

On the Equivalence of the Two Most Favoured Calabi–Yau Compactifications

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Abstract. We discuss the two known multiply connected Calabi-Yau manifolds which give rise to three generations of elementary particles when chosen as the classical vacuum configuration of the $E_8 \times E_8$ heterotic superstring. It is shown that these two manifolds are diffeomorphic.

1. Introduction

In addition to providing a potentially consistent unification of the fundamental forces, superstring theory [1, 2] has the appealing aspect of leading to highly constrained low energy models of particle interactions. Once a vacuum state is chosen much of the low energy structure of such models is determined. There is, of course, a catch [3]. One must first choose a vacuum configuration. Ideally, one hopes that further investigation of the underlying theory will ultimately show that the vacuum state is determined dynamically. For now, however, one does the next best thing by combining the demands of internal consistency and phenomenological viability. Analysis of this sort from a number of viewpoints has shown [4], that this favours vacuum configurations based upon Calabi-Yau compactifications; that is, vacua of the form $M_4 \times K$, where M_4 is four dimensional Minkowski space and K is a three complex dimensional Kähler manifold with vanishing first Chern class. This certainly narrows down the choice, but, alas, there are many Calabi-Yau manifolds to choose from [5, 6]. As noted, though, after a choice of the Calabi-Yau vacuum configuration K is made, much of the resulting low energy phenomenology may be extracted from the topological and cohomological properties of the manifold. In fact, in the case of the $E_8 \times E_8$ Heterotic string, the number of generations of elementary particle multiplets (i.e. 27's of E_6) is given by one half of the Euler characteristic of K , $|\chi(K)/2|$ (after the $SU(3)$ spin connection ω of K

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is identified with an $SU(3)$ subgroup of one of the E_8 factors of the Yang–Mills group), the unbroken gauge group in such a scenario depends upon the fundamental group $\pi_1(K)$ [7], and the Yukawa couplings upon certain products in the cohomology ring of K [8, 9].

Low energy phenomenology may thus be used to adjudicate amongst the many Calabi-Yau manifolds which a priori are equally viable vacuum configurations.

The easiest phenomenological constraint to impose is that of the number of generations; present experimental evidence indicates that Calabi-Yau manifolds with $|\chi(K)| = 6$ are most favoured as they lead to three generations. A recent computer search [10, 11] for three generation Calabi-Yau manifolds realizable as smooth complete intersections in products of complex projective spaces modded out by freely acting projective discrete symmetry groups has shown that such manifolds are very rare. In fact, only one such manifold was found.

This manifold (originally constructed by Yau [5]), which we shall denote by K_1 , is defined by the complete intersection in $CP^3 \times CP^3$ of bidegree (3, 0), (0, 3) and (1, 1) homogeneous polynomials, modded out by a freely acting Z_3 discrete symmetry group. For definiteness, letting the four homogeneous coordinates of the first CP^3 space be denoted by (X_0, \dots, X_3) and similarly for the second with X replaced by Y , K_1 is given by

$$\left(\sum_{i=0}^3 X_i^3 = 0 \right) \times \left(\sum_{i=0}^3 Y_i^3 = 0 \right) \cap \left(\sum_{i=0}^3 X_i Y_i = 0 \right) \quad (1.1)$$

modded out by the Z_3 group G generated by the map g :

$$g: (X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3) \rightarrow (X_0, \alpha^2 X_1, \alpha X_2, \alpha X_3, Y_0, \alpha Y_1, \alpha^2 Y_2, \alpha^2 Y_3), \quad (1.2)$$

where α is a nontrivial cube root of unity. (The defining equations (1.1) correspond to a particular choice of the complex structure for K_1 ; varying these equations in a G -invariant manner yields a space which if nonsingular, is diffeomorphic to K_1 .)

This manifold, has been shown to yield remarkably realistic phenomenology when chosen for the vacuum configuration [12]. In addition to K_1 , two other Calabi-Yau manifolds with $|\chi| = 6$ are known to exist, both constructed by Yau [5]. One of these is simply connected and hence is not amenable to the flux trapping method of gauge symmetry breaking, making it less phenomenologically promising. The other, which we denote by K_2 is as follows [5].

$$K_2 = \widetilde{K'_0} / G_{27}, \quad (1.3)$$

where K'_0 is a bidegree (3, 3) hypersurface in $CP^2 \times CP^2$, G_{27} is a non-freely acting symmetry group of order 27, and the tilde denotes that the resulting fixed points are resolved.

At present therefore, detailed search for Calabi-Yau manifolds which give rise to three generations has provided two multiply connected (and hence of phenomenological interest) possibilities. In this note we show that these ‘two’ possibilities are actually diffeomorphic, with the three generation model analysed in [12] giving the relevant phenomenology for a particular choice of the vacuum moduli.

2. Fundamental Group and Hodge Diamond

At first sight this claim seems unlikely as the Hodge diamonds and fundamental groups of these two manifolds have been reported as being distinct [5]. We now show that they are in fact identical.

Recall from [5] and that fundamental group of K_1 is Z_3 and from [12] that the Hodge diamond for K_1 and its covering space K_0 are given by:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & & 0 & 6 & & 0 \\
 1 & & & 9 & 9 & & 1 \\
 & & & 0 & 6 & & 0 \\
 & & & 0 & 0 & & \\
 & & & & 1 & &
 \end{array} \tag{2.1}$$

and

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & & 0 & 14 & & 0 \\
 1 & & & 23 & 23 & & 1 \\
 & & & 0 & 14 & & 0 \\
 & & & 0 & 0 & & \\
 & & & & 1 & &
 \end{array} \tag{2.2}$$

respectively.

To construct K_2 we choose the nonsingular defining polynomial for the bicubic K'_0 to be

$$\sum_{i=0}^2 x_i^3 y_i^3 + \delta \left(\sum_{i=0}^2 (x_i^3 y_{i+1}^3 + x_i^3 y_{i-1}^3) \right) = 0, \tag{2.3}$$

where δ is a complex number, the x 's and y 's are homogeneous CP^2 coordinates¹, and we mod through by the (non-freely acting) group G_{27} generated by the three maps:

$$\sigma_1: (x_0, x_1, x_2) \times (y_0, y_1, y_2) \rightarrow (x_1, x_2, x_0) \times (y_1, y_2, y_0), \tag{2.4}$$

$$\sigma_2: (x_0, x_1, x_2) \times (y_0, y_1, y_2) \rightarrow (x_0, \alpha x_1, \alpha^2 x_2) \times (y_0, \alpha y_1, \alpha^2 y_2), \tag{2.5}$$

$$\sigma_3: (x_0, x_1, x_2) \times (y_0, y_1, y_2) \rightarrow (x_0, x_1, x_2) \times (y_0, \alpha y_1, \alpha^2 y_2). \tag{2.6}$$

Notice that σ_3 and $\sigma_2^2 \sigma_3$ each have three fixed tori when acting on the manifold K'_0 [5].

To compute the Hodge diamond, we proceed in stages. On K'_0 we apply the Lefschetz Theorem in the form

$$h^{p,q}(K'_0) = h^{p,q}(CP^2 \times CP^2) \quad \text{for } p + q < 3 \tag{2.7}$$

to conclude that $h^{1,1} = 2$, $h^{1,0} = h^{0,1} = 0$, and of course, $h^{0,0} = 1$. From the adjunction formula [13] we have $\chi(K'_0) = -162$, and hence $h^{1,1} - h^{1,2} = -81$.

¹ Throughout this letter, CP^2 homogeneous coordinates are given by lowercase letters, and CP^3 homogeneous coordinates are given by uppercase letters

Thus, we have the Hodge Diamond:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & 0 & & 2 & & 0 \\
 1 & & 83 & & 83 & & 1 \\
 & & 0 & & 2 & & 0 \\
 & & 0 & & 0 & & \\
 & & & & 1 & &
 \end{array} \tag{2.8}$$

The group generated by σ_1 and σ_2 does act freely and hence $K'_0/(\sigma_1, \sigma_2)$ is smooth and has Euler characteristic 18. Also, $h^{1,1}$ remains two, as these are just the pullbacks of the two CP^2 Kähler forms which are invariant under σ_1 and σ_2 . Thus, $h^{1,2}$ on this quotient space is 11.

The final step in the construction is modding out by σ_3 . The easiest way to analyse the resulting cohomology is to split it into a sum of that derived from the non-singular part of the space with that derived from blowing up the singular tori.

For the nonsingular part, V , we employ the same reasoning as above to write the contribution to the Hodge Diamond of K_2 as:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & 0 & & 0 \\
 & & 0 & & 2 & & 0 \\
 1 & & 5 & & 5 & & 1 \\
 & & 0 & & 2 & & 0 \\
 & & 0 & & 0 & & \\
 & & & & 1 & &
 \end{array} \tag{2.9}$$

The value of $\chi(V)$ is -6 .

To analyse the changes in the cohomology which result from blowing up the singularities, we first study these singularities locally. Around any will-be fixed point on V , we may construct a B^4 (the four ball) with boundary being an S^3 . On modding out by σ_3 , this S^3 becomes the Lens space $L(3, 2)$. Thus, to resolve a single fixed point (on any of the singular tori) we must ‘glue’ in a smooth four space Q^4 with $\partial Q^4 = L(3, 2)$.

The construction of such a Q^4 is a well studied problem and has been solved for the general $L(p, q)$ scenario [14]. In this case, we must take Q^4 to be the plum product of two copies of $S^2(-2)$, where $S^2(-2)$ is the disk bundle over S^2 with first Chern number equal to -2 . (In singularity theory resolving spaces of this type are known as (A_2) Hirzebruch–Jung strings [15].)

To address the full problem of blowing up the fixed tori, we must examine the structure of their respective normal bundles in order to understand how they are embedded in the ambient threefold. Consider one such torus (the other five are found by cyclically permuting these coordinates and swapping x and y coordinates):

$$(x_0^3 + \delta(x_1^3 + x_2^3) = 0) \times (1, 0, 0). \tag{2.10}$$

At any point on this torus, we may take (ζ_1, ζ_2) , where $\zeta_i = y_i/y_0$ as suitable normal bundle coordinates, giving us a global frame. The triviality of the normal bundle allows us to extend the resolution procedure described above merely by

extracting a tubular neighborhood around any of the fixed tori of the form

$$T^2 \times B^4 \tag{2.11}$$

on K'_0 , and replacing it by

$$T^2 \times Q^4 \tag{2.12}$$

on K_2 . Viewed on K_2 , this procedure extracts $T^2 \times CL(3, 2)$ (where $CL(3, 2)$ is the cone over $L(3, 2)$) and replaces it by $T^2 \times Q^4$. Since the Euler characteristic is multiplicative, we see that this procedure preserves the $\chi(K_2) = -6$.

However, the individual Hodge numbers are not preserved. The Hodge diamond of T^2 is simply:

$$\begin{array}{cc} & 1 \\ 1 & & 1 \\ & 1 \end{array} \tag{2.13}$$

While, similarly, the Hodge diamond of $T^2 \times Q^4$ is that of $T^2 \times (S^2 \wedge S^2)$ (where the \wedge denotes the plum product, and equality follows because the latter is the deformation retract of $T^2 \times Q^4$). This gives Hodge diamond for $T^2 \times Q^4$:

$$\begin{array}{ccccc} & & 1 & & \\ & & 1 & & 1 \\ & 0 & 3 & 0 & \\ 0 & 2 & 2 & 0 & 0 \\ & 0 & 2 & 0 & \\ & 0 & 0 & & \\ & & 1 & & \end{array} \tag{2.14}$$

On K_2 we must perform this procedure twice (for each of the two pairs of three fixed tori). This yields the net change in the Hodge diamond being:

$$\begin{array}{ccccc} & & 0 & & \\ & & 0 & 0 & \\ & 0 & 4 & 0 & \\ 0 & 4 & 4 & 0 & 0 \\ & 0 & 4 & 0 & \\ & 0 & 0 & & \\ & & 0 & & \end{array} \tag{2.15}$$

Finally, combining this result with that for the non-singular part V of K_2 we find that the resolved Hodge diamond is:

$$\begin{array}{ccccc} & & 1 & & \\ & & 0 & 0 & \\ & 0 & 6 & 0 & \\ 1 & 9 & 9 & 1 & \\ & 0 & 6 & 0 & \\ & 0 & 0 & & \\ & & 1 & & \end{array} \tag{2.16}$$

which is precisely that of K_1 . Furthermore, following from the above, the preimage under σ_1 of each of the resolved tori gives a three sheeted cover, as σ_1 permutes

the tori in two groups of three. Diagonalizing this Z_3 we thus see that the resolution contributes 12 (2, 1) and 12 (1, 1) forms which each comprise three regular representations of Z_3 . Thus, we see that a covering space for K_2 has the Hodge diamond:

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & 0 & & 0 \\
 & & 0 & & 14 & & 0 \\
 1 & & 23 & & 23 & & 1 \\
 & & 0 & & 14 & & 0 \\
 & & & 0 & & 0 \\
 & & & & & & 1
 \end{array} \tag{2.17}$$

Notice that this is precisely that of K_0 . One might fear that this resolution procedure may have effected the vanishing of the first Chern class and Kählerity properties manifest for K'_0 ; it can be shown however that K_2 inherits both of these characteristics [15].

To address the identity of the fundamental group on the blown up threefold, it is simplest to reverse the order of the original construction. Starting from the manifold K'_0 as given in (2.3), we mod through by the group generated by σ_2 and σ_3 . As noted, not only does σ_3 have three fixed tori, but $\sigma_2^2\sigma_3$ has three fixed tori as well. These codimension two fixed point sets then allow us to contract *any* would-be nontrivial loop associated with the passage to the quotient space; i.e. $K'_0/(\sigma_2, \sigma_3)$ is simply connected. Upon resolving the fixed tori using the procedure described above, we then arrive at a smooth simply connected manifold, K''_0 , with Euler characteristic 18, this is the covering space of K_2 . The final step is then to mod through by σ_1 . As this transformation acts freely, when extended to permute the exceptional divisors of the toroidal blow-up's rather than the singular tori themselves, it gives rise to a fundamental group of Z_3 on the quotient manifold K_2 .

The above arguments make our claim that K_1 and K_2 are diffeomorphic a strong possibility. Before making the diffeomorphism explicit, we note the intuitive identification we shall make.

First recall the standard result from algebraic geometry that a cubic hypersurface in CP^3 may be realized as a CP^2 with six points blown up [16]. Then notice that if the CP^3 coordinates are chosen to be cubic functions of the CP^2 coordinates, the bicubic equation in K_2 is mapped to a bilinear equation, i.e. a bidegree (1, 1) relation. Apparently then, the two CP^3 cubic surfaces which are part of the definition of K_1 may arise from the ambient CP^2 spaces (via the resolution of singularities), and the (1, 1) hyperplane arises as the image of the bicubic defining equation. We make this intuitive notion explicit in the next section.

3. Explicit Diffeomorphism

Thus far we have shown that the two $\chi = -6$ manifolds K_1 and K_2 have precisely the same Hodge diamond and fundamental group. This suggests quite strongly, but does not prove, that K_1 and K_2 are diffeomorphic. Before outlining a proof that this is indeed the case, we must discuss a few results from the theory of complex surfaces [15, 17, 18, 14].

Consider a relation, f , from CP^2 to CP^3 given by

$$f:(x_0, x_1, x_2) \rightarrow (x_0^3, x_1^3, x_2^3, x_0x_1x_2) = (X_0, X_1, X_2, X_3). \tag{3.1}$$

This relation is three-to-one except at the three CP^2 points $(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ where it is one-to-one. The image in CP^3 is the singular cubic surface defined by

$$X_3^3 - X_0X_1X_2 = 0. \tag{3.2}$$

This surface is singular at the three CP^3 points $(0, 0, 1, 0)$, $(0, 1, 0, 0)$, and $(1, 0, 0, 0)$ which are precisely the images of the three one-to-one points of f . Now if we mod through by the nonfreely acting Z_3 on CP^2 generated by

$$h:(x_0, x_1, x_2) \rightarrow (x_0, \alpha x_1, \alpha^2 x_2) \tag{3.3}$$

we get a homeomorphism \tilde{f} between singular spaces from f in the obvious way. (The action of h makes the map one-to-one everywhere on CP^2/Z_3 .)

$$\tilde{f}: CP^2/Z_3 \rightarrow (X_3^3 - X_0X_1X_2 = 0) \subset CP^3. \tag{3.4}$$

If we blow up the three singular points on either side of the homeomorphism \tilde{f} , we get a diffeomorphism between Kähler manifolds. The relevant blow-up procedure for each of the points involves an A_2 Hirzebruch–Jung string (as discussed above this is a certain disk bundle over the plumb product of two S^2 's) [14].

It is an important result of the theory of the resolution of rational singularities of complex surfaces that the resolution of the singular points of $(X_3^3 - X_0X_1X_2 = 0) \subset CP^3$ yields a surface which is diffeomorphic to the generic nonsingular cubic surface in CP^3 [19]. This result and the above discussion imply:

$$CP^2/Z_3 \overset{\sim}{\cong} (\text{Nonsingular Cubic Surface}) \subset CP^3 \tag{3.5}$$

(where the Z_3 is generated by the map h above and the tilde denotes resolution).

Equation (3.5) is one form of the fact that the cubic surface is diffeomorphic to the CP^2 with six (nonsingular) points blown-up [16]. (We can move around the six exceptional divisors differentially, and (3.5) depends roughly on them coinciding in three pairs.)

We can apply these facts to show that the covering spaces of K_1 and K_2 are diffeomorphic. To this end it is convenient to slightly deform the $CP^2 \times CP^2$ bicubic polynomial from which K_2 is constructed to the form:

$$B \cong \left(\sum_{i=0}^2 x_i^3 y_i^3 + x_0 x_1 x_2 y_0 y_1 y_2 + \delta \left(\sum_{i=0}^2 (x_i^3 y_{i+1}^3 + x_i^3 y_{i-1}^3) \right) = 0 \right) \subset CP^2 \times CP^2. \tag{3.6}$$

(This change does not effect the construction of K_2 ; it corresponds to a slight change in moduli. To prove K_1 and K_2 are diffeomorphic it suffices to show that any two nonsingular points in their respective moduli spaces are diffeomorphic.)

As discussed above the covering space of K_2 is constructed by resolving the singular curves of a $CP^2 \times CP^2$ bicubic which are introduced by modding out the

nonfreely acting group generated by

$$\sigma_2^2 \sigma_3: (x_0, x_1, x_2) \times (y_0, y_1, y_2) \rightarrow (x_0, \alpha x_1, \alpha^2 x_2) \times (y_0, y_1, y_2) \tag{3.7}$$

and

$$\sigma_3: (x_0, x_1, x_2) \times (y_0, y_1, y_2) \rightarrow (x_0, x_1, x_2) \times (y_0, \alpha y_1, \alpha^2 y_2). \tag{3.8}$$

We write the two generators in this way to exploit the fact that they have precisely the same form as h defined in (3.3) above; $\sigma_2^2 \sigma_3$ acts on the left CP^2 and σ_3 on the right CP^2 . If we ignore the bicubic equation for the moment, the discussion above gives the homeomorphism:

$$F: (CP^2 \times CP^2)/(\sigma_2, \sigma_3) \rightarrow (X_3^3 - X_0 X_1 X_2 = 0) \times (Y_3^3 - Y_0 Y_1 Y_2 = 0) \subset CP^3 \times CP^3 \tag{3.9}$$

with

$$\begin{aligned} F: (x_0, x_1, x_2) \times (y_0, y_1, y_2) &\mapsto (x_0^3, x_1^3, x_2^3, x_0 x_1 x_2) \times (y_0^3, y_1^3, y_2^3, y_0 y_1 y_2) \\ &= (X_0, X_1, X_2, X_3) \times (Y_0, Y_1, Y_2, Y_3). \end{aligned} \tag{3.10}$$

In addition note that the map F carries the bicubic equation to a bidegree (1, 1) polynomial in $CP^3 \times CP^3$. Thus as singular spaces,

$$\begin{aligned} B/(\sigma_2, \sigma_3) &\cong (X_3^3 - X_0 X_1 X_2 = 0) \times (Y_3^3 - Y_0 Y_1 Y_2 = 0) \\ &\cap \left(\sum_{i=0}^3 X_i Y_i + \delta(X_0 Y_1 + X_1 Y_2 + X_2 Y_0 + X_0 Y_2 + X_1 Y_0 + X_2 Y_1 = 0) \right) \\ &\subset CP^3 \times CP^3 \end{aligned} \tag{3.11}$$

Thus, before resolution of the six singular tori to yield the covering space of K_2 , the singular space $B/(\sigma_2, \sigma_3)$ is homeomorphic to a ‘singular point’ in the moduli space of bidegree (3, 0), (0, 3), (1, 1) complete intersections in $CP^3 \times CP^3$. Generic nonsingular points of this moduli space correspond to the covering space of K_1 . Thus to show that the two covering spaces are diffeomorphic, we need to show that resolving the codimension two singularities of the space (3.11) yields a manifold diffeomorphic to a nonsingular manifold with the same degree defining polynomials. (In one lower dimension, this would be analogous to the property of rational surface singularities discussed above.) In general this is not the case for three complex dimensional algebraic varieties [20]. However, in the case (3.11) at hand, we can resolve three of the singular tori by an A_2 resolution of the three singular points of the Y-space cubic ($Y_3^3 - Y_0 Y_1 Y_2 = 0$) thought of as a surface in CP^3 . (We can think of the Y_i as indexing a family of hyperplane sections of this cubic via the bidegree (1, 1) polynomial. The locus where these hyperplanes meet the singular points of the X-space cubic sweeps out three singular tori in the complete intersection. Thus resolving the Y-space CP^3 cubic singular points in this manner is equivalent to resolving the singular torus given in (2.10) (along with its two partners obtained by cyclically permuting the coordinates).

We can clearly repeat this procedure for the X-space cubic and thus the theorems of [19] imply that the resolution of the singular curves in (3.11) yields a manifold which is diffeomorphic to the generic nonsingular bidegree (3, 0), (0, 3), (1, 1) complete intersection in $CP^3 \times CP^3$. This completes the proof that the covering spaces of K_1 and K_2 are diffeomorphic.

To extend this proof to K_1 and K_2 themselves we must consider the nature of the freely acting Z_3 actions relevant to each space. As discussed earlier, the map σ_1 of (2.4) is readily extended to an action on the smooth covering space K_0'' of K_2 ; under the map F this extension is carried to the transformation g :

$$g: (X_0, X_1, X_2, X_3) \times (Y_0, Y_1, Y_2, Y_3) \rightarrow (X_1, X_2, X_0, X_3) \times (Y_1, Y_2, Y_0, Y_3) \quad (3.12)$$

in $CP^3 \times CP^3$. This Z_3 is in turn related by a PGL symmetry of the ambient $CP^3 \times CP^3$ to the one employed in [12]. (From another point of view, a basis can be chosen for the exceptional divisors of K_1 's covering space in which the action of the free Z_3 is identical to that of σ_1 above [21].) More details concerning the equivalence of these Z_3 's shall be presented in [10].

4. Phenomenological Implications

As noted earlier, low energy superstring phenomenology, to a large extent, is determined by the manifold of compactification. Beyond the (topological) Hodge numbers which determine the number of generations and antigerations of elementary particle multiplets, the Yukawa couplings follow from quasitopological computations in the cohomology ring of the chosen manifold. For negative Euler characteristic Calabi–Yau manifolds (such as those discussed in this letter) the Yukawa couplings amongst the antigerations (the (1, 1) forms) are independent of the complex structure imposed on the manifold, while those amongst the generations (the (2, 1) forms) certainly *are* dependent on the complex structure [8]. One diffeomorphism class of Calabi–Yau manifolds thus has the potential to give rise to many possible low energy models. Our result, therefore, does not preclude the possibility of building phenomenologically interesting models based on this diffeomorphism class of compactifications leading to three generations other than that presented in [12], but shows that any such models will differ from that of [12] merely by a change in the choice of the complex structure.

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Note added in proof

After the completion of this work, we received a preprint from R. Schimmrigk [22] presenting a third construction of a $\chi = -6$ Calabi–Yau manifold with $h^{1,1} = 6$, $h^{2,1} = 9$, and $\pi_1 \cong Z_3$. This manifold is built from a bidegree (3, 0), (1, 3) complete intersection in $CP^3 \times CP^2$ by resolving the fixed curves of a Z_3 similar to that generated by h in (3.3) above, and then modding out a freely acting Z_3 . The discussions of this letter can easily be shown to imply that this third manifold is diffeomorphic to K_1 and K_2 .

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