Paolo Perfetti[2]

[1] Dipartimento di Fisica, Università di Lecce, Via per Arnesano, I-73100 Lecce, Italy
[2] Dipartimento di Fisica, Università „La Sapienza", Ple Aldo Moro 5, I-00185 Roma, Italy

Abstract. We study the Sine-Gordon field theory at $\alpha^2 = 8\pi$. We prove that the theory is renormalizable but not superrenormalizable and we show how the perturbative renormalization procedure works in this case where the interaction is not polynomial. To go beyond the perturbative results we investigate the β-functional equation for this theory and discuss in what sense at $\alpha^2 = 8\pi$ the theory is lacking the asymptotic freedom and how it is asymptotic free for $\alpha^2 < 8\pi$ in a appropriate region of the coupling constants.

Introduction

We study the renormalizability and the asymptotic freedom of the field theory model defined by the potential

$$V(\varphi) = \lambda \int_\Lambda d^2x : \cos\alpha\varphi_x : = \frac{\lambda}{2} \sum_{\sigma = \pm 1} \int_\Lambda d^2x : e^{i\alpha\sigma\varphi_x} :, \tag{0.1}$$

where $\lambda \in R \backslash 0$, $\alpha^2 = 8\pi$, Λ is a finite volume in the euclidean $d = 2$ space-time with periodic boundary conditions. The properties of the Sine-Gordon model crucially depend on the α value:

a) $\alpha^2 \in [4\pi, 8\pi)$.

The model is superrenormalizable; there is a finite number of divergences which can be cured with field independent counterterms. Moreover as α^2 approaches 8π the number of divergences tends to infinity. It has been proven in a series of papers [1–4] that in this range of values, for λ sufficiently small, the model exists.

b) $\alpha^2 = 8\pi$

i) The model is only renormalizable. The number of divergences does not depend on the order of the perturbation theory and three types of counterterms are necessary, two of them field dependent. The renormalizability of the model has been proven only in a perturbative sense.

ii) The model is asymptotically free in a region of the (λ, δ) plane where λ is the coupling constant and $(1+\delta)$ is the wave function renormalization constant. The region is specified by the conditions: $\delta \leqq -c\lambda$ if $\lambda > 0$, $\delta \leqq c\lambda$ if $\lambda < 0$, where c is a positive constant that depends on the model. In all other regions of the (λ, δ) plane the model is not asymptotically free. *It is not possible to have the asymptotic freedom starting from a point* (λ, δ) *with* $\delta = 0$.

This paper is devoted to prove statements i) and ii) of b). To our knowledge this is the first time that the renormalization procedure (at the perturbative level) is fully developed for a non-polynomial and non-superrenormalizable theory. Statements ii) shed some light on the properties of the theory beyond the perturbative results and moreover on the Coulomb gas also for the corresponding temperature.

1. Definitions and Notations

The free field φ_x is a gaussian random field whose measure has the covariance

$$C_{xy} = \frac{1}{(2\pi)^2} \int_{R^2} d^2p \, \frac{e^{ip(x-y)}}{p^2+m^2}, \tag{1.1}$$

or better its periodic version

$$C_{xy} = \sum_{\underline{n} \in Z^2} C_{x; y-\underline{n}L}, \quad \text{where } L \text{ is the side of } \Lambda .$$

The properties of the covariances we are interested in are the same in both cases. The field φ_x is very irregular so we do a multiscale decomposition:

$$1 = \lim_{N \to \infty} \left(e^{-(p^2+m^2)} + \sum_{k=1}^{N} \left(e^{-(p^2+m^2)\Upsilon^{-2k}} - e^{-(p^2+m^2)\Upsilon^{-2(k-1)}} \right) \right)$$

$$:= \sum_{k=0}^{\infty} F((p^2+m^2)\Upsilon^{-2k}),$$

$$C_{xy}^{(k)} = \frac{1}{(2\pi)^2} \int d^2p F((p^2+m^2)\Upsilon^{-2k}) \frac{e^{ip(x-y)}}{p^2+m^2}, \qquad \Upsilon > 1 , \tag{1.2}$$

$$C_{xy} = \sum_{k=0}^{\infty} C_{xy}^{(k)} . \tag{1.3}$$

$C_{xy}^{(k)}$ is associated at $\varphi_x^{(k)}$ and one can think of φ_x as decomposed in the following way:

$$\varphi_x = \sum_{k=0}^{\infty} \varphi_x^{(k)} . \tag{1.4}$$

We define the regularized covariance $C_{xy}^{(\leqq N)} = \sum_{k=0}^{N} C_{xy}^{(k)}$ and the associated field

$$\varphi_x^{(\leqq N)} = \sum_{k=0}^{N} \varphi_x^{(k)} .$$

The covariances have the following properties:

$$|\underline{\partial}^r C_{xy}^{(k)}| \leqq B_k \Upsilon^{k|\underline{r}|} e^{-\kappa m \Upsilon^k |x-y|} \quad \kappa \text{ depends on } \Upsilon, \underline{r}, \text{ but not on } k, \tag{1.5}$$

$$\underline{\partial}^r = \prod_{i=1}^{2} \left(\frac{\partial}{\partial x_i}\right)^{r_i} \quad |\underline{r}| = \sum_{i=1}^{2} r_i, \quad |\varphi_x^{(k)} - \varphi_y^{(k)}| \leqq B_k(\Upsilon^k |x-y|), \tag{1.6}$$

$$|\varphi_x^{(k)} - \varphi_y^{(k)} - \underline{\partial}\varphi_y^{(k)}(x-y)| \leqq B_k(\Upsilon^k|x-y|)^2, \tag{1.7}$$

with probability bounded below by $(1-e^{-B_k^2})$.

The scales of smoothness and independence of the field $\varphi_x^{(k)}$ are the same. These properties are summarized by saying that the field $\varphi_x^{(k)}$ is regular on scale Υ^{-k}. The regularized potential is

$$V(\varphi^{(\leqq N)}) = \lambda \int_\Lambda d^2x : \cos\alpha\varphi_x^{(\leqq N)} = \frac{\lambda}{2} \sum_{\sigma=\pm 1} \int_\Lambda d^2x : e^{i\alpha\sigma\varphi_x^{(\leqq N)}} : . \tag{1.8}$$

(W do not specify the value of α to follow, the case $\alpha^2 < 8\pi$ until it is possible.)

We want to compute the effective potential $V_N^{(k)}$ defined through

$$\int P(d\varphi^{(k+1)}) P(d\varphi^{(k+2)}) \ldots P(d\varphi^{(N)}) e^{V_N^{(N)}(\varphi^{(\leqq N)})} := e^{V_N^{(k)}(\varphi^{(\leqq k)})} \tag{1.9}$$

and to study its limit as $N \to \infty$. It is well known (see [1–4]) that when $\alpha^2 > 4\pi$, $\lim_{N\to\infty} V_N^{(k)}(\varphi^{(\leqq k)})$ does not exist.

Before discussing how to renormalize the theory and control the previous limit we recall how to calculate integrals like (1.9):

$$\int P(d\varphi^{(k)})(*) = \mathscr{E}_{(k)}(*), \qquad \mathscr{E}_k^T(x;n) = \frac{d^n}{(d\tau)^n} \ln \mathscr{E}_{(k)}(e^{\tau x})|_{\tau=0}.$$

Of course $\mathscr{E}_k^T(x;0)=0$, $\mathscr{E}_k^T(x;1)=\mathscr{E}_{(k)}(x)$,

$$\mathscr{E}_k^T(x_1, \ldots, x_p; n_1, \ldots, n_p) = \frac{d^{n_1}}{(d\tau_1)^{n_1}}, \ldots, \frac{d^{n_p}}{(d\tau_p)^{n_p}} \ln \mathscr{E}_{(k)} \exp\left(\sum_{j=1}^{p} \tau_j x_j\right)\Big|_{\tau_j=0}.$$

$\mathscr{E}_{(k)}$ and $\mathscr{E}_k^T$ are related by

$$\mathscr{E}_{(k)}(e^x) = \exp \sum_{p=1}^{\infty} \frac{1}{p!} \mathscr{E}_k^T(x;p) \tag{1.10}$$

and we have a "Leibnitz formula"

$$\mathscr{E}_k^T(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_p x_p; n) = \sum_{n_1+n_2+\ldots n_p=n} \frac{\lambda_1^{n_1} \lambda_2^{n_2} \ldots \lambda_p^{n_p}}{n_1! \ldots n_p!} \mathscr{E}_k^T(x_1, \ldots, x_p; n_1, \ldots, n_p). \tag{1.11}$$

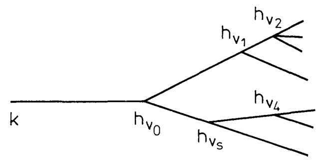


Fig. 1

Applying these formulas to $V_N^{(k)}(\varphi^{(\le k)})$, which we call simply V, we obtain

$$\mathscr{E}_{(N)}(e^V)=\exp\sum_{k=1}^{\infty}\frac{1}{k!}\mathscr{E}_N^T(V;k):=e^{V_N^{(N-1)}(\varphi^{(\le N-1)})}, \tag{1.12}$$

$$\begin{aligned}\mathscr{E}_{(N-1)}\mathscr{E}_{(N)}(e^V)&=\mathscr{E}_{(N-1)}\left(\exp\sum_{k=1}^{\infty}\frac{1}{k!}\mathscr{E}_N^T(V;k)\right)\\&=\exp\sum_{m=1}^{\infty}\frac{1}{m!}\mathscr{E}_{N-1}^T\left(\sum_{k=1}^{\infty}\frac{1}{k!}\mathscr{E}_N^T(V;k)\right)\\&=\sum_{s=1}^{\infty}\sum_{n_1+n_2\ldots+\ldots=s}\frac{1}{n_1!n_2!\ldots}\frac{1}{1!^{n_1}2!^{n_2}\ldots},\end{aligned}$$

$$\mathscr{E}_{N-1}^T(\mathscr{E}_N^T(V;1),\mathscr{E}_N^T(V;2),\ldots;n_1,n_2,\ldots):=e^{V_N^{(N-2)}(\varphi^{(\le N-2)})}. \tag{1.13}$$

The next step is more complicated and does not improve the intuition of this formula. Intuition that can be made easier by introducing a construction developed a few years ago by G. Gallavotti and one of us, (F.N.), which was called "tree expansion." We will not give here the details of the tree expansion but we refer to (Gallavotti [3], Gallavotti-Nicolò [5], and Pordt [6], and we just recall that one obtains it starting from (1.8) with the following recipes:

a) One has to integrate one frequency scale after the other, see (1.9).
b) At each frequency one performs a cumulant expansion in λ, see (1.10).
c) One collects this multiple expansion together.

A tree is just a graphical way of selecting a specific term of this expansion, which is made of truncated expectations of truncated expectations of...on different scales. The point is that for each of these terms one can get very good estimates which take care of the natural length scales of the various factors. A tree is drawn in Fig. 1.

The tree expansion allows us to incorporate pretty well the counterterms needed for the renormalization so that one is also able to use it to study the flow of the running coupling constants. Using this expansion we obtain, starting from (1.8):

$$V_N^{(k)}(\varphi^{(\le k)})=\sum_{n=1}^{\infty}\lambda^n V_{n;N}^{(k)}(\varphi^{(\le k)})), \tag{1.14}$$

$$V_{n;N}^{(k)}(\varphi^{(\le k)})=\sum_{\theta:v(\theta)=n}\sum_{\substack{\sigma_1,\ldots,\sigma_n\\ \sigma_i=\pm1}}\sum_{\substack{\underline{h}\\ \underline{h}\le N}}\int_{\substack{\Lambda x\Lambda x,\ldots,x\Lambda\\ n\,\text{times}}}d^2x_1,\ldots,d^2x_n\,W^{(k)}(\theta;\underline{x};\underline{\sigma};\underline{h}):e^{i\alpha\sum_{j=1}^{n}\sigma_j\varphi_{x_j}^{(\le k)}}:, \tag{1.15}$$

where
$v(\theta)$ number of final lines (points) of the three θ,
σ_i charge of the line i,
$\underline{h}=\{h_v\}_{v\in\theta}$,
h_v frequency of the bifurcation corresponding to the vertex v,
h_{v_0} frequency of the lowest bifurcation,
k: root frequency.

It was proven in [3] that at fixed N and α^2 the following estimate holds:

$$|V_{n;N}^{(k)}(\varphi^{(\leqq k)})|\leqq \lambda^n C(n,k) \sup_\theta \left\{ \sum_{\underline{h}\leqq N} \prod_{\substack{v\in\theta \\ v\neq v_0}} \Upsilon^{-(\varrho_v+2)(h_v-h_{v'})}\Upsilon^{-\varrho_{v_0}(h_{v_0}-k)} \right\} \tag{1.16}$$

with

$$\varrho_v=-\left(\frac{\alpha^2}{4\pi}-2\right)(n_v-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q_v^2, \tag{1.17}$$

where $n_v\geqq 2$ is the number of final lines that merge into v and v' is the bifurcation which immediately follows v (see Fig. 1). $Q_v=\sum_{i=1}^{n}\sigma_i$ is the charge of the n_v points associated to v and it will be often called the charge of the "cluster" v.

It is easy to realize that simple estimates of the truncated expectations give for $C(n,k)$ a very bad n dependence, worst that $n!$, so the estimate of the series would be divergent for every $\lambda\neq 0$. More refined estimates [7, 8] show that for $\alpha^2<6\pi$ the power series in λ is convergent, but up to now the result has not been proven for $\alpha^2\geqq 6\pi$. Anyway before summing over n we have to remove the cut-off N and, because of the sum over $\underline{h}$, it is necessary $\varrho_v+2>0$, $\forall v\neq v_0$ and $\varrho_{v_0}>0$ to have the coefficients of the series finite as $N\to\infty$.

These are essentially the two main problems one is faced to prove the existence of a field theory model. The solution of the second one proves the perturbative renormalizability of the model; the first problem in many cases cannot be avoided, after the renormalization, and simply tells us that the perturbative series is not convergent but only asymptotic and that to prove the existence of the theory beyond the perturbative level, one has some (hard) extra work to do. The next sections are devoted to the study of the second problem. In the last section some steps toward a solution of the first problem are performed.

2. The Perturbative Renormalization

We distinguish two cases a) $\alpha^2\in[4\pi,8\pi)$ and b) $\alpha^2=8\pi$.

a) $\alpha^2\in[4\pi,8\pi)$

This case has been studied in [1–4] (see also [9]) where the model has been proved to exist also constructively. We recall only some results to make more evident the analogies and the differences with case b).

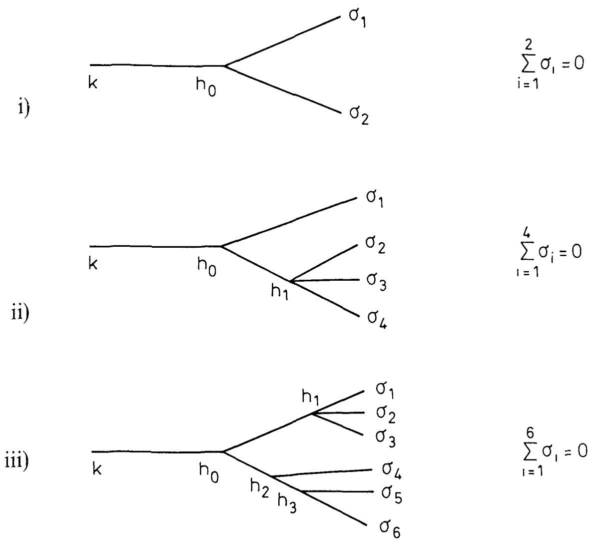


Fig. 2

If $Q^2 > 0$ there are not divergences [see (1.17)]. It can be verified that if $\alpha^2 \in [4\pi, 6\pi)$ the only divergences appears when $n = 2$, i.e. the tree (i) of Fig. 2 diverges (when we sum over the frequencies).

If $\alpha^2 \in [6\pi, \frac{20}{3}\pi)$ divergences are present for $n = 2, 4$. Terms with n odd are forbidden by the condition $Q = 0$. There are, therefore, many trees whose contribution is, before the renormalization, divergent, namely all those with a number of final lines smaller or equal to 4. The tree (ii) is one of them.

If $\alpha^2 \in [\frac{20}{3}\pi, 7\pi)$ the divergences are present when $n = 2, 4, 6$. An example of diverging tree with $n = 6$ is the tree (iii).

This proliferation of diverging trees corresponds to the existence of the thresholds of the Sine-Gordon model. With fixed n it must be $\alpha^2 < 8\pi\left(1 - \frac{1}{n}\right)$ for the tree with n final lines to be convergent.

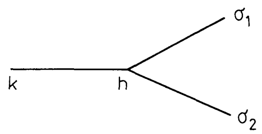
b) $\alpha^2 = 8\pi$

The situation changes drastically: from (1.16) and (1.17) we have

$$\varrho_v = 2(Q_v^2 - 1). \tag{2.1}$$

Therefore if $Q_v = 0$ also the sums over the frequencies of the internal bifurcations can give divergent contributions. Moreover the n dependence in ϱ_{v_0} and the n_v dependence in ϱ_v has disappeared. This is the sign that the theory is only renormalizable. In fact, for fixed Q_v, there are infinitely many trees with arbitrary n_v and the same Q_v. The first consequence immediately appearing is that the divergences of the trees with $n = 2, 3$ drawn in Fig. 3 require to be cured the introduction of field dependent counterterms.

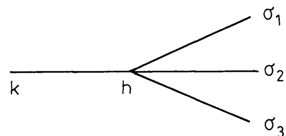
$Q=0$, $\varrho=2$ quadratic divergences



$\sum_{i=1}^{2}\sigma_i=0$

j)

$Q=\pm 1$, $\varrho=0$ logarithmic divergences



$\sum_{i=1}^{3}\sigma_i=\pm 1$

Fig. 3 jj)

Now we estimate the contribution of some of these divergent trees and illustrate the mechanism to cure them. We do not specify the α value to treat together the cases a) and b) as far as possible.

(contribution of the tree j)

$$=\sum_{h=k+1}^{N}\frac{1}{2}\mathscr{E}_{k+1},\ldots,\mathscr{E}_{h-1}\int_{\Lambda\times\Lambda}d^2x_1d^2x_2\mathscr{E}_h^T\left[:\frac{\lambda}{2}e^{i\alpha\sigma_1\varphi_{x_1}^{(\leq h)}}:,:\frac{\lambda}{2}e^{i\alpha\sigma_2\varphi_{x_2}^{(\leq h)}}:\right]. \quad (2.2)$$

Let us take $\sigma_1=+1$ and $\sigma_2=-1$, we get

$$\frac{1}{2}\left(\frac{\lambda}{2}\right)^2\sum_{h=k+1}^{N}\int_{\Lambda\times\Lambda}d^2x_1d^2x_2e^{\alpha^2C_{x_1x_2}^{(\leq h-1)}}(e^{\alpha^2C_{x_1x_2}^{(h)}}-1):e^{i\alpha(\varphi_{x_1}^{(\leq k)}-\varphi_{x_2}^{(\leq k)})}:. \quad (2.3)$$

With standard estimates the previous integral is (upper) bounded by

$$C|\Lambda|\lambda^2\sum_{h=k+1}^{N}\gamma^{h\left(\frac{\alpha^2}{2\pi}-2\right)}, \quad (2.4)$$

which for $\alpha^2<4\pi$ is convergent and for $\alpha^2\geqq 4\pi$ divergent. As we have said, for $\alpha^2<4\pi$ the theory has been constructed with different techniques [10] without using the renormalization group, so we discuss only the $\alpha^2\geqq 4\pi$ case. It is clear that if we had in the previous integral an extra factor $|x-y|^2$ the same calculation would have been given a convergent result for $\alpha^2<8\pi$. To achieve this zero in $|x-y|$ we proceed in this way: we sum to the previous tree a similar one with the opposite charges obtaining

$$\left(\frac{\lambda}{2}\right)^2\sum_{h=k+1}^{N}\int_{\Lambda\times\Lambda}d^2x_1d^2x_2e^{\alpha^2C_{x_1x_2}^{(\leq h-1)}}(e^{\alpha^2C_{x_1x_2}^{(h)}}-1):\cos\alpha(\varphi_{x_1}^{(\leq k)}-\varphi_{x_2}^{(\leq k)}):. \quad (2.5)$$

Then we observe that the substitution of $\cos\alpha(\varphi_{x_1}^{(\leq k)}-\varphi_{x_2}^{(\leq k)})$ with $(\cos\alpha(\varphi_{x_1}^{(\leq k)}-\varphi_{x_2}^{(\leq k)})-1)$ would produce a zero of order $|x-y|^2$ which would guarantee the convergence of (2.4). Therefore, when $\alpha^2<8\pi$, the recipe to make the contribution of this tree finite is to subtract from it a similar contribution with $\cos\alpha(\varphi_{x_1}^{(\leq k)}-\varphi_{x_2}^{(\leq k)})$ replaced by one. It has been proven in [3] that to cure the divergences, (a finite number increasing with α^2), which appear in more complicated trees it is enough to proceed in the same manner replacing in their expressions $\cos\alpha\left(\sum_{j=1}^{n}\sigma_j\varphi_{x_j}^{(\leqq k)}\right)$ with $\left(\cos\alpha\left(\sum_{j=1}^{n}\sigma_j\varphi_{x_j}^{(\leqq k)}\right)-1\right)$ when $\sum_{j=1}^{n}\sigma_j=0$.

When $\alpha^2=8\pi$ this subtraction is not sufficient as a logarithmic divergence remains. This suggests that we define also in this case, as was done in the Φ_4^4 theory, two operations denoted with the symbols R and L which have to be applied to the vertices of a generic tree of the expansion and which modify the factor associated to the vertex itself. The way the R and the L operate is in principle very simple: A truncated expectation with respect to the frequency h_v is associated to each vertex v of a tree operating over functions of the fields $\varphi_{x_j}^{(\leqq h_v)}=\varphi_{x_j}^{(h_v)}+\varphi_{x_j}^{(<h_v)}$, the result being a function of the fields of lower frequency $\varphi_{x_j}^{(<h_v)}$. The R operation is defined on the function of these fields, the L operates as $-L=-(1-R)$ on the field function and moreover changes $\sum_{h_v=h_{v'}+1}^{N}$ into $\sum_{h_v=0}^{h_{v'}}$ (see [5]). The vertex contributions is modified in such a way that applying these operations to all the vertices of a generic tree it becomes finite also when the cutoff is removed. The way the R and L operations have to be defined depends, of course, on the model we are considering.

In this way a new tree expansion has been produced where each tree has appended to each bifurcation the symbol R or the symbol $-L$ in all the possible ways denoting which operation has to be performed on each bifurcation. This new expansion will be called: "renormalized tree expansion" [11]. The proof that the theory is (perturbatively) renormalizable is accomplished in the following two steps:

i) First one proves that each term of the "renormalized tree expansion" is finite in the limit $N\to\infty$.

ii) Second one has to show that this "renormalized tree expansion" can be reobtained by simply starting from a new interaction which differs from the original one because its coupling constants are different (the "bare coupling constants" instead of the physical ones) and by the (possible) presence of some other terms (counterterms) in a finite number.

Of course it is the possibility of proving ii) *that makes the proof of* i) *relevant.* This will be discussed in Sect. 3.

The R and L operations can be defined first for the simplest trees and then in general. Looking at the tree (j) of Fig. 3 we define:

$$R:\cos\alpha(\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)}):=:\cos\alpha(\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)})-1+\frac{\alpha^2}{2}(\underline{\partial}\varphi_{\bar{x}}^{(\leqq k)}(x_1-x_2))^2:,\tag{2.6}$$

$\bar{x}=\dfrac{x_1+x_2}{2}$, $\alpha^2=8\pi$ we still denote by α^2,

$$L:\cos\alpha(\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)}):=:1-\frac{\alpha^2}{2}(\underline{\partial}\varphi_{\bar{x}}^{(\leqq k)}(x_1-x_2))^2:.$$

As

$$\left|\cos\alpha(\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)})-1+\frac{\alpha^2}{2}(\underline{\partial}\varphi_{\bar{x}}^{(\leqq k)}(x_1-x_2))^2\right|$$
$$\leqq O(\alpha^4|\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)}|^4)\leqq B_k\Upsilon^{4k}|x_1-x_2|^4,$$

substituting in the sum (2.4) we get the following estimate:

$$C|\Lambda|\lambda^2 \sum_{h=k+1}^{N} \Upsilon^{h\left(\frac{\alpha^2}{2\pi}-2\right)}\Upsilon^{-4h}, \tag{2.7}$$

which now converges as $N\to\infty$.

Due to the R and L operations a term which contains the gradient of the field squared has been generated. This implies that from now on we have to calculate truncated expectations in which a new type of Wick monomial can be present and that if we want to connect the renormalized tree expansion to the presence of counterterms in the original interaction it will be necessary to include this term in the regularized interaction we start with. We would write, therefore:

$$V(\varphi^{(\leqq N)}, \underline{\lambda}_N(\lambda), N)=\lambda \int_\Lambda d^2x : \cos\alpha\varphi_x^{(\leqq N)} : \\ + d(\lambda, N) \int_\Lambda d^2x :(\underline{\partial}\varphi_x^{(\leqq N)})^2 : + u(\lambda, N) \int_\Lambda d^2x , \tag{2.8}$$

where at the second order in λ,

$$u(\lambda,N)\int_\Lambda d^2x \\ = \left(\frac{\lambda}{2}\right)^2 \sum_{h=0}^{N} \int_{\Lambda\times\Lambda} d^2x_1 d^2x_2 e^{\alpha^2 C^{(\leqq h-1)}_{x_1x_2}}(e^{\alpha^2 C^{(h)}_{x_1x_2}}-1)[-L:\cos\alpha(\varphi_{x_1}^{(\leqq N)}-\varphi_{x_2}^{(\leqq N)}):]_1 \\ = -\left(\frac{\lambda}{2}\right)^2 \sum_{h=0}^{N} \int_{\Lambda\times\Lambda} d^2x_1 d^2x_2 e^{\alpha^2 C^{(\leqq h-1)}_{x_1x_2}}(e^{\alpha^2 C^{(h)}_{x_1x_2}}-1),$$

and

$$d(\lambda,N)\int_\Lambda d^2x :(\underline{\partial}\varphi_x^{(\leqq N)})^2 \\ = \left(\frac{\lambda}{2}\right)^2 \sum_{h=0}^{N} \int_{\Lambda\times\Lambda} d^2x_1 d^2x_2 e^{\alpha^2 C^{(\leqq h-1)}_{x_1x_2}}(e^{\alpha^2 C^{(h)}_{x_1x_2}}-1)[-L:\cos\alpha(\varphi_{x_1}^{(\leqq N)}-\varphi_{x_2}^{(\leqq N)}):]_\partial \\ = \left(\frac{\lambda}{2}\right)^2 \sum_{h=0}^{N} \int_{\Lambda\times\Lambda} d^2x_1 d^2x_2 e^{\alpha^2 C^{(\leqq h-1)}_{x_1x_2}}(e^{\alpha^2 C^{(h)}_{x_1x_2}}-1): \frac{\alpha^2}{2}(\underline{\partial}\varphi_{\bar{x}}^{(\leqq N)}(x_1-x_2))^2 : \\ = \left(\text{introducing the variables } \bar{x}=\frac{x_1+x_2}{2},\ z=x_1-\bar{x}\right) \\ = \left(\frac{\lambda\alpha}{4}\right)^2 \sum_{h=0}^{N} \int d^2z e^{\alpha^2 C^{(\leqq h-1)}_{z0}}(e^{\alpha^2 C^{(h)}_{z0}}-1)|z|^2 \int_\Lambda d^2x :(\underline{\partial}\varphi_x^{(\leqq N)})^2 :. \tag{2.9}$$

The L brings an extra index because the localization of $:\cos\alpha(\varphi_{x_1}^{(\leqq k)}-\varphi_{x_2}^{(\leqq k)}):$ generates two terms and the index near them remembers which one is produced.

Now we consider the $Q=\pm 1$ and $n=3$ case. We start examining the tree (jj) of Fig. 3. Its contribution is:

$$\frac{1}{3!}\mathscr{E}_{k+1},\ldots,\mathscr{E}_{h-1} \\ \times \int_{\Lambda\times\Lambda\times\Lambda} d^2x_1 d^2x_2 d^2x_3 \sum_{h=k+1}^{N} \mathscr{E}_h^T\left[:\frac{\lambda}{2}e^{i\alpha\sigma_1\varphi_{x_1}^{(\leqq h)}}:,:\frac{\lambda}{2}e^{i\alpha\sigma_2\varphi_{x_2}^{(\leqq h)}}:,:\frac{\lambda}{2}e^{i\alpha\sigma_3\varphi_{x_3}^{(\leqq h)}}:\right].$$

To consider a definite case we choose $\sigma_1 = 1, \sigma_2 = -1, \sigma_3 = +1$, and we sum to this contribution that one with all the opposite charges obtaining

$$\begin{aligned}
&\frac{1}{3!}\left(\frac{\lambda}{2}\right)^3 \int_{\Lambda\times\Lambda\times\Lambda} d^2x_1 d^2x_2 d^2x_3 \\
&\times \sum_{h=k+1}^{N} e^{-\alpha^2[C^{(\leqq h-1)}_{x_1x_3} - C^{(\leqq h-1)}_{x_2x_3} - C^{(\leqq h-1)}_{x_1x_2}]} \{(e^{\alpha^2 C^{(h)}_{x_1x_2}} - 1)(e^{\alpha^2 C^{(h)}_{x_2x_3}} - 1) \\
&+ (e^{-\alpha^2 C^{(h)}_{x_1x_3}} - 1)(e^{\alpha^2 C^{(h)}_{x_2x_3}} - 1) \\
&+ (e^{\alpha^2 C^{(h)}_{x_1x_2}} - 1)(e^{-\alpha^2 C^{(h)}_{x_1x_3}} - 1) \\
&+ (e^{-\alpha^2 C^{(h)}_{x_1x_3}} - 1)(e^{\alpha^2 C^{(h)}_{x_2x_3}} - 1)(e^{\alpha^2 C^{(h)}_{x_1x_2}} - 1)\} 2 : \cos\alpha(\varphi^{(\leqq k)}_{x_1} - \varphi^{(\leqq k)}_{x_2} + \varphi^{(\leqq k)}_{x_3}) :,
\end{aligned} \tag{2.10}$$

which, estimated in the usual manner, gives

$$C|\Lambda|\lambda^3 \sum_{h=k+1}^{N} \gamma^{h\left(\frac{\alpha^2}{2\pi} - 4\right)} \approx |\Lambda|\lambda^3 N \to \infty, \quad \text{for} \quad N \to \infty, \tag{2.11}$$

and this is the expected logarithmic divergence.

This divergence can be cured as follows: define

$$R : \cos\alpha(\varphi^{(\leqq k)}_{x_1} - \varphi^{(\leqq k)}_{x_2} + \varphi^{(\leqq k)}_{x_3}) : = : \cos\alpha(\varphi^{(\leqq k)}_{x_1} - \varphi^{(\leqq k)}_{x_2} + \varphi^{(\leqq k)}_{x_3}) - \cos\alpha(\varphi^{(\leqq k)}_{x_1}) : \tag{2.12}$$

which has a zero of order one when all the points shrink.

$$L : \cos\alpha(\varphi^{(\leqq k)}_{x_1} - \varphi^{(\leqq k)}_{x_2} + \varphi^{(\leqq k)}_{x_3}) : = : \cos\alpha(\varphi^{(\leqq k)}_{x_1}) : . \tag{2.13}$$

The introduction of the R operation over the fields of the previous integral gives the following convergent estimate

$$C|\Lambda|\lambda^3 \sum_{h=k+1}^{N} \gamma^{h\left(\frac{\alpha^2}{2\pi} - 4\right)} \gamma^{-h}, \quad \frac{\alpha^2}{2\pi} = 4. \tag{2.14}$$

This is the second new aspect of the model at $\alpha^2 = 8\pi$. Another field dependent counterterm has been generated beyond the gradient squared which, being of the same form as the starting interaction, modifies the original coupling constant. If we want, at this level, to describe the theory in terms of the bare coupling constants we have to add $g(\lambda, N)$ to $u(\lambda, N)$, and $d(\lambda, N)$. $g(\lambda, N)$ is the bare coupling constant associated to the $\int_\Lambda d^2x : \cos\alpha\varphi^{(\leqq N)}_x :$ part of the interaction. At the third order in λ it receives a contribution, from the previous tree, of the following kind:

$$\begin{aligned}
g(\lambda, N) = \lambda + \frac{\lambda^3}{3!8} \sum_{h=0}^{N} \int_{\Lambda\times\Lambda} d^2y d^2z e^{-\alpha^2[C^{(\leqq h-1)}_{z0} - C^{(\leqq h-1)}_{(z-y)0} - C^{(\leqq h-1)}_{y0}]} \\
\times \{2(e^{\alpha^2 C^{(h)}_{y0}} - 1)(e^{\alpha^2 C^{(h)}_{(z-y)0}} - 1) + (e^{-\alpha^2 C^{(h)}_{z0}} - 1)(e^{\alpha^2 C^{(h)}_{(z-y)0}} - 1) \\
+ (e^{\alpha^2 C^{(h)}_{y0}} - 1)(e^{-\alpha^2 C^{(h)}_{z0}} - 1)(e^{\alpha^2 C^{(h)}_{(z-y)0}} - 1)\}.
\end{aligned} \tag{2.15}$$

The crucial fact is that these are the only counterterms we will need, multiplied by the appropriate bare coupling constants, formal series of the physical ones. The next pages are devoted to prove this statement.

We will study the theory in the "renormalized tree expansion" framework and modify the potential written in terms of the physical coupling constant as follows

$$V^{(N)} = \lambda \int_\Lambda d^2x : \cos\alpha\varphi^{(\leqq N)}_x : + \delta \int_\Lambda d^2x : (\partial\varphi^{(\leqq N)}_x)^2 : + \nu \int_\Lambda d^2x . \tag{2.16}$$

This just means that the final lines of the various trees in our expansion have to be thought of as bringing a label denoting to which one of these three terms they are associated. The bare coupling constants will be in this case $u(\lambda, \delta, \upsilon, N)$, $d(\lambda, \delta, N)$, and $g(\lambda, \delta, N)$, three formal series in the physical coupling constants.

A similar calculation to the one performed with δ and υ equal zero [see Eq. (1.15)] gives for the effective potential on scale k:

$$V_N^{(k)}(\varphi^{(\leq k)}) = \sum_{m=1}^{\infty} \sum_{\substack{n_0; n_1 \\ n_0+n_1=m}} \underline{\lambda}^{\underline{s}} V_{\underline{s};N}^{(k)}(\varphi^{(\leq k)}), \quad \text{where} \quad \underline{\lambda} = (\lambda, \delta),\ \underline{s} = (n_0, n_1). \tag{2.17}$$

The effective potential of a fixed order can be written, using the tree expansion, in the following way, which although a little cumbersome is appropriate for the subsequent estimates:

$$\begin{aligned} \sum_{\substack{n_0; n_1 \\ n_0+n_1=m}} \underline{\lambda}^{\underline{s}} V_{\underline{s};N}^{(k)}(\varphi^{(\leq k)}) = & \sum_{\theta : v(\theta)=m} \sum_{n_0=0}^{m} \underline{\lambda}^{\underline{s}} \sum_{\underline{h} : \underline{h} \leq N} \sum_{\underline{\sigma}} \sum_{(\underline{\alpha}, \underline{\beta}, \underline{\gamma})} \sum_{\mathscr{P}}{}^{*} \sum_{\mathscr{G}} \\ & \times \int_{\substack{\Lambda \times \Lambda \times, \dots, \times \Lambda \\ m\text{-times}}} d^2x_1, \dots, d^2x_{n_1} d^2y_1, \dots, d^2y_{n_0} \\ & \times W^{(k)}(\theta, \underline{b}; \underline{y}; \underline{\sigma}; \underline{\alpha}, \underline{\beta}, \underline{\gamma}, \mathscr{P}, \mathscr{G}) : \prod_{j=1}^{n_1} (\partial_{\gamma_j} \varphi_{x_j}^{(\leq k)})^{\beta_j} \\ & \times \prod_{v \in \mathscr{P}_0} \mathrm{Trig}_{\alpha_v} \varphi_{(y_v, \sigma_v)}^{(\leq k)} \prod_{v \in \mathscr{P}_1} e^{i\alpha \varphi_{(y_v, \sigma_v)}^{(\leq k)}} :, \end{aligned} \tag{2.18}$$

where

$$\underline{\beta} = (\beta_1 \dots \beta_{n_1}) \qquad \beta_j = (0, 1); \qquad \underline{\alpha} = \{\alpha_v\}_{v \in \mathscr{P}}, \qquad \alpha_v = \{c, s\};$$

$$\underline{\gamma} = (\gamma_1 \dots \gamma_{n_1}) \qquad \gamma_j = (0, 1); \qquad \mathscr{P} \text{ is a subset of } \{v\}_{v \in \theta}, \qquad \{v\} \supseteqq \mathscr{P}.$$

$\sum_{\mathscr{P}}^{*}$ is the sum over all the possible $\mathscr{P}$ such that for any $v, v' \in \mathscr{P}$: $v \cap v' = \emptyset$,

$$\mathrm{Trig}_c \varphi_{(y_v, \sigma_v)}^{(\leq k)} = \cos \alpha \varphi_{(y_v, \sigma_v)}^{(\leq k)}, \qquad \mathrm{Trig}_s \varphi_{(y_v, \sigma_v)}^{(\leq k)} = \sin \alpha \varphi_{(y_v, \sigma_v)}^{(\leq k)},$$

$\mathscr{G} = \{\mathscr{G}_v\}_{\substack{v \in \theta \\ v \neq v_0}}$, where $\mathscr{G}_v = (n_v^{\partial; e}, z_v)$ is a couple of non-negative integers describing, for any bifurcation (cluster) $v \neq v_0$,

$n_v^{\partial; e}$: the number of $\partial \varphi^{(<h_v)}$ factors present in the term associated to the bifurcation v of the tree after one has integrated over all the frequencies $\geqq h_v$. $n_v^{\partial; e}$ is, therefore, the number of gradφ lines going out from the cluster v (which can become internal half lines for one of the next cluster of lower frequency containing v).

z_v: the number of zeroes present in the field part associated to the bifurcation v (before the integration over the lower frequencies be performed); of course, again, these fields can be integrated but the zeroes will remain in each term of the sum.

Each $\sin \alpha \varphi_{(y_{\hat{v}}, \sigma_{\hat{v}})}^{(\leq h_v)}$ has a zero, if $Q_{\hat{v}} = 0$, when $\hat{v}$ is shrinked to a point (remark that $\hat{v}$ is different, in general, from v). $\prod_{\hat{v} \in \mathscr{P}} \cos \alpha \varphi_{(y_{\hat{v}}, \sigma_{\hat{v}})}^{(\leq h_v)}$ although it does not produce a zero when shrinked to a point gives 1 which in the next truncated expectation acts as a second order zero provided that: i) $|\mathscr{P}| = 1$, ii) $n_v^{\partial; e} = 0$.

We rewrite Eq. (2.17) in a more compact notation with the obvious meaning of the symbols,

$$\begin{aligned} V_N^{(k)}(\varphi^{(\leqq k)}) = \sum_{m=1}^{\infty} \sum_{\substack{n_0; n_1 \\ n_0+n_1=m}} \lambda^s V_{\underline{s};N}^{(k)}(\varphi^{(\leqq k)}) = \sum_{m=1}^{\infty} \sum_{\theta : v(\theta)=m} \sum_{n_0=0}^{m} \underline{\lambda}^{\underline{s}} \sum_{\underline{h}:\underline{h}\leqq N} \sum_{\underline{\sigma}} \\ \times \sum_{\hat{\mathscr{P}}}^{*} \sum_{\mathscr{G}} \int d\underline{x} d\underline{y} : (\underline{\partial}\varphi)_{\hat{\mathscr{P}}} (\mathrm{Trig}\, \varphi)_{\hat{\mathscr{P}}} (e^{i\alpha\varphi})_{\hat{\mathscr{P}}} : \\ \times W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \hat{\mathscr{P}}, \mathscr{G}) . \end{aligned} \tag{2.17'}$$

An estimate similar to Eq. (1.17) is given by the following

Theorem 1 (the unrenormalized case). *The following estimate holds:*

$$\begin{aligned} \int d\underline{x} d\underline{y} \delta(y_1) |W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \hat{\mathscr{P}}, \mathscr{G})| e^{\varrho \Upsilon^{h_0} d(\theta_1, \ldots, \theta_{s_0})} \\ \leqq C(m,k) \prod_{\substack{v \in \theta \\ v \neq v_0}} \Upsilon^{-\eta_v (h_v - h_{v'})} \Upsilon^{-\eta_{v_0}(h_0 - k)} , \end{aligned} \tag{2.18}$$

where $h_0 = h_{v_0}$, ϱ *is a constant* >0, $d^*(\theta_1, \ldots, \theta_{s_0})$ *is the largest distance between* s_0 *points one in each subtree* θ_j.

Differently from (1.17) we have:

$$\eta_v = 2(Q_v^2 - 1) + n_v^{\partial; e} + z_v , \tag{2.19}$$

and if $v = v_0$ $\cos\alpha\varphi^{(<h_{v_0})}$ has not to be considered producing a zero as there are not other truncated expectations of lower frequency. We have defined

$$n_{v_0}^{\partial; e} = \sum_{j=1}^{n_1} \beta_j , \qquad z_{v_0} = n^0(\alpha_v = s, v \in \mathscr{P}) . \tag{2.20}$$

Finally the presence of $\sum_{\underline{\sigma}}$ in (2.17) implies that if $n_v^{\partial; e}$ is even (odd) then z_v is even (odd).

Assuming $|\partial\gamma_j \varphi_{x_j}^{(\leqq k)}| \leqq B_k$, we have the following estimate [see Eq. (1.16)]:

$$|V_{\underline{s};N}^{(k)}(\varphi^{(\leqq k)})| \leqq G(B_k, k, \underline{s}) |\lambda^{n_0} \delta^{n_1}| \sup_{\theta; \hat{\mathscr{P}}; \mathscr{G}} \left\{ \sum_{\underline{h}:\underline{h} \leqq N} \prod_{\substack{v \in \theta \\ v \neq v_0}} \Upsilon^{-\eta_v (h_v - h_{v'})} \Upsilon^{-\eta_{v_0}(h_0 - k)} \right\} . \tag{2.21}$$

The proof of this estimate is similar to that for the $\alpha^2 < 8\pi$ case [see (1.17)]. The result is nearly obvious for a tree with only one bifurcation and then one proceeds inductively in the "level" of the tree which is defined as follows:

Definition 1. *A tree is of level f if starting from its root, there is at least a final point such that to reach it one meets f bifurcations and for all the other final points the number of bifurcations met is smaller or equal to f.*

We do not discuss this proof here as the same proof, although more complicated, will be necessary in the next theorem where we will produce the estimates associated to the renormalized tree expansion.

Looking at the estimate (2.18) we see that there are trees which give a divergent contribution when one sums over their frequencies. In fact from (2.19) we see that:
a) When $Q_v = 0$ and $n_v^{\partial; e} + z_v \leqq 2$ we have a logarithmic or a quadratic divergence (a linear divergence is excluded as $n_v^{\partial; e} + z_v$ is even).

b) When $Q_v=\pm 1$, if $n_v^{\partial;e}=0$ a logarithmic divergence appears.

In all the other cases the contribution associated to v is convergent.

The main difference with the $\alpha^2<8\pi$ case is that there only the lowest bifurcation could produce a divergent sum. Therefore, fixed α^2, only a finite number of trees give a divergent contribution while here, as in any renormalizable but not superrenormalizable theory, an infinite number of trees of any order in the coupling constants produce divergent contributions. The simplest of them have been already discussed and the R and the L operations have been defined for the field factors they produce.

We want now to define the R and the L operations for all the field dependent parts which need it. We start looking at the tree of level 1 starting from the interaction (2.16). Denoting by k the root of the tree the field dependent parts of these trees are:

$$
\begin{aligned}
&:\prod_{j=1}^{q}(\partial_{\gamma_j}\varphi_{x_j}^{(\leqq k)})\,\mathrm{Trig}_{\alpha_{v0}}\varphi_{(\underline{y}_{v0},\,\underline{\sigma}_{v0})}^{(\leqq k)}:\quad \text{if}\quad Q_{v_0}=Q=0\,,\\
&:\prod_{j=1}^{q}(\partial_{\gamma_j}\varphi_{x_j}^{(\leqq k)})\,\mathrm{Trig}_{\alpha_{v0}}\varphi_{(\underline{y}'_{v0},\,\underline{\sigma}'_{v0})}^{(\leqq k)}e^{i\alpha\varphi^{(\leqq k)}_{(\underline{y}''_{v0},\,\underline{\sigma}''_{v0})}}:\quad \text{if}\quad Q_{v_0}=Q\neq 0\,,
\end{aligned}
\tag{2.22}
$$

when $Q=0$ $\alpha_{v_0}=s$ if q is odd and $=c$ otherwise.

A simple application of (2.19) shows that

$$\eta_{v_0}=2(Q^2-1)+q+z_{v_0}\,,\tag{2.23}$$

and the R operation will not be the identity only when $\eta_{v_0}\leqq 0$. In this case they are following: We denote

$$
\varphi_v^{(\leqq k)}=\varphi_{(\underline{y}_v;\,\underline{\sigma}_v)}^{(\leqq k)}\equiv\sum_{j=1}^{n}\sigma_j\varphi_{y_j}^{(\leqq k)}\qquad(\text{here } v=v_0)\,,
$$

$$
\sum_{j=1}^{n}\sigma_j y_j=\hat{y}\,,\qquad \frac{1}{n}\sum_{j=1}^{n}y_j=\bar{y}
$$

$Q=0$.

$$
\begin{aligned}
&R:\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=:\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}-1+\frac{\alpha^2}{2}\left(\partial\varphi_{\bar{y}}^{(\leqq k)}\sum_{j=1}^{n}\sigma_j y_j\right)^2:,\\
&L:\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=:1-\frac{\alpha^2}{2}\left(\partial\varphi_{\bar{y}}^{(\leqq k)}\sum_{j=1}^{n}\sigma_j y_j\right)^2:=:1-\frac{\alpha^2}{2}(\partial\varphi_{\bar{y}}^{(\leqq k)})^2(\hat{y})^2:,
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
R:\partial_\gamma\varphi_x^{(\leqq k)}\sin\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=\;&:\partial_\gamma\varphi_x^{(\leqq k)}[\sin\alpha\varphi_{(\underline{v})}^{(\leqq k)}-\alpha(\partial\varphi_{\bar{y}}^{(\leqq k)})(\hat{y})]:+\alpha:\partial_\gamma\varphi_x^{(\leqq k)}((\partial_\beta\varphi_{\bar{y}}^{(\leqq k)})\\
&-(\partial_\beta\varphi_x^{(\leqq k)}))(\hat{y})_\beta:,\\
L:\partial_\gamma\varphi_x^{(\leqq k)}\sin\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=\;&\frac{\alpha}{2}:(\partial\varphi_x^{(\leqq k)})^2(\hat{y})_\gamma:,
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
R:\partial_\beta\varphi_y^{(\leqq k)}\partial_\gamma\varphi_z^{(\leqq k)}\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=\;&:\partial_\beta\varphi_y^{(\leqq k)}\partial_\gamma\varphi_z^{(\leqq k)}(\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}-1):\\
&+:\partial_\beta\varphi_y^{(\leqq k)}(\partial_\gamma(\varphi_z^{(\leqq k)}-\varphi_y^{(\leqq k)})):,\\
L:\partial_\beta\varphi_y^{(\leqq k)}\partial_\gamma\varphi_z^{(\leqq k)}\cos\alpha\varphi_{(\underline{v})}^{(\leqq k)}:=\;&\tfrac{1}{2}(\partial\varphi_y^{(\leqq k)})^2:\delta_{\beta,\gamma}\,,
\end{aligned}
\tag{2.26}
$$

$$\begin{aligned} R: \partial_\beta \varphi_x^{(\leqq k)} \partial_\gamma \varphi_z^{(\leqq k)} &:=: \partial_\beta \varphi_x^{(\leqq k)} (\partial_\gamma (\varphi_z^{(\leqq k)} - \varphi_x^{(\leqq k)})):, \\ L: \partial_\beta \varphi_x^{(\leqq k)} \partial_\gamma \varphi_z^{(\leqq k)} &:= \tfrac{1}{2} : (\underline{\partial} \varphi_x^{(\leqq k)})^2 : \delta_{\beta,\gamma}. \end{aligned} \tag{2.27}$$

$Q = \sigma = \pm 1$.

The only factor we have to consider is $:\cos\alpha \left(\sum_{j=1}^{n} \sigma_j \varphi_{x_j}^{(\leqq k)} \right):$ with $n > 2$ odd and $\sum_{j=1}^{n} \sigma_j = \pm 1$:

$$\begin{aligned} R: \cos\alpha \left(\sum_{j=1}^{n} \sigma_j \varphi_{x_j}^{(\leqq k)} \right) &:=: \cos\alpha \left(\sum_{j=1}^{n} \sigma_j \varphi_{x_j}^{(\leqq k)} \right) - \cos\alpha \varphi_x^{(\leqq k)}:, \\ L: \cos\alpha \left(\sum_{j=1}^{n} \sigma_j \varphi_{x_j}^{(\leqq k)} \right) &:=: \cos\alpha \varphi_x^{(\leqq k)}:. \end{aligned} \tag{2.28}$$

The L operation $(= 1 - R)$ acts in the way we have listed under the integral sign after we use the invariance of the covariances for translation and rotation of $\frac{\pi}{2}$ around each axis.

This completes the definition of the R (L) operation for the trees of level 1. After the R operation has been applied all the right-hand side of Eqs. (2.24)–(2.27) have $n_v^{\partial; e} + z_v > 2$. For a generic tree one has to investigate which other field expressions can appear which still need the R operation that is have $n_v^{\partial; e} + z_v \leqq 2$. It is easy to realize there are few of them:

$$\begin{aligned} R: \partial_\beta \varphi_x^{(\leqq k)} \cos\alpha \varphi_{(v)}^{(\leqq k)} \sin\alpha \varphi_{(v')}^{(\leqq k)} &:=: \partial_\beta \varphi_x^{(\leqq k)} (\cos\alpha \varphi_{(v)}^{(\leqq k)} - 1) \sin\alpha \varphi_{(v')}^{(\leqq k)} \\ &\quad + : \partial_\beta \varphi_x^{(\leqq k)} [\sin\alpha \varphi_{(v)}^{(\leqq k)} - \alpha (\underline{\partial} \varphi_y^{(\leqq k)}) (\hat{y})]: \\ &\quad + \alpha : \partial_\beta \varphi_x^{(\leqq k)} (\underline{\partial} (\varphi_y^{(\leqq k)} - \varphi_x^{(\leqq k)})) \hat{y}:, \\ L: \partial_\beta \varphi_x^{(\leqq k)} \cos\alpha \varphi_{(v)}^{(\leqq k)} \sin\alpha \varphi_{(v')}^{(\leqq k)} &:= \frac{\alpha}{2} : (\underline{\partial} \varphi_x^{(\leqq k)})^2 \hat{y}_\beta :. \end{aligned} \tag{2.29}$$

With this definition of the R and L operations we are able to write for the effective potential the renormalized tree expansion and show that it is finite, at each order uniformly in the cutoff. We proceed again as in the case of the unrenormalized expansion, first we give the general expression which we prove by induction, then we prove the estimate analogous to (2.18). Now, due to the presence of the R and the L's the estimate give convergent results when summed over the various frequencies.

To write the explicit expression for the effective potential first we list all the possible Wick functions of the fields which can appear in the effective potential. A Wick function is a Wick monomial or a Wick monomial to which the R operation has been applied and we denote it by $:\mathscr{F}_{a;q}^{v;Q_v}(\varphi_v^{(\leqq h)}):$, where Q_v is the charge of the cluster v, a labels the different functions and q is the value of $n^{\partial; e}$. We denote $\omega = 2Q_v^2 + q + z$, where z is the order of zero of the function.

Wick functions:

$Q_v=0,$	a		ω
	1	$:\cos\alpha\varphi^{(\leqq h)}_{(v)}:$	0
	2	$:\cos\alpha\varphi^{(\leqq h)}_{(v)}-1:$	2
	3	$:\cos\alpha\varphi^{(\leqq k)}_{(v)}-1+\frac{\alpha^2}{2}(\underline{\partial}\varphi^{(\leqq k)}_{(\bar{y})}\hat{y})^2:$	4
	4	$:\sin\alpha\varphi^{(\leqq h)}_{(v)}:$	1
	5	$:[\sin\alpha\varphi^{(\leqq h)}_{(v)}-\alpha(\underline{\partial}\varphi^{(\leqq h)}_{\bar{y}})(\hat{y})]:$	3
	6	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j}):$	q_v
	7	$:\partial_\beta\varphi^{(\leqq h)}_x\partial_\gamma\varphi^{(\leqq h)}_z:-\frac{1}{2}:(\underline{\partial}\varphi^{(\leqq h)}_x)^2:\delta_{\beta,\gamma}$	3
	8	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})\cos\alpha\varphi^{(\leqq h)}_{(v)}:$	q_v
	9	$:\prod_{j=1}^{2}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})(\cos\alpha\varphi^{(\leqq h)}_{(v)}-1):$	4
	10	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})\sin\alpha\varphi^{(\leqq h)}_{(v)}:$	q_v+1
	11	$:\partial_\beta\varphi^{(\leqq h)}_x[\sin\alpha\varphi^{(\leqq h)}_{(v)}-\alpha(\underline{\partial}\varphi^{(\leqq h)}_{\bar{y}})(\hat{y})]:$	4
$Q_v=\pm 1.$			ω
	1	$:e^{i\alpha\sigma\varphi^{(\leqq h)}_{(y)}}\cos\alpha\varphi^{(\leqq h)}_{(v)}:$	2
	2	$:e^{i\alpha\sigma\varphi^{(\leqq h)}_{(y)}}(\cos\alpha\varphi^{(\leqq h)}_{(v)}-1):$	4
	3	$:e^{i\alpha\sigma\varphi^{(\leqq h)}_{(y)}}\sin\alpha\varphi^{(\leqq h)}_{(v)}:$	3
	4	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})e^{i\alpha\sigma\varphi^{(\leqq h)}_{(y)}}\cos\alpha\varphi^{(\leqq h)}_{(v)}:$	q_v+2
	5	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})e^{i\alpha\sigma\varphi^{(\leqq h)}_{(y)}}\sin\alpha\varphi^{(\leqq h)}_{(v)}:$	q_v+3
$\lvert Q_v\rvert\geqq 2.$			
	1	$:e^{i\alpha\varphi^{(\leqq h)}_{(v)}}:$	$2Q_v^2$
	2	$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j})e^{i\alpha\sigma\varphi^{(\leqq h)}_{(v)}}:$	$2Q_v^2+q_v$

(2.30)

Remarks. i) In the generic Wick function all the coordinates are associated to the same v, both the x's of the grad fields and the y's of the trigonometric functions.

ii) The factors we have listed are associated to a vertex (bifurcation) of the generic tree. In other words the final field expression of a generic term of the effective potential will be constructed by the product of these factors each one associated to a specific vertex. Looking at Eq. (2.17) we see that they are the generalisation of the factors

$$:\mathrm{Trig}_c\,\varphi^{(\leqq k)}_{(y_v,\sigma_v)}:=:\cos\alpha\varphi^{(\leqq k)}_{(y_v,\sigma_v)}:,\qquad :\mathrm{Trig}_s\,\varphi^{(\leqq k)}_{(y_v,\sigma_v)}:=:\sin\alpha\varphi^{(\leqq k)}_{(y_v,\sigma_v)}:,\qquad :e^{i\alpha\varphi^{(\leqq h)}_{(v)}}:$$

and

$$:\prod_{j=1}^{q_v}(\partial_{\gamma_j}\varphi^{(\leqq h)}_{x_j}):.$$

This is due to the fact that in the renormalized tree expansion the R or the L operations act at each bifurcation modifying the Wick monomials and introducing appropriate zeroes to guarantee the convergence of all the sums. These zeroes once introduced must be kept in all the next scales. This is the reason [see also (5)] why the field dependence has to be written in terms of the Wick functions $:\mathscr{F}^{v;Q_v}_{a;q}(\varphi^{(\leqq h)}_v):$'s.

Notice that if a single $\mathscr{F}^{v;Q_v}_{a;q}(\varphi^{(\leqq h)}_v)$ describes the field dependence of a vertex v of a tree then $\omega=\eta_v+2$. Therefore if the normalized expansion with these definitions of the R and the L must produce finite results at any order the field expressions must be products of Wick functions such that $\eta_v>0$ always. This is obvious, just by definition, for the tree of level 1 and will be proved by induction for the generic tree. The next theorem is the generalization of Theorem 1 to the renormalized tree expansion. To prove it we will need a canonical way of decomposing the generic $\mathscr{F}^{v;Q_v}_{a;q}(\varphi^{(\leqq h)}_v)$ in a part which depends on $\varphi^{(<h)}_v$ and a part which depends only on $\varphi^{(h)}_v$.

Lemma 1. *For any $\mathscr{F}^{v;Q_v}_{a;q}(\varphi^{(\leqq h)}_v)$ the following decomposition holds:*

$$\mathscr{F}^{v;Q_v}_{a;q}(\varphi^{(\leqq h)}_v)=\sum_b\mathscr{F}^{v;Q_v}_{b;q}(\varphi^{<h)}_v)\mathscr{F}^{v;Q_v}_{c(b);q}(\varphi^{(h)}_v),\tag{2.31}$$

where $c(b)$ is a well defined function depending on b which is unique if we require the right-hand side to be symmetric under the interchange of $\varphi^{(<h)}_v$ with $\varphi^{(h)}_v$. All the addends in the right-hand side have at least the same order of zeroes as the left-hand side.

The proof of this lemma is just a matter of trivial computation and we do not include it here.

Theorem 2. *The effective potential on scale k with a fixed cutoff N, $V^{(k)}_N(\varphi^{(\leqq k)})$ can be written as a formal series in the physical coupling constants in the following way:*

$$\begin{aligned}V^{(k)}_N(\varphi^{(\leqq k)})&=\sum_{m=1}^{\infty}\sum_{\substack{n_0;n_1\\ n_0+n_1=m}}\underline{\lambda}^s V^{(k)}_{\underline{s};N}(\varphi^{(\leqq k)})\\&=\sum_{m=1}^{\infty}\sum_{\theta:v(\theta)=m}\sum_{n_0=0}^{m}\underline{\lambda}^{\underline{s}}\sum_{\underline{h}:\underline{h}\leqq N}\sum_{\underline{\sigma}}\int d\underline{x}d\underline{y}\\&\quad\times\sum_{\mathscr{P}}{}^{*}\sum_{\underline{a}}\chi(\eta_{v_0}>0):\Big(\prod_{v_s\in\mathscr{P}}\mathscr{F}^{v_s;Q_{v_s}}_{a_s;q_s}(\varphi^{(\leqq k)})\Big)\\&\quad\times W^{(k)}(\theta,\underline{h};\underline{x},\underline{y};\underline{\sigma};\mathscr{P},\underline{a})\\&=\sum_{m=1}^{\infty}\sum_{\theta:v(\theta)=m}\sum_{n_0=0}^{m}\underline{\lambda}^{\underline{s}}\sum_{h:h\leqq N}\sum_{\underline{\sigma}}\int d\underline{x}d\underline{y}\\&\quad\times\sum_{\mathscr{P}}{}^{*}\sum_{\underline{a}}W^{(k)}(\theta,\underline{h};\underline{x},\underline{y};\underline{\sigma};\mathscr{P},\underline{a},\varphi^{(\leqq k)}),\end{aligned}\tag{2.32}$$

where $\mathscr{P}$ *is a subset of* $\{v\}_{v\in\theta}$, $\{v\}\supseteq\mathscr{P}$. $\sum_{\mathscr{P}}^*$ *is the sum over all the possible* $\mathscr{P}$ *such that for any* $v,v'\in\mathscr{P}: v\cap v'=\emptyset$ *and* $\chi(\eta_{v_0}>0)$ *reminds us that we are doing the renormalized tree expansion. Moreover all the final lines merge in some* $v\in\mathscr{P}$. *The coefficients* $W^{(k)}(\theta,\underline{h};\underline{x},\underline{y};\mathscr{P},\underline{a})$ *satisfy the following bounds: We define the norm*

$$\begin{aligned}&\|W^{(k)}(\theta,\underline{h},\ldots,\mathscr{P},\underline{a})\|_{\{Q_v\}}\\ &=\sum_{\substack{\underline{\sigma}\\ \{Q_v\}\text{ fixed}}}\int d\underline{x}d\underline{y}\,|W^{(k)}(\theta,\underline{h};\underline{x},\underline{y};\underline{\sigma};\mathscr{P},\underline{a})|\\ &\quad\times\delta(y_1)e^{\varrho\Upsilon^{h_0}d^*(\theta_1,\ldots,\theta_{s_0})}|(\text{zeroes})|,\end{aligned}$$

where $d^*(\theta_1,\ldots,\theta_{s_0})$ *is the maximum distance between* s_0 *points one for each cluster,* $\varrho>0$ *will be chosen later on (see after Eq.* (2.38)*) and*

$$|(\text{zeroes})|:=\prod_{v_s\in\mathscr{P}}|\Upsilon^k\hat{y}_{v_s}|^{z_s}.$$

We remark that this factor just takes into account the zeroes which are present in the field dependent part of Eq. (2.32). *The following bound is satisfied uniformly in* N:

$$\|W^{(k)}(\theta,\underline{h},\ldots,\mathscr{P},\underline{a})\|_{\{Q_v\}}\leqq c(k,n)\prod_{\substack{v\in\theta\\ v\neq v_0}}\Upsilon^{-\chi_v(h_v-h_{v'})}\Upsilon^{-\eta_{v_0}h_{v_0}},\tag{2.34}$$

where $\chi_v>0$, η_{v_0} *has been defined in Eq.* (2.23) *and* $c(n,k)=c(n)\Upsilon^{z_{v_0}k}$ *with* $c(n)$ *a constant whose n-dependence will be left unspecified. This k dependence originates from our definition of the (zeroes) factors.*

From (2.34) *the following estimate also follows:*

$$\|W^{(k)}(\theta,\underline{h},\ldots)\|:=\sup_{\{Q_v\};\{\mathscr{P};\underline{a}\}}\|W^{(k)}(\theta,\underline{h},\ldots,\mathscr{P},\underline{a})\|_{\{Q_v\}}\leqq c(k,n)\prod_{v\in\theta}\Upsilon^{-\chi_v(h_v-h_{v'})}\tag{2.35}$$

with $\chi_v>0$ *for any* $v\in\theta$.

Some comments are appropriate at this point.

i) This theorem says that the formal series in the coupling constants of the effective potential has finite coefficients uniformly in N. The same results immediately follows for the Schwinger functions.

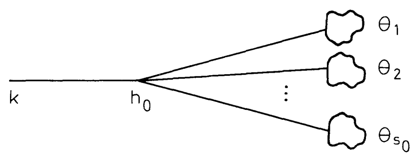
ii) Differently from the polynomial case we do not discuss here the n dependence of the order n coefficients. We will consider again this aspect when we discuss this model from the constructive point of view.

iii) The crucial point of the theorem is to prove that the number of different types of Wick functions which are needed in the field parts of the different tree contributions is independent from the tree level. This implies that the $R(L)$ operation is different from the identity only in the finite number of cases we have listed.

Proof. The proof of this theorem is long but does not require any computation. We proceed inductively. The result is true for the level 1, we assume it true for all the levels $\leqq f$ and we prove it for the trees of level $f+1$.

Let θ be a tree of level $f+1$, v_0 its lowest bifurcation of frequency h_0, s_0 the number of subtrees merging in v_0 and k its root frequency,

Fig. 4



The following expression holds:

$$\sum_{\mathscr{P}:\underline{a}}^{*} W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \mathscr{P}, \underline{a}, \varphi^{(\leqq k)}) = \left(\prod_{t=1}^{s_0} \sum_{\mathscr{P}^{(t)}}^{*} \sum_{\underline{a}^{(t)}} \chi(\eta_{v_1} > 0) \right)$$
$$\times \prod_{t=1}^{s_0} W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)})$$
$$\times R\mathscr{E}_{>k}\mathscr{E}^{T}_{h_0}\left(: \prod_{v_s \in \mathscr{P}^{(1)}} \mathscr{F}^{v_s; Q_{v_s}}_{a_s; q_s}(\varphi^{(\leqq h_0)}) :, \ldots, : \prod_{v_s \in \mathscr{P}^{(s_0)}} \mathscr{F}^{v_s; Q_{v_s}}_{a_s; q_s}(\varphi^{(\leqq h_0)}) : \right), \tag{2.36}$$

where $W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)})$ satisfy the estimates (2.33) by the inductive assumption.

As in the norm (2.34) there is the factor |(zeroes)| which takes care of the zeroes in the field dependent part. It is crucial that all the manipulations performed on the field dependent parts must preserve the zeroes appearing in each $\prod_{v_s \in \mathscr{P}^{(t)}} \mathscr{F}^{v_s; Q_{v_s}}_{a_s; q_s}(\varphi^{(\leqq h_0)})$ in each term of the sum one gets performing the various expectations. The decomposition of Lemma 1 has this property. Therefore applying (2.31) we obtain:

$$R\mathscr{E}_{>k}\mathscr{E}^{T}_{h_0}(\ ) = \sum_{\underline{b}^{(1)}; \ldots; \underline{b}^{(s_0)}} R\mathscr{E}_{>k}\left(: \prod_{v_s \in \mathscr{P}^{(1)}} \mathscr{F}^{v_s; Q_{v_s}}_{b_s; q_s}(\varphi^{(<h_0)}) :, \ldots, : \prod_{v_s \in \mathscr{P}^{(s_0)}} \mathscr{F}^{v_s; Q_{v_s}}_{b_s; q_s}(\varphi^{(<h_0)}) : \right)$$
$$\times \mathscr{E}^{T}_{h_0}\left(: \prod_{v_s \in \mathscr{P}^{(1)}} \mathscr{F}^{v_s; Q_{v_s}}_{c(b_s); q_s}(\varphi^{(h_0)}) :, \ldots, : \prod_{v_s \in \mathscr{P}^{(s_0)}} \mathscr{F}^{v_s; Q_{v_s}}_{c(b_s); q_s}(\varphi^{(h_0)}) : \right). \tag{2.37}$$

Let z_s be the order of zero of $\mathscr{F}^{v_s; Q_{v_s}}_{a_s; q_s}(\varphi^{(\leqq h_0)})$, then in the decomposition (2.31) $\mathscr{F}^{v_s; Q_{v_s}}_{c(b_s); q_s}(\varphi^{(h_0)})$ has a zero of order $w_s \leqq z_s$. We have for $\mathscr{E}^{T}_{h_0}(\ )$ of (2.37) the following estimate:

$$|\mathscr{E}^{T}_{h_0}(\ )| \leqq C(n, s_0) e^{-\kappa \Upsilon^{h_0} d(\theta_1, \ldots, \theta_{s_0})} \prod_{t=1}^{s_0} \prod_{v_s \in \theta_t}^{s} \Upsilon^{\bar{n}^{\hat{c}; e}_{v_s} h_0} |\Upsilon^{h_0} \hat{y}_{v_s}|^{ds}, \tag{2.38}$$

where $\sum_{v_s \in \theta_t}^{s} \bar{n}^{\partial; e}_{v_s}$ is the number of the $\partial\varphi$ fields coming out from θ_t which are contracted in v_0, $d(\theta_1, \ldots, \theta_{s_0})$ is the distance between the s_0 clusters $\theta_1, \ldots, \theta_{s_0}$. $\kappa > 0$ and d_s is the zero order of the subcluster v_s appearing in $\mathscr{E}^{T}_{h_0}(\ )$. In the definition of norm (2.33) ϱ has been chosen $\ll \kappa$. The proof of this estimate is very simple as we are not interested to an optimal estimate for $C(n, s_0)$ and we do not report it here.

We rewrite (2.36) in the following way:

$$\sum_{\mathscr{P};\underline{a}}^{*} W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \mathscr{P}, \underline{a}, \varphi^{(\leqq k)})$$
$$= \left(\left(\prod_{t=1}^{s_0} \sum_{\mathscr{P}^{(t)}}^{*} \sum_{\underline{a}^{(t)}} \chi(\eta_{v_1} > 0) \right) \sum_{\underline{b}^{(1)};\ldots;\underline{b}^{(s_0)}} \right) \prod_{t=1}^{s_0} W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)}) \mathscr{E}^T_{h_0}()$$
$$\times R\mathscr{E}_{>k}\left(: \prod_{v_s \in \mathscr{P}^{(t)}} \mathscr{F}^{v_s;Q_v}_{b_s;q_s}(\varphi^{(<h_0)}):, \ldots, : \prod_{v_s \in \mathscr{P}^{(s_0)}} \mathscr{F}^{v_s;Q_{v_s}}_{b_s;q_s}(\varphi^{(<h_0)}): \right). \tag{2.36'}$$

We need to rewrite the field dependent part according to the following

Lemma 2. *The following decomposition holds:*

$$R\mathscr{E}_{>k}\left(: \prod_{v_s \in \mathscr{P}^{(1)}} \mathscr{F}^{v_s;Q_{v_s}}_{b_s;q_s}(\varphi^{(<h_0)}):, \ldots, : \prod_{v_s \in \mathscr{P}^{(s_0)}} \mathscr{F}^{v_s;Q_{v_s}}_{b_s;q_s}(\varphi^{(<h_0)}): \right)$$
$$= \sum_{\mathscr{P}}^{*} \sum_{\underline{a}} : \left(\prod_{v_s \in \mathscr{P}} \mathscr{F}^{v_s;Q_{v_s}}_{a_s;q_s}(\varphi^{(\leqq k)}) \right) : \mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma}) \tag{2.39}$$

with $\mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma})$ *satisfying the following estimate:*

$$|\mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma})| \leqq \Upsilon^{-2Q^2 h_0} \prod_{t=1}^{s_0} \Upsilon^{2Q_t^2 h_0} |(\text{zeroes}) \text{ of } \mathscr{k}|. \tag{2.40}$$

Proof. The proof of this lemma is trivial. It is enough to recall how one can rearrange everything inside a couple of Wick dots. Some examples will be given in the Appendix. Using this lemma we can write:

$$W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \mathscr{P}, \underline{a})$$
$$= \left(\left(\prod_{t=1}^{s_0} \sum_{\mathscr{P}^{(t)}}^{*} \sum_{\underline{a}^{(t)}} \chi(\eta_{v_0} > 0) \right) \sum_{\underline{b}^{(1)};\ldots;\underline{b}^{(s_0)}} \delta(\mathscr{P}, \underline{a} \text{ fixed}) \right) \mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma})$$
$$\times \prod_{t=1}^{s_0} W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)}) \mathscr{E}^T_{h_0}(). \tag{2.41}$$

To estimate the norm of this expression we can neglect the (finite) multiple sum and estimate the norm of the generic term of it. Therefore we are left with the norm of

$$W^*(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \mathscr{P}, \underline{a}) := \mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma}) \prod_{t=1}^{s_0} \chi(\eta_{v_0} > 0)$$
$$\times W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)}) \mathscr{E}^T_{h_0}(\cdot) \tag{2.42}$$

uniformly in the indices on which the multiple sums are performed.

We have

$$\| W^{*(k)}(\theta, \underline{h}; \ldots; \mathscr{P}, \underline{a}) \|_{\{Q_v\}}$$
$$= \prod_{t=1}^{s_0} \left(\sum_{\substack{\underline{\sigma}^{(t)} \\ \{Q_v\}_{\theta_t} \text{fixed}}} \int d\underline{x}^{(t)} d\underline{y}^{(t)} \delta(y_1^{(t)}) \right)$$
$$\times e^{\varrho \Upsilon^{h_0} d^*(\theta_1, \ldots, \theta_{s_0})} \prod_{t=1}^{s_0} e^{-\varrho \Upsilon^{h_t} d^*(\theta_1^{(t)}, \ldots, \theta_{s_t}^{(t)})} \mathscr{E}^T_{h_0} |(\cdot)|$$
$$\times \prod_{t=1}^{s_0} [e^{\varrho \Upsilon^{h_t} d^*(\theta_1^{(t)}, \ldots, \theta_{s_t}^{(t)})} | W^{(h_0)}(\theta, \underline{h}^{(t)}; \underline{x}^{(t)}, \underline{y}^{(t)}; \underline{\sigma}^{(t)}, \mathscr{P}^{(t)}, \underline{a}^{(t)}) \chi(\eta_{v_t} > 0)]$$
$$\times |\mathscr{k}^{(<h_0)}_{\mathscr{P};\underline{a}}(\underline{x}, \underline{y}, \underline{\sigma})| \left| (\text{zeroes}) \text{ of } \left(\prod_{v_s \in \mathscr{P}} \mathscr{F}^{v_s;Q_{v_s}}_{a_s;q_s}(\varphi^{(\leqq k)}) \right) \right|. \tag{2.43}$$

Using the estimate (2.38) we have:

$$\begin{aligned}
&\| W^{*(k)}(\theta, \underline{h}; \ldots; \mathscr{P}, \underline{a})\|_{\{Q_v\}} \\
&\leqq \int dy_1^{(1)}, \ldots, dy_1^{(s_0)} \cdot \delta(y_1^{(1)}) e^{-\frac{\kappa}{2}\Upsilon^{h_0} d(y_1^{(1)}, \ldots, y_1^{(s_0)})} \\
&\times \prod_{t=1}^{s_0} \Bigg[\sum_{\substack{\underline{\sigma}^{(t)} \\ \{Q_v\}\theta_t \text{fixed}}} \int d\underline{x}^{(t)} d\underline{y}^{(t)} \delta(y_1^{(t)}) e^{\varrho \Upsilon^{h_t} d^*(\theta_1^{(t)}, \ldots, \theta_{s_t}^{(t)})} |W^{(h_0)}(\theta, \underline{h}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)})| \chi(\eta_{v_t} > 0) \\
&\times |(\text{zoeroes}) \text{ of } (\mathscr{E}_{h_0}^T(\cdot), (t))| \prod_{v_s \in \theta_t}^{s} \Upsilon^{\bar{n}_{v_s}^{\hat{c}_i} e h_0} \Bigg] \\
&\times |k_{\mathscr{P}; \underline{a}}^{(<h_0)}(\underline{x}, \underline{y}, \underline{\sigma})| \, \left| (\text{zeroes}) \text{ of } \left(\prod_{v_s \in \mathscr{P}} \mathscr{F}_{a_s; q_s}^{v_s; Q_{v_s}}(\varphi^{(\leqq k)}) \right) \right|,
\end{aligned} \tag{2.44}$$

where $|(\text{zeroes}) \text{ of } (\mathscr{E}_{h_0}^T(\cdot)), (t))|$ is the product of zeroes of $\mathscr{E}_{h_0}^T(\cdot)$ associated to the subtree θ_t.

From this expression it is possible to reexpress the right-hand side in terms of the norms associated to the subtrees $\theta_1, \ldots, \theta_{s_0}$, and then proving the theorem, using the inductive assumptions on the norms, provided that:

a) one can reconstruct all the zeroes present in the definition of the previous norms.

b) One shows that also the dependence on h_0 produces a convergent sum over h_0.

To prove a) we observe that, due to the decomposition (2.31) of Lemma 1 and to Eq. (2.38) of Lemma 2 the zeroes present in the definition of the norms for the subtrees θ_t are partially in the $\mathscr{E}_{h_0}^T(\cdot)$ factor [we have called them $|(\text{zeroes}) \text{ of } (\mathscr{E}_{h_0}^T(\cdot)), (t))|$] and partially in the $R\mathscr{E}_{>k}(\ )$ part of Eq. (2.36). This second factor is then decomposed in Lemma 2 in such a way that these zeroes are partially contained in $k_{\mathscr{P}; \underline{a}}^{(<h_0)}(\underline{x}, \underline{y}, \underline{\sigma})$ and partially in $\left(\prod_{v_s \in \mathscr{P}} \mathscr{F}_{a_s; q_s}^{v_s; Q_{v_s}}(\varphi^{(\leqq k)}) \right)$. One has still to observe that from these last two factors the zeroes appear as $|\Upsilon^{h_0} \hat{y}_{v_s}|^{d_{s'}}$ when they are in $k_{\mathscr{P}; \underline{a}}^{(<h_0)}(\underline{x}, \underline{y}, \underline{\sigma})$ and as $|\Upsilon^k \hat{y}_{v_s}|^{d_{s''}}$ when they come from the second factor and of course

$$d_{s'} + d_{s''} := d_s = z_s - w_s. \tag{2.44}$$

The zeroes produced by the second factor give an extra factor, beyond the part needed from the norms of the subtrees, proportional to $(\Upsilon^{-(h_0-k)})^{d_{s''}}$.

Taking the zeroes correctly into account we obtain the following estimate:

$$\begin{aligned}
&[(\text{r.h.s.}) \text{ of } (2.43)] \\
&\leqq \int dy_1^{(1)}, \ldots, dy_1^{(s_0)} \cdot \delta(y_1^{(1)}) e^{-\frac{\kappa}{2}\Upsilon^{h_0} d(y_1^{(1)}, \ldots, y_1^{(s_0)})} \Upsilon^{-2Q^2 h_0} \prod_{t=1}^{s_0} \Upsilon^{2Q_t^2 h_0} \\
&\times \prod_{t=1}^{s_0} \prod_{v_s \in \theta_t} \Upsilon^{\bar{n}_{v_s}^{\hat{c}_i} e h_0} \\
&\times \prod_{t=1}^{s_0} \Upsilon^{-(\eta_{v_t} - z_{v_t}) h_0} \left[\prod_{t=1}^{s_0} \Upsilon^{(\eta_{v_t} - z_{v_t}) h_0} \| W^{(h_0)}(\theta_t, \underline{h}^{(t)}; \mathscr{P}^{(t)}, \underline{a}^{(t)}) \| \right] \\
&\times \prod_{v_s \in \theta_t}^{s} \Upsilon^{-(h_0-k) d_s''},
\end{aligned} \tag{2.45}$$

the factor $\left[\prod_{t=1}^{s_0} \Upsilon^{(\eta_{v_t}-z_{v_t})h_0} \|W^{(h_0)}(\theta_t,\underline{h}^{(t)};\mathscr{P}^{(t)},\underline{a}^{(t)})\|\right]$ is estimated by the inductive assumption and gives

$$\left[\prod_{t=1}^{s_0} \Upsilon^{(\eta_{v_t}-z_{v_t})h_0} \|W^{(h_0)}(\theta_t,\underline{h}^{(t)};\mathscr{P}^{(t)},\underline{a}^{(t)})\|\right] \leqq \prod_{t=1}^{s_0}\left[\Upsilon^{\eta_{v_t}(h_{v_t}-h_0)}\prod_{v\neq v_t}\Upsilon^{\chi_v(h_v-h_{v'})}\right]$$
$$\leqq \prod_{\substack{v\in\theta\\ v\neq v_0}} \Upsilon^{\chi_v(h_v-h_{v'})} \tag{2.46}$$

and $\chi_v\geqq 0$.

The estimate of the remaining part gives

$$\int dy_1^{(1)},\dots,dy_1^{(s_0)}\cdot\delta(y_1^{(1)})e^{-\frac{\kappa}{2}\Upsilon^{h_0}d(y_1^{(1)},\dots,y_1^{(s_0)})}\Upsilon^{-2Q^2h_0}\prod_{t=1}^{s_0}\Upsilon^{2Q_t^2h_0}$$
$$\times\prod_{t=1}^{s_0}\prod_{v_s\in\theta_t}\Upsilon^{\bar{n}_{v_s}^{\partial;e}h_0}\prod_{t=1}^{s_0}\Upsilon^{-(\eta_{v_t}-z_{v_t})h_0}\prod_{v_s\in\theta_t}{}_s\,\Upsilon^{-(h_0-k)d_s''}$$
$$\times c(n)\Upsilon^{kz_{v_0}}\Upsilon^{-2Q^2h_0+\sum_{t=1}^{s_0}Q_t^2h_0}\left[\prod_{t=1}^{s_0}\Upsilon^{-2(Q_t^2-1)h_0}\Upsilon^{-n_{v_t}^{\partial;e}h_0}\right]\Upsilon^{-2h_0(s_0-1)}$$
$$\times\prod_{t=1}^{s_0}\Upsilon^{\bar{n}_{v_t}^{\partial;e}h_0}\Upsilon^{-h_0z_{v_0}}, \tag{2.47}$$

where $z_{v_0}:=\sum_s d_s''$, $\bar{n}_{v_t}^{\partial;e}:=\sum_{v_s\in\theta_t}\bar{n}_{v_s}^{\partial;e}$, $\Upsilon^{-2h_0(s_0-1)}$ is the volume factor and $\sum_t(n_{v_t}^{\partial;e}-\bar{n}_{v_t}^{\partial;e})=n_{v_0}^{\partial;e}$.

Collecting all together we get

$$[(\text{r.h.s.})\text{ of }(2.43)]\leqq c(n)\Upsilon^{kz_{v_0}}\Upsilon^{-[2(Q^2-1)+n_{v_0}^{\partial;e}+z_{v_0}]h_0}\prod_{\substack{v\in\theta\\ v\neq v_0}}\Upsilon^{\chi_v(h_v-h_{v'})}$$
$$\leqq c(n)\Upsilon^{kz_{v_0}}\Upsilon^{-\eta_{v_0}h_0}\prod_{\substack{v\in\theta\\ v\neq v_0}}\Upsilon^{\chi_v(h_v-h_{v'})}, \tag{2.48}$$

and η_v is positive due to the presence of the R operation in the last bifurcation.

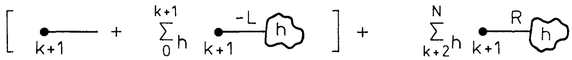
3. The Running Coupling Constants for the Sine-Gordon Model at $\alpha^2=8\pi$

From the results of the previous section it follows easily that the theory is renormalizable in the perturbative sense, namely we can write the interaction:

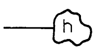
$$V^{(N)}=g(\lambda,N)\int_\Lambda d^2x:\cos\alpha\varphi_x^{(\leqq N)}:+d(\lambda,N)\int_\Lambda d^2x:(\underline{\partial}\varphi_x^{(\leqq N)})^2:u(\lambda,\upsilon,N)\int_\Lambda d^2x, \tag{3.1}$$

where $g(\lambda,N)=g(\lambda,\delta=0,N)$, $d(\lambda,N)=d(\lambda,\delta=0,N)$ and $u(\lambda,\upsilon,N)=u(\lambda,\delta=0,\upsilon,N)$ are the bare coupling constants such that if we compute the effective potential $V_N^{(\leqq k)}$ as a power series in λ, each order is made by a finite number of terms whose coefficients are uniformly bounded in N.

This result is proven in this way: Let $V_N^{(\leqq k+1)}(\varphi^{(\leqq k+1)}) := V_N^{(\leqq k+1)}$ be the effective potential at the level $k+1$ computed with the renormalized tree expansion which we describe, graphically, in the following way:

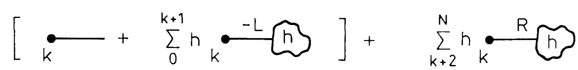


$$:= \sum_{\alpha=1}^{3} \lambda^{(\alpha)}_{(k+1;N)} I^{(\alpha)}_{(\varphi^{(\leqq k+1)})} + W_N^{(k+1)}, \tag{3.2}$$

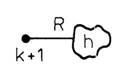
where  denotes the sum over all the trees with all the possible frequencies except the last one $h_{v_0} = h$ fixed and with R or $-L$ at the internal vertices. $V_N^{(\leqq k+1)}$ has been decomposed in a local part which we denote by $\sum_{\alpha=1}^{3} \lambda^{(\alpha)}_{(k+1;N)} I^{(\alpha)}(\varphi^{(\leqq k+1)})$ and the non-local one. We have

$$\begin{aligned} I^{(1)} &= \int_\Lambda d^2x : \cos\alpha\varphi_x^{(\leqq k)} : = \frac{1}{2} \sum_{\sigma=\pm1} \int_\Lambda d^2x : e^{i\alpha\sigma\varphi_x^{(\leqq k)}} \\ I^{(2)} &= \int_\Lambda d^2x : (\underset{\sim}{\partial}\varphi_x^{(\leqq k)})^2 :, \\ I^{(3)} &= \int_\Lambda d^2x \quad \text{and} \quad W_N^{(N)} \equiv 0 . \end{aligned} \tag{3.3}$$

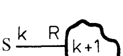
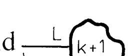
We compute $V_N^{(\leqq k)}$ performing the standard tree expansion (the non-renormalized one):

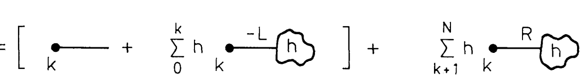
$V_N^{(\leqq k)} =$ 

$$+ \sum_{p=2}^{\infty} \frac{1}{p!} \mathscr{E}^T_{k+1}(V_N^{(k+1)}(\varphi^{(\leqq k+1)}), \ldots, V_N^{(k+1)}(\varphi^{(\leqq k+1)}); p)$$

$= \mathscr{E}_{k+1}\left\{ \sum_{\alpha=1}^{3} \lambda^{(\alpha)}_{(k+1;N)} I^{(\alpha)}(\varphi^{(\leqq k+1)}) \right\} + \mathscr{E}_{k+1}\left\{ \sum_{h=k+2}^{N} \right.$  $\left. \right\}$

$$\begin{aligned} &+ \sum_{p=2}^{\infty} \frac{1}{p!} R\mathscr{E}^T_{k+1}(V_N^{(k+1)}(\varphi^{(\leqq k+1)}), \ldots, V_N^{(k+1)}(\varphi^{(\leqq k+1)}); p) \\ &+ \sum_{p=2}^{\infty} \frac{1}{p!} L\mathscr{E}^T_{k+1}(V_N^{(k+1)}(\varphi^{(\leqq k+1)}), \ldots, V_N^{(k+1)}(\varphi^{(\leqq k+1)}); p) . \end{aligned} \tag{3.4}$$

The last two terms can be represented as  and  and collecting with the previous ones, we obtain:

$V^{(\leqq k)} =$ 

$$= \sum_{\alpha=1}^{3} \lambda^{(\alpha)}_{(k;N)} I^{(\alpha)}(\varphi^{(\leqq k)}) + W_N^{(k)} . \tag{3.5}$$

Therefore the proof that a finite renormalized tree expansion implies the perturbative renormalization is achieved. $\lambda_{(k)}^{(\alpha)}=\lim_{N\to\infty}\lambda_{(k;N)}^{(\alpha)}$ are the running coupling constants.

From Eq. (3.4) and the fact that $V_N^{(\leq k+1)}$ has the same expression as $V_N^{(\leq k)}$ with $k+1$ substituted by k everywhere, it follows that we can express $V_N^{(\leq k)}$ with a new tree expansion where at each bifurcation there is a R label and the final lines bring instead of the physical coupling constants the running coupling constants with the frequency of the first bifurcation where they merge. This expansion will be called: "the running coupling constants tree expansion." To be useful one has to know, at least formally, the running coupling constants. They are defined as solutions of the following β-functional equations whose derivation we recall [5], [12]. From Eqs. (3.4), (3.5) we get:

$$\begin{aligned}&\sum_{\alpha=1}^{3}(\lambda_{(k+1;N)}^{(\alpha)}-\lambda_{(k;N)}^{(\alpha)})I^{(\alpha)}(\varphi^{(\leq k)})\\&\quad=-\sum_{p=2}^{\infty}\frac{1}{p!}L\mathscr{E}_{k+1}^{T}(V_N^{(k+1)}(\varphi^{(\leq k+1)}),\dots,V_N^{(k+1)}(\varphi^{(\leq k+1)});p).\end{aligned}\tag{3.6}$$

The right-hand side can be written, using the "r.c.c. tree expansion"

$$\begin{aligned}&\sum_{\alpha=1}^{3}\sum_{m=2}^{\infty}\sum_{\theta:v(\theta)=m}\sum_{n=0}^{m}\sum_{\sigma_1,\dots,\sigma_n}\sum_{\underline{\alpha}}\sum_{(\mathscr{P};\underline{a})}\sum_{\substack{\underline{h}:h\leq N\\ h_{v_0}=k+1}}\\&\qquad\times\int_{\substack{\Lambda x\Lambda x,\dots,x\Lambda\\ m\text{-times}}}d^2x_1,\dots,d^2x_m\,W^{(k+1)}(\theta;\underline{x};\underline{\sigma};\underline{h};\underline{\alpha};\mathscr{P};\underline{a})\\&\qquad\times L^{(\alpha)}(:P(\varphi,\underline{\partial}\varphi):)\prod_{j=1}^{m}\lambda_{(h_j;N)}^{(\alpha_j)},\end{aligned}\tag{3.7}$$

where $\sum_{(\mathscr{P};\underline{a})}$ is the sum over all the possible final field dependences for a tree with $v(\theta)=m$ and $\sum_{\underline{\alpha}}$ sums over the different final lines. $:P(\varphi,\underline{\partial}\varphi):$ are complicated polynomials in $\underline{\partial}\varphi$ and $e^{i\alpha\varphi}$, depending on $\mathscr{P};\underline{a}$ whose expression is explicitly given in Theorem 2. $L^{(\alpha)}(:P(\varphi^{(\leq k)},\underline{\partial}\varphi^{(\leq k)}):)$ is $=I^{(\alpha)}(\varphi^{(\leq k)})$ or $=0$ depending on the local part of the polynomial $P(\varphi,\underline{\partial}\varphi)$.

We write (3.7) in the following compact way:

$$(3.7):=\sum_{\alpha=1}^{3}I^{(\alpha)}(\varphi^{(\leq k)})\beta^{(\alpha)}\left(\{\lambda^{(\gamma)}(h;N)\}_{\substack{h:h\leq N\\ h\geq k+1}}\right)$$

and the β-functional equations are:

$$(\lambda_{(k+1;N)}^{(\alpha)}-\lambda_{(k;N)}^{(\alpha)})=\beta^{(\alpha)}\left(\{\lambda^{(\gamma)}(h;N)\}_{\substack{h:h\leq N\\ h\geq k+1}}\right),\quad \alpha=1,2,3.\tag{3.8}$$

If we try to solve Eq. (3.8) perturbatively fixing as initial data

$$(\lambda_{(0;N)}^{(1)},\lambda_{(0;N)}^{(2)},\lambda_{(0;N)}^{(3)})=(\lambda,\delta,v)$$

(the physical coupling constants) we find, obviously, that $\lambda_{(N;N)}^{(\alpha)}$ is the bare coupling constant with index α. Moreover (3.7) and (3.8) produce a tree expansion also for the running coupling constants.

To go beyond the perturbative approach one looks for a non-perturbative solution of Eqs. (3.8). This is a complicated task as the right-hand side of (3.8) is defined through a tree expansion as a formal series which in general will not be convergent. Nevertheless there are chances that the right-hand side has a rigorous meaning if we stay in the small field region (see [13]). In this case we can keep the first few terms of the series and neglect the others provided the physical coupling constants are chosen enough small. Therefore we look at the solutions of the β-functional equations, truncated at the second order, in the $N\to\infty$ limit. In particular we are interested in the solution $\lambda^{(1)}_{(k)}$ such that:

$$\lim_{k\to\infty} \lambda^{(1)}_{(k)}=0\,, \tag{3.9}$$

which corresponds to a asymptotically free theory. The β-functional equation at the second order has the following graphical representation:

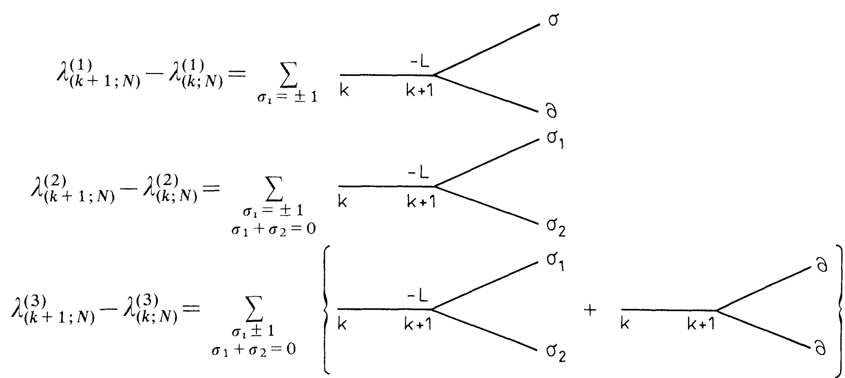


Fig. 5

The rules explained in Sect. 1 must be used to calculate these expressions that become (changing k into $k-1$)

$$\begin{aligned}
\lambda^{(1)}_{(k;N)}-\lambda^{(1)}_{(k-1;N)}&=a(k)\lambda^{(1)}_{(k;N)}\lambda^{(2)}_{(k;N)}\,,\\
\lambda^{(2)}_{(k;N)}-\lambda^{(2)}_{(k-1;N)}&=b(k)\,(\lambda^{(1)}_{(k;N)})^2+e(k)\,(\lambda^{(2)}_{(k;N)})^2\,,\\
\lambda^{(3)}_{(k;N)}-\lambda^{(3)}_{(k-1;N)}&=-c(k)\,(\lambda^{(1)}_{(k;N)})^2-d(k)\,(\lambda^{(2)}_{(k;N)})^2\,,
\end{aligned} \tag{3.10}$$

with

$$\begin{aligned}
a(k)&=\frac{\alpha^2}{2}\int d^2z((\partial z C^{(\leq k)}_{z0})^2-(\partial_z C^{(\leq k-1)}_{z0})^2)\,,\\
b(k)&=\left(\frac{\alpha}{4}\right)^2\int d^2z e^{\alpha^2 C^{(\leq k-1)}_{z0}}(e^{\alpha^2 C^{(k)}_{z0}}-1)\,|z|^2\,,\\
c(k)&=-\frac{1}{4}\int d^2z e^{\alpha^2 C^{(\leq k-1)}_{z0}}(e^{\alpha^2 C^{(k)}_{z0}}-1)\,,\\
d(k)&=-\sum_{i;\,j:1}^{2}\int d^2z((\partial_{z_i}\partial_{z_j}C^{(\leq k)}_{z0})^2-(\partial_{z_i}\partial_{z_j}C^{(\leq k-1)}_{z0})^2)\,,
\end{aligned} \tag{3.11}$$

$e(k)=0$ because of the invariance properties of the covariances.

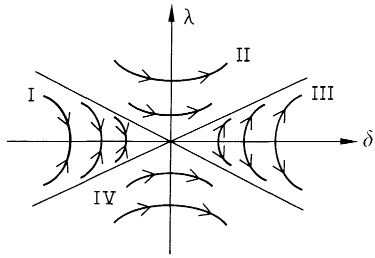


Fig. 6

Observe that, differently from $\lambda^{(1)}_{(k;N)}$ and $\lambda^{(2)}_{(k;N)}$, the $\lambda^{(3)}_{(k;N)}$ r.c.c. are dimensional quantities. Defining $\lambda^{(3)}_{(k;N)} := \Upsilon^{2k}\lambda^{(3)}_{a(k;N)}$, where λ_a is the adimensional constant the β-equation becomes:

$$\lambda^{(3)}_{a(k,N)} - \lambda^{(3)}_{a(k-1;N)} = -(1-\Upsilon^{-2})\lambda^{(3)}_{a(k-1;N)} - \Upsilon^{-2k}c(k)(\lambda^{(1)}_{(k;N)^2} - \Upsilon^{-2k}d(k)(\lambda^{(2)}_{(k;N)})^2 . \tag{3.12}$$

We neglect the third equation which is associated to the vacuum counterterm and we study these equations with $a(k)$, $b(k)$, $c(k)$, $d(k)$ substituted by their $k\to\infty$ limits we call a, b, c, d. We perform the limit $N\to\infty$ and finally the limit $\gamma\to 1$ to transform the finite difference equations into differential ones. Defining $t := k\log\gamma$, Eq. (3.10) becomes:

$$\dot{\lambda} = a\lambda\delta,\ \dot{\delta} = b\lambda^2, \quad \text{where} \quad \lambda^{(1)}_{(k)} = \lambda(t),\ \lambda^{(2)}_{(k)} = \delta(t); \quad \dot{\lambda} = \frac{d\lambda}{dt},\ \dot{\delta} = \frac{d\delta}{dt}. \tag{3.13}$$

This system is easily solved (see also [14] where similar results have been obtained) and the solutions in the (λ, δ) plane are hyperbolae.

The horizontal line is a line of fixed points stable when $\delta < 0$ and unstable when $\delta > 0$. The plane is divided in four regions by the separatrices of the equation:

$$\delta = \pm c\lambda \quad \text{with} \quad c = \left(\frac{b}{a}\right)^{1/2}. \tag{3.14}$$

In regions II, III, IV the solutions do not fulfill Eq. (3.9) and therefore the theory is not asymptotically free. Every point that begins in these regions goes to infinity moving along the hyperboles. Starting in region I, viceversa, the point does not go to infinity but to a stable fixed point $(0, \delta \leqq 0)$. It is easy to see that in region I $(\lambda(t), \delta(t))$ tends to a fixed point with exponential rate (in t) if we start inside the region while if the initial point is on a separatrix it tends to the origin as $1/t$.

Although one would be tempted to conclude that the theory is asymptotically free at $\alpha^2 = 8\pi$ this is not correct for the following reason: the value of α in the interaction part $\int_\Lambda d^2x : \cos\alpha\varphi_x :$ can be modified by a simple redefinition of the field φ. Therefore $\alpha^2 = 8\pi$ implies also that the wave function renormalization constants is fixed to 1.

If the theory is defined through the running coupling constants we would like therefore, on the finite scale k, we decide to choose as the physical one, the effective potential which has a local part $\int_\Lambda d^2x : \cos\alpha\varphi_x :$ and not a local part

$$I^{(2)}(\varphi^{(\leqq k)} = \int_\Lambda d^2x : (\partial\varphi^{(\leqq N)}_x)^2 : .$$

This is possible if and only if the running coupling constant $\lambda^{(2)}_{(k)} := \delta(t_0)$ can be chosen equal zero. This means we need to solve Eqs. (3.13) choosing as initial conditions a point $(\lambda, 0)$. This is certainly possible but the solution does not satisfy the condition $\lim_{k\to\infty} \lambda^{(1)}_{(k)} = 0$, that is the point $(\lambda, 0)$ goes to infinity (see Fig. 5) and not to a stable fixed point. This can be rephrased by saying that the theory is not asymptotically free at $\alpha^2 = 8\pi$.

The asymptotic free region I of Fig. 5 has to be interpreted in the following way: Let us choose a fixed value $\delta_0 < 0$ and assume we start from the point (λ, δ_0). Then if $|\lambda| < c^{-1}|\delta_0|$ the theory described is asymptotically free and corresponds to a value $\alpha^2 = 8\pi(1 + |\delta_0|)^{-1}$. This suggests an approach different from that developed in (2) to study constructively the theory for $\alpha^2 < 8\pi$ and moreover tells us which is the maximum value of λ for which this is possible.

Appendix

In this appendix we explain, through an example, the content of Theorem 2 and of the lemmas we need to prove it. We consider the contribution $I(\theta_0)$ of the trees of the following kind with $\sigma_1 + \sigma_2 = 0$ and $\sigma_3 + \sigma_4 = 0$:

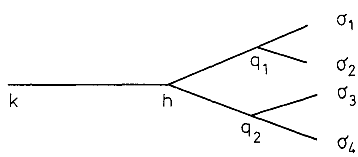


Fig. 7

$$I(\theta_0) = \frac{1}{2}\left(\frac{\lambda}{2}\right)^4 \sum_{h=k+1}^{N} \sum_{q_1;q_2=h+1}^{N}$$
$$\times \int_{\Lambda\times\Lambda\times\Lambda\times\Lambda} d^2x_1 d^2x_2 d^2x_3 d^2x_4 e^{\alpha^2 C^{(\le q_1-1)}_{x_1x_2}} (e^{\alpha^2 C^{(q_1)}_{x_1x_2}} - 1) e^{\alpha^2 C^{(\le q_2-1)}_{x_3x_4}} (e^{\alpha^2 C^{(q_2)}_{x_3x_4}} - 1)$$
$$\times R\mathscr{E}_{>k} \cdot \mathscr{E}^T_h \Big[:\cos\alpha(\varphi^{(\le h)}_{x_1} - \varphi^{(\le h)}_{x_2}) - 1 + \frac{\alpha^2}{2} (\underline{\partial}\varphi^{(\le h)}_{\bar{x}}(x_1 - x_2)^2):;$$
$$\times :\cos\alpha(\varphi^{(\le h)}_{x_3} - \varphi^{(\le h)}_{x_4}) - 1 + \frac{\alpha^2}{2} (\underline{\partial}\varphi^{(\le h)}_{\bar{x}'}(x_3 - x_4)^2): \Big]. \tag{1.a}$$

This is just the term in the expansion (2.32) associated with the tree θ_0, where

$$\sum_{\mathscr{P}}{}^{*} \sum_{\underline{a}} W^{(k)}(\theta, \underline{h}; \underline{x}, \underline{y}; \underline{\sigma}; \mathscr{P}, \underline{a}, \varphi^{(\le k)})$$

is written as in Eq. (2.36) with $s_0 = 2$ and

$$\begin{aligned} W^{(h)}(\theta_1, q_1; x_1, x_2; \underline{\sigma}; \mathscr{P}, \underline{a}) &= \left(\frac{\lambda}{2}\right)^2 e^{\alpha^2 C^{(\le q_1-1)}_{x_1x_2}} (e^{\alpha^2 C^{(q_1)}_{x_1x_2}} - 1), \\ W^{(h)}(\theta_2, q_2; x_3, x_4; \underline{\sigma}; \mathscr{P}, \underline{a}) &= \left(\frac{\lambda}{2}\right)^2 e^{\alpha^2 C^{(\le q_2-1)}_{x_3x_4}} (e^{\alpha^2 C^{(q_2)}_{x_3x_4}} - 1). \end{aligned} \tag{2.a}$$

To arrive at the same expression as in the first term of the right-hand side of Eq. (2.32), one has to compute explicitly the factor

$$R\mathscr{E}_{>k} \cdot \mathscr{E}^T_h \Big[:\cos\alpha(\varphi^{(\le h)}_{x_1} - \varphi^{(\le h)}_{x_2}) - 1 + \frac{\alpha^2}{2} (\underline{\partial}\varphi^{(\le h)}_{\bar{x}}(x_1 - x_2)^2):;:\cos\alpha(\varphi^{(\le h)}_{x_3} - \varphi^{(\le h)}_{x_4})$$
$$- 1 + \frac{\alpha^2}{2} (\underline{\partial}\varphi^{(\le h)}_{\bar{x}'}(x_3 - x_4)^2): \Big]. \tag{3.a}$$

To do that one has to use first Lemma 1 and then Lemma 2. Using Lemma 1 we get, defining

$$
\begin{aligned}
\varphi_{x_1}^{(\leqq h)} - \varphi_{x_2}^{(\leqq h)} &:= \Delta\varphi_{12}^{(\leqq h)}, \qquad \varphi_{x_3}^{(\leqq h)} - \varphi_{x_4}^{(\leqq h)} := \Delta\varphi_{34}^{(\leqq h)}, \\
\cos\alpha(\varphi_{x_i}^{(\leqq h)} - \varphi_{x_j}^{(\leqq h)}) &:= C\Delta\varphi_{ij}^{(\leqq h)}, \qquad \sin\alpha(\varphi_{x_i}^{(\leqq h)} - \varphi_{x_j}^{(\leqq h)}) := S\Delta\varphi_{ij}^{(\leqq h)},
\end{aligned}
\tag{4.a}
$$

$$
\begin{aligned}
[(3.\mathrm{a})] = \mathscr{E}_{>k} \Big\{ &: C\Delta_{12}^{(<h)} - 1 :: C\Delta\varphi_{34}^{(<h)} - 1 : \mathscr{E}_h^T [: C\Delta_{12}^{(h)} :: C\Delta\varphi_{34}^{(h)} :] \\
&+ : C\Delta\varphi_{12}^{(<h)} - 1 : \mathscr{E}_h^T \left[: C\Delta\varphi_{12}^{(h)} :: C\Delta\varphi_{34}^{(h)} + \frac{\alpha^2}{2} (\underline{\partial}\varphi_{\bar{x}'}^{(h)}(x_3 - x_4)^2 : \right] \\
&+ : C\Delta\varphi_{34}^{(<h)} - 1 : \mathscr{E}_h^T \left[: C\Delta\varphi_{34}^{(h)} :: C\Delta\varphi_{12}^{(h)} + \frac{\alpha^2}{2} (\underline{\partial}\varphi_{\bar{x}}^{(h)}(x_1 - x_2)^2 : \right] \\
&+ \mathscr{E}_h^T \left[: C\Delta\varphi_{34}^{(h)} + \frac{\alpha^2}{2} (\underline{\partial}\varphi_{\bar{x}'}^{(h)}(x_3 - x_4)^2 :: C\Delta\varphi_{12}^{(h)} + \frac{\alpha^2}{2} (\underline{\partial}\varphi_{\bar{x}}^{(h)}(x_1 - x_2)^2 :: \right] \\
&+ : S\Delta\varphi_{12}^{(\leqq h)} - \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_1 - x_2) :: S\Delta\varphi_{34}^{(<h)} - \underline{\partial}\varphi_{\bar{x}'}^{(<h)}(x_3 - x_4) : \\
&\times \mathscr{E}_h^{\mathrm{T}} [: S\Delta\varphi_{12}^{(h)} :: S\Delta\varphi_{34}^{(h)} :] \\
&+ : S\Delta\varphi_{12}^{(<h)} - \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_1 - x_2) :: \underline{\partial}\varphi_{\bar{x}'}^{(<h)}(x_3 - x_4) : \\
&\times \mathscr{E}_h^{\mathrm{T}} [: S\Delta\varphi_{12}^{(h)} :: S\Delta\varphi_{34}^{(h)} - \underline{\partial}\varphi_{\bar{x}'}^{(h)}(x_3 - x_4) :] \\
&+ : S\Delta\varphi_{34}^{(<h)} - \underline{\partial}\varphi_{\bar{x}'}^{(<h)}(x_3 - x_4) :: \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_1 - x_2) : \\
&\times \mathscr{E}_h^{\mathrm{T}} [: S\Delta\varphi_{34}^{(h)} :: S\Delta\varphi_{12}^{(h)} - \underline{\partial}\varphi_{\bar{x}}^{(h)}(x_1 - x_2) :] \\
&+ : \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_3 - x_4) :: \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_1 - x_2) : \\
&\times \mathscr{E}_h^{\mathrm{T}} [: S\Delta\varphi_{34}^{(h)} - \underline{\partial}\varphi_{\bar{x}'}^{(h)}(x_3 - x_4) :: S\Delta\varphi_{12}^{(h)} - \underline{\partial}\varphi_{\bar{x}}^{(h)}(x_1 - x_2) :] \Big\} .
\end{aligned}
\tag{5.a}
$$

What is left now is an easy application of Lemma 2; we give the result only for a couple of factors. We define $\tilde{C} := \langle \Delta\varphi_{12}^{(<h)} \Delta\varphi_{34}^{(<h)} \rangle$, then:

$$
\begin{aligned}
&\mathscr{E}_{>k} \cdot (: C\Delta\varphi_{12}^{(<h)} - 1 :: C\Delta\varphi_{34}^{(<h)} - 1 :) \\
&\quad = \frac{1}{2} \sinh \frac{\alpha^2}{2} \tilde{C} \left[e^{\frac{\alpha^2}{2}\tilde{C}} : C(\Delta\varphi_{12}^{(\leqq k)} - \Delta\varphi_{34}^{(\leqq k)}) : - e^{-\frac{\alpha^2}{2}\tilde{C}} : C(\Delta\varphi_{12}^{(\leqq k)} + \Delta\varphi_{34}^{(\leqq k)}) : \right] \\
&\qquad + : (C\Delta\varphi_{12}^{(\leqq k)} - 1)(C\Delta\varphi_{34}^{(\leqq k)} - 1) : ,
\end{aligned}
\tag{6.a}
$$

$$
\begin{aligned}
&\mathscr{E}_{>k} \cdot (: S\Delta\varphi_{12}^{(<h)} - \underline{\partial}\varphi_{\bar{x}}^{(<h)}(x_1 - x_2) :: S\Delta\varphi_{34}^{(<h)} - \underline{\partial}\varphi_{\bar{x}'}^{(<h)}(x_3 - x_4) :) \\
&\quad = \tfrac{1}{2}(e^{\alpha^2\tilde{C}} - \alpha^2\tilde{C} - 1) : C(\Delta\varphi_{12}^{(\leqq k)} - \Delta\varphi_{34}^{(\leqq k)}) : \\
&\qquad + \tfrac{1}{2}(1 - \alpha^2\tilde{C} - e^{-\alpha^2\tilde{C}}) : C(\Delta\varphi_{12}^{(\leqq k)} + \Delta\varphi_{34}^{(\leqq k)}) : \\
&\qquad + \alpha^2\tilde{C} : (C\Delta\varphi_{12}^{(\leqq k)} - 1)(C\Delta\varphi_{34}^{(\leqq k)} - 1) : \\
&\qquad + : (S\Delta\varphi_{12}^{(\leqq k)} - \underline{\partial}\varphi_{\bar{x}}^{(\leqq k)}(x_1 - x_2))(S\Delta\varphi_{34}^{(\leqq k)} - \underline{\partial}\varphi_{\bar{x}'}^{(\leqq k)}(x_3 - x_4)) : .
\end{aligned}
\tag{7.a}
$$

Finally on each of these terms the R operation has to be applied following the previous rules.

References

1. Benfatto, G., Gallavotti, G., Nicolò, F.: On the massive Sine-Gordon equation in the first few regions of collapse. Commun. Math. Phys. **83**, 387 (1982)
2. Nicolò, F.: On the massive Sine-Gordon equation in the higher regions of collapse. Commun. Math. Phys. **88**, 681 (1983)
3. Gallavotti, G.: Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods. Rev. Mod. Phys. **57**, (1985)
4. Nicolò, F., Renn, J., Steinmann, A.: On the massive Sine-Gordon equation in all regions of collapse. Commun. Math. Phys. **105**, 291 (1986)
5. Gallavotti, G., Nicolò, F.: Renormalization theory in four dimensional scalar fields (I). Commun. Math. Phys. **100**, 545 (1985);
   Renormalization theory in four dimensional scalar fields (II). Commun. Math. Phys. **101**, 247 (1985)
6. Pordt, A.: Mayer expansion for euclidean lattice field theory: Convergence properties and relation with perturbation theory. Preprint ISSN 0418-9833 Institut für Theoretische Physik, Universität Hamburg 1985
7. Benfatto, G.: An iterated Mayer expansion for the Yukawa gas. J. Stat. Phys. **41**, 671 (1985)
8. Benfatto, G., Gallavotti, G., Nicolò, F.: On the analyticity of the pressure in the hierarchical dipole gas. (Submitted to J. Stat. Phys.)
9. Marino, E.C.: Nucl. Phys. B **251**, 227 (1985)
10. Fröhlich, J.: Quantum Sine-Gordon equation and quantum solitons in two space-time dimensions. In: Renormalization Theory. Velo, G., Wightman, A. (eds.), Dordrecht: Reidel 1976, Classical and quantum statistical mechanics in one and two dimensions: Two component Yukawa and Coulomb systems. Commun. Math. Phys. **47**, 233 (1976)
11. Nicolò, F.: Some applications of the Renormalization Group to the scalar field theories, Contribution to the volume 36 of the R.C.P. Institut de Recherche Mathematique Avancée, Strasbourg 1986
12. Felder, G., Gallavotti, G.: Perturbation theory and non renormalizable scalar fields. Commun. Math. Phys. **102**, 549 (1986)
13. Gawedzki, K., Kupiainen, A.: Asymptotic freedom beyond perturbation theory. In: Critical Phenomena, Random Systems, Gauge Theories. Osterwalder, K., Stora, R. (eds.). Les Houches Summer School. New York: Elsevier 1984
14. Josè, J.V., Kadanoff, L.P., Kirkpatrick, S., Nelson, D.R.: Phys. Rev. B **16**, 1217 (1977)

Communicated by K. Gawedzki

August 25, 1988