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Nearly One Dimensional Singularities of Solutions to the Navier–Stokes Inequality

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Abstract. There exists a function u satisfying (1) u is a weak solution to the Navier–Stokes equations of incompressible fluid flow in three-space with an external force that reduces the speed at every point, (2) the internal singularities of u have Hausdorff dimension close to 1.

Section 1. Introduction

The theorem below is an improvement on Theorem 1.1 of [4]. The difference between the two theorems is in the size of the singular set S. This set consisted of only one point in [4]. In the present paper, the Hausdorff dimension of S is nearly one. In what follows, the laplacian Δ and the gradient ∇ of a function defined on a subset of $R^3 \times R$ will involve only the R^3 variables.

Theorem. *If $\zeta < 1$ then there exists a Cantor set $S \subset R^3 \times \{1\}$ and there exist functions $u: R^3 \times [0, \infty) \to R^3$ and $p: R^3 \times [0, \infty) \to R$ satisfying the following properties:*

$$\text{there is a compact set } K \subset R^3 \text{ such that } u(x,t) = 0 \text{ for all } x \notin K, \tag{1.1}$$

$$\text{for fixed } t, \text{ the function } u_t: R^3 \to R^3 \text{ defined by } u_t(x) = u(x,t) \text{ is a } C^\infty \text{ function}, \tag{1.2}$$

$$\sum_{i=1}^{3} (\partial u_i/\partial x_i)(x,t) = 0, \tag{1.3}$$

$$p(x,t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} (\partial u_j/\partial x_i)(y,t)(\partial u_i/\partial x_j)(y,t)(4\pi|x-y|)^{-1}\, dy, \tag{1.4}$$

$$\text{there exists } M < \infty \text{ such that } \|u_t\|_2 < M \text{ for all } t\ (u_t \text{ defined in } (1.2)), \tag{1.5}$$

$$|\nabla u|^2, |u|^3 \text{ and } |u||p| \text{ are integrable}, \tag{1.6}$$

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if $\phi: R^3 \times (0,\infty) \to R$ *is a* C^∞ *function with compact support and* $\phi \geqq 0$, *then*

$$\int_0^\infty \int_{R^3} |\nabla u|^2 \phi \leqq \int_0^\infty \int_{R^3} (2^{-1}|u|^2 + p) u \cdot \nabla \phi + \int_0^\infty \int_{R^3} 2^{-1}|u|^2 \left(\frac{\partial \phi}{\partial t} + \Delta \phi \right), \tag{1.7}$$

u is not essentially bounded on any neighborhood of any point in S, (1.8)

the Hausdorff dimension of S is greater than ζ. (1.9)

The introduction of [4] contains a discussion of the heuristics of this type of theorem. Briefly, we are interested in solutions to

$$\frac{\partial}{\partial t} u_i = -\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \Delta u_i + f_i,$$

$$\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0, \quad \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i} = 0, \quad \sum_{i=1}^{3} f_i u_i \leqq 0. \tag{1.10}$$

This says that u is a solution to the Navier–Stokes equations of incompressible fluid flow with an external force f that is divergence free and pushes against the flow at every point of space-time. Properties (1.3), (1.4), (1.7) say that (1.10) is satisfied in a weak sense. The right definition of weak solution is obtained by multiplying (1.10) by a nonnegative test function ϕ and integrating.

It can be shown that the theorem in this paper is nearly optimal. The proof of Theorem 2.1 of [3] implies that the intersection of the singular set with any hyperplane of the form $R^3 \times \{t\}$ cannot have dimension greater than 1. Therefore, the conditions stated in the theorem (which include $S \subset R^3 \times \{1\}$) always force $\dim(S) \leqq 1$. Furthermore, L. Caffarelli, R. Kohn and L. Nirenberg proved, in a slightly different context, that it is not possible to increase the dimension of S beyond 1 by relaxing the requirement $S \subset R^3 \times \{1\}$. This result appears in [1].

Section 2. Preliminaries

We recall some of the notation of [4]. The set of C^∞ functions with compact support from U into R^n will be denoted by $C_c^\infty(U, R^n)$. The support of a function f will be written $\operatorname{spt}(f)$. We set $P = \{(x_1, x_2) \in R^2 : x_2 > 0\}$. If $c = (c_1, c_2) \in R^2$ and $c_1^2 + c_2^2 = 1$, then $R_c: R^3 \to R^3$ is the rotation

$$R_c(x_1, x_2, x_3) = (x_1, c_1 x_2 - c_2 x_3, c_1 x_3 + c_2 x_2)$$

about the x_1 axis. If $f \in C_c^\infty(P, R)$, $v = (v_1, v_2) \in C_c^\infty(P, R^2)$, $f \geqq 0$ and $f(x) > |v(x)|$ holds for all $x \in \operatorname{spt}(v)$, then $u[v,f] \in C_c^\infty(R^3, R^3)$ is defined by

$$u[v,f](x_1, x_2, 0) = (v_1(x_1,x_2), v_2(x_1,x_2), ((f(x_1,x_2))^2 - |v(x_1,x_2)|^2)^{1/2})$$

if $(x_1, x_2) \in P$,

$$u[v,f](R_c(x_1,x_2,0)) = R_c(u[v,f](x_1,x_2,0)) \quad \text{if} \quad c \in R^2, |c| = 1, \quad (x_1,x_2) \in P,$$

and $u[v,f](x_1,0,0) = 0$. Under the same hypotheses, we define $p^*[v,f]: R^3 \to R$

and $p[v,f]:P\to R$ by means of

$$p^*[v,f](x)=\int_{R^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[v,f]\right)(y)\left(\frac{\partial}{\partial x_j}u_i[v,f]\right)(y)(4\pi|x-y|)^{-1}\,dy,$$

and $p[v,f](x_1,x_2)=p^*[v,f](x_1,x_2,0)$. If $f\in C_c^\infty(P,R)$, we set

$$\begin{aligned}L(f)(x_1,x_2)=&\,\Delta f(x_1,x_2)+x_2^{-1}(\partial f/\partial x_2)(x_1,x_2)\\&-x_2^{-2}f(x_1,x_2)\quad\text{if}\quad(x_1,x_2)\in P.\end{aligned}$$

If A and B are sets then we set $A\sim B=\{x\in A:x\notin B\}$. If f_i is a function defined on a subset of $R^3\times R$, then we will write $f_{i,t}(x)\equiv f_i(x,t)$ for appropriate $(x,t)\in R^3\times R$. This notation is used in Sect. 3 and Sect. 5 for the functions h_i^m, $q_i^{m,k}$ and h_i. However, this does not apply to the function $F_{i,j}^{m,n}$ in Lemma 3.1.

The functions $u[v,f], p^*[v,f], p[v,f]$ will only be used when v satisfies

$$x_2\frac{\partial}{\partial x_1}v_1(x_1,x_2)+x_2\frac{\partial}{\partial x_2}v_2(x_1,x_2)+v_2(x_1,x_2)=0.$$

This equation implies $\operatorname{div}(u[v,f])=0$.

Lemma 2.1. *We have $p[v,f]=p[-v,f]$ if the left side makes sense.*

Proof. If $u:R^3\to R^3$ satisfies $u(R_c(x))=R_c(u(x))$, then the identity

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}(x)\frac{\partial u_i}{\partial x_j}(x)=\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}(R_c(x))\frac{\partial u_i}{\partial x_j}(R_c(x))$$

follows because we can write the left side in the invariant form

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}\,\frac{\partial u_i}{\partial x_j}=\operatorname{div}(u\cdot\nabla u)-u\cdot\nabla(\operatorname{div}(u)).$$

If $(x_1,x_2)\in P$, then we conclude

$$\begin{aligned}&\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[v,f]\right)(x_1,x_2,0)\left(\frac{\partial}{\partial x_j}u_i[v,f]\right)(x_1,x_2,0)\\&=\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[-v,f]\right)(x_1,x_2,0)\left(\frac{\partial}{\partial x_j}u_i[-v,f]\right)(x_1,x_2,0).\end{aligned}$$

If $y=(y_1,y_2,y_3)=R_c(x_1,x_2,0)$ for some $(x_1,x_2)\in P$, then all this implies

$$\begin{aligned}&\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[v,f]\right)(y)\left(\frac{\partial}{\partial x_j}u_i[v,f]\right)(y)\\&=\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_i}u_j[-v,f]\right)(y)\left(\frac{\partial}{\partial x_j}u_i[-v,f]\right)(y).\end{aligned}$$

We get $p^*[v,f]=p^*[-v,f]$ and, consequently, $p[v,f]=p[-v,f]$.

Lemma. 2.2. *The identity $p[v,f]+p[w,g]=p[v+w,f+g]$ is valid if* $\operatorname{spt}(f)$ *and* $\operatorname{spt}(g)$ *are disjoint and the left side makes sense.*

Proof. This is immediate from the definition.

Section 3. An Oscillatory Process

Lemma 3.1. *If M is a positive integer and T is a positive real number, then there exist C^∞ functions $a_i^{m,k}: R \to [-1,1]$ for $i \in \{1,2\}, m \in \{1,2,\dots,M\}, k \in \{1,2,3,\dots\}$ such that* $\operatorname{spt}(a_i^{m,k}) \subset [0,T]$ *and the following property holds: Suppose that $G_i^m: P \times [0,T] \to R$ and $F_{i,j}^{m,n}: P \times [0,T] \times [-1,1] \to R$ are uniformly continuous and bounded functions for $\{i,j\} \subset \{1,2\}, \{m,n\} \subset \{1,2,\dots,M\}$ such that the identity*

$$F_{i,j}^{m,n}(x,t,1) = F_{i,j}^{m,n}(x,t,-1) \tag{3.1}$$

is satisfied. Suppose, also, that the functions $H_i^{m,k}: P \times [0,T] \to R$, $H_i^m: P \times [0,T] \to R$ are defined by

$$H_i^{m,k}(x,s) = \int_0^s a_i^{m,k}(t)\left(G_i^m(x,t) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{i,j}^{m,n}(x,t,a_j^{n,k}(t)) \right) dt,$$

$$H_1^m(x,s) = 0, \quad H_2^m(x,s) = \int_0^s 2^{-1}(F_{2,1}^{m,m}(x,t,1) - F_{2,1}^{m,m}(x,t,0))\,dt. \tag{3.2}$$

Then $H_i^{m,k}$ converges to H_i^m uniformly as k approaches ∞.

Proof. Let $b_i^{m,k}: [0,T] \to [-1,1]$ and $c^{m,k}: [0,T] \to [0,1]$ be defined almost everywhere by the identities

$$\begin{aligned}
&b_1^{m,k}(t) = 1 \quad \text{if} \quad (4p-4)4^{-m}k^{-1}T < t < (4p-3)4^{-m}k^{-1}T,\\
&b_1^{m,k}(t) = -1 \quad \text{if} \quad (4p-3)4^{-m}k^{-1}T < t < (4p-2)4^{-m}k^{-1}T,\\
&b_1^{m,k}(t) = 0 \quad \text{if} \quad (4p-2)4^{-m}k^{-1}T < t < (4p)4^{-m}k^{-1}T,\\
&b_2^{m,k}(t) = 1 \quad \text{and} \quad c^{m,k}(t) = 1 \quad \text{if} \quad (4p-4)4^{-m}k^{-1}T < t < (4p-2)4^{-m}k^{-1}T,\\
&b_2^{m,k}(t) = -1 \quad \text{and} \quad c^{m,k}(t) = 0 \quad \text{if} \quad (4p-2)4^{-m}k^{-1}T < t < (4p)4^{-m}k^{-1}T
\end{aligned} \tag{3.3}$$

for $p = 1,2,3,\dots,4^{m-1}k$. We choose functions $a_i^{m,k}$ satisfying

$$\begin{gathered}
a_i^{m,k}: R \to [-1,1],\ a_i^{m,k} \text{ is } C^\infty, \quad \operatorname{spt}(a_i^{m,k}) \subset [0,T],\\
\text{the Lebesgue measure of } \{t \in [0,T]: a_i^{m,k}(t) \neq b_i^{m,k}(t)\} \text{ is less than } k^{-1}.
\end{gathered} \tag{3.4}$$

Let G_i^m and $F_{i,j}^{m,n}$ satisfy the hypotheses of the lemma. For every positive integer k, we can define $\varepsilon(k)$ to be the smallest element of the interval $[0,\infty]$ satisfying

$$\begin{aligned}
|F_{i,j}^{m,n}(x,t,a) - F_{i,j}^{m,n}(x,t',a)| &\leqq \varepsilon(k) \quad \text{if} \quad |t-t'| \leqq k^{-1}T,\\
|G_i^m(x,t) - G_i^m(x,t')| &\leqq \varepsilon(k) \quad \text{if} \quad |t-t'| \leqq k^{-1}T.
\end{aligned} \tag{3.5}$$

The uniform continuity hypothesis implies

$$\varepsilon(k) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.6}$$

Our hypotheses also allow us to find N with the properties

$$N < \infty, \quad |F_{i,j}^{m,n}(x,t,a)| \leqq N, \quad |G_i^m(x,t)| \leqq N. \tag{3.7}$$

Now we fix $k \in \{1, 2, 3, \dots\}$ and $s \in [0, T]$. We define

$$t_q = qk^{-1}T \quad \text{if} \quad q \in \{0, 1, 2, \dots, k\}, \; S_q = \text{the interval } (t_{q-1}, t_q),$$
$$Q \text{ is an integer satisfying } s \in [t_Q, t_{Q+1}] \text{ and } 0 \leqq Q \leqq k-1. \tag{3.8}$$

The identities in (3.3) imply

$$\begin{aligned}
&\int_{S_q} b_i^{m,k}(t)\,dt = 0, \quad \int_{S_q} b_1^{m,k}(t) c^{n,k}(t)\,dt = 0,\\
&\int_{S_q} b_2^{m,k}(t) c^{n,k}(t)\,dt = 0 \quad \text{if} \quad m \neq n,\\
&\int_{S_q} b_2^{m,k}(t) c^{m,k}(t)\,dt = 2^{-1}k^{-1}T \quad \text{for} \quad q = 1, 2, \dots, k.
\end{aligned} \tag{3.9}$$

Using (3.1) and (3.3) we find

$$\begin{aligned}
&F_{i,1}^{m,n}(x, t_q, b_1^{n,k}(t)) = c^{n,k}(t) F_{i,1}^{m,n}(x, t_q, 1) + (1 - c^{n,k}(t)) F_{i,1}^{m,n}(x, t_q, 0),\\
&F_{i,2}^{m,n}(x, t_q, b_2^{n,k}(t)) = F_{i,2}^{m,n}(x, t_q, 1)
\end{aligned} \tag{3.10}$$

for almost all $t \in [0, T]$. From (3.9)–(3.10) we get

$$\begin{aligned}
&\int_{S_q} b_1^{m,k}(t)\left(G_1^m(x, t_q) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{1,j}^{m,n}(x, t_q, b_j^{n,k}(t)) \right) dt = 0,\\
&\int_{S_q} b_2^{m,k}(t)\left(G_2^m(x, t_q) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{2,j}^{m,n}(x, t_q, b_j^{n,k}(t)) \right) dt\\
&\qquad = 2^{-1}k^{-1}T(F_{2,1}^{m,m}(x, t_q, 1) - F_{2,1}^{m,m}(x, t_q, 0)).
\end{aligned} \tag{3.11}$$

Using (3.2), (3.7), (3.8), (3.4), $|b_i^{m,k}| \leqq 1$ and (3.5) we find

$$\begin{aligned}
&\left| H_i^{m,k}(x, s) - \sum_{q=1}^{Q} \int_{S_q} b_i^{m,k}(t)\left(G_i^m(x, t_q) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{i,j}^{m,n}(x, t_q, b_j^{n,k}(t)) \right) dt \right|\\
&\leqq |H_i^{m,k}(x, s) - H_i^{m,k}(x, t_Q)|\\
&\quad + \left| H_i^{m,k}(x, t_Q) - \int_0^{t_Q} b_i^{m,k}(t)\left(G_i^m(x, t) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{i,j}^{m,n}(x, t, b_j^{n,k}(t)) \right) dt \right|\\
&\quad + \left| \int_0^{t_Q} b_i^{m,k}(t)\left(G_i^m(x, t) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{i,j}^{m,n}(x, t, b_j^{n,k}(t)) \right) dt \right.\\
&\quad \left. - \sum_{q=1}^{Q} \int_{S_q} b_i^{m,k}(t)\left(G_i^m(x, t_q) + \sum_{j=1}^{2} \sum_{n=1}^{M} F_{i,j}^{m,n}(x, t_q, b_j^{n,k}(t)) \right) dt \right|\\
&\leqq (2M+1)N(k^{-1}T) + 2(2M+1)Nk^{-1} + (2M+1)\varepsilon(k)T\\
&= (2M+1)(Nk^{-1}T + 2Nk^{-1} + \varepsilon(k)T).
\end{aligned} \tag{3.12}$$

Similarly, we can use (3.2), (3.7), (3.8), (3.5) to obtain

$$\left| H_2^m(x, s) - \sum_{q=1}^{Q} 2^{-1}k^{-1}T(F_{2,1}^{m,m}(x, t_q, 1) - F_{2,1}^{m,m}(x, t_q, 0)) \right| \leqq |H_2^m(x, s) - H_2^m(x, t_Q)|$$

$$+\left|H_2^m(x,t_Q)-\sum_{q=1}^{Q}2^{-1}k^{-1}T(F_{2,1}^{m,m}(x,t_q,1)-F_{2,1}^{m,m}(x,t_q,0))\right|$$

$$\leqq Nk^{-1}T+\varepsilon(k)T. \tag{3.13}$$

Now (3.11)–(3.13) give us

$$|H_1^{m,k}(x,s)|\leqq(2M+1)(Nk^{-1}T+2Nk^{-1}+\varepsilon(k)T),$$

$$|H_2^{m,k}(x,s)-H_2^m(x,s)|\leqq(2M+1)(Nk^{-1}T+2Nk^{-1}+\varepsilon(k)T)+Nk^{-1}T+\varepsilon(k)T.$$

The conclusion follows from the above, (3.2) and (3.6).

Lemma 3.2 *Suppose that $v,\xi,T,\delta,M,C_i^m,h_i^m,v_i^m,g_i^m$ satisfy (3.14)–(3.24):*

$$v,\xi,T,\delta,M \text{ are positive, } M \text{ is an integer, } m\in\{1,2,\dots,M\},\ i\in\{1,2\}, \tag{3.14}$$

$$C_i^m\subset P, C_i^m \text{ is compact, the sets } C_i^m \text{ are disjoint,} \tag{3.15}$$

$$h_i^m:P\times(-\delta,T+\delta)\to R \text{ is } C^\infty,\quad v_i^m\in C_c^\infty(P,R^2),\quad g_i^m\in C_c^\infty(P,R), \tag{3.16}$$

$$\text{if } x\notin C_i^m \text{ then } h_i^m(x,t)=0, \tag{3.17}$$

$$h_i^m\geqq 0,\quad \text{if } x\in\operatorname{spt}(g_i^m),\quad \text{then}\quad h_i^m(x,t)>|v_i^m(x)|, \tag{3.18}$$

$$0\leqq g_i^m\leqq 1,\quad \operatorname{spt}(v_i^m)\subset\{x:g_i^m(x)=1\}, \tag{3.19}$$

$$x_2\frac{\partial}{\partial x_1}v_{i1}^m(x_1,x_2)+x_2\frac{\partial}{\partial x_2}v_{i2}^m(x_1,x_2)+v_{i2}^m(x_1,x_2)=0, \tag{3.20}$$

where $v_i^m=(v_{i1}^m,v_{i2}^m)$,

$$(h_1^m(x,s))^2=(h_1^m(x,0))^2-2s\delta g_1^m(x), \tag{3.21}$$

$$(h_2^m(x,s))^2=(h_2^m(x,0))^2-2s\delta g_2^m(x)-\int_0^s v_2^m(x)\cdot\nabla(p[v_1^m,h_{1,t}^m]-p[0,h_{1,t}^m])(x)\,dt, \tag{3.22}$$

where $h_{i,t}^m(x)=h_i^m(x,t)$,

$$L(h_{i,t}^m)(x)\geqq 0\quad \text{if}\quad g_i^m(x)<1, \tag{3.23}$$

$$vu[av_i^m,h_{i,t}^m](x)\cdot\Delta(u[av_i^m,h_{i,t}^m])(x)\geqq -\delta/4\quad \text{if}\quad |a|\leqq 1\quad \text{and}\quad t\in[0,T]. \tag{3.24}$$

Then there exist $\eta>0$ and C^∞ functions $u:R^3\times(-\eta,T+\eta)\to R^3$ and $p:R^3\times(-\eta,T+\eta)\to R$ satisfying (3.25)–(3.30):

$$u(x_1,x_2,x_3,t)=0\quad \text{if}\quad (x_1,(x_2^2+x_3^2)^{1/2})\notin\bigcup_{i=1}^{2}\bigcup_{m=1}^{M}C_i^m, \tag{3.25}$$

$$\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=0, \tag{3.26}$$

$$p(x,t)=\int_{R^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_i}(y,t)\frac{\partial u_i}{\partial x_j}(y,t)(4\pi|x-y|)^{-1}\,dy, \tag{3.27}$$

$$\frac{\partial}{\partial t}2^{-1}|u(x,t)|^2 \leqq -u(x,t)\cdot\nabla(2^{-1}|u|^2+p)(x,t)+\nu u(x,t)\cdot\Delta u(x,t) \quad \textit{if} \quad t\in[0,T], \tag{3.28}$$

$$|u(x_1,x_2,x_3,0)| = \sum_{m=1}^{M}(h_1^m+h_2^m)(x_1,(x_2^2+x_3^2)^{1/2},0) \quad \textit{if} \quad (x_2,x_3)\neq(0,0) \tag{3.29}$$

$$\left||u(x_1,x_2,x_3,t)| - \sum_{m=1}^{M}(h_1^m+h_2^m)(x_1,(x_2^2+x_3^2)^{1/2},t)\right| \leqq \xi \tag{3.30}$$

if $t\in[0,T]$ *and* $(x_2,x_3)\neq(0,0)$.

Proof. Assumptions (3.19), (3.18), (3.17), (3.15) imply

$$\operatorname{spt}(v_i^m)\subset\{x{:}g_i^m(x)=1\}\subset\operatorname{spt}(g_i^m)\subset\operatorname{spt}(h_{i,t}^m)\subset C_i^m. \tag{3.31}$$

Let $Q_i^{m,k}{:}P\times(-\delta,T+\delta)\to R$ be defined for $k=1,2,3,\dots$ by the equation

$$\begin{aligned} Q_i^{m,k}(x,s)-2^{-1}(h_i^m(x,0))^2+s\delta g_i^m(x) &= -\int_0^s a_i^{m,k}(t)v_i^m(x)\cdot\nabla(2^{-1}(h_{i,t}^m)^2 \\ &\quad + \sum_{j=1}^{2}\sum_{n=1}^{M} p[a_j^{n,k}(t)v_j^n, h_{j,t}^n])(x)\,dt, \end{aligned} \tag{3.32}$$

where the $a_i^{m,k}$ are the functions that were found in Lemma 3.1. Observe that the properties $|a_i^{m,k}|\leqq 1$, (3.18), (3.31), (3.15), (3.16) imply that this definition makes sense. Using Lemma 3.1, Lemma 2.1, (3.15), (3.16), (3.18) and (3.31) with

$$G_i^m(x,t)=v_i^m(x)\cdot\nabla(2^{-1}(h_{i,t}^m)^2)(x), \quad F_{i,j}^{m,n}(x,t,a)=v_i^m(x)\cdot\nabla(p[av_j^n,h_{j,t}^n])(x),$$

we discover that, when we restrict (x,s) to $P\times[0,T]$, we have

$$Q_1^{m,k}(x,s)-2^{-1}(h_1^m(x,0))^2+s\delta g_1^m(x) \text{ converges uniformly to } 0,$$

$$Q_2^{m,k}(x,s)-2^{-1}(h_2^m(x,0))^2+s\delta g_2^m(x) \text{ converges uniformly to}$$

$$-\int_0^s 2^{-1}v_2^m(x)\cdot\nabla(p[v_1^m,h_{1,t}^m]-p[0,h_{1,t}^m])(x)\,dt$$

as k goes to ∞. Combining this with (3.21), (3.22), we find

$$Q_i^{m,k}\to 2^{-1}(h_i^m)^2 \quad \text{uniformly on} \quad P\times[0,T] \quad \text{as} \quad k\to\infty. \tag{3.33}$$

Using (3.33), (3.18), (3.16), we find $\varepsilon>0$ and an integer k_0 satisfying

$$Q_i^{m,k}(x,s)\geqq 2^{-1}(|v_i^m(x)|+\varepsilon)^2 \quad \text{if} \quad k\geqq k_0, \quad x\in\operatorname{spt}(g_i^m), \quad s\in[0,T]. \tag{3.34}$$

From (3.32), (3.21), (3.22) we conclude

$$Q_i^{m,k}(x,s)=2^{-1}(h_i^m(x,s))^2 \quad \text{if} \quad x\notin\operatorname{spt}(v_i^m). \tag{3.35}$$

We pause for an informal discussion. We will have to deal with the square root of $2Q_i^{m,k}$ for $k\geqq k_0$. This square root will be called $q_i^{m,k}$. In order to verify that $q_i^{m,k}$ is defined and that it satisfies the right properties, we will use (3.19) to write P in the form

$$P = (P \sim \operatorname{spt}(v_i^m)) \cup \{x: g_i^m(x) > 0\} \equiv V_1 \cup V_2,$$

and note the following: If $x \in P \sim \operatorname{spt}(v_i^m)$, then (3.35), (3.18) imply $q_i^{m,k}(x,t) = h_i^m(x,t)$. If $x \in \operatorname{spt}(g_i^m)$, then (3.34) implies $q_i^{m,k}(x,t) \geqq |v_i^m(x)| + \varepsilon$ for $t \in [0,T]$ and $k \geqq k_0$. In the first case, properties of $q_i^{m,k}$ will follow from corresponding properties of h_i^m. In the second case, we can use the boundedness of $q_i^{m,k}$ away from zero (and away from $|v_i^m|$), the compactness of $\operatorname{spt}(g_i^m)$ and properties of $Q_i^{m,k}$ to obtain results for $q_i^{m,k}$. This type of argument will be used repeatedly. It will be called the $V_1 - V_2$ argument.

The precise statement of all this is the following: There exist $\delta_k > 0$ and $q_i^{m,k}$ for $k \geqq k_0$ satisfying (3.36)–(3.39):

$$q_i^{m,k}: P \times (-\delta_k, T + \delta_k) \to R \quad \text{is a } C^\infty \text{ function,} \tag{3.36}$$

$$q_i^{m,k} \geqq 0, \quad Q_i^{m,k}(x,s) = 2^{-1}(q_i^{m,k}(x,s))^2 \quad \text{if} \quad s \in (-\delta_k, T + \delta_k), \tag{3.37}$$

$$q_i^{m,k}(x,s) = h_i^m(x,s) \quad \text{if} \quad x \notin \operatorname{spt}(v_i^m) \quad \text{and} \quad s \in (-\delta_k, T + \delta_k), \tag{3.38}$$

$$q_i^{m,k}(x,s) \geqq |v_i^m(x)| + \varepsilon \quad \text{if} \quad x \in \operatorname{spt}(g_i^m) \quad \text{and} \quad s \in (-\delta_k, T + \delta_k). \tag{3.39}$$

Using (3.17), (3.31), (3.38) we get

$$q_i^{m,k}(x,s) = 0 \quad \text{if} \quad x \notin C_i^m. \tag{3.40}$$

The $V_1 - V_2$ argument and (3.33) yield

$$q_i^{m,k} \to h_i^m \quad \text{uniformly on} \quad P \times [0,T] \quad \text{as} \quad k \to \infty. \tag{3.41}$$

The $V_1 - V_2$ argument and (3.32), (3.31), (3.15), (3.16), (3.18) yield $N < \infty$ satisfying

$$\left|\frac{\partial}{\partial x_j} q_i^{m,k}(x,s)\right| \leqq N, \quad \left|\frac{\partial^2}{\partial x_j \partial x_n} q_i^{m,k}(x,s)\right| \leqq N,$$

$$\left|\frac{\partial^3}{\partial x_j \partial x_n \partial x_r} q_i^{m,k}(x,s)\right| \leqq N \quad \text{if} \quad s \in [0,T]. \tag{3.42}$$

The mean value theorem and (3.41), (3.42), (3.15)–(3.17) give us

$$\frac{\partial}{\partial x_j} q_i^{m,k} \to \frac{\partial}{\partial x_j} h_i^m \quad \text{and} \quad \frac{\partial^2}{\partial x_j \partial x_n} q_i^{m,k} \to \frac{\partial^2}{\partial x_j \partial x_n} h_i^m$$

$$\text{uniformly on} \quad P \times [0,T] \quad \text{as} \quad k \to \infty. \tag{3.43}$$

From (3.32) and (3.37) we get

$$\frac{\partial}{\partial t} 2^{-1}(q_i^{m,k}(x,t))^2 = -\delta g_i^m(x) - a_i^{m,k}(t) v_i^m(x) \cdot \nabla(2^{-1}(h_{i,t}^m)^2$$

$$+ \sum_{j=1}^{2} \sum_{n=1}^{M} p[a_j^{n,k}(t) v_j^n, h_{j,t}^n])(x) \tag{3.44}$$

for $-\delta_k < t < T + \delta_k$. Using (3.36), (3.37), (3.40), (3.15), (3.16), (3.39), (3.19) and

$|a_i^{m,k}| \leqq 1$, we conclude

$$p^*[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}], p[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}], u[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}] \quad \text{make sense.} \tag{3.45}$$

Now we will show the existence of $k_1 \geqq k_0$ satisfying

$$\begin{aligned}\frac{\partial}{\partial t}2^{-1}(q_i^{m,k}(x,t))^2 \leqq &-(\delta/2)g_i^m(x) - a_i^{m,k}(t)v_i^m(x)\cdot\nabla(2^{-1}(q_{i,t}^{m,k})^2 \\ &+ \sum_{j=1}^{2}\sum_{n=1}^{M} p[a_j^{n,k}(t)v_j^n, q_{j,t}^{n,k}])(x) \\ &\text{if} \quad k \geqq k_1 \quad \text{and} \quad (x,t)\in P\times[0,T].\end{aligned} \tag{3.46}$$

If $x \notin \operatorname{spt}(v_i^m)$, then the long terms of (3.44) and (3.46) are zero. Thus, (3.46) follows in this case from (3.44), (3.45) and $(\delta/2)g_i^m \geqq 0$ (see (3.14), (3.19)). If we restrict our attention to $x \in \operatorname{spt}(v_i^m)$ then (3.45), $|a_i^{m,k}| \leqq 1$, (3.41), (3.43), (3.38), (3.39), (3.16), (3.19) gives us $k_1 \geqq k_0$ satisfying

$$\begin{aligned}&\Big| a_i^{m,k}(t)v_i^m(x)\cdot\nabla(2^{-1}(q_{i,t}^{m,k})^2 + \sum_{j=1}^{2}\sum_{n=1}^{M} p[a_j^{n,k}(t)v_j^n, q_{j,t}^{n,k}])(x) \\ &\qquad - a_i^{m,k}(t)v_i^m(x)\cdot\nabla(2^{-1}(h_{i,t}^m)^2 + \sum_{j=1}^{2}\sum_{n=1}^{M} p[a_j^{n,k}(t)v_j^n, h_{j,t}^n])(x)\Big| \\ &\leqq \delta/2 = (\delta/2)g_i^m(x)\end{aligned}$$

if $k \geqq k_1$, $x \in \operatorname{spt}(v_i^m)$ and $t \in [0,T]$. Consequently, (3.46) follows from (3.44) in this case also. We use (3.45) to define

$$\begin{aligned}u^k(x,t) &= \sum_{i=1}^{2}\sum_{m=1}^{M} u[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}](x), \\ p^k(x,t) &= \sum_{i=1}^{2}\sum_{m=1}^{M} p^*[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}](x) \quad \text{for } k \geqq k_1 \quad \text{and} \quad x \in R^3.\end{aligned} \tag{3.47}$$

Using (3.47), (3.40), (3.15) and Lemma 2.2, we get

$$p^k(x,t) = \int_{R^3} \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_j^k}{\partial x_i}(y,t)\frac{\partial u_i^k}{\partial x_j}(y,t)(4\pi|x-y|)^{-1}dy. \tag{3.48}$$

The general identity $|u[v,f](x_1,x_2,0)| = f(x_1,x_2)$, which is valid whenever $(x_1,x_2)\in P$ and $u[v,f]$ makes sense, and properties (3.47), (3.40), (3.15), (3.31) imply

$$|u^k(x_1,x_2,0,t)|^2 = \sum_{i=1}^{2}\sum_{m=1}^{M} (q_i^{m,k}(x_1,x_2,t))^2,$$

$$v_i^m(x_1,x_2)\cdot\nabla(2^{-1}(q_{i,t}^{m,k})^2)(x_1,x_2) = \sum_{r=1}^{2} v_{ir}^m(x_1,x_2)\frac{\partial}{\partial x_r}(2^{-1}|u^k|^2)(x_1,x_2,0,t)$$

if $k \geqq k_1$ and $(x_1,x_2)\in P$. If we also have $t\in[0,T]$, then we may apply (3.46), (3.47) and $(\partial/\partial x_3)(2^{-1}|u^k|^2 + p^k)(x_1,x_2,0,t) = 0$ (which follows from the rotational symmetry of $|u^k|^2$ and p^k about the x_1 axis) to obtain the inequality

$$\frac{\partial}{\partial t}2^{-1}|u^k(x_1,x_2,0,t)|^2 = \sum_{i=1}^{2}\sum_{m=1}^{M}\frac{\partial}{\partial t}2^{-1}(q_i^{m,k}(x_1,x_2,t))^2$$

$$\leqq -(\delta/2)\left(\sum_{i=1}^{2}\sum_{m=1}^{M} g_i^m(x_1,x_2)\right)$$

$$-\sum_{i=1}^{2}\sum_{m=1}^{M}\sum_{r=1}^{2} a_i^{m,k}(t)v_{ir}^m(x_1,x_2)\frac{\partial}{\partial x_r}(2^{-1}|u^k|^2+p^k)(x_1,x_2,0,t)$$

$$= -(\delta/2)\left(\sum_{i=1}^{2}\sum_{m=1}^{M} g_i^m(x_1,x_2)\right)$$

$$-u^k(x_1,x_2,0,t)\cdot\nabla(2^{-1}|u^k|^2+p^k)(x_1,x_2,0,t),$$

where the gradient in the last line is taken with respect to all three space variables. Rotational symmetry yields

$$\frac{\partial}{\partial t}2^{-1}|u^k(x,t)|^2 \leqq -(\delta/2)\left(\sum_{i=1}^{2}\sum_{m=1}^{M} g_i^m(x_1,(x_2^2+x_3^2)^{1/2})\right)$$

$$-u^k(x,t)\cdot\nabla(2^{-1}|u^k|^2+p^k)(x,t)$$

$$\text{if}\quad k\geqq k_1,\quad x=(x_1,x_2,x_3),\quad (x_2,x_3)\neq(0,0)\quad\text{and}\quad t\in[0,T]. \tag{3.49}$$

For $k\geqq k_1$ and $x\in R^3$ we will use the notation

$$f_{i,t}^{m,k}(x)\equiv u[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}](x)\cdot\Delta(u[a_i^{m,k}(t)v_i^m, q_{i,t}^{m,k}])(x). \tag{3.50}$$

Using $|a_i^{m,k}|\leqq 1$, (3.24), (3.41), (3.43), (3.38), (3.39), (3.31), (3.15), (3.16) we can prove the existence of an integer $k_2\geqq k_1$ satisfying

$$\nu f_{i,t}^{m,k}(x)\geqq -\delta/2\quad\text{if}\quad t\in[0,T]\quad\text{and}\quad k\geqq k_2. \tag{3.51}$$

If $(x_1,x_2)\in P$ and $g_i^m(x_1,x_2)<1$ then the consequence $(x_1,x_2)\notin\operatorname{spt}(v_i^m)$ (see (3.19)) and (3.50), (3.38), (3.18), (3.23), (3.14) yield

$$\nu f_{i,t}^{m,k}(x_1,x_2,0)=\nu u[0,h_{i,t}^m](x_1,x_2,0)\cdot\Delta(u[0,h_{i,t}^m])(x_1,x_2,0)$$

$$=\nu h_{i,t}^m(x_1,x_2)L(h_{i,t}^m)(x_1,x_2)\geqq 0.$$

Combining this with the rotational symmetry of $f_{i,t}^{m,k}$ about the x_1 axis and (3.14), (3.19) we obtain

$$\nu f_{i,t}^{m,k}(x_1,x_2,x_3)\geqq 0\geqq -(\delta/2)g_i^m(x_1,(x_2^2+x_3^2)^{1/2})$$

$$\text{if}\quad (x_2,x_3)\neq(0,0)\quad\text{and}\quad g_i^m(x_1,(x_2^2+x_3^2)^{1/2})<1. \tag{3.52}$$

Now (3.51), (3.52), (3.19) give us

$$\nu f_{i,t}^{m,k}(x_1,x_2,x_3)\geqq -(\delta/2)g_i^m(x_1,(x_2^2+x_3^2)^{1/2})$$

$$\text{if}\quad k\geqq k_2,(x_2,x_3)\neq(0,0)\quad\text{and}\quad t\in[0,T]. \tag{3.53}$$

Using (3.53), (3.50), (3.47), (3.40), (3.15) we find

$$\nu u^k(x,t)\cdot\Delta u^k(x,t) \geqq -(\delta/2)\left(\sum_{i=1}^{2}\sum_{m=1}^{M} g_i^m(x_1,(x_2^2+x_3^2)^{1/2})\right)$$

$$\text{if} \quad k \geqq k_2, \quad (x_2,x_3) \neq (0,0) \quad \text{and} \quad t \in [0,T]. \tag{3.54}$$

Finally, we show that the conclusions of Lemma 3.2 are satisfied when $\eta = \delta_k, u = u^k$, $p = p^k$ and k is sufficiently large. We have that (3.40), (3.47) imply (3.25); (3.20) implies (3.26); (3.48) implies (3.27); (3.49), (3.54) and $u(x_1,0,0,t)=0$ imply (3.28); (3.15), (3.40), (3.37), (3.32), (3.18) imply (3.29); (3.15), (3.40), (3.41) imply (3.30).

Section 4. The Geometric Building Blocks

To begin with, we use Lemma 6.2 of [4]. This lemma says that we can construct U, K, f, z, F, A, B, C, D such that (4.1)–(4.10) are satisfied:

$$U = \{(x_1,x_2) \in R^2 : |x_1| < 1, 1/8 < x_2 < 7/8\}, \tag{4.1}$$

$$f \in C_c^\infty(P,R), \quad z = (z_1,z_2) \in C_c^\infty(P,R^2), \quad F = (F_1,F_2) \in C^\infty(R^2,R^2), \tag{4.2}$$

$$K \text{ is compact, } \operatorname{spt}(z) \subset K \subset U, \tag{4.3}$$

$$f \geqq 0, \quad f(x) = 0 \quad \text{if} \quad x \notin U, \quad f(x) > |z(x)| \quad \text{if} \quad x \in U, \tag{4.4}$$

$$L(f)(x) \geqq 0 \quad \text{if} \quad x \notin K, \quad L(f)(x) > 0 \quad \text{if} \quad x \in U \sim K, \tag{4.5}$$

$$x_2\frac{\partial z_1}{\partial x_1}(x_1,x_2) + x_2\frac{\partial z_2}{\partial x_2}(x_1,x_2) + z_2(x_1,x_2) = 0. \tag{4.6}$$

$$\nabla(p[0,f] - p[z,f])(x) = F(x) \quad \text{if} \quad x \in P, \tag{4.7}$$

$$A, B, C, D \text{ are real numbers, } B > 0, \quad C > 0, \quad D > 0, \tag{4.8}$$

$$F_1(A,0) = B, \quad \text{if} \quad x \in R, \quad \text{then} \quad |F_1(x,0)| \leqq B \quad \text{and} \quad F_2(x,0) = 0, \tag{4.9}$$

$$\lim_{x\to\infty} x^4 F_1(x,0) = D, \quad \text{if} \quad x \in R^2, \quad \text{then } |F(x)| \leqq C|x|^{-4} \quad \text{and}$$
$$|\nabla F(x)| \leqq C|x|^{-5}. \tag{4.10}$$

We will use the following definitions:

$$\begin{aligned}
F^{\alpha,\rho,\sigma}(x_1,x_2) &= (\sigma^2/\rho)F((x_1-\alpha)/\rho, x_2/\rho),\\
f^{\alpha,\rho,\sigma}(x_1,x_2) &= \sigma f((x_1-\alpha)/\rho, x_2/\rho),\\
z^{\alpha,\rho,\sigma}(x_1,x_2) &= \sigma z((x_1-\alpha)/\rho, x_2/\rho),\\
U^{\alpha,\rho} &= \{x \in R^2 : ((x_1-\alpha)/\rho, x_2/\rho) \in U\},\\
K^{\alpha,\rho} &= \{x \in R^2 : ((x_1-\alpha)/\rho, x_2/\rho) \in K\}
\end{aligned} \tag{4.11}$$

if $\alpha \in R$, $\rho > 0$, $\sigma > 0$. With the aid of (4.4), (4.7), we conclude:

$$\operatorname{spt}(f) = \operatorname{closure}(U), \quad \operatorname{spt}(f^{\alpha,\rho,\sigma}) = \operatorname{closure}(U^{\alpha,\rho}), \tag{4.12}$$

$$\nabla(p[0,f^{\alpha,\rho,\sigma}] - p[z^{\alpha,\rho,\sigma}, f^{\alpha,\rho,\sigma}])(x) = F^{\alpha,\rho,\sigma}(x) \quad \text{if} \quad x \in P. \tag{4.13}$$

Properties (4.2), (4.8), (4.9), (4.10) say that all of the hypotheses of Sect. 4 of [4] are satisfied. In what follows, we will use some results of that section (where, of course, the definition of $f^{\alpha,\rho,\sigma}$ is the same as in (4.11)).

Lemma. 4.1. *There exist real numbers $a', a'', r', r'', s', s''$, E such that conditions (4.14)–(4.17) hold:*

$$0 < E < r''/8, \quad r'' < r'/8, \quad r' < 1/8, \quad a' = A - r'A, \quad a'' = A - r''A, \tag{4.14}$$

if $x_1 \in R$ and $|x_2| \leqq E$, then

$$(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1, x_2) > -(1.01)B, \tag{4.15}$$

if $|x_1 - A| \leqq (10^4 C/D)E$ and $|x_2| \leqq E$, then

$$(F_1^{a',r',s'} + F_1^{a'',r'',s''} + F_1)(x_1, x_2) \geqq (6.99)B, \tag{4.16}$$

if $x \in R^2$ and $|x| > 2|A|$, then

$$|(F^{a',r',s'} + F^{a'',r'',s''} + F)(x)| \leqq 2C|x|^{-4}. \tag{4.17}$$

Proof. This is the initial step of the construction carried out in Sect. 4 of [4]. We use (4.8), (4.10), (4.11), (4.12), (4.13) and Lemma 4.4 of that paper.

Lemma 4.2. *There exists $M < \infty$ with the following property: If $m \geqq M$, $|x_1 - m| \leqq 10^4 C/D$ and $|x_2| \leqq 1$, then $|F_1(x_1, x_2) - m^{-4}D| \leqq 10^{-3} m^{-4} D$.*

Proof. This is done in (4.14)–(4.17) of [4].

Now we introduce a constant X that was not used in [4]. Let X satisfy

$$X > 2, \quad X > 4|A|, \quad X > 4(10^4 C/D)E, \quad 2CX^{-4}\left(\sum_{n=-\infty}^{\infty} ||n| - 1/2|^{-4}\right) < (.005)B. \tag{4.18}$$

In (4.18)–(4.21) of [4] we choose ε small enough to satisfy

$$\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E/10 > 2|A| + 1, \quad 2C(\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E/10)^{-4} < 10^{-3}\varepsilon^2 B, \tag{4.19}$$

$$0 < \varepsilon < .01, \quad \varepsilon^{-1} 10^3 C/D > M, \quad \varepsilon^{-1} 10^2 C/D > 10^4 C/D, \quad (.99)(1+\varepsilon^2)^{1/2} < 1, \tag{4.20}$$

and we defined a, r, s, d as follows:

$$r = \varepsilon^{-1}(1+\varepsilon^2)^{1/2} E, \quad a = -r(\varepsilon^{-1} 10^3 C/D),$$
$$d = r(10^4 C/D), \quad s = [((1.02)B/(.999))(\varepsilon^{-1} 10^3 C/D)^4 (r/D)]^{1/2}. \tag{4.21}$$

At this point, we come to a sharpening of the construction of [4]. Recalling that $\zeta < 1$ is the given number of Sect. 1, we see that ε can be chosen so that (4.22)–(4.24) are also satisfied:

$$\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E(10^3 C/D)/(4X) \text{ is a positive integer,} \tag{4.22}$$

$$((.48)\varepsilon)^{\zeta}(\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E(10^3 C/D)/(4X) + 1) > 1, \tag{4.23}$$

$$20C(\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E(10^3 C/D)/(4X) + 1)(\varepsilon^{-1}(1+\varepsilon^2)^{1/2} E/10)^{-4} \leqq 10^{-3}\varepsilon^2 B. \tag{4.24}$$

Lemma. 4.3. *If* $-d \leqq x_1 \leqq d$ *and* $0 \leqq x_2 \leqq r$, *then*

$$|F_2^{a,r,s}(x_1,x_2)| \leqq (.002)B\varepsilon \quad \text{and} \quad (1.02)B \leqq F_1^{a,r,s}(x_1,x_2) \leqq (1.03)B.$$

Proof. This is Lemma 4.5 of [4].

Lemma 4.4. *There exists* $v_2 = (v_{21}, v_{22}) \in C_c^\infty(P, R^2)$ *that satisfies the six properties listed below:*

$$\operatorname{spt}(v_2) \subset (-d, d) \times (10^{-3}\varepsilon r, r), \tag{4.25}$$

$$\operatorname{spt}(v_2) \quad \text{and} \quad [r-d, d-r] \times [E, r/10] \quad \text{are disjoint}, \tag{4.26}$$

$$x_2 \frac{\partial}{\partial x_1} v_{21}(x_1,x_2) + x_2 \frac{\partial}{\partial x_2} v_{22}(x_1,x_2) + v_{22}(x_1,x_2) = 0, \tag{4.27}$$

$$|v_2(x)| \leqq 1, \quad v_{21}(x) \geqq -\varepsilon^2, \quad |v_{22}(x)| \leqq \varepsilon/2, \tag{4.28}$$

$$\text{if } |x_1| \leqq d-r \quad \text{and} \quad (.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r, \quad \text{then} \quad v_2(x_1,x_2) = (1,0), \tag{4.29}$$

$$\text{if } |x_1| \leqq d-r \quad \text{and } 0 < x_2 < E, \quad \text{then } v_{21}(x_1,x_2) \geqq 0$$
$$\text{and } v_{22}(x_1,x_2) = 0. \tag{4.30}$$

Proof. This is Lemma 4.8 and (4.40)–(4.43) of [4], where v_2 is called w. Note that the (v_1, v_2) in (4.40) of [4] has nothing to do with the v_2 here.

Now we make the definition

$$\tau = (.48)\varepsilon, \quad Y = r(10^3 C/D)/(4X). \tag{4.31}$$

From (4.19)–(4.24) and (4.31) we obtain

$$YX = d/40, \quad Y \text{ is a positive integer}, \quad \tau^\zeta(Y+1) > 1, \quad 0 < \tau < 1, \tag{4.32}$$

$$r/10 > 2|A| + 1, \quad 20C(Y+1)(r/10)^{-4} \leqq 10^{-3}\varepsilon^2 B. \tag{4.33}$$

Using (4.20), (4.21), (4.31) we find

$$(.99)r\varepsilon < E < r/100, \quad \text{hence} \quad \tau d \leqq (10^4 C/D)E. \tag{4.34}$$

Also, (4.8), (4.10) and (4.21) yield

$$C/D \geqq 1, \quad \text{hence} \quad d \geqq 10^4 r. \tag{4.35}$$

Lemma 4.5. *The* $2(Y+1)$ *closed intervals*

$$[\tau(a-r) + A + nX, \quad \tau(a+r) + A + nX], \quad n = 0, 1, \ldots, Y \quad \text{and} \tag{4.36}$$

$$[-\tau d + A + n'X, \quad \tau d + A + n'X], \quad n' = 0, 1, \ldots, Y \tag{4.37}$$

are pairwise disjoint. Furthermore, all of these intervals are contained in the open interval $(r-d, d-r)$.

Proof. All we have to show is

$$2\tau r < X, \quad 2\tau d < X, \tag{4.38}$$

$$r - d < \tau(a-r) + A, \tag{4.39}$$

$$\tau(a+r)+A+YX< -\tau d+A, \tag{4.40}$$

$$\tau d+A+YX<d-r. \tag{4.41}$$

Inequality (4.40) says that the rightmost interval in (4.36) is to the left of the leftmost interval in (4.37). From (4.31), (4.21), (4.20), (4.18) we get

$$2\tau d=2(.48)\varepsilon(\varepsilon^{-1}(1+\varepsilon^2)^{1/2}E)(10^4C/D)<(10^4C/D)E<X.$$

This fact and (4.35) imply (4.38). Now (4.33), (4.31), (4.20), (4.35), (4.21) imply

$$\begin{aligned}-A+(1+\tau)r-d&<r/20+r+(.48)r/100-d< -(.999)d\\ &= -(.999)r(10^4C/D)<(.48)\varepsilon(-r(\varepsilon^{-1}10^3C/D))=\tau a,\end{aligned}$$

which gives (4.39). Using (4.32), (4.21), (4.31), (4.20), (4.35) we get

$$YX=(1/4)r(10^3C/D)=\tau r((1.92)^{-1}\varepsilon^{-1}10^3C/D)<\tau r(\varepsilon^{-1}10^3C/D-1-10^4C/D).$$

The above and (4.21) imply (4.40). Finally, (4.32), (4.35), (4.20), (4.31), (4.33) yield

$$YX=d/40<d-r-r/10-(.48)\varepsilon d<d-r-A-\tau d,$$

which gives (4.41).

Lemma 4.6. *The* $3(Y+1)$ *closed rectangles*

$$[-1+nX,\quad 1+nX]\times[1/8,7/8], \tag{4.42}$$

$$[a'-r'+n'X,\quad a'+r'+n'X]\times[r'/8,7r'/8], \tag{4.43}$$

$$[a''-r''+n''X,\quad a''+r''+n''X]\times[r''/8,7r''/8], \tag{4.44}$$

where $n,n',n''\in\{0,1,\dots,Y\}$, *are disjoint. In addition, all of these rectangles are contained in the open rectangle* $(r-d,d-r)\times(E,r/10)$. *Finally, the intervals* $[a-r,a+r]$ *and* $[-d,d]$ *are disjoint.*

Proof. From $2r''<2r'<2<X$ (see (4.14), (4.18)) we conclude that *each* of (4.42)–(4.44) consists of $Y+1$ disjoint rectangles. The inequalities $7r''/8<r'/8$, $7r'/8<1/8$ (see (4.14)) imply that rectangles from different lines in (4.42)–(4.44) are disjoint. In order to prove the second assertion of the lemma, we observe that (4.14), (4.33), (4.35) imply

$$\begin{gathered}|a'|+r'<|A|+1,\quad |a''|+r''<|A|+1,\\ 1+d/40\leqq|A|+1+d/40<r/10+d/40<d-r.\end{gathered}$$

These facts and (4.32) imply

$$1+YX<d-r,\quad |a'|+r'+YX<d-r,\quad |a''|+r''+YX<d-r.$$

Therefore, the projections of the rectangles in (4.42)–(4.44) on the x_1 axis are contained in $(r-d,d-r)$. The second assertion follows from this fact, (4.14) and (4.33). Since (4.20), (4.21), (4.35) imply

$$a+r= -r(\varepsilon^{-1}10^3C/D)+r< -10r(10^4C/D)+r= -10d+r< -d,$$

we obtain the third assertion of the lemma.

We set

$$F^* = F^{a',r',s'} + F^{a'',r'',s''} + F. \tag{4.45}$$

Lemma 4.7. *If $(x_1, x_2) \in R^2$ and $j \in \{0, 1, \ldots, Y\}$ satisfies the condition*

$$|x_1 - jX| = \min\{|x_1 - nX| : n = 0, 1, \ldots, Y\}, \tag{4.46}$$

then we have $\sum_{\substack{n=0\\ n \neq j}}^{Y} |F^*(x_1 - nX, x_2)| \leqq (.005)B.$

Proof. If $n \neq j$, then $|x_1 - nX| \geqq (|n - j| - 1/2)X$. The conclusion follows from (4.45), (4.17) and the second and fourth inequalities of (4.18).

Lemma 4.8. *If $(x_1, x_2) \in R^2$ and $|x_2| \leqq E$, then* $\sum_{n=0}^{Y} F_1^*(x_1 - nX, x_2) > -(1.015)B.$

Proof. Let $j \in \{0, 1, \ldots, Y\}$ satisfy (4.46). Lemma 4.7, (4.45) and (4.15) give us the conclusion.

Lemma 4.9. *If $(x_1, x_2) \in R^2$, $j \in \{0, 1, \ldots, Y\}$, $|x_1 - jX - A| \leqq (10^4 C/D)E$ and $|x_2| \leqq E$, then* $\sum_{n=0}^{Y} F_1^*(x_1 - nX, x_2) \geqq (6.985)B.$

Proof. From (4.18) we obtain $|x_1 - jX| \leqq |x_1 - jX - A| + |A| \leqq (10^4 C/D)E + X/4 < X/4 + X/4 = X/2$, which implies (4.46). Now Lemma 4.7, (4.45) and (4.16) yield the desired conclusion.

Lemma 4.10. *If $(x_1, x_2) \in R^2$ and $|(x_1 - nX, x_2)| > r/10$ holds for all $n \in \{0, 1, \ldots, Y\}$, then* $\sum_{n=0}^{Y} |F^*(x_1 - nX, x_2)| \leqq 10^{-3} \varepsilon^2 B.$

Proof. Since (4.33) implies $|(x_1 - nX, x_2)| > r/10 > 2|A|$, we can use (4.17), (4.45), and the second part of (4.33) to conclude

$$\sum_{n=0}^{Y} |F^*(x_1 - nX, x_2)| \leqq 2C \sum_{n=0}^{Y} |(x_1 - nX, x_2)|^{-4} \leqq 2C \sum_{n=0}^{Y} (r/10)^{-4} \leqq 10^{-3} \varepsilon^2 B.$$

Lemma 4.11. *If $h: P \to R$ is defined by*

$$h(x_1, x_2) = v_2(x_1, x_2) \cdot \left(F^{a,r,s}(x_1, x_2) + \sum_{n=0}^{Y} F^*(x_1 - nX, x_2)\right),$$

where v_2 is the function in Lemma 4.4, then

(a) *if $j \in \{0, 1, \ldots, Y\}$, $|x_1 - jX - A| \leqq \tau d$ and $(.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r$, then $h(x_1, x_2) \geqq (8.005)B$,*
(b) *if $|x_1| \leqq d - r$ and $(.02)\varepsilon r \leqq x_2 \leqq (.98)\varepsilon r$, then $h(x_1, x_2) \geqq (.005)B$,*
(c) *if $x \in P$ then $h(x) \geqq -\varepsilon^2(1.032)B$.*

Proof. Using (4.34) we find that the hypotheses of (a) imply

$$|x_1 - jX - A| \leqq \tau d \leqq (10^4 C/D)E, \quad 0 < x_2 < E < r.$$

Also, the last part of Lemma 4.5 gives us $x_1 \in (r-d, d-r)$. All this, Lemma 4.3 and Lemma 4.9 give us

$$F_1^{a,r,s}(x_1,x_2) + \sum_{n=0}^{Y} F_1^*(x_1 - nX, x_2) \geqq (1.02)B + (6.985)B = (8.005)B.$$

The conclusion of (a) follows from the above and (4.29).

Suppose that the hypotheses of (b) hold. As in (a), we get $0 < x_2 < E < r$. Now Lemma 4.3 and Lemma 4.8 yield

$$F_1^{a,r,s}(x_1,x_2) + \sum_{n=0}^{Y} F_1^*(x_1 - nX, x_2) \geqq (1.02)B - (1.015)B = (.005)B.$$

As before, the conclusion of (b) follows from (4.29).

Now we prove (c). Since B, ε are positive (see (4.8), (4.20)), we may assume $(x_1,x_2) \in \operatorname{spt}(v_2)$. We will distinguish two cases:

Case I. $|(x_1 - jX, x_2)| > r/10$ for all $j \in \{0, 1, \ldots, Y\}$.
In this case, we use (4.25), (4.28), Lemma 4.3, Lemma 4.10 and the positivity of B, ε to conclude

$$\begin{aligned} h(x_1,x_2) &= v_{21}(x_1,x_2) F_1^{a,r,s}(x_1,x_2) + v_{22}(x_1,x_2) F_2^{a,r,s}(x_1,x_2) \\ &\quad + v_2(x_1,x_2) \cdot \left(\sum_{n=0}^{Y} F^*(x_1 - nX, x_2) \right) \\ &\geqq -\varepsilon^2(1.03)B - (\varepsilon/2)(.002)B\varepsilon - 10^{-3}\varepsilon^2 B = -\varepsilon^2(1.032)B. \end{aligned}$$

Case II. $|(x_1 - jX, x_2)| \leqq r/10$ for some $j \in \{0, 1, \ldots, Y\}$.
From (4.32), (4.35) we get

$$|x_1| \leqq |(x_1,x_2)| \leqq |jX| + r/10 \leqq YX + r/10 = d/40 + r/10 < d - r.$$

Now, the assumption $(x_1,x_2) \in \operatorname{spt}(v_2)$ and (4.26) imply $x_2 \notin [E, r/10]$. From this fact and $|x_2| \leqq |(x_1 - jX, x_2)| \leqq r/10$ we get $x_2 < E$. The assumption $(x_1,x_2) \in P$ gives $x_2 > 0$. All this and (4.30) imply $v_{21}(x_1,x_2) \geqq 0$, $v_{22}(x_1,x_2) = 0$. The above, (4.8), (4.25), Lemma 4.3 and Lemma 4.8 give us

$$\begin{aligned} h(x_1,x_2) &= v_{21}(x_1,x_2) \left(F_1^{a,r,s}(x_1,x_2) + \sum_{n=0}^{Y} F_1^*(x_1 - nX, x_2) \right) \\ &\geqq v_{21}(x_1,x_2)((1.02)B - (1.015)B) \geqq 0 > -\varepsilon^2(1.032)B. \end{aligned}$$

This concludes the proof of Lemma 4.11.

We recall (4.11) and set

$$f_1 = f^{a,r,s} + \sum_{n=0}^{Y} (f^{a'+nX,r',s'} + f^{a''+nX,r'',s''} + f^{nX,1,1}),$$

$$v_1 = z^{a,r,s} + \sum_{n=0}^{Y} (z^{a'+nX,r',s'} + z^{a''+nX,r'',s''} + z^{nX,1,1}),$$

$$U_1 = U^{a,r} \cup \bigcup_{n=0}^{Y} (U^{a'+nX,r'} \cup U^{a''+nX,r''} \cup U^{nX,1}),$$

$$K_1 = K^{a,r} \cup \bigcup_{n=0}^{Y} (K^{a'+nX,r'} \cup K^{a''+nX,r''} \cup K^{nX,1}). \tag{4.47}$$

Lemma 4.12. *The sets* $U^{a,r}, U^{a'+n'X,r'}, U^{a''+n''X,r''}, U^{nX,1}$ *(where* n, n', n'' *range over* $\{0, 1, \ldots, Y\}$*) have disjoint closures.*

Proof. This follows from (4.1), (4.11) and all three assertions of Lemma 4.6.

With the aid of Lemma 4.12, (4.12) and (4.1)–(4.6) we conclude:

$$K_1 \subset U_1 \subset P, K_1 \text{ is compact, } U_1 \text{ is open, closure } (U_1) \subset P, \tag{4.48}$$

$$f_1 \in C_c^\infty(P, R), \quad v_1 \in C_c^\infty(P, R^2), \quad \operatorname{spt}(v_1) \subset K_1, \tag{4.49}$$

$$f_1 \geqq 0, \quad f_1(x) = 0 \quad \text{if} \quad x \notin U_1, \quad f_1(x) > |v_1(x)| \quad \text{if} \quad x \in U_1, \tag{4.50}$$

$$x_2 \frac{\partial}{\partial x_1} v_{11}(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} v_{12}(x_1, x_2) + v_{12}(x_1, x_2) = 0, \tag{4.51}$$

$$L(f_1)(x) \geqq 0 \quad \text{if} \quad x \notin K_1, \quad L(f_1)(x) > 0 \quad \text{if} \quad x \in U_1 \sim K_1. \tag{4.52}$$

Lemma 4.13. *If* $(x_1, x_2) \in P$, *then*

$$\nabla(p[0, f_1] - p[v_1, f_1])(x_1, x_2) = F^{a,r,s}(x_1, x_2) + \sum_{n=0}^{Y} F^*(x_1 - nX, x_2).$$

Proof. This follows from (4.11)–(4.13), (4.45), (4.47), Lemma 2.2 and Lemma 4.12.

Now we adopt the construction in (6.42)–(6.45) of [4]. Recall that the function w of [4] is called v_2 here. This gives us U_2, K_2, k satisfying (4.53)–(4.56):

$$U_2 = ((-d, d) \times (10^{-3}\varepsilon r, r)) \sim ([r - d, d - r] \times [E, r/10]), \tag{4.53}$$

$$\operatorname{spt}(v_2) \subset K_2 \subset U_2, K_2 \quad \text{is compact,} \quad k \in C_c^\infty(P, R), \quad 0 \leqq k \leqq 1, \tag{4.54}$$

$$k(x) = 0 \quad \text{if} \quad x \notin U_2, k(x) > 0 \quad \text{if} \quad x \in U_2, k(x) = 1 \quad \text{if} \quad x \in \operatorname{spt}(v_2), \tag{4.55}$$

$$L(k)(x) \geqq 0 \quad \text{if} \quad x \notin K_2, L(k)(x) > 0 \quad \text{if} \quad x \in U_2 \sim K_2. \tag{4.56}$$

We choose a constant μ satisfying

$$\mu^2 > 10(\| v_2 \|_\infty^2 + 1), \quad \mu > 100 \| f_1 \|_\infty, \tag{4.57}$$

and we set

$$f_2 = \mu k, \quad T = (\mu^2 - \| v_2 \|_\infty^2 - 1)/(\varepsilon^2 (1.032) B). \tag{4.58}$$

Lemma 4.14. *We have the following four properties*:

$$\operatorname{spt}(f_1) \subset ([a - r, a + r] \times [r/8, 7r/8]) \cup ((r - d, d - r) \times (E, r/10)), \tag{4.59}$$

$$\operatorname{spt}(f_2) = ([-d, d] \times [10^{-3}\varepsilon r, r]) \sim ((r - d, d - r) \times (E, r/10)), \tag{4.60}$$

$$\operatorname{spt}(f_1) \text{ and } \operatorname{spt}(f_2) \text{ are disjoint,} \tag{4.61}$$

$$\| f_2 \|_\infty = \mu, \quad f_2(x) = \mu \quad \text{if} \quad x \in \operatorname{spt}(v_2). \tag{4.62}$$

Proof. Inclusion (4.59) follows from the second part of Lemma 4.6, (4.1), (4.12) and (4.47). Identity (4.60) is a consequence of (4.53), (4.55), (4.58). The last part of

Lemma 4.6 and (4.59), (4.60) give us (4.61). Finally, (4.54), (4.55), (4.58) yield (4.62).

Lemma 4.15. *There exists $\theta > 0$ such that (4.63), (4.64) hold:*

$$(\mu^2 + T(.005)B)^{1/2} - \theta > \tau^{-1} \cdot \| f_1 \|_\infty, \tag{4.63}$$

$$(\mu^2 + T(8.005)B)^{1/2} - \theta > \tau^{-1}(\| f_1 \|_\infty + \| f_2 \|_\infty). \tag{4.64}$$

Proof. Using (4.58), (4.57) we find

$$TB = (\mu^2 - \| v_2 \|_\infty^2 - 1)/(\varepsilon^2(1.032)) > (.9)\mu^2/(\varepsilon^2(1.032)).$$

This inequality and (4.31), (4.57), (4.62) yield

$$T(.005)B > 10^{-4}\mu^2/((.48)^2\varepsilon^2) > \tau^{-2} \| f_1 \|_\infty^2,$$
$$T(8.005)B > (1.01)^2\mu^2/((.48)^2\varepsilon^2) = \tau^{-2}(\mu/100 + \mu)^2 > \tau^{-2}(\| f_1 \|_\infty + \| f_2 \|_\infty)^2.$$

Lemma 4.16. *If $x \in U_2$, then*

$$(f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x) > |v_2(x)|^2.$$

Proof. If $x \notin \operatorname{spt}(v_2)$, then the inequality reduces to $(f_2(x))^2 > 0$, which is a consequence of $x \in U_2$ and (4.55), (4.57), (4.58). If $x \in \operatorname{spt}(v_2)$, then Lemma 4.13, (4.62), part (*c*) of Lemma 4.11 and (4.58) give us

$$\begin{aligned}
&(f_2(x))^2 - Tv_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x) \\
&= \mu^2 + Tv_2(x) \cdot \left(F^{a,r,s}(x) + \sum_{n=0}^{Y} F^*(x_1 - nX, x_2) \right) \\
&\geqq \mu^2 - T\varepsilon^2(1.032)B = \| v_2 \|_\infty^2 + 1 > |v_2(x)|^2.
\end{aligned}$$

Lemma 4.17. *If $(y_1, y_2) \in \operatorname{spt}(f_1) \cup \operatorname{spt}(f_2)$, $z = (z_1, z_2) \in P$, $z_1 = \tau y_1 + A + jX$ for some $j \in \{0, 1, \ldots, Y\}$, $(.02)\varepsilon r \leqq z_2 \leqq (.98)\varepsilon r$ and*

$$Q = (f_2(z))^2 - Tv_2(z) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(z),$$

then $Q > (\tau^{-1}(f_1 + f_2)(y_1, y_2) + \theta)^2$.

Proof. Parts (4.59), (4.60) of Lemma 4.14 and the last part of Lemma 4.5 give us

$$y_1 \in [a - r, a + r] \cup [-d, d], \quad \text{hence} \quad z_1 \in (r - d, d - r). \tag{4.65}$$

Hence (4.29), (4.62) yield

$$z \in (r - d, d - r) \times [(.02)\varepsilon r, \ (.98)\varepsilon r] \subset \operatorname{spt}(v_2), \quad \text{hence} \quad f_2(z) = \mu. \tag{4.66}$$

Now (4.66) and Lemma 4.13 imply

$$Q = \mu^2 + Tv_2(z_1, z_2) \cdot \left(F^{a,r,s}(z_1, z_2) + \sum_{n=0}^{Y} F^*(z_1 - nX, z_2) \right). \tag{4.67}$$

From (4.65) we get two cases: $y_1 \in [a - r, a + r]$ and $y_1 \in [-d, d]$. Using (4.67), (4.66), part (b) of Lemma 4.11 and (4.63), we get

$$Q \geqq \mu^2 + T(.005)B > (\tau^{-1} \| f_1 \|_\infty + \theta)^2 \geqq (\tau^{-1} f_1(y_1, y_2) + \theta)^2.$$

The conclusion follows in the case $y_1 \in [a - r, a + r]$ because (4.60) and the third part of Lemma 4.6 imply $(y_1, y_2) \notin \mathrm{spt}(f_2)$. If $y_1 \in [-d, d]$ then (4.67), the hypothesis $z_1 = \tau y_1 + A + jX$, part (a) of Lemma 4.11 and (4.64) yield

$$Q \geqq \mu^2 + T(8.005)B > (\tau^{-1}(\|f_1\|_\infty + \|f_2\|_\infty) + \theta)^2 \geqq (\tau^{-1}(f_1 + f_2)(y_1, y_2) + \theta)^2.$$

The conclusion also follows in this case.

Lemma 4.18. *If* $(x_1, x_2, x_3) \in R^3, (x_1, (x_2^2 + x_3^2)^{1/2}) \in \mathrm{spt}(f_1) \cup \mathrm{spt}(f_2), j \in \{0, 1, \ldots, Y\}$ *and*

$$z = (z_1, z_2) = (\tau x_1 + A + jX, ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2}),$$

then $(z_1, z_2) \in P$ (i.e. $z_2 > 0$) *and*

$$((f_2(z))^2 - T v_2(z) \cdot \nabla (p[v_1, f_1] - p[0, f_1])(z))^{1/2}$$
$$> \tau^{-1}(f_1 + f_2)(x_1, (x_2^2 + x_3^2)^{1/2}) + \theta.$$

Proof. In view of Lemma 4.17, all we have to show is $(.02)\varepsilon r \leqq z_2 \leqq (.98)\varepsilon r$. Since Lemma 4.14 implies $(x_2^2 + x_3^2)^{1/2} \leqq r$, we can use (4.31) to obtain that $(\tau x_2 + \varepsilon r/2, \tau x_3) = ((.48)\varepsilon x_2 + \varepsilon r/2, (.48)\varepsilon x_3)$ is contained in the circular disc with center $(\varepsilon r/2, 0)$ and radius $(.48)\varepsilon r$. This disc is contained in the annulus with center $(0, 0)$ and radii $\varepsilon r/2 - (.48)\varepsilon r = (.02)\varepsilon r$ and $\varepsilon r/2 + (.48)\varepsilon r = (.98)\varepsilon r$.

Section 5. The Cantor Set

We will use the construction in Sect. 4. In order to apply a result from ref. [4], we will first establish (5.1)–(5.7) for $i = 1, 2$. Using (4.48), (4.53), (4.54), $\varepsilon > 0$ and $r > 0$ (see (4.20), (4.33)), we get

$$K_i \subset U_i \subset P,\ K_i \text{ is compact},\ U_i \text{ is open, closure } (U_i) \subset P. \tag{5.1}$$

From (4.49), (4.54), (4.57), (4.58) and Lemma 4.4 we obtain

$$f_i \in C_c^\infty(P, R), \quad v_i = (v_{i1}, v_{i2}) \in C_c^\infty(P, R^2). \tag{5.2}$$

The identity $\mathrm{spt}(f_i) = \mathrm{closure}(U_i)$ follows from (4.50), (4.55), (4.57) and (4.58). Therefore, we can use (4.61), (5.2), (4.49), (4.54) to write

$$\mathrm{closure}(U_1) \text{ and } \mathrm{closure}(U_2) \text{ are disjoint compact sets, } \mathrm{spt}(v_i) \subset K_i. \tag{5.3}$$

If $x \in \mathrm{spt}(v_2)$, then (4.58), (4.55), (4.57) imply $f_2(x) = \mu k(x) = \mu > |v_2(x)|$. If $x \in U_2$ and $x \notin \mathrm{spt}(v_2)$, then (4.58), (4.57), (4.55) give us $f_2(x) > 0 = |v_2(x)|$. These computations and (4.50), (4.54), (4.55), (4.58) yield

$$f_i \geqq 0, \quad f_i(x) = 0 \quad \text{if} \quad x \notin U_i, \quad f_i(x) > |v_i(x)| \quad \text{if} \quad x \in U_i. \tag{5.4}$$

Identities (4.27) and (4.51) imply

$$x_2 \frac{\partial}{\partial x_1} v_{i1}(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} v_{i2}(x_1, x_2) + v_{i2}(x_1, x_2) = 0. \tag{5.5}$$

Finally, Lemma 4.16, (4.52) and (4.56)–(4.58) give

$$(f_2(x))^2 - T v_2(x) \cdot \nabla (p[v_1, f_1] - p[0, f_1])(x) > |v_2(x)|^2 \quad \text{if} \quad x \in U_2, \tag{5.6}$$

$$L(f_i)(x) \geqq 0 \quad \text{if} \quad x \notin K_i, \quad L(f_i)(x) > 0 \quad \text{if} \quad x \in U_i \sim K_i. \tag{5.7}$$

Lemma (4.15), (4.57), (4.58) and (4.8) give us $T > 0$ and $\theta > 0$. These facts and (5.1)–(5.7) are the hypotheses of Sect. 3 of [4]. Therefore, we may use Lemma 3.1 of [4] to obtain $\delta, g_1, g_2, h_1, h_2$ satisfying (5.8)–(5.15):

$$\delta > 0, \quad g_i \in C_c^\infty(P, R), \quad h_i : P \times (-\delta, T + \delta) \to R \text{ is a } C^\infty \text{ function,} \tag{5.8}$$

$$\operatorname{spt}(g_i) \subset U_i, \quad \operatorname{spt}(v_i) \subset \{x : g_i(x) = 1\}, \quad 0 \leqq g_i(x) \leqq 1 \quad \text{if} \quad x \in P, \tag{5.9}$$

$$h_i \geqq 0, \quad \text{if} \quad x \in \operatorname{spt}(g_i) \quad \text{then} \quad h_i(x,t) > |v_i(x)|, \tag{5.10}$$

$$h_i(x,t) = f_i(x) \quad \text{if} \quad x \notin \operatorname{spt}(g_i), \tag{5.11}$$

$$(h_1(x,s))^2 = (f_1(x))^2 - 2s\delta g_1(x), \tag{5.12}$$

$$(h_2(x,s))^2 = (f_2(x))^2 - 2s\delta g_2(x) - \int_0^s v_2(x) \cdot \nabla(p[v_1, h_{1,t}] - p[0, h_{1,t}])(x)\, dt,$$

where $h_{i,t}(x) = h_i(x,t)$, (5.13)

$$h_2(x,T) + \theta > ((f_2(x))^2 - T v_2(x) \cdot \nabla(p[v_1, f_1] - p[0, f_1])(x))^{1/2} \quad \text{if} \quad x \in P, \tag{5.14}$$

$$L(h_{i,t})(x) \geqq 0 \quad \text{if} \quad g_i(x) < 1 \quad (\text{where } h_{i,t}(x) = h_i(x,t) \text{ as in (5.13)}). \tag{5.15}$$

Properties (5.4), (5.9), (5.11) imply

$$h_i(x,t) = 0 \quad \text{if} \quad x \notin \operatorname{closure}(U_i) = \operatorname{spt}(f_i). \tag{5.16}$$

Also, (5.9) and (5.10) yield

$$h_i(x,t) > |v_i(x)| \quad \text{if} \quad x \in \operatorname{spt}(v_i). \tag{5.17}$$

In view of (5.2), (5.3), (5.8), (5.10), (5.16), (5.17), we may fix $v > 0$ such that the following inequality holds:

$$\begin{aligned} &v u[a v_i, h_{i,t}](x) \cdot \Delta(u[a v_i, h_{i,t}])(x) \geqq -\delta/4 \\ &\text{if } (x,t) \in P \times [0,T], \quad a \in [-1,1] \quad \text{and} \quad i \in \{1,2\}. \end{aligned} \tag{5.18}$$

We recall (4.8), (4.18), (4.21), (4.31) and make the following definitions:

$$\beta_n : R \to R \text{ is given by } \beta_n(x) = \tau x + A + nX \quad \text{for} \quad n \in \{0, 1, \ldots, Y\}, \tag{5.19}$$

$$M(Z) = \{m = (m_1, m_2, \ldots, m_Z) : m_i \in \{0, 1, \ldots, Y\}\} \quad \text{for} \quad Z = 1, 2, 3, \ldots, \tag{5.20}$$

$$\pi_m = \beta_{m_1} \circ \beta_{m_2} \circ \cdots \circ \beta_{m_Z} \quad \text{if} \quad m = (m_1, m_2, \ldots, m_Z) \in M(Z), \tag{5.21}$$

$$Q = [a - r, a + r] \cup [-d, d]. \tag{5.22}$$

If $(x_1, x_2) \in P$, $\quad t \in (-\delta, T + \delta)$, $\quad i \in \{1, 2\}$, $\quad m = (m_1, m_2, \ldots, m_Z) \in M(Z)$,

we set

$$\begin{aligned} &f_i^m(x_1, x_2) = f_i(\pi_m^{-1}(\tau^Z x_1), x_2), \quad v_i^m(x_1, x_2) = v_i(\pi_m^{-1}(\tau^Z x_1), x_2), \\ &g_i^m(x_1, x_2) = g_i(\pi_m^{-1}(\tau^Z x_1), x_2), \quad h_i^m(x_1, x_2, t) = h_i(\pi_m^{-1}(\tau^Z x_1), x_2, t). \end{aligned} \tag{5.23}$$

Lemma 5.1. *If* $Z \in \{1, 2, 3, \ldots\}$, $\{m, m'\} \subset M(Z)$ *and* $m \neq m'$, *then* $\pi_m(Q)$ *and* $\pi_{m'}(Q)$ *are disjoint sets.*

Proof. Since (4.33) implies $d-r<d$, we can use (5.22) and Lemma 4.5 to obtain

$$\beta_j(Q)\subset Q, \tag{5.24}$$

$$\beta_j(Q) \text{ and } \beta_{j'}(Q) \text{ are disjoint if } j\neq j'. \tag{5.25}$$

Let I be the smallest integer i such that $m_i\neq m_i'$. Inclusion (5.24) gives

$$(\beta_{m_I}\circ\beta_{m_{I+1}}\circ\cdots\circ\beta_{m_Z})(Q)\subset\beta_{m_I}(Q), (\beta_{m_I'}\circ\beta_{m_{I+1}'}\circ\cdots\circ\beta_{m_Z'})(Q)\subset\beta_{m_I'}(Q).$$

Therefore, (5.25) implies that the sets

$$(\beta_{m_I}\circ\cdots\circ\beta_{m_Z})(Q) \text{ and } (\beta_{m_I'}\circ\cdots\circ\beta_{m_Z'})(Q) \tag{5.26}$$

are disjoint. We obtain the conclusion by applying the one-to-one function $\beta_{m_1}\circ\cdots\circ\beta_{m_{I-1}}=\beta_{m_1'}\circ\cdots\circ\beta_{m_{I-1}'}$ to both sets in (5.26).

Lemma 5.2. *If* $Z\in\{1,2,3,\dots\}$, *then the sets* $\operatorname{spt}(f_i^m)$ (*where* $i\in\{1,2\}$, $m\in M(Z)$) *are disjoint.*

Proof. From Lemma 4.14, $d-r<d$ and (5.22) we get

$$\{x_1:(x_1,x_2)\in\operatorname{spt}(f_i) \text{ for some } x_2 \text{ and some } i\}\subset Q.$$

Now (5.23) yields

$$\{x_1:(x_1,x_2)\in\operatorname{spt}(f_i^m) \text{ for some } x_2 \text{ and } i\}\subset\{x_1:\tau^Z x_1\in\pi_m(Q)\}.$$

The above and Lemma 5.1 imply that $\operatorname{spt}(f_i^m)$ and $\operatorname{spt}(f_{i'}^{m'})$ are disjoint if $m\neq m'$. Finally, (4.61) and (5.23) imply that $\operatorname{spt}(f_1^m)$ and $\operatorname{spt}(f_2^m)$ are disjoint for every $m\in M(Z)$.

Lemma 5.3. *If* Z *is a positive integer,* $m\in M(Z), n\in M(Z+1)$, $m_i=n_i$ *for* $i\leqq Z$, $(x_1,x_2,x_3)\in R^3$ *and*

$$(x_1,(x_2^2+x_3^2)^{1/2})\in\operatorname{spt}(f_1^n)\cup\operatorname{spt}(f_2^n), \tag{5.27}$$

then $((\tau x_2+\varepsilon r/2)^2+(\tau x_3)^2)^{1/2}>0$ *and*

$$h_2^m(\tau x_1,((\tau x_2+\varepsilon r/2)^2+(\tau x_3)^2)^{1/2},T)>\tau^{-1}(f_1^n+f_2^n)(x_1,(x_2^2+x_3^2)^{1/2}).$$

Proof. We set $j=n_{Z+1}$ and

$$z=(z_1,z_2)=(\beta_j(\pi_n^{-1}(\tau^{Z+1}x_1)),((\tau x_2+\varepsilon r/2)^2+(\tau x_3)^2)^{1/2}).$$

Since (5.27) and (5.23) imply

$$(\pi_n^{-1}(\tau^{Z+1}x_1),(x_2^2+x_3^2)^{1/2})\in\operatorname{spt}(f_1)\cup\operatorname{spt}(f_2),$$

we can use Lemma 4.18 and (5.19), (5.14), (5.23) to conclude $(z_1,z_2)\in P$ (which means $((\tau x_2+\varepsilon r/2)^2+(\tau x_3)^2)^{1/2}=z_2>0$) and

$$\begin{aligned}h_2(z,T)&>((f_2(z))^2-Tv_2(z)\cdot\nabla(p[v_1,f_1]-p[0,f_1])(z))^{1/2}-\theta\\&>\tau^{-1}(f_1+f_2)(\pi_n^{-1}(\tau^{Z+1}x_1),(x_2^2+x_3^2)^{1/2})\\&=\tau^{-1}(f_1^n+f_2^n)(x_1,(x_2^2+x_3^2)^{1/2}).\end{aligned}$$

We also have $\pi_n=\beta_{n_1}\circ\cdots\circ\beta_{n_Z}\circ\beta_j=\beta_{m_1}\circ\cdots\circ\beta_{m_Z}\circ\beta_j=\pi_m\circ\beta_j$, which implies $\pi_n^{-1}=$

$\beta_j^{-1} \circ \pi_m^{-1}$ and $\beta_j \circ \pi_n^{-1} = \pi_m^{-1}$. Hence we can use (5.23) and say

$$h_2^m(\tau x_1, z_2, T) = h_2(\pi_m^{-1}(\tau^Z \tau x_1), z_2, T) = h_2(\beta_j(\pi_n^{-1}(\tau^{Z+1} x_1)), z_2, T) = h_2(z, T).$$

The conclusion follows from this identity and the previous inequality.

Lemma 5.4. *There exists $\xi_Z > 0$ such that*

$$\sum_{m \in M(Z)} (h_1^m + h_2^m)(\tau x_1, ((\tau x_2 + \varepsilon r/2)^2 + (\tau x_3)^2)^{1/2}, T)$$
$$\geqq \xi_Z + \sum_{n \in M(Z+1)} \tau^{-1}(f_1^n + f_2^n)(x_1, (x_2^2 + x_3^2)^{1/2})$$

holds if $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \bigcup_{n \in M(Z+1)} (\operatorname{spt}(f_1^n) \cup \operatorname{spt}(f_2^n))$.

Proof. This follows from Lemma 5.2 (applied to $M(Z+1)$), Lemma 5.3, the compactness of $\operatorname{spt}(f_i^n)$ and $h_i^m \geqq 0$ (see (5.2), (5.10), (5.23)).

Lemma 5.5. *For every positive integer Z, there exist $\eta_Z > 0$ and C^∞ functions $u^Z: R^3 \times (-\eta_Z, T + \eta_Z) \to R^3$ and $p^Z: R \times (-\eta_Z, T + \eta_Z) \to R$ satisfying the following:*

$$u^Z(x_1, x_2, x_3, t) = 0 \quad \text{if} \quad (x_1, (x_2^2 + x_3^2)^{1/2}) \notin \bigcup_{m \in M(Z)} (\operatorname{spt}(f_1^m) \cup \operatorname{spt}(f_2^m)), \tag{5.28}$$

$$\sum_{i=1}^{3} \frac{\partial u_i^Z}{\partial x_i} = 0, \tag{5.29}$$

$$p^Z(x, t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j^Z}{\partial x_i}(y, t) \frac{\partial u_i^Z}{\partial x_j}(y, t) (4\pi |x - y|)^{-1} \, dy, \tag{5.30}$$

$$\frac{\partial}{\partial t} 2^{-1} |u^Z(x, t)|^2 \leqq -u^Z(x, t) \cdot \nabla(2^{-1}|u^Z|^2 + p^Z)(x, t) + \nu u^Z(x, t) \cdot \Delta u^Z(x, t) \quad \text{if } t \in [0, T], \tag{5.31}$$

$$|u^Z(x_1, x_2, x_3, 0)| = \sum_{m \in M(Z)} (h_1^m + h_2^m)(x_1, (x_2^2 + x_3^2)^{1/2}, 0)$$
$$= \sum_{m \in M(Z)} (f_1^m + f_2^m)(x_1, (x_2^2 + x_3^2)^{1/2}) \quad \text{if } (x_2, x_3) \neq (0, 0), \tag{5.32}$$

$$\Big| |u^Z(x_1, x_2, x_3, t)| - \sum_{m \in M(Z)} (h_1^m + h_2^m)(x_1, (x_2^2 + x_3^2)^{1/2}, t) \Big| \leqq \xi_Z \quad \text{if } t \in [0, T] \text{ and } (x_2, x_3) \neq (0, 0). \tag{5.33}$$

Proof. We set $C_i^m = \operatorname{spt}(f_i^m)$. If $x \notin C_i^m$, then the identity $h_i^m(x, t) = 0$ follows from (5.16) and (5.23). In addition, (5.19), (5.21), (5.23) imply that $f_i^m, v_i^m, g_i^m, h_i^m$ are *translations* of f_i, v_i, g_i, h_i. The property $h_i^m(x, 0) = f_i^m(x)$ is a consequence of (5.12), (5.13), (5.4), (5.10) and (5.23). Using these properties, (5.1)–(5.23) and Lemma 5.2, we find that the hypotheses of Lemma 3.2 are satisfied when the set $M(Z)$ is placed in one-to-one correspondence with the set $\{1, 2, \ldots, M\}$ for some M, and the obvious identification is made. The conclusions follow from Lemma 3.2.

Lemma 5.6. *If Z is a positive integer then every $(x_1, x_2, x_3) \in R^3$ satisfies*

$$|u^Z(\tau x_1, \tau x_2 + \varepsilon r/2, \tau x_3, T)| \geqq \tau^{-1} |u^{Z+1}(x_1, x_2, x_3, 0)|.$$

Proof. If $(x_1, (x_2^2 + x_3^2)^{1/2}) \in \bigcup_{n \in M(Z+1)} (\operatorname{spt}(f_1^n) \cup \operatorname{spt}(f_2^n))$, then the inequality follows from Lemma 5.4 and (5.32), (5.33). Otherwise, we can use (5.28) to conclude $u^{Z+1}(x_1, x_2, x_3, 0) = 0$.

Using the notation

$$\begin{aligned} a_Z &= \tau^{-Z}(\tau^0 + \tau^1 + \tau^2 + \cdots + \tau^{Z-1})\varepsilon r/2, \\ b_Z &= (\tau^0 + \tau^2 + \tau^4 + \cdots + \tau^{2Z-2})T, \end{aligned} \tag{5.34}$$

we recall Lemma 5.5 and set

$$\begin{aligned} v^Z(x_1, x_2, x_3, t) &= \tau^{-Z} u^Z(\tau^{-Z} x_1, \tau^{-Z} x_2 - a_Z, \tau^{-Z} x_3, (t - b_Z)\tau^{-2Z}), \\ q^Z(x_1, x_2, x_3, t) &= \tau^{-2Z} p^Z(\tau^{-Z} x_1, \tau^{-Z} x_2 - a_Z, \tau^{-Z} x_3, (t - b_Z)\tau^{-2Z}) \\ \text{if } Z \in \{1, 2, 3, \ldots\}, \quad &(x_1, x_2, x_3) \in R^3 \quad \text{and} \quad b_Z - \tau^{2Z}\eta_Z < t < b_{Z+1} + \tau^{2Z}\eta_Z. \end{aligned} \tag{5.35}$$

This definition supersedes the earlier definitions of v_i^m and q_i^m. From (5.29)–(5.31), (5.35) we obtain

$$\sum_{i=1}^{3} \frac{\partial v_i^Z}{\partial x_i} = 0, \tag{5.36}$$

$$q^Z(x, t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_j^Z}{\partial x_i}(y, t) \frac{\partial v_i^Z}{\partial x_j}(y, t)(4\pi |x - y|)^{-1}\, dy, \tag{5.37}$$

$$\frac{\partial}{\partial t} 2^{-1} |v^Z(x, t)|^2 \leqq -v^Z(x, t) \cdot \nabla(2^{-1}|v^Z|^2 + q^Z)(x, t) + \nu v^Z(x, t) \cdot \Delta v^Z(x, t) \quad \text{if } b_Z \leqq t \leqq b_{Z+1}. \tag{5.38}$$

Lemma 5.7. *If Z is a positive integer, then $|v^Z(x, b_{Z+1})| \geqq |v^{Z+1}(x, b_{Z+1})|$.*

Proof. We use (5.34), (5.35) and Lemma 5.6 to derive

$$\begin{aligned} &|v^Z(x_1, x_2, x_3, b_{Z+1})| \\ &\quad = \tau^{-Z} |u^Z(\tau(\tau^{-(Z+1)} x_1), \tau(\tau^{-(Z+1)} x_2 - a_{Z+1}) + \varepsilon r/2, \tau(\tau^{-(Z+1)} x_3), T)| \\ &\quad \geqq \tau^{-Z} \tau^{-1} |u^{Z+1}(\tau^{-(Z+1)} x_1, \tau^{-(Z+1)} x_2 - a_{Z+1}, \tau^{-(Z+1)} x_3, 0)| \\ &\quad = |v^{Z+1}(x_1, x_2, x_3, b_{Z+1})|. \end{aligned}$$

Lemma 5.8. *If $v^Z(x, t) \neq 0$ for some Z, x, t then $x_1 \in (r - d, d - r)$, $|x_2| \leqq \tau r + (1 - \tau)^{-1}(\varepsilon r/2)$ and $|x_3| \leqq \tau r$.*

Proof. If $v^Z(x_1, x_2, x_3, t) \neq 0$, then (5.35) and (5.28) imply

$$(\tau^{-Z} x_1, ((\tau^{-Z} x_2 - a_Z)^2 + (\tau^{-Z} x_3)^2)^{1/2}) \in \operatorname{spt}(f_1^m) \cup \operatorname{spt}(f_2^m)$$

for some $m \in M(Z)$. Then (5.23) yields

$$(\pi_m^{-1}(x_1), ((\tau^{-Z}x_2 - a_Z)^2 + (\tau^{-Z}x_3)^2)^{1/2}) \in \operatorname{spt}(f_1) \cup \operatorname{spt}(f_2).$$

Therefore, Lemma 4.14 and $d - r < d$ (see (4.33)) give us

$$\pi_m^{-1}(x_1) \in [a-r, a+r] \cup [-d, d], \quad |\tau^{-Z}x_2 - a_Z| \leqq r, \quad |\tau^{-Z}x_3| \leqq r.$$

The above, (5.21), (5.22), (5.24), the last part of Lemma 4.5, (5.19), $0 < \tau < 1$ (see (4.32)) and (5.34) yield

$$\begin{aligned} & x_1 \in (\beta_{m_1} \circ \cdots \circ \beta_{m_Z})([a-r, a+r] \cup [-d, d]) \\ & \quad \subset \beta_{m_1}([a-r, a+r] \cup [-d, d]) \subset (r-d, d-r), \\ & |x_2| \leqq |\tau^Z r| + |\tau^Z a_Z| \leqq \tau r + (1-\tau)^{-1} \varepsilon r/2, |x_3| \leqq \tau^Z r \leqq \tau r. \end{aligned}$$

Lemma 5.9. *If Z is a positive integer, $\phi: R^3 \times R \to R$ is a C^∞ function with compact support and $\phi \geqq 0$, then*

$$\begin{aligned} & \int_{R^3} 2^{-1}|v^Z(x, b_{Z+1})|^2 \phi(x, b_{Z+1})\,dx - \int_{R^3} 2^{-1}|v^Z(x, b_Z)|^2 \phi(x, b_Z)\,dx \\ & \quad + \int_{b_Z}^{b_{Z+1}} \int_{R^3} \nu |\nabla v^Z(x,t)|^2 \phi(x,t)\,dx\,dt \\ & \leqq \int_{b_Z}^{b_{Z+1}} \int_{R^3} (2^{-1}|v^Z|^2 + q^Z)(x,t) v^Z(x,t) \cdot \nabla \phi(x,t)\,dx\,dt \\ & \quad + \int_{b_Z}^{b_{Z+1}} \int_{R^3} 2^{-1}|v^Z(x,t)|^2 \left(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \right)(x,t)\,dx\,dt. \end{aligned}$$

Proof. The inequality follows when we multiply (5.38) by $\phi(x,t)$, integrate over $R^3 \times [b_Z, b_{Z+1}]$ and use (5.36).

Recalling $0 < \tau < 1$ (see (4.32)) and (5.34)–(5.35), we define $u: R^3 \times [T, \infty) \to R^3$ and $p: R^3 \times [T, \infty) \to R$ using the following formulas:

$$\begin{aligned} & u(x,t) = v^Z(x,t), \quad p(x,t) = q^Z(x,t) \quad \text{if} \quad b_Z \leqq t < b_{Z+1}, \\ & u(x,t) = 0, \quad p(x,t) = 0 \quad \text{if} \quad t \geqq T(1-\tau^2)^{-1} = \lim_{Z \to \infty} b_Z. \end{aligned} \tag{5.39}$$

From (5.37) we conclude

$$p(x,t) = \int_{R^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_i}(y,t) \frac{\partial u_i}{\partial x_j}(y,t) (4\pi|x-y|)^{-1}\,dy. \tag{5.40}$$

Lemma 5.10. *If Z, ϕ satisfy the hypotheses of Lemma 5.9 and* $\operatorname{spt}(\phi) \subset R^3 \times (T, \infty)$, *then*

$$\begin{aligned} & \int_{R^3} 2^{-1}|u(x, b_{Z+1})|^2 \phi(x, b_{Z+1})\,dx + \int_T^{b_{Z+1}} \int_{R^3} \nu |\nabla u(x,t)|^2 \phi(x,t)\,dx\,dt \\ & \leqq \int_T^{b_{Z+1}} \int_{R^3} (2^{-1}|u|^2 + p)(x,t) u(x,t) \cdot \nabla \phi(x,t)\,dx\,dt \\ & \quad + \int_T^{b_{Z+1}} \int_{R^3} 2^{-1}|u(x,t)|^2 \left(\frac{\partial \phi}{\partial t} + \nu \Delta \phi \right)(x,t)\,dx\,dt. \end{aligned}$$

Proof. We use Lemma 5.9 with Z replaced by W, add the resulting inequalities for $W=1,2,\dots,Z$, and apply $\phi(x,b_1)=\phi(x,T)=0$ and Lemma 5.7, which implies

$$\begin{aligned}
&\sum_{W=1}^{Z}\int_{R^3} 2^{-1}|v^W(x,b_{W+1})|^2\phi(x,b_{W+1})\,dx\\
&\geqq \sum_{W=1}^{Z}\int_{R^3} 2^{-1}|v^{W+1}(x,b_{W+1})|^2\phi(x,b_{W+1})\,dx\\
&=\sum_{W=2}^{Z+1}\int_{R^3} 2^{-1}|v^W(x,b_W)|^2\phi(x,b_W)\,dx\\
&=\sum_{W=1}^{Z}\int_{R^3} 2^{-1}|v^W(x,b_W)|^2\phi(x,b_W)\,dx\\
&\quad+\int_{R^3} 2^{-1}|u(x,b_{Z+1})|^2\phi(x,b_{Z+1})\,dx.
\end{aligned}$$

Lemma 5.11. *The function* ∇u *is square integrable and* $\int_{R^3}|u(x,t)|^2\,dx$ *is a non-increasing real-valued function of* t.

Proof. Properties (5.36), (5.38) and Lemma 5.8 imply the usual energy inequality for v^Z on the set $R^3\times[b_Z,b_{Z+1}]$. The conclusion is pieced together from this fact, (5.39) and Lemma 5.7.

Lemma 5.12. *If* $T'=T(1-\tau^2)^{-1}$, *then* $\int_T^{T'}\int_{R^3}|u(x,t)|^3\,dx\,dt$ *and* $\int_T^{T'}\int_{R^3}|u(x,t)||p(x,t)|\,dx\,dt$ *are finite.*

Proof. This follows from Lemma 5.11, Lemma 5.8 and (5.40) in a standard way. See, for example, Lemma 3.2 and Lemma 3.6 of [3].

Lemma 5.13. *If* $\phi: R^3\times R\to R$ *is a* C^∞ *function with compact support,* $\phi\geqq 0$ *and* $\operatorname{spt}(\phi)\subset R^3\times(T,\infty)$, *then*

$$\begin{aligned}
\int_T^\infty\int_{R^3} v|\nabla u(x,t)|^2\phi(x,t)\,dx\,dt &\leqq \int_T^\infty\int_{R^3}(2^{-1}|u|^2+p)(x,t)u(x,t)\cdot\nabla\phi(x,t)\,dx\,dt\\
&\quad+\int_T^\infty\int_{R^3}2^{-1}|u(x,t)|^2\left(\frac{\partial\phi}{\partial t}+v\Delta\phi\right)(x,t)\,dx\,dt.
\end{aligned}$$

Proof. This is a consequence of the three previous lemmas and the identity $u(x,t)=0$ for $t\geqq T(1-\tau^2)^{-1}=\lim_{Z\to\infty} b_Z$ (see (5.39)).

We set

$$S_Z=\bigcup_{m\in M(Z)}\pi_m[-d,d],\quad S=\bigcap_{Z=1}^{\infty}S_Z, \tag{5.41}$$

$$S'=\{(s,(1-\tau)^{-1}\varepsilon r/2,0,(1-\tau^2)^{-1}T): s\in S\}. \tag{5.42}$$

Lemma 5.14. *The Hausdorff dimension of S' is greater than ζ.*

Proof. Lemma 4.5, (5.19) and (4.33) give us $\beta_k[-d,d] \subset (r-d, d-r) \subset [-d,d]$, which implies $(\pi_m \circ \beta_k)[-d,d] \subset \pi_m[-d,d]$. This fact and (5.20), (5.21) yield

$$S_{Z+1} \subset S_Z \subset [-d,d], S_Z \text{ is compact.} \tag{5.43}$$

Using (5.21) we also get

$$S_{Z+1} = \bigcup_{j=0}^{Y} \beta_j(S_Z). \tag{5.44}$$

Let V be an open set containing $\bigcup_{j=0}^{Y} \beta_j(S)$. There is an open set W containing S such that $\beta_j(W) \subset V$ holds for all j. From (5.41), (5.43) we obtain the existence of Z satisfying $S_Z \subset W$. For this Z, we can use (5.41), (5.44) to conclude $S \subset S_{Z+1} = \bigcup_{j=0}^{Y} \beta_j(S_Z) \subset \bigcup_{j=0}^{Y} \beta_j(W) \subset V$. Since V is an arbitrary open set about $\bigcup_{j=0}^{Y} \beta_j(S)$, we conclude $S \subset \bigcup_{j=0}^{Y} \beta_j(S)$. The opposite inclusion is immediate from (5.41), (5.43) and (5.44). These properties, (5.41) and (5.43) give us

$$S = \bigcup_{j=0}^{Y} \beta_j(S), \; S \text{ is compact.} \tag{5.45}$$

Since (5.22) and (5.25) imply that the sets $\beta_j[-d,d]$ are disjoint, we can use (5.41), (5.43) to conclude

$$\{\beta_j(S) : j = 0, 1, \ldots, Y\} \text{ is a family of disjoint sets.} \tag{5.46}$$

A theorem of Moran [2, Theorem II] says that (5.45), (5.46) and (5.19) imply $(Y+1)\tau^{\dim(S)} = 1$, where dim is the Hausdorff dimension. This identity, (4.32) and (5.42) yield $\zeta < \dim(S) = \dim(S')$.

Lemma 5.15. *If $s' \in S'$ and V is an open neighborhood of s' in $R^3 \times (T, \infty)$, then u is not essentially bounded on V.*

Proof. Definition (5.42) implies $s' = (s, (1-\tau)^{-1}\varepsilon r/2, 0, (1-\tau^2)^{-1}T)$ for some $s \in S$. Let N be a given positive number. Using (4.32), (5.34), (4.57), (4.62), we conclude $0 < \tau < 1$, $\lim_{Z\to\infty} \tau^Z a_Z = (1-\tau)^{-1}\varepsilon r/2$, $\lim_{Z\to\infty} b_Z = (1-\tau^2)^{-1}T$, $\|f_2\|_\infty = \mu > 0$. Also, (5.19), (5.21) yield length $(\pi_n[-d,d]) = \tau^Z(2d)$ if $n \in M(Z)$. All this and (5.41) imply the existence of some $Z \in \{1, 2, 3, \ldots\}$ and some $n \in M(Z)$ satisfying

$$\pi_n[-d,d] \times [\tau^Z a_Z, \tau^Z a_Z + \tau^Z r] \times \{0\} \times \{b_Z\} \subset V \tag{5.47}$$

and $\tau^{-Z}\|f_2\|_\infty > N$. This last part, (5.4) and (4.60) enables us to write

$$\tau^{-Z} f_2(x_1, x_2) > N, \text{ hence } x_1 \in [-d,d], \quad 0 < x_2 \leqq r \tag{5.48}$$

for some $(x_1, x_2) \in \mathrm{spt}(f_2)$. From (5.47), (5.48) we get

$$(\pi_n(x_1), \tau^Z a_Z + \tau^Z x_2, 0, b_Z) \in V. \tag{5.49}$$

In addition, (5.35), (5.32), $x_2 > 0$, (5.4), (5.23), (5.48) imply

$$\begin{aligned} |v^Z(\pi_n(x_1), \tau^Z a_Z + \tau^Z x_2, 0, b_Z)| &= \tau^{-Z} |u^Z(\tau^{-Z}\pi_n(x_1), x_2, 0, 0)| \\ &= \sum_{m \in M(Z)} \tau^{-Z}(f_1^m + f_2^m)(\tau^{-Z}\pi_n(x_1), x_2) \geqq \tau^{-Z} f_2^n(\tau^{-Z}\pi_n(x_1), x_2) \\ &= \tau^{-Z} f_2(x_1, x_2) > N. \end{aligned}$$

Since v^Z is C^∞, we find that $|v^Z| \geqq N$ holds on a neighborhood of $(\pi_n(x_1),\ \tau^Z a_Z + \tau^Z x_2, 0, b_Z)$. Combining this with (5.39) and (5.49), we obtain that $|u| \geqq N$ holds on a set $W \subset V$, where W has positive Lebesgue measure.

Using Lemma 5.5, (5.35), (5.39), Lemma 5.8, (5.36), (5.40), Lemma 5.11, Lemma 5.12, Lemma 5.13, Lemma 5.14 and Lemma 5.15, we find that all of the properties listed in the theorem of Sect. 1 are satisfied when we make the following changes: S is replaced by S', $[0, \infty)$ is replaced by $[T, \infty)$, $R^3 \times \{1\}$ is replaced by $R^3 \times \{(1-\tau^2)^{-1}T\}$, and (1.7) is replaced by Lemma 5.13. The proof is completed with a change of scale.

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