

The Internal Symmetry Group of a Connection on a Principal Fiber Bundle with Applications to Gauge Field Theories

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Abstract. The internal symmetry group of a connection on a principal fiber bundle P is studied. It is shown that this group is a smooth proper Lie transformation group of P , which, if P is connected, is also free. Moreover, this group is shown to be isomorphic to the centralizer of the holonomy group of the connection. Several examples and applications of these results to gauge field theories are given.

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Introduction

Let $P(M, G)$ denote a principal fiber bundle with structure group G over a connected manifold M , let $\text{AUT}(P)$ denote the group of automorphisms of P , and let $\text{Aut}(P)$ denote the normal subgroup of automorphisms of P that cover the identity diffeomorphism of M . Let ω be the connection 1-form of a connection on P . If $F \in \text{AUT}(P)$, let $F^*\omega$ denote the pullback of ω by F . Then

$$\text{AUT}_\omega(P) = \{F \in \text{AUT}(P) | F^*\omega = \omega\}$$

is the *symmetry group of ω* , and

$$I_\omega(P) = \{F \in \text{Aut}(P) | F^*\omega = \omega\} = \text{Aut}(P) \cap \text{AUT}_\omega(P)$$

is the *internal symmetry group* of ω . In this paper we prove that if P is a connected manifold, then with respect to the compact open topology, $I_\omega(P)$ is a finite dimensional Lie transformation group of P that acts smoothly, properly, and freely on P (see Theorem 6.2). This is in contrast to the fact that $\text{AUT}_\omega(P)$ is in general an infinite-dimensional group. Another result is that for each $p \in P$, there is a Lie group isomorphism

$$i_p: I_\omega(P) \rightarrow C_G(\text{Hol}_p(\omega)), \quad F \mapsto u_F(p),$$

where $\text{Hol}_p(\omega) \subseteq G$ is the holonomy group of ω with reference point p , $C_G(\text{Hol}_p(\omega))$ is the centralizer of $\text{Hol}_p(\omega)$ in G , and $u_F: P \rightarrow G$ is a map determined by F . Thus the determination of $I_\omega(P)$ is completely reduced to finding the holonomy group $\text{Hol}(\omega)$ and an algebraic problem. We give several examples and applications of these results to gauge field theories.

It is of interest to compare these results to the results regarding the classical affine group of a linear connection Γ on the frame bundle LM of M . Let ω be the corresponding connection 1-form. Then a diffeomorphism f of M is an *affine transformation* if the induced mapping $Tf: TM \rightarrow TM$ maps horizontal curves to horizontal curves. This condition is equivalent to $\hat{f}^*\omega = \omega$, where $\hat{f}: LM \rightarrow LM$ denotes the natural lift of $f: M \rightarrow M$ to the frame bundle LM . Thus if $\text{Diff}(M)$ denotes the group of diffeomorphisms of M , then

$$A_\omega(M) = \{f \in \text{Diff}(M) \mid \hat{f}^*\omega = \omega\}$$

is the *group of affine transformations* of ω .

Now let θ denote the canonical \mathbb{R}^n -valued 1-form on LM , and let

$$\text{AUT}_\theta(LM) = \{F \in \text{AUT}(LM) \mid F^*\theta = \theta\}.$$

Then $F \in \text{AUT}_\theta(LM)$ if and only if $F = \hat{f}$ for some $f \in \text{Diff}(M)$ (Kobayashi-Nomizu [1963], p. 226, or Kobayashi [1972], p. 40). Thus

$$\text{AUT}_\theta(LM) = \{\hat{f} \in \text{AUT}_\theta(LM) \mid f \in \text{Diff}(M)\} \approx \text{Diff}(M)$$

and the affine group $A_\omega(M)$ is isomorphic to

$$\begin{aligned} A_\omega(M) &\approx \text{AUT}_{(\theta, \omega)}(LM) = \{F \in \text{AUT}(LM) \mid F^*\theta = \theta \text{ and } F^*\omega = \omega\} \\ &= \text{AUT}_\theta(LM) \cap \text{AUT}_\omega(LM). \end{aligned}$$

Thus at the bundle level, affine transformations are bundle automorphisms that preserve *both* the connection form ω and the canonical form θ , so that the classical group of affine transformations is actually the *simultaneous symmetry group* of both ω and θ . Although in general both $\text{AUT}_\theta(LM)$ and $\text{AUT}_\omega(LM)$ are infinite dimensional, the intersection $\text{AUT}_{(\theta, \omega)}(LM) \approx A_\omega(M)$ is a finite dimensional Lie group.

It is interesting that historically affine transformations were first considered as transformations of M whose natural lifts preserved ω , rather than as more general bundle automorphisms of LM that preserve ω , thereby relegating the interesting infinite dimensional group $\text{AUT}_\omega(LM)$ to the shelf of mathematical obscurities.

On a general principal fiber bundle there is no canonical 1-form θ . However, there are still interesting “reductions” of $\text{AUT}_\omega(P)$ resulting in a finite dimensional

group. Firstly, we may intersect $\text{AUT}_\omega(P)$ with $\text{Aut}(P)$, thereby obtaining the internal symmetry group $I_\omega(P)$. Thus in this respect $\text{Aut}(P)$ plays the role $\text{AUT}_\theta(LM)$ did for the affine connection on the frame bundle LM .

Another possibility is to put a Riemannian metric g on M and consider the group

$$I_{(g, \omega)}(P) = \{F \in \text{AUT}(P) \mid F^*\omega = \omega \text{ and } f^*g = g\},$$

where f is the diffeomorphism of M induced by the automorphism F . Thus $I_{(g, \omega)}(P)$ is the *group of symmetries of ω that cover isometries of g* . This also is a finite dimensional Lie transformation group of P acting smoothly, properly, and effectively (though not freely) on P (see Theorem 7.1). This group is of interest in gauge field theories (see Fischer [1982, 1985, 1988] and the references therein).

On a frame bundle LM we can compare the internal symmetry group

$$I_\omega(LM) = \text{Aut}(LM) \cap \text{AUT}_\omega(LM)$$

with the affine transformation group

$$A_\omega(M) \approx \text{AUT}_{(\theta, \omega)}(LM) = \text{Aut}_\theta(LM) \cap \text{AUT}_\omega(LM).$$

But if $F \in I_\omega(LM) \cap \text{AUT}_{(\theta, \omega)}(LM)$, F must cover the identity diffeomorphism id_M of M [since $I_\omega(LM) \subseteq \text{Aut}(LM)$], and F must be the lift of a diffeomorphism of M [since $\text{AUT}_{(\theta, \omega)} \subseteq \text{AUT}_\theta(LM)$]. Thus $F = \text{id}_{LM}$, the identity diffeomorphism of LM , and so

$$I_\omega(LM) \cap \text{AUT}_{(\theta, \omega)}(LM) = \{\text{id}_{LM}\}.$$

Thus the internal symmetry group $I_\omega(LM)$ has only a trivial intersection with $\text{AUT}_{(\theta, \omega)}(LM) \approx A_\omega(M)$. Thus $I_\omega(LM)$ and $\text{AUT}_{(\theta, \omega)}(L, M)$ are respectively the “vertical” and “horizontal” components of $\text{AUT}_\omega(LM)$. $I_\omega(LM)$ itself has no classical analog.

In considering the symmetry (or automorphism) group of a geometric structure (see Kobayashi-Nomizu [1963], p. 306, and [1969], p. 392), attention is usually focused on the automorphism group of a G -structure on M , i.e., a reduction of LM to a G -principal fiber bundle, $G \subseteq GL(n)$. The automorphism group is then the subgroup of $\text{Diff}(M)$ consisting of those maps f whose natural lift \hat{f} maps the G -structure to itself. In this regard the affine group $A_\omega(M)$ of a linear connection ω is somewhat of an exception inasmuch as a connection is not a G -structure on M (i.e., it is not a reduction of LM), and so $A_\omega(M)$ is not an automorphism group of a G -structure. Similarly, a connection ω on a general principal fiber bundle $P(M, G)$ is a geometric structure on P (or on M), but is not a G -structure on M , and so the internal symmetry group $I_\omega(P)$ is not an automorphism group of a G -structure. However, $I_\omega(P)$ is a symmetry group of a geometric structure, and so our result that $I_\omega(P)$ is a Lie transformation group adds another geometric structure to the list enumerated in Kobayashi-Nomizu [1969], p. 332, of geometric structures whose automorphism groups are Lie groups.

1. Notation and Preliminaries

Throughout this paper, M will denote a *connected* C^∞ n -dimensional manifold (Hausdorff and second countable, and hence paracompact), G a (second countable)

Lie group, and $P(M, G)$ a principal fiber bundle (PFB) with total space P , base space M , structure group G , projection map

$$\pi : P \rightarrow M,$$

and right action

$$\Phi : P \times G \rightarrow P, \quad (p, a) \mapsto p \cdot a.$$

All of the above paraphernalia will be denoted by $P(M, G)$.

For $p \in P$, let

$$\Phi_p : G \rightarrow P, \quad a \mapsto p \cdot a,$$

denote the orbit map through p , and for $a \in G$, let

$$R_a = \Phi_a : P \rightarrow P, \quad p \mapsto p \cdot a$$

denote the diffeomorphism of P corresponding to $a \in G$. Let $G(P) = \{R_a : P \rightarrow P | a \in G\}$ denote the Lie group of diffeomorphisms of P induced by the action Φ .

Let e be the identity in G , let $\mathfrak{G} = T_e G$ the Lie algebra of G (taken with the usual Lie algebra structure of left-invariant vector fields on G), and let

$$\text{Ad} : G \rightarrow GL(\mathfrak{G}), \quad a \mapsto \text{Ad}(a) : \mathfrak{G} \rightarrow \mathfrak{G}$$

denote the adjoint representation of G on \mathfrak{G} . Let $V_p = \ker T_p \pi \subseteq T_p P$ denote the vertical subspace of the tangent space $T_p P$, and let $T_e \Phi_p : \mathfrak{G} \rightarrow V_p \subseteq T_p P$ denote the derivative at the identity $e \in G$ of the orbit map $\Phi_p : G \rightarrow P$. For $A \in \mathfrak{G}$, let A^* be the fundamental vector field on P generated by A . For $p \in P$, A^* is defined by $A^*(p) = T_e \Phi_p \cdot A$.

Let $\text{Diff}(P)$ and $\text{Diff}(M)$ denote the groups of C^∞ diffeomorphisms of P and M , respectively, with composition of diffeomorphisms as the group structure. Let

$$\text{AUT}(P) = \{F \in \text{Diff}(P) | F \circ R_a = R_a \circ F \text{ for all } a \in G\}$$

denote the group of automorphisms of P . The condition $F \circ R_a = R_a \circ F$ can also be written $F(p \cdot a) = F(p) \cdot a$ for all $p \in P$ and $a \in G$. For $F \in \text{AUT}(P)$, F maps fibers to fibers and so induces a diffeomorphism of the base manifold M . Let $f = \hat{\pi}(F) \in \text{Diff}(M)$ denote this induced diffeomorphism. Moreover, the map

$$\hat{\pi} : \text{AUT}(P) \rightarrow \text{Diff}(M), \quad F \mapsto \hat{\pi}(F) = f,$$

is a group homomorphism. Its kernel is

$$\text{Aut}(P) = \{F \in \text{AUT}(P) | \hat{\pi}(F) = \text{id}_M\},$$

the normal subgroup of automorphisms of P covering the identity diffeomorphism id_M of M . The group $\text{Aut}(P)$ has two other representations, which we briefly describe.

Firstly, let $C^\infty(P, G)$ denote the C^∞ maps from P to G , and let

$$C_{\text{Aut}}^\infty(P, G) = \{u \in C^\infty(P, G) | u(p \cdot a) = a^{-1} \cdot u(p) \cdot a \text{ for all } p \in P \text{ and } a \in G\}.$$

Then $C_{\text{Aut}}^\infty(P, G)$ is a group with respect to pointwise multiplication of functions.

Secondly, consider the left action of G on itself by inner automorphisms $G \times G \rightarrow G$, $(a, b) \mapsto a \cdot b \cdot a^{-1}$. Let $G_{\text{Aut}}(M) = (P \times G)/G$ denote the corresponding

associated fiber bundle over M with standard fiber G . For $x \in M$, the fiber at x , denoted $G_{\text{Aut}}(M)_{x}$, is the Lie group of diffeomorphisms of the fiber $\pi^{-1}(x) \subseteq P$ that commute with the action of G on $\pi^{-1}(x)$. Thus $G_{\text{Aut}}(M)$ is a bundle of Lie groups over M . Let $C^\infty(G_{\text{Aut}}(M))$ denote the space of smooth cross-sections of $G_{\text{Aut}}(M)$, taken with its group structure of pointwise multiplication of sections. Then the following proposition is basic (see e.g., Trautman [1980], or Fischer [1985]).

1.1. Proposition. *The three groups $\text{Aut}(P)$, $C^\infty_{\text{Aut}}(P; G)$ and $C^\infty(G_{\text{Aut}}(M))$ are naturally isomorphic with each other. The first of these isomorphisms (i.e., group bijections) is given by*

$$\text{Aut}(P) \rightarrow C^\infty_{\text{Aut}}(P, G), \quad F \mapsto u_F,$$

where for $p \in P$, $u_F(p)$ is defined by the equation

$$F(p) = p \cdot u_F(p).$$

The second of these isomorphisms is given by

$$C^\infty_{\text{Aut}}(P, G) \rightarrow C^\infty(G_{\text{Aut}}(M)), \quad u \mapsto u_M,$$

where for $x \in M$, $u_M(x)$ is defined by

$$u_M(x) = \{(p \cdot a, a^{-1} \cdot u(p) \cdot a) \mid a \in G\} \in (P \times G)/G,$$

where $p \in \pi^{-1}(x)$, and the above definition is independent of the choice of p . \square

We shall have occasion to use the viewpoint represented by all three groups. However, since we shall be working primarily on P , we shall use mostly either $\text{Aut}(P)$ or $C^\infty_{\text{Aut}}(P, G)$, depending on convenience. In Sect. 2, convenience will dictate strongly that $C^\infty_{\text{Aut}}(P, G)$ be primarily used, while in Sect. 6, $\text{Aut}(P)$ will be primarily used.

Let $\mathfrak{X}(P)$ and $\mathfrak{X}(M)$ denote the Lie algebras of C^∞ vector fields on P and M , respectively, taken with the usual bracket of vector fields as the Lie algebra structure. Let

$$\mathfrak{X}_G(P) = \{Z \in \mathfrak{X}(P) \mid (R_a)_* Z = Z \text{ for all } a \in G\}$$

denote the Lie subalgebra of G -invariant vector fields on P . Then $\pi : P \rightarrow M$ induces a Lie algebra homomorphism $\pi_* : \mathfrak{X}_G(P) \rightarrow \mathfrak{X}(M)$, $Z \mapsto \pi_* Z = X$, whose kernel is given by

$$\mathfrak{X}_G^{\text{vert}}(P) = \{Z \in \mathfrak{X}_G(P) \mid \pi_* Z = 0\},$$

the G -invariant vertical vector fields on P . As above, $\mathfrak{X}_G^{\text{vert}}(P)$ has two other representations. Let $C^\infty(P, \mathfrak{G})$ denote the C^∞ maps from P to \mathfrak{G} , and let

$$C^\infty_{\text{Ad}}(P, \mathfrak{G}) = \{\varphi \in C^\infty(P, \mathfrak{G}) \mid \varphi(p \cdot a) = \text{Ad}(a^{-1}) \cdot \varphi(p) \text{ for all } P \in P \text{ and } a \in G\}.$$

Then $C^\infty_{\text{Ad}}(P, \mathfrak{G})$ is a Lie algebra using the pointwise bracket of maps, i.e., $[\varphi_1, \varphi_2](p) = [\varphi_1(p), \varphi_2(p)]$. Also, consider the adjoint action of G on \mathfrak{G} , $G \times \mathfrak{G} \rightarrow \mathfrak{G}$, $(a, A) \rightarrow \text{Ad}(a) \cdot A$, and let $\mathfrak{G}_{\text{Ad}}(M) = (P \times \mathfrak{G})/G$ denote the corresponding associated vector bundle over M with standard fiber \mathfrak{G} . Then $\mathfrak{G}_{\text{Ad}}(M)$ is a

bundle of Lie algebras over M . Let $C^\infty(\mathfrak{G}_{\text{Ad}}(M))$ denote the space of C^∞ cross-sections. Then $C^\infty(\mathfrak{G}_{\text{Ad}}(M))$ is a Lie algebra with respect to the pointwise bracket of sections. Corresponding to Proposition 1.1 is the following.

1.2. Proposition. *The three Lie algebras $\mathfrak{X}_G^{\text{vert}}(P)$, $C_{\text{Ad}}^\infty(P, \mathfrak{G})$, and $C^\infty(\mathfrak{G}_{\text{Ad}}(M))$ are naturally isomorphic with each other. These Lie algebra isomorphisms (i.e., Lie algebra bijections) are given respectively, by*

$$\mathfrak{X}_G^{\text{vert}}(P) \rightarrow C_{\text{Ad}}^\infty(P; \mathfrak{G}), \quad T \mapsto \varphi_T,$$

where for $p \in P$, $\varphi_T(p)$ is defined by the equation $T_e \Phi_p \cdot \varphi_T(p) = T(p)$, and by

$$C_{\text{Ad}}^\infty(P, \mathfrak{G}) \rightarrow C^\infty(\mathfrak{G}_{\text{Ad}}(M)), \quad \varphi \mapsto \varphi_M,$$

where for $x \in M$, $\varphi_M(x)$ is defined by

$$\varphi_M(x) = \{(p \cdot a, \text{Ad}(a^{-1}) \cdot \varphi(p)) \mid a \in G\} \in (P \times \mathfrak{G})/G,$$

where $p \in \pi^{-1}(x)$, and which is independent of the choice of p . \square

Let $\exp: \mathfrak{G} \rightarrow G$ denote the exponential map of the Lie algebra \mathfrak{G} . Then \exp induces an exponential map

$$\text{EXP}: C_{\text{Ad}}^\infty(P, \mathfrak{G}) \rightarrow C_{\text{Aut}}^\infty(P, G),$$

where if $\varphi \in C_{\text{Ad}}^\infty(P; \mathfrak{G})$, EXP is defined pointwise by $(\text{EXP } \varphi)(p) = \exp(\varphi(p))$. The resulting $u = \text{EXP } \varphi$ is in $C_{\text{Aut}}^\infty(P, G)$, since

$$\begin{aligned} u(p \cdot a) &= (\text{EXP } \varphi)(p \cdot a) = \exp(\varphi(p \cdot a)) = \exp(\text{Ad}(a^{-1}) \cdot \varphi(p)) \\ &= a^{-1} \cdot \exp(\varphi(p)) \cdot a = a^{-1} \cdot u(p) \cdot a. \end{aligned}$$

We let

$$\text{Exp}: \mathfrak{X}_G^{\text{vert}}(P) \rightarrow \text{Aut}(P), \quad T \mapsto \text{Exp } T$$

denote the corresponding exponential map on $\mathfrak{X}_G^{\text{vert}}(P)$, defined by

$$(\text{Exp } T)(p) = p \cdot (\text{EXP } \varphi_T)(p) = p \cdot (\exp \varphi_T(p)).$$

Note that if $\varphi \in C_{\text{Ad}}^\infty(P, \mathfrak{G})$, or $T \in \mathfrak{X}_G^{\text{vert}}(P)$, then $u_\lambda = \text{EXP}(\lambda\varphi) \in C_{\text{Aut}}^\infty(P, G)$ and $F_\lambda = \text{Exp } \lambda T \in \text{Aut}(P)$ are defined for all $\lambda \in \mathbb{R}$. Thus, in particular, if $T \in \mathfrak{X}_G^{\text{vert}}(P)$, T is a complete vector field on P with flow $F_\lambda = \text{Exp } \lambda T$.

It is important to note that, in general, the diffeomorphism $R_a: P \rightarrow P$, $a \in G$, is not an automorphism of P . Indeed, $R_a \in \text{Aut}(P)$ if and only if a is in the center $Z(G)$ of G , where the center

$$Z(G) = \{a \in G \mid a \cdot b = b \cdot a \text{ for all } b \in G\}$$

is a closed normal subgroup of G , and hence a Lie subgroup. Note that $Z(G) \subseteq \ker \text{Ad}$, and if G is connected, then we have equality.

The Lie algebra of $Z(G)$ is

$$z(G) = \{A \in \mathfrak{G} \mid \text{Ad}(b) \cdot A = A \text{ for all } b \in G\}.$$

On the other hand, the center of the Lie algebra \mathfrak{G} is defined to be

$$z(\mathfrak{G}) = \{A \in \mathfrak{G} \mid [A, B] = 0 \text{ for all } B \in \mathfrak{G}\},$$

so that $z(G) \subseteq z(\mathfrak{G})$. If G is connected, then $z(G) = z(\mathfrak{G})$.

Let

$$Z_G(P) = \{R_a | a \in Z(G)\} = G(P) \cap \text{Aut}(P),$$

a normal subgroup of $\text{Aut}(P)$. Similarly, let $Z_G(P) = \{A^* \in \mathfrak{X}_G^{\text{vert}}(P) | A \in z(G)\}$, where A^* denotes the fundamental vector field on P associated with $A \in \mathfrak{G}$, denote the corresponding ideal of $\mathfrak{X}_G^{\text{vert}}(P)$. Note that if $A \in z(G)$, then $\text{Ad}(a) \cdot A = A$ for all $a \in G$, and so

$$(R_a)_* A^* = (\text{Ad}(a^{-1}) \cdot A)^* = A^*,$$

so that A^* is indeed in $\mathfrak{X}_G^{\text{vert}}(P)$. For $A \notin z(G)$, $A^* \in \mathfrak{X}^{\text{vert}}(P)$, but is not G -invariant. Similarly, let

$$Z(P, G) = \{u : P \rightarrow Z(G) | u \text{ is constant on } P\},$$

and

$$Z(P, \mathfrak{G}) = \{\varphi : P \rightarrow z(G) | \varphi \text{ is constant on } P\},$$

the corresponding normal subgroup and ideal of $C_{\text{Aut}}^\infty(P, G)$ and $C_{\text{Ad}}^\infty(P, \mathfrak{G})$, respectively. Note that if $u \equiv \text{constant} \in Z(G)$, then $u(p \cdot a) = a^{-1} \cdot u(p) \cdot a = u(p)$, so u is in $C_{\text{Aut}}^\infty(P, G)$, and similarly, if $\varphi \equiv \text{constant} \in z(G)$, then

$$\varphi(p \cdot a) = \text{Ad}(a^{-1}) \cdot \varphi(p) = \varphi(p),$$

so φ is in $C_{\text{Ad}}^\infty(P, \mathfrak{G})$.

Let

$$Z(C_{\text{Aut}}^\infty(P, G)) = \{u \in C_{\text{Aut}}^\infty(P, G) | u \cdot u_1 = u_1 \cdot u \text{ for all } u_1 \in C_{\text{Aut}}^\infty(P, G)\}$$

denote the center of $C_{\text{Aut}}^\infty(P, G)$. Since for fixed p and varying u_1 , $u_1(p)$ spans G , $u \in Z(C_{\text{Aut}}^\infty(P, G))$ if and only if u takes values only in $Z(G)$ if and only if u is constant on the fibers of P . Such a u may however vary from fiber to fiber. Thus

$$Z(C_{\text{Aut}}^\infty(P, G)) = C_{\text{Aut}}^\infty(P, Z(G)) = \{u \in C_{\text{Aut}}^\infty(P, G) | u(p) \in Z(G) \text{ for all } p \in P\},$$

which in general is an infinite dimensional subgroup of $C_{\text{Aut}}^\infty(P, G)$. If however G is a semi-simple Lie group, then its center $Z(G)$ is a discrete subgroup of G . Thus if $u \in C_{\text{Aut}}^\infty(P, G)$, u is constant on fibers, and by continuity of u and connectivity of M , u is then constant on P . Thus in this case, the centers of $C_{\text{Aut}}^\infty(P, G)$ and G are isomorphic

$$Z(C_{\text{Aut}}^\infty(P, G)) = Z(P, G) \approx Z_G(P) \approx Z(G),$$

and are zero dimensional Lie groups (see also Daniel-Viallet [1980]).

For $0 \leq k \leq n = \dim M$, let $C^\infty(A^k(P) \otimes \mathfrak{G})$ denote the space of smooth Lie-algebra valued k -forms on P , let

$$C_{\text{Ad}}^\infty(A^k(P) \otimes \mathfrak{G}) = \{\varphi \in C^\infty(A^k(P) \otimes \mathfrak{G}) | R_a^* \varphi = \text{Ad}(a^*) \cdot \varphi \text{ for all } a \in G\}$$

denote the Lie algebra-valued k -forms of type $(\text{Ad}, \mathfrak{G})$, and let

$$A^k(P, \mathfrak{G}) = C_{\text{Ad, hor}}^\infty(A^k(P) \otimes \mathfrak{G})$$

$$= \{\varphi \in C_{\text{Ad, hor}}^\infty(A^k(P) \otimes \mathfrak{G}) | \varphi(Z_1, \dots, Z_k) = 0 \text{ whenever one of } Z_i \text{'s is vertical}\}$$

denote the space of horizontal (or tensorial) Lie-algebra valued k -forms of type $(\text{Ad}, \mathfrak{G})$. Since a zero-form on P is always horizontal, $C_{\text{Ad}}^\infty(P, \mathfrak{G}) = A^0(P, \mathfrak{G})$.

A connection 1-form ω on P is an element of $C^\infty(A^1(P) \otimes \mathfrak{G})$ such that

- (1) $R_a^* \omega = \text{Ad}(a^{-1}) \cdot \omega$ for all $a \in G$;
- (2) $\omega(A^*) = A$ for all $A \in \mathfrak{G}$.

We let $\mathcal{C}(P)$ denote the space of connection 1-forms on P , naturally identifiable with the space of connections on P . Since M is paracompact, $\mathcal{C}(P) \neq \emptyset$ (see Kobayashi-Nomizu [1967], p. 67). Also, $\mathcal{C}(P)$ is an affine space, and is the affine space associated to the vector space $A^1(P, \mathfrak{G})$.

For a connection 1-form ω , we let

$$D : C_{\text{Ad}}^\infty(P, \mathfrak{G}) \rightarrow A^1(P, \mathfrak{G}), \quad \varphi \mapsto D\varphi$$

denote the *gauge covariant exterior derivative*, where

$$D\varphi = (d\varphi)_{\text{hor}} = d\varphi + [\omega \wedge \varphi],$$

and where the *wedge bracket* $[\omega \wedge \varphi] \in C_{\text{Ad}}^\infty(A^1(P) \otimes \mathfrak{G})$ is defined pointwise by $[\omega \wedge \varphi] \cdot Z = [\omega(Z), \varphi]$ for $Z \in \mathfrak{X}(P)$ (see Fischer [1985] for more on this terminology). Also, we let

$$\Omega = D\omega = (d\omega)_{\text{hor}} = d\omega + \frac{1}{2}[\omega \wedge \omega] \in A^2(P, \mathfrak{G})$$

denote the *curvature 2-forms of ω* , where the wedge bracket

is given by

$$[\omega \wedge \omega] \in C_{\text{Ad}}^\infty(A^2(P) \otimes \mathfrak{G})$$

$$\begin{aligned} [\omega \wedge \omega] \cdot (Z_1, Z_2) &= [\omega(Z_1), \omega(Z_2)] - [\omega(Z_2), \omega(Z_1)] \\ &= 2[\omega(Z_1), \omega(Z_2)] \end{aligned}$$

for $Z_1, Z_2 \in \mathfrak{X}(P)$.

For $F \in \text{AUT}(P)$, and $\omega \in \mathcal{C}(P)$, we let $F^* \omega$ denote the pull-back of ω by F . Thus if $p \in P$ and $Z_p \in T_p P$, $F^* \omega$ is defined by

$$F^* \omega(p) \cdot Z_p = \omega(F(p)) \cdot (T_p F \cdot Z_p).$$

An easy check shows that $F^* \omega \in \mathcal{C}(P)$, and indeed the map $F^* : \mathcal{C}(P) \rightarrow \mathcal{C}(P)$ is an affine transformation of $\mathcal{C}(P)$. Thus $\text{AUT}(P)$ acts on $\mathcal{C}(P)$ on the right by pull-back as a group of affine transformations,

$$\mathcal{C}(P) \times \text{Aut}(P) \rightarrow \mathcal{C}(P), \quad (\omega, F) \mapsto F^* \omega.$$

We shall be interested in the restricted action by $\text{Aut}(P)$. Note that if $a \in Z(G)$, then $R_a \in \text{Aut}(P)$ and $R_a^* \omega = (\text{Ad}(a^{-1}) \cdot \omega) = \omega$, so that $Z_G(P) \subseteq I_\omega(P)$ for any connection ω . Thus $\text{Aut}(P)$ does not act effectively on $\mathcal{C}(P)$. However since $Z_G(P)$ is a normal subgroup of $\text{Aut}(P)$, this is easily remedied by considering the quotient group $\mathcal{A}(P) = Z_G(P) \backslash \text{Aut}(P)$ of right cosets of $Z(G)$, and the induced action

$$\mathcal{C}(P) \times \mathcal{A}(P) \rightarrow \mathcal{C}(P), \quad (\omega, [F]) \mapsto F^* \omega,$$

where $[F] = Z(G) \circ F$. Under the assumption that P is a connected manifold, the resulting action is then effective (see Fischer [1985]).

Let $\theta \in C^\infty(A^1(G) \otimes \mathfrak{G})$ denote the Maurer-Cartan form on G ; i.e., the unique left invariant Lie algebra valued 1-form that satisfies

- (1) $\theta(e) = \text{id}_{\mathfrak{G}} : \mathfrak{G} \rightarrow \mathfrak{G}$, the identity transformation of \mathfrak{G} .
- (2) $\tilde{L}_a^* \theta = \theta$ for all $a \in G$, where $\tilde{L}_a : G \rightarrow G, b \mapsto a \cdot b$ denotes left translation on G .

Thus θ is defined by

$$\theta(a) = T_a \tilde{L}_{a^{-1}} : T_a G \rightarrow \mathfrak{G}, \quad v_a \mapsto T_a \tilde{L}_{a^{-1}} \cdot v_a.$$

Note θ is now different from the canonical form θ discussed in the introduction.

The following proposition is then a global version of the local transformation law for connection 1-forms under a gauge transformation $u \in C_{\text{Aut}}^\infty(P, G)$ (see Kobayashi-Nomizu [1963], p. 66 or Bleecker [1981], p. 49).

1.3. Proposition. *Let $\omega \in \mathcal{C}(P)$, $F \in \text{Aut}(P)$, and $u_F \in C_{\text{Aut}}^\infty(P, G)$ correspond to F as in Proposition 1.2. Let θ be the Maurer-Cartan form on G . Then*

$$F^*\omega = \text{Ad}(u_F^{-1}) \cdot \omega + u_F^*\theta,$$

where $u_F^{-1}(p) \equiv (u_F(p))^{-1}$. \square

Thus for $u \in C_{\text{Aut}}^\infty(P, G)$, we define $u^*\omega = \text{Ad}(u^{-1}) \cdot \omega + u^*\theta$, thereby generating the equation $F^*\omega = u_F^*\omega$.

There is another interesting interpretation of the equation $u^*\omega = \text{Ad}(u^{-1}) \cdot \omega + u^*\theta$. For $u \in C_{\text{Aut}}^\infty(P; G)$, we would like to define a gauge covariant derivative of u . Since u takes values in G , which in general is not a linear space, we must modify the usual definition of gauge covariant derivative. The ingredient needed is a means of identifying the tangent spaces of G with \mathfrak{G} . This ingredient is provided by θ . Thus a reasonable definition of the gauge covariant derivative $Du \in A^1(P, \mathfrak{G})$ of u is given by

$$Du(p) \cdot Z_p = \theta(u(p)) \cdot (T_p u \cdot Z_{\text{hor}}) = (u^*\theta)(p) \cdot Z_{\text{hor}} = (u^*\theta)_{\text{hor}} \cdot Z_p,$$

where $p \in P$, $Z_p \in T_p P$, and Z_{hor} is the horizontal projection of Z_p . Thus $Du \equiv (u^*\theta)_{\text{hor}}$. A calculation as in Proposition 1.3 then shows that

$$Du = (u^*\theta)_{\text{hor}} = u^*\theta + \text{Ad}(u^{-1}) \cdot \omega - \omega = u^*\omega - \omega.$$

Thus in particular $u^*\omega = \omega$ if and only if $Du = 0$.

For $T \in \mathfrak{X}_G^{\text{vert}}(P)$ and $\omega \in \mathcal{C}(P)$, let $L_T \omega \in A^1(P, \mathfrak{G})$ denote the Lie derivative of ω with respect to T . That $L_T \omega$ is horizontal is the infinitesimal version of the fact that the difference between two connections is horizontal. Moreover, using the connection ω , the bijection

$$\mathfrak{X}_G^{\text{vert}}(P) \rightarrow C_{\text{Ad}}^\infty(P, \mathfrak{G}), \quad T \mapsto \varphi_T$$

of Proposition 1.2 can be written as $T \mapsto \varphi_T = \omega(T)$, although this correspondence is independent of the connection. The following relationship between $L_T \omega$ and $D\varphi_T$ is of importance.

1.4. Lemma. *Let $T \in \mathfrak{X}_G^{\text{vert}}(P)$ and $\varphi_T = \omega(T) \in C_{\text{Ad}}^\infty(P, \mathfrak{G})$. Then $L_T \omega = D\varphi_T$.*

Proof. $L_T \omega = di_T \omega + i_T d\omega = d\varphi_T + i_T d\omega$. Since $\Omega = D\omega$ is horizontal,

$$\begin{aligned} 0 &= i_T D\omega = i_T(d\omega + \frac{1}{2}[\omega \wedge \omega]) = i_T d\omega + \frac{1}{2}([\omega(T) \wedge \omega] - [\omega \wedge \omega(T)]) \\ &= i_T d\omega + [\varphi_T \wedge \omega]. \end{aligned}$$

Hence

$$L_T \omega = d\varphi_T + i_T d\omega = d\varphi_T - [\varphi_T \wedge \omega] = d\varphi_T + [\omega \wedge \varphi_T] = D\varphi_T. \quad \square$$

For $\omega \in \mathcal{C}(P)$ and $p \in P$, let $\text{Hol}_p(\omega) \subseteq G$ denote the holonomy group of ω with reference point $p \in P$. Then $\text{Hol}_p(\omega)$ is a Lie subgroup of G (Kobayashi-Nomizu [1963], p. 73), although it is not necessarily a closed subgroup. Note however that $\text{Hol}_p(\omega)$ has at most a countable number of components (since M is connected and

paracompact), and thus $\text{Hol}_p(\omega)$ satisfies the second axiom of countability as required by our definition of Lie group. Since M is connected, all of the holonomy groups are conjugate in G to each other, and we denote $\text{Hol}(\omega)$ as any one of them.

For $p \in P$, we let $P(p)$ denote the holonomy bundle (or sub-bundle) through p , i.e., the set of points in P that can be joined to p by a horizontal curve. Then $P(p) = Q(M, \text{Hol}_p(\omega))$ is a $\text{Hol}_p(\omega)$ -PFB over M , and the connection ω reduces to a connection on $Q(M, \text{Hol}_p(\omega))$ (Kobayashi-Nomizu [1963], p. 83).

Note that the holonomy sub-bundle $P(p)$ need only be an immersed sub-bundle. Although in general the closure of an immersed submanifold need not be a closed embedded submanifold, the closure $\overline{P(p)}$ of $P(p)$ will be a closed embedded submanifold in P , and in fact the closed sub-bundle $\overline{P(p)} = \overline{Q(M, \text{Hol}_p(\omega))} = \overline{Q(M, \overline{\text{Hol}_p(\omega)})}$ will be a reduced sub-bundle of $P(M, G)$ with structure group $\overline{\text{Hol}_p(\omega)}$, the closure of $\text{Hol}_p(\omega)$ in G . Moreover, the connection ω reduces to a connection in $Q(M, \overline{\text{Hol}_p(\omega)})$. These remarks follow easily from the fact that the closure H of a Lie subgroup H of G is a closed subgroup and hence an embedded submanifold.

In a similar vein as the above, we have the following lemma.

1.5. Lemma. *Let G be a topological group, S a subset of G , and*

$$C_G(S) = \{a \in G \mid a \cdot s = s \cdot a \text{ for all } s \in S\}$$

the centralizer of S in G . Then $C_G(S)$ is a closed subgroup of G . Also, $C_G(\overline{S}) = \overline{C_G(S)}$, where \overline{S} denotes the closure of S in G . Thus $C_G(S) = C_G(\overline{S}) = \overline{C_G(S)}$, where $\overline{C_G(S)}$ denotes the closure of $C_G(S)$ in G .

Proof. Let $\{a_n\}$ be a sequence in $C_G(S)$ such that $a_n \rightarrow a \in \overline{C_G(S)}$. Then for all n and $s \in S$, $a_n \cdot s = s \cdot a_n$, and so by continuity of the group operation, $a_n \cdot s \rightarrow a \cdot s$ and $s \cdot a_n \rightarrow s \cdot a$. Thus $a \cdot s = s \cdot a$ and so $a \in C_G(S)$ and thus $C_G(S)$ is closed.

Similarly, let $\{s_n\}$ be a sequence in S such that $s_n \rightarrow s \in \overline{S}$. Then for all n and $a \in C_G(S)$, $a \cdot s_n = s_n \cdot a$, and so as above $a \cdot s = s \cdot a$. Thus $a \in C_G(\overline{S})$, and so $C_G(S) \subseteq C_G(\overline{S})$. The reverse inclusion follows by definition of the centralizer. \square

Remark. In particular, the center of a topological group is closed.

We return now to G being a Lie group.

2. The Lie Group $I_\omega(P)$

Let ω be a connection on the PFB $P(M, G)$. Let

$$I_\omega(P) = \{F \in \text{Aut}(P) \mid F^*\omega = \omega\}$$

denote the *internal symmetry group* of ω . We denote the corresponding subgroup of $C_{\text{Aut}}^\infty(P, G)$ as

$$I_\omega(P, G) = \{u \in C_{\text{Aut}}^\infty(P, G) \mid u^*\omega = \omega\},$$

where $u^*\omega = \text{Ad}(u^{-1}) \cdot \omega + u^*\theta$. Let

$$\mathcal{I}_\omega(P) = \{T \in \mathfrak{X}_G^{\text{vert}}(P) \mid L_T\omega = 0\}$$

denote the Lie algebra of *infinitesimal symmetries* of ω . That $\mathcal{I}_\omega(P)$ is a Lie algebra follows from the identity

$$L_{[T_1, T_2]}\omega = L_{T_1}L_{T_2}\omega - L_{T_2}L_{T_1}\omega$$

for $T_1, T_2 \in \mathfrak{X}_G^{\text{vert}}$. Note that since infinitesimal symmetries are in $\mathfrak{X}_G^{\text{vert}}(P)$, they are automatically complete vector fields on P .

Since $L_T \omega = D\varphi_T$, the corresponding Lie algebra of $C_{\text{Ad}}^\infty(P, \mathfrak{G})$ is

$$\mathcal{J}_\omega(P, \mathfrak{G}) = \{\varphi \in C_{\text{Ad}}^\infty(P, \mathfrak{G}) \mid D\varphi = 0\} = \ker D$$

That $\mathcal{J}_\omega(P, \mathfrak{G})$ is a Lie algebra follows from the identity

$$D([\varphi_1, \varphi_2]) = [D\varphi_1 \wedge \varphi_2] + [\varphi_1 \wedge D\varphi_2]$$

for $\varphi_1, \varphi_2 \in C_{\text{Ad}}^\infty(P, \mathfrak{G})$.

For $p \in P$, let $\text{Hol}_p(\omega)$ be the holonomy group with reference point p , and let

$$C_G(\text{Hol}_p(\omega)) = \{a \in G \mid a \cdot b = b \cdot a \text{ for all } b \in \text{Hol}_p(\omega)\}$$

denote the centralizer of $\text{Hol}_p(\omega)$ in G . Although $\text{Hol}_p(\omega)$ need not be a closed subgroup of G , by Lemma 1.5, $C_G(\text{Hol}_p(\omega))$ is a closed subgroup of G , and hence a Lie subgroup. Its Lie algebra is

$$C_G(\text{Hol}_p(\omega)) = \{A \in \mathfrak{G} \mid \text{Ad}(b) \cdot A = A \text{ for all } b \in \text{Hol}_p(\omega)\}.$$

The following classical result will be our main workhorse for this section.

2.1. Proposition. *Let $P(M, G)$ be a PFB, let $\omega \in \mathcal{C}(P)$, and let $u \in I_\omega(P, G)$. Then*

- (1) *u is constant on every holonomy sub-bundle $P(p)$;*
- (2) *for $p \in P$, $u(p) \in C_G(\text{Hol}_p(\omega))$;*
- (3) *u is determined by its value at a single point. In particular, if $u(p_0) = e$ for some $p_0 \in P$, then $u \equiv e$ identically on P .*

Conversely if $u \in C_{\text{Aut}}^\infty(P, G)$ is constant on a particular holonomy subbundle $P(p_0)$, then $u^ \omega = \omega$, and (1), (2), and (3) above hold.*

Proof. (1): If $u \in I_\omega(P, G)$, then $u^* \omega = \text{Ad}(u^{-1}) \cdot \omega + u^* \theta = \omega$. For $p \in P$ and $p_1 \in P(p)$, let $\bar{c}(\lambda)$, $\lambda \in [0, 1]$, denote a horizontal curve from p to p_1 . Then $\omega(\bar{c}(\lambda)) \cdot \bar{c}'(\lambda) = 0$ for $\lambda \in [0, 1]$, and so $u^* \theta(\bar{c}'(\lambda)) \cdot \bar{c}'(\lambda) = 0$. Let $\lambda \mapsto a(\lambda) = (u \circ \bar{c})(\lambda)$, a curve in G . Then

$$\begin{aligned} u^* \theta(\bar{c}'(\lambda)) \cdot \bar{c}'(\lambda) &= \theta(u(\bar{c}(\lambda))) \cdot T_{\bar{c}(\lambda)} u \cdot \bar{c}'(\lambda) = \theta(u \circ \bar{c})(\lambda) \cdot (u \circ \bar{c})'(\lambda) \\ &= \theta(a(\lambda)) \cdot a'(\lambda) = 0. \end{aligned}$$

Since

$$\theta(a(\lambda)) = T_{a(\lambda)} \tilde{L}_{(a(\lambda))^{-1}} : T_{a(\lambda)} G \rightarrow \mathfrak{G}$$

is an isomorphism, $a'(\lambda) = (u \circ \bar{c})'(\lambda) = 0$. Thus $(u \circ \bar{c})(\lambda)$ is constant and so $u(p_1) = u(p)$.

(2): For any $a \in \text{Hol}_p(\omega)$, $p \cdot a \in P(p)$, and since u is constant on $P(p)$, $u(p) = u(p \cdot a) = a^{-1} \cdot u(p) \cdot a$. Thus $u(p) \in C_G(\text{Hol}_p(\omega))$.

(3): If $u_1 \in I_\omega(P)$ and $u_1(p_0) = u(p_0)$ for some $p_0 \in P$, then by (1), u_1 is constant on $P(p_0)$, and hence u and u_1 agree on $P(p_0)$. Since $P(p_0)$ intersects each fiber of P at least once, u_1 agrees with u on at least one point in each fiber, and thus by their transformation properties, on the entire fiber. Thus $u_1 = u$ on P .

Conversely, if $u \in C_{\text{Aut}}^\infty(P, G)$ is constant on a particular holonomy subbundle $P(p_0)$, then since $P(p_0 \cdot a) = P(p_0) \cdot a$,

$$u(P(p_0 \cdot a)) = u(P(p_0) \cdot a) = a^{-1} \cdot u(P(p_0)) \cdot a,$$

and so u is constant on every holonomy sub-bundle. Thus for $p \in P$, let $\bar{X} \in H_p \subseteq T_p P$ be a vector at p which is horizontal with respect to the connection ω , so that $\omega(p) \cdot \bar{X} = 0$. Thus

$$(u^*\omega)(p) \cdot \bar{X} = \text{Ad}(u^{-1}(p)) \cdot (\omega(p) \cdot \bar{X}) + u^*\theta(p) \cdot \bar{X} = \theta(u(p)) \cdot (T_p u \cdot \bar{X}).$$

By the Holonomy Reduction Theorem (Kobayashi-Nomizu [1963], p. 83), the connection ω on P reduces to a connection on $P(p)$. Thus the horizontal space H_p is contained in the tangent space $T_p(P(p))$. Since u is constant on $P(p)$, $T_p u \cdot \bar{X} = 0$. Thus $(u^*\omega)(p) \cdot \bar{X} = 0$, and so $H_p = \ker \omega(p) \subseteq \ker(u^*\omega)(p)$. Since $u^*\omega$ is also a connection on P , $\dim \ker \omega(p) = \dim \ker(u^*\omega)(p)$, and so $\ker \omega(p) = \ker(u^*\omega)(p)$. Thus $\omega(p)$ and $(u^*\omega)(p)$ have the same horizontal subspace at p , and since p was arbitrary, $u^*\omega = \omega$. Note that all connections determine the same vertical subspace $V_p = \ker T_p \pi$, reflected by the requirement that $\omega(A^*) = A$ for all $A \in \mathfrak{G}$. \square

Remarks. 1. Thus Proposition 2.1 can be summarized by

$$I_\omega(P, G) = \{u \in C_{\text{Aut}}^\infty(P, G) \mid u \text{ is constant on the holonomy sub-bundles of } P\}.$$

2. Property (3) can be rephrased in terms of $I_\omega(P)$ as follows:

2.1.1. Proposition. *If $F \in I_\omega(P)$ fixes a point $p_0 \in P$, $F(p_0) = p_0$, then $F = \text{id}$, the identity automorphism of P .*

3. As we have seen in Sect. 1, $Du = (u^*\theta)_{\text{hor}} = u^*\omega - \omega$. Thus $u \in I_\omega(P, G)$ if and only if $Du = 0$ if and only if u is a gauge covariant constant if and only if (1), (2), and (3) above hold. \square

The infinitesimal version of Proposition 2.1 is the following.

2.2. Proposition. *Let $P(M, G)$ be a PFB, let $\varphi \in \mathcal{I}_\omega(P, \mathfrak{G}) = \ker D$. Then*

- (1) φ is constant on every holonomy sub-bundle $P(p)$;
- (2) for $p \in P$, $\varphi(p) \in c_G(\text{Hol}_p(\omega)) = \{A \in \mathfrak{G} \mid \text{Ad}(b) \cdot A = A \text{ for all } b \in \text{Hol}_p(\omega)\}$;
- (3) φ is determined by its value at a single point. In particular, if $\varphi(p_0) = 0$ for some $p_0 \in P$, then $\varphi \equiv 0$ on P .

Conversely, if $\varphi \in C_{\text{Ad}}^\infty(P, \mathfrak{G})$ is constant on a particular holonomy subbundle, then $D\varphi = 0$, and (1), (2), and (3) above hold.

Proof. If $D\varphi = (d\varphi)_{\text{hor}} = d\varphi + [\omega, \varphi] = 0$, then on a horizontal curve $\bar{c}(\lambda)$,

$$D\varphi(\bar{c}(\lambda)) \cdot \bar{c}'(\lambda) = d\varphi(\bar{c}(\lambda)) \cdot \bar{c}'(\lambda) = (\varphi \circ \bar{c})'(\lambda) = 0.$$

Thus φ is constant on horizontal curves and hence is constant on every holonomy sub-bundle. The proof now proceeds as in Proposition 2.1. Alternately, the proposition can be proven by taking curves in $I_\omega(P, G)$ and using Proposition 2.1. \square

Remark. In terms of $\mathcal{I}_\omega(P) = \{T \in \mathfrak{X}_G^{\text{vert}}(P) \mid L_T \omega = 0\}$, property (3) can be rephrased as

2.2.1. Proposition. *If $T \in \mathcal{I}_\omega(P)$ satisfies $T(p_0) = 0$ for some $p_0 \in P$, then $T \equiv 0$. \square*

Using Proposition 2.1, we can now derive a “formula” for $I_\omega(P, G)$.

2.3. Theorem. *Let $P(M, G)$ be a PFB, let $\omega \in \mathcal{C}(P)$, and let $p_0 \in P$. Then the evaluation map at p_0 ,*

$$\text{eval}_{p_0} : I_\omega(P, G) \rightarrow C_G(\text{Hol}_{p_0}(\omega)), \quad u \mapsto u(p_0),$$

is a group isomorphism. Similarly, the evaluation map

$$\text{eval}'_{p_0} : \mathcal{J}_\omega(P, G) \rightarrow c_G(\text{Hol}_{p_0}(\omega)), \quad \varphi \mapsto \varphi(p_0)$$

is a Lie algebra isomorphism.

Proof. By the definition of the group structure in $C_{\text{Aut}}^\infty(P, G)$,

$$\text{eval}_{p_0}(u_1 \cdot u_2) = u_1(p_0) \cdot u_2(p_0) = (\text{eval}_{p_0}(u_1)) \cdot (\text{eval}_{p_0}(u_2)),$$

and so eval_{p_0} is a group homomorphism. It is injective by property (3) of Proposition 2.1.

To show surjectivity, let $a \in C_G(\text{Hol}_{p_0}(\omega))$, and define a function $u : P \rightarrow G$ such that

$$u(p) = \begin{cases} a & \text{if } p \in P(p_0) \\ b^{-1} \cdot a \cdot b & \text{if } p \in P(p_0 \cdot b) \end{cases}$$

u is well-defined, for if $p \in P(p_0) \cap P(p_0 \cdot b)$, then $b \in \text{Hol}_{p_0}(\omega)$ [and $P(p_0) = P(p_0 \cdot b)$], and thus $b^{-1} \cdot a \cdot b = a$ since $a \in C_G(\text{Hol}_{p_0}(\omega))$. To find the transformation properties of u , let $p \in P(p_0 \cdot b)$, so $u(p) = b^{-1} \cdot a \cdot b$. Then for $c \in G$,

$$p \cdot c \in P(p_0 \cdot b) \cdot c = P(p_0 \cdot b \cdot c),$$

so

$$u(p \cdot c) = (b \cdot c)^{-1} \cdot a \cdot (b \cdot c) = c^{-1} \cdot (b^{-1} \cdot a \cdot b) \cdot c = c^{-1} \cdot u(p) \cdot c.$$

If $\text{Hol}_{p_0}(\omega)$ is a closed subgroup of G , $P(p_0)$ is an embedded submanifold, and since by construction u is constant on the holonomy sub-bundles, it follows that u is a C^∞ function on P . Thus $u \in C_{\text{Aut}}^\infty(P, G)$, and by Proposition 2.1, $u^*\omega = \omega$.

If $\text{Hol}_{p_0}(\omega)$ is not closed in G , then by Lemma 1.5, $C_G(\text{Hol}_p(\omega)) = C_G(\overline{\text{Hol}_{p_0}(\omega)})$. Thus if a centralizes $\text{Hol}_{p_0}(\omega)$, a centralizes $\overline{\text{Hol}_{p_0}(\omega)}$. Thus u defined above is constant on the embedded submanifold $\overline{P(p_0)} = Q(M, \overline{\text{Hol}_{p_0}(\omega)})$. Thus, as above, u is a C^∞ function on P , $u \in C_{\text{Aut}}^\infty(P, G)$, and as u is constant on the holonomy sub-bundles, $u^*\omega = \omega$. \square

Remark. Another approach to the surjectivity of eval_{p_0} that avoids the complication of non-closed holonomy groups is as follows. Let $x_0 = \pi(p_0)$, and let Hol_{x_0} denote the holonomy group with reference point x_0 . Then Hol_{x_0} is a Lie subgroup of $G_{x_0} = (G_{\text{Aut}}(M))_{x_0}$, the group of automorphisms of the fiber P_{x_0} . If $a \in C_G(\text{Hol}_{p_0})$, let $a_{x_0} \in G_{x_0}$ be the corresponding element in G_{x_0} . Then $a_{x_0} \in C_{G_{x_0}}(\text{Hol}_{x_0})$. We construct a smooth section $u_M \in C^\infty(G_{\text{Aut}}(M))$ as follows. If $x_1 \in M$, let $c : [0, 1] \rightarrow M$, $\lambda \rightarrow c(\lambda)$, be a smooth curve from x_0 to x_1 , let $\bar{c}^1(\lambda) = c(1 - \lambda)$, let $\tau_c : P_{x_0} \rightarrow P_{x_1}$ denote parallel translation of the fiber P_{x_0} along c , and let $\tau_c^{-1} : P_{x_1} \rightarrow P_{x_0}$ denote its inverse. Define $u_M(x_1)$ by

$$u_M(x_1) = \tau_c \circ a_{x_0} \circ \tau_c^{-1}.$$

Then $u_M(x_1)$ is independent of the curve c from x_0 to x_1 since if c_1 and c_2 are two such curves,

$$\begin{aligned} \tau_{c_1} \circ a_{x_0} \circ \tau_{c_1^{-1}} &= \tau_{c_2} \circ (\tau_{c_2^{-1}} \circ \tau_{c_1}) \circ a_{x_0} \circ (\tau_{c_1^{-1}} \circ \tau_{c_2}) \circ \tau_{c_2^{-1}} \\ &= \tau_{c_2} \circ \tau_\gamma \circ a_{x_0} \circ \tau_\gamma^{-1} \circ \tau_{c_2^{-1}} = \tau_{c_2} \circ a_{x_0} \circ \tau_{c_2^{-1}}, \end{aligned}$$

where $\gamma = c_1 \cdot c_2^{-1}$ is the closed composite curve based at x_0 , and the last equality follows since $\tau_\gamma \in \text{Hol}_{x_0}$ and $a_{x_0} \in C_{G_{x_0}}(\text{Hol}_{x_0})$. Let $u \in C_{\text{Aut}}^\infty(P, G)$ correspond to $u_M \in C^\infty(G_{\text{Aut}}(M))$. Then by construction of u_M , $u(p_0) = a$ and u is constant on the holonomy sub-bundles of P , and so $u^*\omega = \omega$. \square

By Lemma 1.5, $C_G(\text{Hol}_{p_0}(\omega))$ is a closed subgroup of G , and hence a closed Lie subgroup. In particular, $C_G(\text{Hol}_{p_0}(\omega))$ is an embedded submanifold in G . We put a Lie group structure on $I_\omega(P, G)$ by declaring that the group isomorphism

$$\text{eval}_{p_0} : I_\omega(P, G) \rightarrow C_G(\text{Hol}_{p_0}(\omega))$$

is a Lie group isomorphism. With respect to this Lie group structure,

$$\text{eval}'_{p_0} = T_e \text{eval}_{p_0} : \mathcal{I}_\omega(P, \mathfrak{G}) \rightarrow \mathfrak{c}_G(\text{Hol}_{p_0}(\omega))$$

is the induced Lie algebra isomorphism.

Clearly, for different $p \in P$, the Lie group structures induced on $I_\omega(P, G)$ are all isomorphic. Moreover, any other second countable Lie group structure on $I_\omega(P, G)$ must be isomorphic to that induced by eval_{p_0} (see e.g., Helgason [1962], p. 109, or Warner [1971], p. 95). Thus the Lie group structure on $I_\omega(P, G)$ is unique (up to isomorphism).

Using the group bijection between $I_\omega(P)$ and $I_\omega(P, G)$, we carry the Lie group structure of $I_\omega(P, G)$ back to $I_\omega(P)$. With respect to this structure, the map

$$I_\omega(P) \rightarrow C_G(\text{Hol}_{p_0}(\omega)), \quad F \mapsto u_F(p_0)$$

is a Lie group isomorphism, where u_F corresponds to F by Proposition 1.1. Thus we have the following “formula” for $I_\omega(P)$, $I_\omega(P) \approx C_G(\text{Hol}(\omega))$, where $\text{Hol}(\omega)$ denotes any one of the holonomy groups $\text{Hol}_p(\omega)$, $p \in P$.

Remark. If $a \in C_G(\text{Hol}_{p_0}(\omega))$, $p_0 \in P$, $F \in I_\omega(P)$ is determined by the equation $F(p_0) = p_0 \cdot a$ [since $F \in I_\omega(P)$, F is uniquely determined by its value at a point]. Note, however, that although R_a solves this equation, R_a is not the corresponding internal symmetry of ω unless $a \in Z(G)$, and then $R_a \in \text{Aut}(P)$. Thus $I_\omega(P)$, although isomorphic to the subgroup $C_G(\text{Hol}_{p_0}(\omega))$ of G , does not in general have a representation as a subgroup of $G(P) = \{R_a | a \in G\}$. Indeed $I_\omega(P)$ is a subgroup of $\text{Aut}(P)$, and $G(P)$ is a subgroup of $\text{Diff}(P)$.

3. Some Examples

We now give some examples of the formula $I_\omega(P) \approx C_G(\text{Hol}(\omega))$.

3.1. Example. Let $P(M, G)$ be a PFB with Abelian structure group G . Then $Z(G) = G$, and so for all $a \in G$, $R_a \in \text{Aut}(P)$. Let $\omega \in \mathcal{C}(P)$ with holonomy group $\text{Hol}(\omega) \subseteq G$. Note that since the holonomy group of ω with respect to different

points are all conjugate to each other in G , $\text{Hol}(\omega)$ is independent of reference point.

Since G is abelian, $C_G(\text{Hol}(\omega)) = G$, and so $I_\omega(P) \approx G$. But since for all $a \in G$, $R_a \in \text{Aut}(P)$, and $R_a^* \omega = \omega$, $\{R_a | a \in G\} \subseteq I_\omega(P)$, so that $I_\omega(P) = G(P) = \{R_a | a \in G\} \approx G$. Thus in the Abelian case, for any connection ω , $I_\omega(P)$ is actually isomorphic to the structure group of the bundle. The Lie algebra of $I_\omega(P)$ is $\mathcal{I}_\omega(P) = \{A^* | A \in \mathfrak{G}\} = \mathfrak{G}(P) \approx \mathfrak{G}$. Similarly,

$$I_\omega(P, G) = \{u : P \rightarrow G | u \text{ is constant on } P\}$$

with Lie algebra

$$\mathcal{I}_\omega(P, \mathfrak{G}) = \{\varphi : P \rightarrow \mathfrak{G} | \varphi \text{ is constant on } P\}.$$

In physical field theories, the Abelian case is of interest inasmuch as electromagnetism can be formulated in terms of a connection ω on a $U(1)$ -PFB $P(M, U(1))$ over a Lorentz manifold (M, g) . For any such connection the internal symmetry group is then given by $I_\omega(P) \approx U(1)$. Since $I_\omega(P)$ leads to global conservation laws, in this case (or more generally, in any Abelian case), the structure group $U(1)$ gives rise to global conservation laws for the resulting field theory (see also Sect. 4), and Fischer [1985]).

3.2. *Example.* Let $P(M, G) = M \times G$ be the product bundle and let ω_0 be the canonical flat connection on P . Thus if $\pi_2 : M \times G \rightarrow G$, $(x, a) \mapsto a$, denotes projection onto the second factor, then $\omega_0 = \pi_2^* \theta$, where θ is the Maurer-Cartan form on G . Then $\text{Hol}(\omega_0) = \{e\}$ (again independent of reference point), and $C_G(\text{Hol}(\omega_0)) = C_G(\{e\}) = G$, so that $I_{\omega_0}(P) \approx G$ is again isomorphic to the structure group of the bundle. In this case however, $I_{\omega_0}(P)$ is not equal to $G(P)$, since the right translations $R_a : P \rightarrow P$ are not automorphisms of the bundle [unless $a \in Z(G)$]. However, on a product bundle, G can be represented as a group of left translations

$$a \mapsto L_a = (\text{id}_M \times \tilde{L}_a) : M \times G \rightarrow M \times G, \quad (x, b) \mapsto (x, ab),$$

where \tilde{L}_a is left translation on G by a , and this action of G on $M \times G$ commutes with the right action of G on $M \times G$,

$$L_a \circ R_b(x, c) = (x, acb) = R_b \circ L_a(x, c).$$

Thus for all $a \in G$, $L_a \in \text{Aut}(P)$, and also

$$L_a^* \omega_0 = L_a^* \pi_2^* \theta = (\pi_2 \circ L_a)^* \theta = (\tilde{L}_a \circ \pi_2)^* \theta = \pi_2^* \tilde{L}_a^* \theta = \pi_2^* \theta = \omega_0,$$

since θ is invariant by left translations. Hence $L_a \in I_{\omega_0}(P)$. Thus $I_{\omega_0}(P)$ can be displayed concretely as the group of left translations of $M \times G$,

$$I_{\omega_0}(M \times G) = \{L_a : M \times G \rightarrow M \times G | a \in G\} = G_L,$$

where G_L is defined only for product bundles, or for bundles isomorphic to product bundles.

The Lie algebra of G_L is given by

$$\mathfrak{G}_L = \{(0, \tilde{A}) \in \mathfrak{X}_G^{\text{vert}}(M \times G) | A \in \mathfrak{G}\},$$

where \tilde{A} is the left-invariant vector field on G corresponding to $A \in \mathfrak{G}$. Thus for $a \in G$, $\tilde{A}(a) = T_e \tilde{L}_a \cdot A$.

3.3. *Example.* As a slightly more general example, let ω be a connection on a PFB $P(M, G)$ whose holonomy group $\text{Hol}(\omega) = \{e\}$. Then for $p_0 \in P$, the holonomy bundle $P(p_0)$ is an $\{e\}$ -PFB over M , or a global cross-section of $P(M, G)$. Thus $P(M, G)$ is bundle isomorphic to the product bundle $M \times G$ (but not canonically) and ω is isomorphic to the canonical flat connection ω_0 on $M \times G$.

Then as above $I_\omega(P) \approx G$, and using an isomorphism of $P(M, G)$ with $M \times G$, $I_\omega(P)$ can be represented as a group of “left translations” on $P(M, G)$.

As a particular subcase, if M is simply connected and ω is a flat connection on $P(M, G)$, then $\text{Hol}(\omega) = \{e\}$, and so $I_\omega(P) \approx G$. This case is of interest for Yang-Mills field theories over a simply connected spacetime M , often taken to be either \mathbb{R}^4 or S^4 . In this case, if the connection ω on $P(M, G)$ is flat, then $I_\omega(P) \approx G$. Ignoring the distinction between left and right translations, the structure group G is said to induce the global conservation laws of the associated Lagrangian field theory (see also Sect. 4). \square

We now give some examples where we must compute the centralizer $C_G(\text{Hol}(\omega))$.

3.4. *Example.* Let $M = S^n$ taken with its usual metric g_0 and orientation as the unit sphere in \mathbb{R}^{n+1} . Let $L(S^n)$ denote its frame bundle, and let ω_0 be the Levi-Civita connection of g_0 on $L(S^n)$. Then $\text{Hol}(\omega_0) = \text{SO}(n)$ (see Poor [1981], p. 64, for the calculation), so that

$$I_{\omega_0}(L(S^n)) \approx C_{GL(n)}(\text{SO}(n)).$$

Let $\mathbb{R}^* = \mathbb{R} - \{0\}$. Then we have the following.

3.4.1. Lemma. For $n \in \mathbb{Z}^+, n \neq 2$,

$$C_{GL(n)}(\text{SO}(n)) = Z(GL(n)) = \{\lambda I \mid \lambda \in \mathbb{R}^*\} \approx \mathbb{R}^*.$$

For $n = 2$,

$$C_{GL(2)}(\text{SO}(2)) = \mathbb{R}^* \cdot \text{SO}(2) = \{\lambda \cdot a \mid \lambda \in \mathbb{R}^* \text{ and } a \in \text{SO}(2)\}.$$

Proof. Since $\text{SO}(n)$ acts irreducibly on \mathbb{R}^n , if $A \in GL(n)$ commutes with every element of $\text{SO}(n)$, then by Schur’s lemma (see e.g., Kobayashi-Nomizu [1969], p. 277), if n is odd, then $A = \lambda I, \lambda \in \mathbb{R}^*$, or if $n = 2m$ is even,

$$A = \begin{pmatrix} aI_m & bI_m \\ -bI_m & aI_m \end{pmatrix}, \quad a^2 + b^2 \neq 0.$$

Since for all $n \in \mathbb{Z}^+, Z(GL(n)) \subseteq C_{GL(n)}(\text{SO}(n))$, for n odd, we have equality. If n is even, let $B \in \text{SO}(2n)$. Then a conjugate of B can be put into the standard form

$$B(\theta_1, \dots, \theta_m) = \begin{pmatrix} B(\theta_1) & & \\ & \ddots & \\ & & B(\theta_m) \end{pmatrix},$$

where $B(\theta_i) = \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}$. A matrix computation shows that if $m \geq 2$ and A above commutes with $B(\theta_1, \dots, \theta_m)$ for all θ_i , then $b = 0$. Hence in this case $Z(GL(n)) = C_{GL(n)}(\text{SO}(n))$. If $n = 2$, all elements A of the above form commute with $\text{SO}(2)$, so

that

$$C_{GL(2)}(\text{SO}(2)) = \mathbb{R}^+ \cdot \text{SO}(2) = \mathbb{R}^* \cdot \text{SO}(2).$$

For $\lambda \in \mathbb{R}^*$, denote by λ also the map

$$\lambda : L(S^n) \rightarrow L(S^n), \quad p \mapsto \lambda p,$$

where if $p = (X_1, \dots, X_n)$ is a frame at x , $\lambda p = (\lambda X_1, \dots, \lambda X_n)$ is also a frame at x . Thus for $n \neq 2$,

$$I_{\omega_0}(L(S^n)) = \{ \lambda : L(S^n) \rightarrow L(S^n) \mid \lambda \in \mathbb{R}^* \} \approx \mathbb{R}^*,$$

with a similar result for $n = 2$.

Now let $0^+(S^n) \subseteq L(S^n)$ denote the $\text{SO}(n)$ bundle of positively oriented orthonormal frames on S^n , and let ω'_0 denote the Levi-Civita connection of g_0 on $0^+(S^n)$. Then ω'_0 is the reduction of ω_0 to $0^+(S^n)$, and so $\text{Hol}(\omega'_0) = \text{Hol}(\omega_0) = \text{SO}(n)$. Thus

$$I_{\omega'_0}(0^+(S^n)) \approx C_{\text{SO}(n)}(\text{SO}(n)) = Z(\text{SO}(n)) = \begin{cases} \{I\} & \text{for } n \text{ odd} \\ \{\pm I\} & \text{for } n \text{ even } \geq 4 \\ \text{SO}(2) & \text{for } n = 2. \end{cases} \quad \square$$

Remark. The above example shows that the internal symmetry group of a connection can change when the connection is reduced to a sub-bundle (although it need not necessarily change; see Example 3.6). For comparison's sake, note that the classical affine group $A_\omega(M)$ is independent of any reduction of ω to a sub-bundle. In the case at hand, for $n \geq 2$,

$$A_{\omega_0}(S^n) = A_{\omega'_0}(S^n) = I_{g_0}(S^n) = O(n+1).$$

Also, we remark that $A_{\omega_0}(S^n) \subseteq \text{Diff}(S^n)$, whereas $I_{\omega_0}(L(S^n)) \subseteq \text{Aut}(L(S^n))$, so that

$$\text{AUT}_{(\theta, \omega_0)}(L(S^n)) \cap I_{\omega_0}(L(S^n)) = \{\text{id}_{L(S^n)}\},$$

where θ is here the canonical 1-form on $L(S^n)$.

3.5. Example. Let $P(M, G)$ be a PFB with $\dim M \geq 2$, let H be a Lie subgroup of G , and let $i : Q(M, H) \rightarrow P(M, G)$ be a reduction of G to H . Assume that Q is a connected manifold. Then there exists a connection ω_0 on $Q(M, H)$ such that $\text{Hol}(\omega_0) = H$ (see Kobayashi-Nomizu [1963], p. 90). Also ω_0 induces a unique connection ω_1 on $P(M, G)$ such that $\omega_0 = i^*\omega_1$ and $\text{Hol}(\omega_1) = \text{Hol}(\omega_0) = H$. Now

$$I_{\omega_0}(Q) \approx C_H(\text{Hol}(\omega_0)) = C_H(H) = Z(H),$$

and

$$I_{\omega_1}(P) \approx C_G(\text{Hol}(\omega_0)) = C_G(H). \quad \square$$

We give two special cases of this example.

3.6. Example. Let M be a $2m$ -dimensional orientable manifold, $m \geq 1$, J an almost complex structure on M ($J^2 = -I$), and g a Hermitian metric on M , $g(JX, JY) = g(X, Y)$. Consider M with the natural orientation induced by J and let $C(M) = P(M, GL(m, \mathbb{C}))$ denote the complex linear frame bundle;

$0^+(M) = P(M, SO(2m))$ the bundle of oriented orthonormal frames and $U(M) = C(M) \cap 0^+(M) = P(M, U(m))$ the unitary frame bundle.

In the real representation of $GL(m, \mathbb{C})$,

$$GL(m, \mathbb{C}) \rightarrow GL(2m, \mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B are real $m \times m$ matrices,

$$U(m) = GL(m, \mathbb{C}) \cap O(2m) = GL(m, \mathbb{C}) \cap SO(2m),$$

the latter equation following from the fact that $GL(m, \mathbb{C})$ is connected and hence lies in $GL^0(2m, \mathbb{R})$, the connected component of the identity of $GL(2m, \mathbb{R})$.

Let $i: U(M) \rightarrow 0^+(M)$ denote the bundle reduction of $0^+(M)$ to $U(M)$ induced by the almost complex structure J . Since M is connected and since $U(m)$ is connected ($m \geq 1$), it follows that the total space $U(M)$ is connected. Also, $\dim M = 2m \geq 2$, and so there exists a connection ω_0 on $U(M)$ such that $\text{Hol}(\omega_0) = U(m)$. This connection then induces a unique connection ω_1 on $0^+(M)$ such that $\omega_0 = i^*\omega_1$, and such that $\text{Hol}(\omega_0) = \text{Hol}(\omega_1) = U(m)$. For these connections,

$$I_{\omega_0}(U(M)) \approx C_{U(m)}(U(m)) = Z(U(m)) = \{e^{i\theta} I_m \mid \theta \in \mathbb{R}\} \approx U(1),$$

and

$$I_{\omega_1}(0^+(M)) \approx C_{SO(2m)}(U(m)),$$

where $U(m)$ is taken as a subgroup of $SO(2m)$ in the real representation.

3.6.1. Lemma.

$$C_{SO(2m)}(U(m)) = Z(U(m)) \approx U(1).$$

Proof. In the real representation,

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in U(m),$$

where I_m is the $m \times m$ real identity matrix. Let $A \in C_{SO(2m)}(U(m))$. Since A commutes with J , $A \in GL(m, \mathbb{C})$. Thus $A \in GL(m, \mathbb{C}) \cap SO(2m) = U(m)$, and so $A \in U(m)$. Since $Z(U(m)) \subseteq C_{SO(2m)}(U(m))$, we have equality. \square

Thus in the above example, the internal symmetry groups of ω_0 and ω_1 are isomorphic,

$$I_{\omega_1}(0^+(M)) \approx I_{\omega_0}(U(M)) \approx Z(U(m)) \approx U(1).$$

3.7. Example. Let $G = SU(n)$, $n \geq 2$, and let $P(M, G) = M \times SU(n)$ be the product bundle, $\dim M \geq 2$. Let H be a closed connected proper subgroup of $SU(n)$ such that H acts irreducibly on \mathbb{C}^n [for example, in the real representation, $SO(n) \subseteq SU(n)$, $n \geq 2$, acts irreducibly on \mathbb{C}^n]. Let $i: M \times H \rightarrow M \times SU(n)$ denote the reduction of $SU(n)$ to H . Let ω_0 be a connection on $M \times H$ such that $\text{Hol}(\omega_0) = H$, and let ω_1 be the connection pushed over to $M \times SU(n)$. Then $\text{Hol}(\omega_1) = \text{Hol}(\omega_0) = H$, and so

$$I_{\omega_1}(M \times SU(n)) \approx C_{SU(n)}(H) = Z(SU(n)) = \left\{ e^{\frac{2\pi ik}{n}} I_n \mid k = 0, 1, \dots, n-1 \right\},$$

since by Schur's Lemma (valid for any field), if H acts irreducibly on \mathbb{C}^n , then $C_{\text{SU}(n)}(H) = Z(\text{SU}(n))$. ω_1 is an example of a generic reducible connection (see Sect. 5).

3.8. *Example.* Let $P(M, G)$ be a PFB such that the total space P is connected, and such that G is a discrete Lie group. Then $\mathfrak{G} = \{0\}$, and $\pi: P \rightarrow M$ is a regular covering manifold of M . Let

$$\text{Cov}(P) = \{F \in \text{Diff}(P) \mid \pi \circ F = F\}$$

denote the covering transformations of P . Then

$$\text{Cov}(P) \approx G \approx \pi_1(M, x_0) / \pi_* (\pi_1(P, p_0)),$$

where π_1 denotes the first homotopy group of M (respectively P) based at $x_0 \in M$ [respectively $p_0 \in \pi^{-1}(x_0)$]. Also, $\text{Aut}(P) \subseteq \text{Cov}(P)$, and so $\text{Aut}(P) \approx Z(G)$.

Since $\mathfrak{G} = \{0\}$, there exists only one connection $\omega_0 = 0$ on P , which is flat. Moreover, since every curve in P is horizontal, $\text{Hol}(\omega_0) = G$. Thus

$$I_{\omega_0}(P) \approx C_G(\text{Hol}(\omega)) \approx Z(G),$$

so that $I_{\omega_0}(P) \approx Z(G)$. Note that since $\omega_0 = 0$, $I_{\omega_0}(P)$ is the entire group $\text{Aut}(P)$, which is, however, only the 0-dimensional Lie group $Z(G)$. Also, see Proposition 5.4. \square

Finally, we give an example where the holonomy group is not a closed subgroup of the structure group G .

3.9. *Example.* Let $P(S^1, S^1) = S^1 \times S^1$ be the product bundle, where $S^1 = \text{U}(1) = \{e^{i\omega} \mid \omega \in \mathbb{R}\}$. Identify $S^1 \times S^1$ with the square E of length 2π in \mathbb{R}^2 with opposite edges identified. Let λ be a fixed irrational real number, and for $(x, y) \in E \subseteq \mathbb{R}^2$, take as the horizontal subspace the line with slope λ . Let ω_0 denote the corresponding connection 1-form. Then since any connection over a 1-dimensional manifold is flat, ω_0 is flat, and its holonomy group is the discrete subgroup of $\text{U}(1)$,

$$\text{Hol}(\omega_0) = \{e^{in\lambda} \mid n \in \mathbb{Z}\} \approx Z.$$

Thus $\text{Hol}(\omega_0)$ is not a closed Lie subgroup of $\text{U}(1)$. Since the structure group is Abelian, by Example 3.1, $I_{\omega_0}(P) = \text{U}(1)$.

If λ is a fixed rational number, and if the horizontal subspaces are taken as the lines with slope λ , then the holonomy sub-bundles are the n -fold covering manifolds $S^1 \rightarrow S^1, z \rightarrow z^n$, for some $n \in \mathbb{Z}^+$ depending on λ , and the holonomy group is

$$\text{Hol}(\omega) = \left\{ e^{\frac{2\pi ik}{n}} \mid k = 0, 1, \dots, n-1 \right\} \approx Z_n,$$

the cyclic group of order n .

Thus the possible holonomy groups for connections on $S^1 \times S^1$ are Z and Z_n , $n \in \mathbb{Z}^+$, and each of these holonomy groups is attained.

Remark. In contrast, on the “cylindrical” PFB $P(S^1, \mathbb{R}) = S^1 \times \mathbb{R}$, the holonomy sub-bundles of each connection produces either a spiral or circular foliation of $S^1 \times \mathbb{R}$. Thus the possible holonomy groups are integer multiples of the vertical distance $d \geq 0$ attained in one spiral. Thus the possible holonomy groups are

$$Zd = \{nd | n \in \mathbb{Z}, d \geq 0\} \approx \begin{cases} \{0\} & \text{if } d = 0 \\ \mathbb{Z} & \text{if } d > 0 \end{cases} \quad \square$$

4. Connections with Internal Symmetry Group as Large as Possible

From the realization of $I_\omega(P)$ as a closed subgroup of G by the map

$$\text{eval}_{p_0} : I_\omega(P) \rightarrow C_G(\text{Hol}_{p_0}(\omega)) \subseteq G, \quad F \mapsto u_F(p_0),$$

we see that $I_\omega(P)$ cannot exceed G . Thus we have the general estimate on the dimension of $I_\omega(P)$, $\dim I_\omega(P) \leq \dim G = k$.

It is of interest to know when $I_\omega(P)$ is as large as possible, i.e., when $I_\omega(P)$ is isomorphic to G . In this case we shall say that $I_\omega(P)$ (or that ω) is *maximal*. If the weaker condition $\dim I_\omega(P) = k$ holds, we shall say that $I_\omega(P)$ has *maximal dimension*. Again, we note that $I_\omega(P) \approx G$ does not imply that $I_\omega(P)$ is equal to $\{R_a | a \in G\}$, but merely that the internal symmetry group is isomorphic to G (see last paragraph of Sect. 2).

In Lagrangian field theories, there is some confusion over the role played by the symmetry group $I_\omega(P)$ and by the structure group G of the PFB $P(M, G)$. The symmetry group $I_\omega(P)$ generates global internal conservation laws, whereas in general the structure group G does not. Thus it is of considerable interest to know when these groups are isomorphic. Thus, although the cases discussed below are special and “rare,” they are many of the cases that have been considered in great detail classically, often resulting in the unfortunate confusion between the different roles played by $I_\omega(P)$ and G . (See Fischer [1985] for more details regarding these differing roles.)

The formula of Theorem 2.3 gives an easy criterion in terms of the holonomy group for when $I_\omega(P) \approx G$.

4.1. Proposition. *Let $P(M, G)$ be a PFB, and let $\omega \in \mathcal{C}(P)$. Then $I_\omega(P) \approx G$ if and only if $\text{Hol}(\omega) \subseteq Z(G)$.*

Proof. If $\text{Hol}(\omega) \subseteq Z(G)$, then $C_G(\text{Hol}(\omega)) = G$. Thus $I_\omega(P) \approx C_G(\text{Hol}(\omega)) = G$.

Conversely, if $I_\omega(P) \approx G$, then $C_G(\text{Hol}(\omega)) = G$. Thus for all $a \in G$, $a \cdot b = b \cdot a$ for all $b \in \text{Hol}(\omega)$. Thus if $b \in \text{Hol}(\omega)$, then $b \in Z(G)$, and so $\text{Hol}(\omega) \subseteq Z(G)$. \square

Examples 3.1, 3.2, and 3.3 are special cases of the above criterion. In Example 3.1, G is Abelian and so $\text{Hol}(\omega) \subseteq Z(G) = G$ for all connections $\omega \in \mathcal{C}(P)$. In Examples 3.2 and 3.3, $\text{Hol}(\omega) = \{e\} \subseteq Z(G)$ for all Lie groups G , and so again the above criterion is satisfied. Using Example 3.3 and Proposition 4.1, we can also give the following criterion for maximality of flat connections.

4.2. Proposition. *Let ω be a flat connection on the PFB $P(M, G)$.*

- (i) *If M is simply connected, then $I_\omega(G) \approx G$.*
- (ii) *If G is connected and $\text{Hol}(\omega)$ is a normal subgroup of G , then $I_\omega(P) \approx G$.*

Proof. (i) If ω is flat and M is simply connected, then $\text{Hol}(\omega) = \{e\}$, and so $I_\omega(P) \approx G$ (see Example 3.3).

(ii) If ω is flat, $\text{Hol}(\omega)$ is discrete. But a discrete normal subgroup of a connected topological group is central (see e.g., Greenberg-Harper [1981], p. 18), and so $\text{Hol}(\omega) \subseteq Z(G)$. Thus by Proposition 4.1, $I_\omega(P) \approx G$. \square

We can also give a criterion for when $I_\omega(P)$ has maximal dimension. Let G^0 denote the connected component of the identity of G .

4.3. Proposition. *Let $P(M, G)$ be a PFB, let $k = \dim G$, and let $\omega \in \mathcal{C}(P)$. Then $\dim I_\omega(P) = k$ if and only if $\text{Hol}(\omega) \subseteq C_G(G^0)$. In particular, if G is connected, then $I_\omega(P) \approx G$ ($I_\omega(P)$ is maximal) if and only if $\dim I_\omega(P) = k$ ($I_\omega(P)$ has maximal dimension).*

Proof. From Theorem 2.3 and Lemma 1.5, $I_\omega(P) \approx C_G(\text{Hol}(\omega)) \subseteq G$, and $C_G(\text{Hol}(\omega))$ is a closed subgroup and hence a closed submanifold of G . Thus if $\dim I_\omega(P) = k = \dim G$, then $C_G(\text{Hol}(\omega)) \supseteq G^0$. Thus if $a \in G^0$, $a \cdot b = b \cdot a$ for all $b \in \text{Hol}(\omega)$, so if $b \in \text{Hol}(\omega)$, then b commutes with all $a \in G^0$, and so $b \in C_G(G^0)$. Thus $\text{Hol}(\omega) \subseteq C_G(G^0)$.

Conversely, if $\text{Hol}(\omega) \subseteq C_G(G^0)$, then if $b \in \text{Hol}(\omega)$, then b commutes with all elements in G^0 , $b \cdot a = a \cdot b$ for all $a \in G^0$. Thus if $a \in G^0$, a commutes with all $b \in \text{Hol}(\omega)$. Thus

$$G^0 \subseteq C_G(\text{Hol}(\omega)) \subseteq G,$$

and so $\dim C_G(\text{Hol}(\omega)) = k$. \square

4.4. Example. For an example of a connection with maximal dimension but which is not maximal, let $P = S^1 \times O(2)$ and let ω_0 be a connection on P such that the horizontal subspaces on each component of P have a fixed irrational slope λ (see Example 3.9). Then $\text{Hol}(\omega_0) = \{e^{in\lambda} | n \in \mathbb{Z}\}$, and $\overline{\text{Hol}(\omega_0)} = \text{SO}(2)$. Thus

$$I_{\omega_0}(P) \approx C_{O(2)}(\text{Hol}(\omega_0)) = C_{O(2)}(\overline{\text{Hol}(\omega_0)}) = C_{O(2)}(\text{SO}(2)) = \text{SO}(2).$$

Thus $\dim I_{\omega_0}(P) = 1$, but $I_{\omega_0}(P) \neq O(2)$, so ω_0 has maximal dimension but is not maximal. \square

In Example 3.3 we saw that if M is simply connected and ω is flat, then $I_\omega(P) \approx G$. As a partial converse, we have the following.

4.5. Proposition. *Let $P(M, G)$ be a PFB with G semi-simple, and let $\omega \in \mathcal{C}(P)$.*

- (1) *If $I_\omega(P) \approx G$, then ω is flat.*
- (2) *If G is connected and $\dim I_\omega(P) = k = \dim G$, then ω is flat.*

Proof. If $I_\omega(P) \approx G$, then by Proposition 4.1, $\text{Hol}(\omega) \subseteq Z(G)$. Since G is semi-simple, its center $Z(G)$ is a discrete subgroup of G . Thus $\text{Hol}(\omega)$ is a discrete subgroup of G , and so by the Ambrose-Singer Theorem (see Kobayashi-Nomizu [1963], p. 89), the connection is flat.

If G is connected and $\dim I_\omega(P) = k$, then by Proposition 4.3, $I_\omega(P) \approx G$, and so by (1) the connection is flat. \square

In Yang-Mills field theories, G is often assumed to be compact, connected, and semi-simple (see e.g., Mitter [1980] or Singer [1978]). With these assumptions (even without G being compact), a maximal dimension connection must be flat, thereby indicating the ‘‘scarcity’’ of maximal dimension connections.

5. Connections with Internal Symmetry Group as Small as Possible and Irreducible Connections

We have seen that for all $a \in Z(G)$, $R_a^* \omega = \omega$ for any connection $\omega \in \mathcal{C}(P)$. Thus the inclusion $Z(G) \subseteq I_\omega(P)$ holds for all connections ω , so that $Z(G)$ is the smallest possible symmetry group for any connection. We shall say that the connection ω is *generic* if $I_\omega(P) = Z(G)$. Thus if ω is generic, $I_\omega(P)$ is as small as possible. By Theorem 2.3, a criterion for ω to be generic is given in terms of its holonomy group as follows:

ω is generic if and only if $C_G(\text{Hol}(\omega)) = Z(G)$.

We let

$$\mathcal{C}_0(P) = \{ \omega \in \mathcal{C}(P) \mid I_\omega(P) = Z(P) \}$$

denote the space of generic connections on P . The term generic is justified by the fact that if $\mathcal{C}_0(P) \neq \emptyset$, then $\mathcal{C}_0(P)$ is an open dense subset of $\mathcal{C}(P)$ under the assumption that P is a compact connected manifold [see Singer [1978] for this result for irreducible connections (see below)].

In the special case when G is Abelian, $C_G(\text{Hol}(\omega)) = G = Z(G)$ for any connection ω (see Example 3.1), so that $\mathcal{C}_0(P) = \mathcal{C}(P)$. Thus in the Abelian case, every connection is both generic ($I_\omega(P) = Z(G)$) and maximal ($I_\omega(P) = G$), since $Z(G) = G$.

Returning to the general case, we remark that although $I_\omega(P) = Z(G)$ is constant as a function of $\omega \in \mathcal{C}_0(P)$, the holonomy group $\text{Hol}(\omega)$ may depend on $\omega \in \mathcal{C}_0(P)$. For example, if $P = S^1 \times S^1$ (see Example 3.9), then $\mathcal{C}_0(P) = \mathcal{C}(P)$, but $\text{Hol}(\omega)$ depends on $\omega \in \mathcal{C}(P)$.

We shall also need the following result regarding generic connections.

5.1. Proposition. *Let $P(M, G)$ be a PFB, and assume $\mathcal{C}_0(P) \neq \emptyset$. Let $\omega \in \mathcal{C}_0(P)$. If for some $p \in P$, $\text{Hol}_p(\omega)$ is Abelian, then G is Abelian.*

Proof. Let $H = \text{Hol}_p(\omega)$ be Abelian. If $a \in H$, $a \cdot b = b \cdot a$ for all $b \in H$, and so $a \in C_G(H)$. Thus $H \subseteq C_G(H)$. Since $\omega \in \mathcal{C}_0(P) \neq \emptyset$, $C_G(H) = Z(G)$. Thus $H \subseteq Z(G)$, and so $C_G(H) = G$. Thus $Z(G) = G$, and so G is Abelian. \square

Remark. Note that under the conditions of the proposition, ω is also maximal. \square

We now go on to study the irreducible connections, which turn out to be a large sub-class of the space of generic connections. Unfortunately, there are at least three definitions of an irreducible connection in the literature.

If ω is a connection on $P(M, G)$, then by the Holonomy Reduction Theorem (see Kobayashi-Nomizu [1963], p. 83), ω can be reduced to a connection on the holonomy sub-bundle $P(p_0) = Q(M, \text{Hol}_{p_0}(\omega))$. Thus it is reasonable to define a connection to be irreducible if $\text{Hol}(\omega) = G$, and this is in fact done by several authors (see e.g. Atiyah-Jones [1978], p. 101, Daniel-Viallet [1980], p. 186, or Rawnsley [1978], p. 35). However, if $\text{Hol}_{p_0}(\omega)$ is not a closed subgroup of G , $P(p_0)$ will only be an immersed and not an embedded submanifold of P . The connection ω could then be reduced to a connection on the closed sub-bundle $\bar{P}(p_0) = Q(M, \overline{\text{Hol}_{p_0}(\omega)})$. Thus if one wishes to rule out reductions of ω to immersed sub-bundles as not being “proper” reductions, then an irreducible connection should

be defined as a connection such that $\overline{\text{Hol}(\omega)} = G$. This weaker definition is chosen by other authors (see e.g. Atiyah, Hitchin, and Singer [1978], p. 442). Finally, for a matrix Lie group $G \subseteq GL(n, \mathbb{R})$, Singer [1978], p. 9, defines a connection to be irreducible if its holonomy group $\text{Hol}(\omega) \subseteq G$ acts irreducibly on \mathbb{R}^n .

Since for a general PFB this last definition requires a representation of the structure group G on \mathbb{R}^n , we do not take it over to define an irreducible connection on $P(M, G)$. Since non-closed holonomy groups are a fact of life in the theory of connections, we will consider reductions to immersed sub-bundles as being *bona fide* reductions. Thus we end up with the following.

5.2. *Definition.* Let $\omega \in \mathcal{C}(P)$ be a connection on $P(M, G)$. Then ω is irreducible if $\text{Hol}(\omega) = G$, and ω is weakly irreducible if $\overline{\text{Hol}(\omega)} = G$. ω is irreducible if $\text{Hol}(\omega) \neq G$, and ω is weakly reducible if $\overline{\text{Hol}(\omega)} \neq G$.

Let

$$\mathcal{C}_I(P) = \{\omega \in \mathcal{C}(P) \mid \text{Hol}(\omega) = G\}$$

denote the space of irreducible connections on P , and let

$$\mathcal{C}_W(P) = \{\omega \in \mathcal{C}(P) \mid \overline{\text{Hol}(\omega)} = G\}$$

denote the space of weakly irreducible connections on P . Thus

$$\mathcal{C}_I(P) \subseteq \mathcal{C}_W(P) \subseteq \mathcal{C}(P).$$

5.3. **Proposition.** If ω is a weakly irreducible connection on $P(M, G)$, then

$$I_\omega(P) = Z(G) = \{R_a : P \rightarrow P \mid a \in Z(G)\}.$$

Thus every weakly irreducible connection is generic, and in particular, every irreducible connection is generic.

Proof. Since ω is weakly irreducible, $\overline{\text{Hol}(\omega)} = G$. Thus from Theorem 2.3 and Lemma 1.5,

$$I_\omega(P) \approx C_G(\text{Hol}(\omega)) = C_G(\overline{\text{Hol}(\omega)}) = C_G(G) = Z(G).$$

Since for any connection, $Z(G) \subseteq I_\omega(P)$, it follows that $I_\omega(P) = Z(G)$. \square

From the above proposition, we now have the following inclusions:

$$\mathcal{C}_I(P) \subseteq \mathcal{C}_W(P) \subseteq \mathcal{C}_0(P) \subseteq \mathcal{C}(P).$$

In general, the inclusion $\mathcal{C}_W(P) \subseteq \mathcal{C}_0(P)$ is proper. For example, if G is Abelian, $\mathcal{C}_0(P) = \mathcal{C}(P)$ (see Example 3.1), and so any weakly reducible connection ($\overline{\text{Hol}(\omega)} \neq G$) will be in $\mathcal{C}_0(P)$ but not in $\mathcal{C}_W(P)$. Another less trivial example is the canonical connection ω_0 on the frame bundle $L(S^n)$ with $n \geq 3$ (see Example 3.4). Here $\text{Hol}(\omega_0) = \text{SO}(n)$, and so ω_0 is reducible and so weakly reducible to the proper closed sub-bundle $0^+(S^n)$, but $I_{\omega_0}(L(S^n)) = Z(G)$, and so $\omega_0 \in \mathcal{C}_0(L(S^n))$ is generic, but $\omega_0 \notin \mathcal{C}_W(P)$.

The above inclusion is proper even in the case when G is compact, connected, and semi-simple, as shown in Example 3.7, where a generic connection ω_1 is given on $M \times \text{SU}(n)$, $\dim M \geq 2$, $n \geq 2$. By construction, ω_1 is reducible and so strongly reducible to a connection ω_0 on the proper closed sub-bundle, $M \times H$.

Regarding the existence of irreducible connections, we have the following.

5.4. Proposition. *Let $P(M, G)$ be a PFB where the total space P is connected and $\dim M \geq 2$. Then there exists an irreducible connection on P , so that $\mathcal{C}_I(P) \neq \phi$, $\mathcal{C}_W(P) \neq \phi$, and $\mathcal{C}_0(P) \neq \phi$.*

Proof. By a theorem of Kobayashi-Nomizu [1963], p. 90, there exists a connection ω_0 such that $\text{Hol}(\omega_0) = G$. This ω_0 is irreducible, and so $\omega_0 \in \mathcal{C}_I(P) \subseteq \mathcal{C}_W(P) \subseteq \mathcal{C}_0(P)$. \square

The one-dimensional case is an interesting curiosity. The variety of possibilities, listed below, is somewhat unexpected.

5.5. Proposition. *Let $\dim M = 1$, so that M is either diffeomorphic to \mathbb{R} or to S^1 , and let $P(M, G)$ be a PFB over M .*

If $M \approx \mathbb{R}$, then $P(M, G) \approx \mathbb{R} \times G$. If G is Abelian, then $\mathcal{C}_0(P) = \mathcal{C}(P) \neq \phi$. If G is not Abelian, then $\mathcal{C}_I(P) = \mathcal{C}_W(P) = \mathcal{C}_0(P) = \phi$. If $G \neq \{e\}$ (G Abelian or not), then $\mathcal{C}_I(P) = \mathcal{C}_W(P) = \phi$. If $G = \{e\}$, then $\mathcal{C}_I(P) = \mathcal{C}_W(P) = \mathcal{C}_0(P) = \mathcal{C}(P) = \{\omega_0\} \neq \phi$, where $\omega_0 = 0$ is the zero connection on $P(\mathbb{R}\{e\})$ (see Example 3.8).

If $M \approx S^1$, and if G is Abelian, then $\mathcal{C}_0(P) = \mathcal{C}(P) \neq \phi$. If G is not Abelian then $\mathcal{C}_I(P) = \mathcal{C}_W(P) = \mathcal{C}_0(P) = \phi$. If $P(S^1, G)$ is connected and $\dim G = 0$, then G is isomorphic to either Z or Z_n for some $n \in \mathbb{Z}^+$, and $P(S^1, G)$ is bundle isomorphic to either covering manifold $\mathbb{R} \rightarrow S^1; \lambda \mapsto e^{i\lambda}$, for $G \approx Z$, or $S^1 \rightarrow S^1; z \mapsto z^n$ for $G \approx Z_n$, and where $S^1 = \{z \in \mathbb{C} | z\bar{z} = 1\}$. In either case

$$\mathcal{C}_W(P) = \mathcal{C}_I(P) = \mathcal{C}_0(P) = \mathcal{C}(P) = \{\omega_0\} \neq \phi,$$

where $\omega_0 = 0$.

If G is Abelian and $\dim G \geq 1$, then $\mathcal{C}_I(P) = \phi$, but whether $\mathcal{C}_W(P)$ is empty or not depends on the bundle $P(S^1, G)$.

Proof. From Example 3.1 and Sect. 5, if G is Abelian, then $\mathcal{C}_0(P) = \mathcal{C}(P) \neq \phi$, for any PFB $P(M, G)$.

If $\dim M = 1$, then the curvature 2-form of connection ω on $P(M, G)$ must vanish, so that any connection on $P(M, G)$ must be flat.

Now let $M \approx \mathbb{R}$. Since \mathbb{R} is contractable, $P(\mathbb{R}, G)$ is bundle isomorphic to $\mathbb{R} \times G$. Let $\omega \in \mathcal{C}(P)$. Since ω is flat and \mathbb{R} is simply connected, $\text{Hol}(\omega) = \{e\}$. Now suppose $\mathcal{C}_0(P) \neq \phi$. Let $\omega_0 \in \mathcal{C}_0(P)$, so that $Z(G) = C_G(\text{Hol}(\omega_0)) = C_G(\{e\}) = G$, and so G is Abelian. Thus if G is not Abelian, $\mathcal{C}_0(P) = \phi$, and so $\mathcal{C}_W(P) = \mathcal{C}_I(P) = \mathcal{C}_0(P) = \phi$.

Since $\text{Hol}(\omega) = \{e\}$ for every connection $\omega \in \mathcal{C}(P)$, every connection is reducible to the zero connection $\omega_0 = 0$ on the closed holonomy sub-bundle $P(p_0) = Q(M, \{e\})$. Thus if $G \neq \{e\}$, then $\mathcal{C}_W(P) \neq \phi$, and so also $\mathcal{C}_I(P) = \phi$. If $G = \{e\}$, then

$$\mathcal{C}_W(P) = \mathcal{C}_I(P) = \mathcal{C}_0(P) = \mathcal{C}(P) = \{\omega_0\} \neq \phi.$$

Now let $M \approx S^1$. For $p_0 \in P(S^1, G)$, let $x_0 = \pi(p_0) \in S^1$. For $\omega \in \mathcal{C}(P)$, let

$$h: \pi_1(S^1, x_0) \rightarrow \text{Hol}_{p_0}(\omega)$$

denote the holonomy homomorphism (see Kobayashi-Nomizu [1963], p. 93).

Since $\pi_1(S^1, x_0) \approx Z$ and since h is surjective, $\text{Hol}_{p_0}(\omega) \approx \frac{Z}{\ker h}$ is isomorphic to

either Z or $Z_n = \frac{Z}{nZ}$ for some $n \in Z^+$. Thus $\text{Hol}_{p_0}(\omega)$ is Abelian, and so by Proposition 5.1, if $\mathcal{C}_0(P) \neq \phi$, then G is Abelian. Thus if G is not Abelian, $\mathcal{C}_0(P) = \phi$, and so $\mathcal{C}_W(P) = \mathcal{C}_I(P) = \mathcal{C}_0(P) = \phi$.

If $\dim G = 0$, G is a discrete Lie group, and so if P is connected, by Example 3.8, $P \rightarrow S^1$ is a regular covering manifold of S^1 . But every connected covering manifold of S^1 is equivalent to either $\mathbb{R} \rightarrow S^1, \lambda \rightarrow e^{i\lambda}$, or $S^1 \rightarrow S^1, z \rightarrow z^n$, for some $n \in Z^+$ (see Spanier [1966], p. 80). Thus G must be either Z or Z_n for some $n \in Z^+$. In either case, there is only one connection $\omega_0 = 0$ on P , and $\text{Hol}(\omega_0) = G$ (see Example 3.8). Thus

$$\mathcal{C}_I(P) = \mathcal{C}_W(P) = \mathcal{C}_0(P) = \mathcal{C}(P) = \{\omega_0\} \neq \phi.$$

Now assume $\dim G \geq 1$, G Abelian. Since every connection ω on $P(S^1, G)$ is flat, the holonomy groups must be discrete subgroups of G . Thus if $\dim G \geq 1$, $\text{Hol}(\omega) \neq G$ for any connection ω on P . Hence $\mathcal{C}_I(P) = \phi$.

Example 3.9 with $P = S^1 \times S^1$ and with connection $\overline{\omega_0}$ determined by horizontal subspaces with irrational slope gives $\overline{\text{Hol}(\overline{\omega_0})} = S^1$, and hence $\overline{\omega_0} \in \mathcal{C}_W(P) \neq \phi$. On the other hand, the holonomy group of any connection on $P = S^1 \times \mathbb{R}$ is a discrete subgroup of \mathbb{R} (see Example 3.9), and so $\overline{\text{Hol}(\omega)} \neq G$ for any connection ω on P . Hence $\mathcal{C}_W(P) = \phi$. Thus whether $\mathcal{C}_W(P)$ is empty or not depends on the bundle $P(S^1, G)$. \square

We summarize the above results in the following tables, with $\omega_0 = 0$.

Subspaces of $\mathcal{C}(P)$			
Group G	$\mathcal{C}_I(P)$	$\mathcal{C}_W(P)$	$\mathcal{C}_0(P)$
$G = \{e\}$	$\{\omega_0\}$	$\{\omega_0\}$	$\{\omega_0\}$
G Abelian, $G \neq \{e\}$	ϕ	ϕ	$\mathcal{C}_0(P) = \mathcal{C}(P) \neq \phi$
G not Abelian	ϕ	ϕ	ϕ

Case 1, $M \approx \mathbb{R}$

Subspaces of $\mathcal{C}(P)$			
Group G	$\mathcal{C}_I(P)$	$\mathcal{C}_W(P)$	$\mathcal{C}_0(P)$
$P(S^1, G)$ connected, $\dim G = 0 \Rightarrow G \approx Z$ or Z_n , $n \in Z^+$	$\{\omega_0\}$ $\{\omega_0\}$	$\{\omega_0\}$ $\{\omega_0\}$	$\{\omega_0\}$ $\{\omega_0\}$
G Abelian, $\dim G \geq 1$	ϕ	depends on $P(S, G)$	$\mathcal{C}_0(P) = \mathcal{C}(P) \neq \phi$
G not Abelian	ϕ	ϕ	ϕ

Case 2, $M \approx S^1$

6. $I_\omega(P)$ as a Lie Group of Transformations

In Sect. 2 we showed that $I_\omega(P, G)$, and hence $I_\omega(P)$ has a Lie group structure. In this section we show that $I_\omega(P)$ is a Lie transformation group that acts properly and freely on P .

Let $P(M, G)$ be a PFB over M , let g be a Riemannian metric on M (M is paracompact and hence admits Riemannian metrics), let $\omega \in \mathcal{C}(P)$, and let γ be a positive-definite inner product on \mathfrak{G} . We are not assuming that γ is adjoint-invariant. Indeed, such a γ does not exist on a general Lie algebra. Using g, ω , and γ , we construct a Riemannian metric \hat{g} on P by $\hat{g} = \pi^*g + \gamma \cdot (\omega \otimes \omega)$, so that

$$g(p) \cdot (Z_1, Z_2) = g(x)(X_1, X_2) + \gamma \cdot (\omega(Z_1), \omega(Z_2)),$$

where $p \in P, Z_1, Z_2 \in T_pP, x = \pi(p)$, and $X_1 = T_p\pi \cdot Z_1, X_2 = T_p\pi \cdot Z_2 \in T_xM$. Note that if γ is not Ad-invariant, then \hat{g} is not G -invariant ($R_a^*\hat{g} \neq \hat{g}$ for all $a \in G$). However, we shall see below that \hat{g} is $I_\omega(P)$ -invariant. This observation is the key to Theorem 6.2 below.

Let
$$I_{\hat{g}}(P) = \{H \in \text{Diff}(P) \mid H^*\hat{g} = \hat{g}\}$$

denote the group of isometries of P . Note that an isometry need not be an automorphism of P . A classical result of Myers and Steenrod ([1939]) states that if P is connected, then with respect to the compact open topology, $I_{\hat{g}}(P)$ is a Lie transformation group of P ; i.e., $I_{\hat{g}}(P)$ has a Lie group structure such that the action $I_{\hat{g}}(P) \times P \rightarrow P, (H, p) \mapsto H(p)$ is a C^∞ map. Also, the Lie algebra of $I_{\hat{g}}(P)$ consists of the complete Killing vector fields on P ,

$$\mathcal{K}_{\hat{g}}(P) = \{W \in \mathfrak{X}(P) \mid W \text{ is complete and } L_W g = 0\}.$$

As part of this classical result, it is shown that if $\{H_n\}$ is a sequence in $I_{\hat{g}}(P)$ such that $H_n(p_0) \rightarrow p_1$, then there exists an $H \in I_{\hat{g}}(P)$ [with $H(p_0) = p_1$] and a subsequence $\{H_{n_k}\}$ of $\{H_n\}$ that converges to H in the compact open topology (see Helgason [1962], p. 167 or Kobayashi-Nomizu [1963], p. 47). This latter result can be rephrased as follows.

6.1. Proposition. *If P is connected, then the action $I_{\hat{g}}(P) \times P \rightarrow P, (H, p) \mapsto H(p)$ is a proper action.*

Proof. The above action is proper if the map $I_{\hat{g}}(P) \times P \rightarrow P \times P, (H, p) \mapsto (p, H(p))$ is a proper map, i.e., if the inverse image of a compact set is compact. This is equivalent to showing that if $\{p_n\}$ is a sequence that converges in P , and $\{H_n\}$ is a sequence in $I_{\hat{g}}(P)$ such that $\{H_n(p_n)\}$ converges in P , then $\{H_n\}$ has a convergent subsequence in $I_{\hat{g}}(P)$. Thus let $(p_n, H_n(p_n)) \rightarrow (p_0, p_1) \in P \times P$. Let d be the metric on P associated with \hat{g} . Then since \hat{g} is $I_{\hat{g}}(P)$ -invariant, so is d . Thus

$$d(H_n(p_0), p_1) \leq d(H_n(p_0), H_n(p_n)) + d(H_n(p_n), p_1) = d(p_0, p_n) + d(H_n(p_n), p_1).$$

Since $p_n \rightarrow p_0$ and $H_n(p_n) \rightarrow p_1$, the right-hand side converges to zero. Hence $H_n(p_0) \rightarrow p_1$. Thus by the classical result, there exists a convergent subsequence $\{H_{n_k}\}$ in $I_{\hat{g}}(P)$. Hence the action is proper. \square

Our main theorem is now the following.

6.2. Theorem. Let $P(M, G)$ be a PFB, let ω be a connection on $P(M, G)$, and let

$$I_\omega(P) = \{F \in \text{Aut}(P) \mid F^*\omega = \omega\}$$

denote the internal symmetry group of ω . Assume that the total space P is a connected manifold. Then with respect to the compact open topology on $I_\omega(P)$,

(1) $I_\omega(P)$ is a Lie transformation group of P , i.e., the action $\Psi: I_\omega(P) \times P \rightarrow P$, $(F, p) \mapsto F(p)$ is a C^∞ action.

(2) The Lie algebra of $I_\omega(P)$ is:

$$\mathcal{I}_\omega(P) = \{T \in \mathfrak{X}_G^{\text{vert}}(P) \mid L_T\omega = D\varphi_T = 0\}.$$

(3) The action Ψ is proper and free.

(4) $\dim I_\omega(P) \leq \dim \mathfrak{G}$.

(5) For $p \in P$, the orbits

$$\mathcal{O}_p = \{F(p) \mid F \in I_\omega(P)\}$$

are closed submanifolds of P , and the orbit maps $\Psi_p: I_\omega(P) \rightarrow \mathcal{O}_p \subseteq P$, $F \mapsto F(p)$ are diffeomorphisms onto the orbits. The differentiable structure on $I_\omega(P)$ is given by the embedding of $I_\omega(P)$ onto the closed submanifold \mathcal{O}_p .

(6) The orbit space

$$I_\omega(P) \backslash P = \{\mathcal{O}_p \mid p \in P\}$$

in the quotient topology has the structure of a C^∞ manifold such that the orbit projection map $P \rightarrow I_\omega(P) \backslash P$, $p \mapsto \mathcal{O}_p$ is a submersion, and is also a left principal fiber bundle with total space P , base space $I_\omega(P) \backslash P$, and structure group $I_\omega(P)$.

(7) For $p \in P$, the map

$$I_\omega(P) \rightarrow C_G(\text{Hol}_p(\omega)), \quad F \mapsto u_F(p)$$

is a Lie group isomorphism.

(8) If the structure group G of the PFB $P(M, G)$ is compact, then $I_\omega(P)$ is a compact Lie transformation group.

Proof. (1) For $F \in \text{AUT}(P)$, let $f = \hat{\pi}(F) \in \text{Diff}(M)$, so that $\pi \circ F = f \circ \pi$. Then

$$F^*\hat{g} = F^*(\pi^*g) + F^*(\gamma \cdot (\omega \otimes \omega)) = \pi^*(f^*g) + \gamma \cdot (F^*\omega \otimes F^*\omega),$$

where F^* pulls through the second term because of the bilinearity of γ . Thus if $F \in I_\omega(P)$, $\hat{\pi}(F) = \text{id}_M$, and $F^*\omega = \omega$, so that $F^*\hat{g} = \pi^*g + \gamma \cdot (\omega \otimes \omega) = \hat{g}$. Thus $F \in I_{\hat{g}}(P)$, and so $I_\omega(P) \subseteq I_{\hat{g}}(P)$.

Moreover, in the compact open topology, $F^*\omega = \omega$ is clearly a closed condition, and so $I_\omega(P)$ is a closed subgroup of $I_{\hat{g}}(P)$. From the classical result stated above, $I_{\hat{g}}(P)$ is a Lie transformation group. Since any closed subgroup of a Lie transformation group is also a Lie transformation group, it follows that $I_\omega(P)$ is a Lie transformation group.

(2) Let F_λ be a one-parameter group of transformations in $I_\omega(P)$, and let

$$T = \left. \frac{dF_\lambda}{d\lambda} \right|_{\lambda=0} \in \mathfrak{X}_G^{\text{vert}}(P)$$

be its generator. Then $F_\lambda^*\omega = \omega$, and so $L_T\omega = 0$. Thus $T \in \mathcal{I}_\omega(P)$.

Conversely, let $T \in \mathcal{I}_\omega(P) \subseteq \mathfrak{X}_G^{\text{vert}}(P)$. Then since G -invariant vertical vector fields are automatically complete, T is complete. Let $F_\lambda \in \text{Aut}(P)$ be its flow. Since $L_T \omega = 0$, $F_\lambda^* \omega = \omega$ (see Kobayashi-Nomizu [1963], p. 33), and so $F_\lambda \in I_\omega(P)$.

(3) The action is proper since the action of $I_\mathfrak{g}(P)$ on P is proper (see Proposition 6.1), and since the restriction of a proper action to any closed subgroup is also a proper action.

Since P is connected, M is necessarily connected. Thus the action is free by Proposition 2.1 and Remark 2 following it. Note that the action of $I_\mathfrak{g}(P)$ on P is not necessarily free.

(4) Fix $p \in P$. Then the map $\mathcal{I}_\omega(P) \rightarrow \mathfrak{G}$, $T \mapsto \omega(p) \cdot T(p)$ is an injection by Proposition 2.1 (and Remark 2 following it) and so $\dim I_\omega(P) = \dim \mathcal{I}_\omega(P) \leq \dim \mathfrak{G}$.

(5) and (6) follow for any smooth, proper and free action (see Abraham-Marsden [1978], pp. 266 and 276). That the differentiable structure on $I_\omega(P)$ is induced by the embedding $\Phi_p : I_\omega(P) \rightarrow \mathcal{C}_p$, $F \mapsto F(p)$, follows from the corresponding result for $I_\mathfrak{g}(P)$ (see Kobayashi [1969], pp. 15 and 41).

(7) Since $C_G(\text{Hol}_p(\omega))$ is a closed subgroup of G , the map

$$C_G(\text{Hol}_p(\omega)) \rightarrow \mathcal{C}_p \subseteq P, \quad a \mapsto p \cdot a,$$

is a diffeomorphism onto the orbit. Hence the Lie group structure on $I_\omega(P)$ induced by the embedding $I_\omega(P) \rightarrow \mathcal{C}_p$, $F \mapsto F(p)$, coincides with the Lie group structure on $I_\omega(P)$ induced by the group isomorphism

$$I_\omega(P) \rightarrow C_G(\text{Hol}_p(\omega)), \quad F \mapsto u_F(p).$$

Thus (7) is a restatement of Theorem 2.3, again using the fact that M is connected. We also note again that all second countable Lie group structures on a Lie group are equivalent (see discussion following Proposition 2.3).

(8) Since $C_\mathfrak{g}(\Phi_p(\omega))$ is a closed subgroup of G , if G is compact, so is $C_G(\Phi_p(\omega))$, and hence also $I_\omega(P)$. \square

Remarks. 1. A similar but modified formulation of the above theorem is available for the “action”

$$I_\omega(P, G) \times P \rightarrow G, \quad (u, p) \mapsto u(p).$$

Thus $I_\omega(P, G)$ has a Lie group structure such that the above “action” is C^∞ , it is free in the sense that if $u(p) = e$, then $u \equiv e$, and it is proper in the sense that

$$I_\omega(P, G) \times P \rightarrow P \times G, \quad (u, p) \mapsto (p, u(p))$$

is a proper map.

2. As we have noted, if P is connected, then M is connected. However, G need not be connected. On the other hand, if both M and G are connected, then P is connected.

3. Note that if G is compact, $\text{Hol}(\omega)$ need not be compact (see Example 3.9). Thus it is of interest that $I_\omega(P) \approx C_G(\text{Hol}(\omega))$ is compact [since $C_G(\text{Hol}(\omega))$ is a closed subgroup of G]. Note the analogy with Riemannian geometry, where if M is compact, then the isometry group $I_\mathfrak{g}(M)$ of a Riemannian metric g is compact. Thus in this regard, compactness of the structure group G , causing $I_\omega(P)$ to be compact, is analogous to compactness of M , causing $I_\mathfrak{g}(M)$ to be compact.

4. If P has a finite number of components, then $I_{\hat{g}}(P)$ is still a Lie transformation group, but its action on P is no longer proper. However, if M is connected, the action of $I_{\omega}(P)$ on P is still a proper action, inasmuch as $F \in I_{\omega}(P)$ is determined by its value at a point.

7. The Lie Group of Transformations $I_{(g, \omega)}(P)$

As noted in the introduction, there are several finite dimensional subgroups of the infinite dimensional group $\text{AUT}_{\omega}(P)$ which are Lie groups. If P is a frame bundle, then we can consider the Lie group of affine transformations of ω ,

$$A_{\omega}(M) \approx \text{AUT}_{(\theta, \omega)}(P) = \text{AUT}_{\theta}(P) \cap \text{AUT}_{\omega}(P) \equiv I_{(\theta, \omega)}(P),$$

where θ is now the canonical 1-form on $P=L(M)$. For a general PFB, we considered in the previous section the Lie group $I_{\omega}(P) = \text{Aut}(P) \cap \text{AUT}_{\omega}(P)$. In this section we put a Riemannian metric g on the base manifold M and consider the larger subgroup

$$I_{(g, \omega)}(P) = \{F \in \text{AUT}(P) | F^*\omega = \omega \text{ and } f^*g = g\} \supseteq I_{\omega}(P),$$

where $f = \hat{\pi}(F) \in \text{Diff}(M)$ is the diffeomorphism of M induced by the automorphism F of P . Thus $I_{(g, \omega)}(P)$ is the group of automorphisms of ω that cover isometries of g .

Regarding $I_{(g, \omega)}(P)$ we have the following result.

7.1. Theorem. *Let $P(M, G)$ be a PFB, let ω be a connection on P , and let g be a Riemannian metric on M . Assume that the total space P is a connected manifold. Then with respect to the compact open topology on $I_{(g, \omega)}(P)$,*

(1) $I_{(g, \omega)}(P)$ is a Lie transformation group of P , i.e., the action Ψ :

$$I_{(g, \omega)}(P) \times P \rightarrow P, \quad (F, p) \mapsto F(p)$$

is a C^{∞} action.

(2) The Lie algebra of $I_{(g, \omega)}(P)$ is

$$\begin{aligned} \mathcal{L}_{(g, \omega)}(P) = \{Z \in \mathfrak{X}_G(P) | Z \text{ is a complete vector field on } P, \\ \text{and } L_Z\omega = 0 \text{ and } L_Xg = 0, \text{ where } X = \pi_*Z \in \mathfrak{X}(M)\}. \end{aligned}$$

(3) The action Ψ is a proper action.

(4) $\dim I_{(g, \omega)}(P) \leq \frac{1}{2}n(n+1) + k$, where $k = \dim G$.

(5) The isotropy group at an arbitrary point $p_0 \in P$,

$$(I_{(g, \omega)}(P))_{p_0} = \{F \in I_{(g, \omega)}(P) | F(p_0) = p_0\},$$

is compact.

(6) For $p \in P$, the orbits $\mathcal{O}_p = \{F(p) | F \in I_{(g, \omega)}(P)\}$ are closed submanifolds of P , and the maps

$$\frac{I_{(g, \omega)}(P)}{(I_{(g, \omega)}(P))_p} \rightarrow \mathcal{O}_p \subseteq P, \quad [F] \mapsto F(p)$$

are diffeomorphisms onto the orbits.

(7) If $P(M, G)$ is compact, then $I_{(g, \omega)}(P)$ is a compact Lie transformation group.

Proof. (1) and (3). As in Sect. 6 and Theorem 6.2, let γ be a positive-definite inner product on \mathfrak{G} . Then $\hat{g} = \pi^*g + \gamma \cdot (\omega \otimes \omega)$ is a Riemannian metric on P . If $F \in I_{(g, \omega)}(P)$,

$$F^*\hat{g} = \pi^*(f^*g) + \gamma \cdot (F^*\omega \otimes F^*\omega) = \hat{g},$$

so $I_{(g, \omega)}(P)$ is a closed subgroup of $I_{\hat{g}}(P)$. Thus $I_{(g, \omega)}(P)$ is a Lie transformation group on P , and the action is proper.

(2): If F_λ is a one-parameter subgroup of $I_{(g, \omega)}(P)$ with generator $Z \in \mathfrak{X}_{\mathfrak{G}}(P)$, then Z is a complete vector field and $F_\lambda^*\omega = \omega$ and $f_\lambda^*g = g$, where $f_\lambda = \hat{\pi}(F_\lambda)$. Thus $L_Z\omega = 0$ and $L_Xg = 0$.

Conversely, if $Z \in \mathfrak{X}_{(g, \omega)}(P)$, then by definition of $\mathfrak{X}_{(g, \omega)}(P)$, Z is complete and $L_Z\omega = 0$ and $L_Xg = 0$. Let F_λ be the flow of Z and $f_\lambda = \hat{\pi}(F_\lambda)$ the flow of $X = \pi_*Z$. Then $F_\lambda^*\omega = \omega$ and $f_\lambda^*g = g$. Thus $F_\lambda \in I_{(g, \omega)}(P)$.

(4): Note that since $\dim P = n + k$, we have the immediate rough estimate

$$\dim I_{(g, \omega)}(P) \leq \frac{1}{2}(n + k)(n + k + 1).$$

This can be refined as follows. For $Z \in \mathfrak{X}_{(g, \omega)}(P)$, let $X = \pi_*Z \in \mathfrak{X}_g(M)$, the space of complete Killing vector fields on M . Let $O(M)$ denote the orthonormal frame bundle of M , and for $p \in P$ and $u \in O(M)$, consider the map

$$\mathfrak{X}_{(g, \omega)}(P) \rightarrow \mathfrak{G} \times T_u(O(M)), \quad Z \mapsto (\omega(p) \cdot Z(p), \hat{X}(u)),$$

where $\hat{X}(u)$ denotes the natural lift of X to $O(M)$. This map is injective, for if $\omega(p) \cdot Z(p) = 0$ and $\hat{X}(u) = 0$, then $X(x) = 0$ and $T_xX = 0$, where x is the base point of u . Then since X is a Killing vector field, $X = 0$. Thus Z is a vertical vector field, and so $Z \in \mathfrak{X}_\omega(P)$. Since $\omega(p) \cdot Z(p) = 0$, $Z = 0$. Since the above map is injective,

$$\dim \mathfrak{X}_{(g, \omega)}(P) \leq \dim G + \dim T_u(O(M)) = k + \frac{1}{2}n(n + 1).$$

(5): Since the action is proper, the isotropy subgroups are compact. Alternatively, $(I_{(g, \omega)}(P))_{p_0}$ is a closed subgroup of the compact isotropy group $(I_{\hat{g}}(P))_{p_0} = \{F \in I_{\hat{g}}(P) | F(p_0) = p_0\}$.

(6): This is a consequence of the properness of the action (Abraham-Marsden [1978]).

(7): If $P(M, G)$ is compact, $I_{\hat{g}}(P)$ is a compact transformation group. Since $I_{(g, \omega)}(P)$ is a closed subgroup, it is also compact. \square

Remarks. 1. Note that in contrast to $\mathfrak{X}_\omega(P)$, where infinitesimal vertical symmetries of ω are automatically complete, infinitesimal symmetries Z of $\mathfrak{X}_{(g, \omega)}$ (i.e., $L_Z\omega = 0$ and $L_Xg = 0$) are not necessarily complete, so the Lie algebra $\mathfrak{X}_{(g, \omega)}(P)$ must be restricted to the space of complete infinitesimal symmetries.

2. Another difference between the groups $I_\omega(P)$ and $I_{(g, \omega)}(P)$ is that the action of $I_{(g, \omega)}(P)$ on P is not necessarily free. This is because if $F \in I_{(g, \omega)}(P)$ with $F(p_0) = p_0$, then $f(x_0) = x_0$, where $f = \hat{\pi}(F) \in I_g(M)$ and $x_0 = \pi(p_0)$. But an isometry which fixes a point need not be the identity. Thus F need not cover the identity, and thus Proposition 2.1 cannot be applied. Thus F fixing a point $p_0 \in P$ is not sufficient to cause F to be the identity (see Example 7.2 below).

3. Since the action of $I_{(g, \omega)}(P)$ is not necessarily free, the orbit space $\frac{P}{I_{(g, \omega)}(P)}$ is

not necessarily a manifold. However, since the action is proper, it is a Hausdorff space. \square

Here's an example of the non-freeness of the action of $I_{(g, \omega)}(P)$ on P .

7.2. *Example.* Let S^n be the standard unit sphere in \mathbb{R}^{n+1} with standard metric g_0 , x_0 the north pole, and $f \in I_{g_0}(S^n)$, an isometry fixing x_0 with $f \neq \text{id}_M$. Let $P(S^n, G) = S^n \times G$ be the product bundle, and let $\omega_0 = \pi_2^* \theta$ the canonical flat connection on P (see Example 3.2). Let

$$F : S^n \times G \rightarrow S^n \times G, \quad (x, a) \mapsto (f(x), a).$$

Then $F \in \text{AUT}(P)$, F covers f , and

$$F^* \omega_0 = F^* \pi_2^* \theta = (\pi_2 \circ F)^* \theta = \pi_2^* \theta = \omega_0,$$

so that $F \in I_{(g_0, \omega_0)}(P)$. Also F fixes (x_0, a) , $F(x_0, a) = (f(x_0), a) = (x_0, a)$, but $F \neq \text{id}$ on $S^n \times G$. \square

For $Z \in \mathfrak{X}_G(P)$ and $\omega \in \mathcal{C}(P)$, the Lie derivative $L_Z \omega \in A^1(P, \mathfrak{G})$ can be further examined. Let $Z = T + \bar{X}$ be the decomposition of Z into its vertical and horizontal part. Here $T \in \mathfrak{X}_G^{\text{vert}}(P)$ is defined by $T(p) = T_e \Phi_p \cdot (\omega(p) \cdot Z(p))$, and \bar{X} is the horizontal lift of $X = \pi_* Z \in \mathfrak{X}(M)$.

7.3. Proposition. *Let $Z \in \mathfrak{X}_G(P)$, $\omega \in \mathcal{C}(P)$, and let $Z = T + \bar{X}$ the decomposition of Z into G -invariant vertical and horizontal vector fields. Let $\varphi_T = \omega(T) \in C_{\text{Ad}}^\infty(P, \mathfrak{G})$ correspond to T according to Proposition 1.1, and let $\Omega = D\omega \in A^2(P, \mathfrak{G})$ be the curvature of ω . Then $L_Z \omega = D\varphi_T + i_{\bar{X}} \Omega$.*

Proof. By Lemma 1.4, $L_T \omega = D\varphi_T$. Since $\omega(\bar{X}) = 0$,

$$L_{\bar{X}} \omega = i_{\bar{X}} d\omega + di_{\bar{X}} \omega = i_{\bar{X}} d\omega,$$

and also

$$i_{\bar{X}} \Omega = i_{\bar{X}} d\omega + \frac{1}{2} i_{\bar{X}}([\omega \wedge \omega]) = i_{\bar{X}} d\omega.$$

Thus

$$L_Z \omega = L_{T + \bar{X}} \omega = L_T \omega + L_{\bar{X}} \omega = D\varphi_T + i_{\bar{X}} d\omega = D\varphi_T + i_{\bar{X}} \Omega. \quad \square$$

Thus the Lie algebra $\mathcal{I}_{(g, \omega)}(P)$ can be described by

$$\mathcal{I}_{(g, \omega)}(P) = \{Z \in \mathfrak{X}_G(P) \mid Z \text{ is complete and } D\varphi_T + i_{\bar{X}} \Omega = 0 \text{ and } L_X g = 0\},$$

where $\varphi_T = \omega(T) = \omega(Z)$, and $X = \pi_* Z$. This description is important in Yang-Mills field theories (see e.g. Forgacs-Manton [1980]; Jackiw-Manton [1979]; Fischer [1985]).

Finally we remark that in Lagrangian gauge field theories, the Lie group $I_{(g, \omega)}(P)$ plays an important role as the generator of global conservation laws for matter fields coupled to the fixed parameter fields (g, ω) . This is analogous to the role played by $I_\omega(P)$ as the generator of global *internal* conservation laws, and to the role played by $I_g(M)$ as the generator of the global spacetime conservation laws. Here g is usually a Lorentz signature metric (see Fischer [1984, 1985] for more details).

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