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The Decomposition Property and Equilibrium States of Ferromagnetic Lattice Systems

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Abstract. We discuss classical lattice gas models with a finite number of different particles and ferromagnetic type interaction between them. We make the set of particle types into a finite abelian group and explore the algebraic structure of such a system. We present a criterion for using the Peierls argument to establish the existence of phase transitions. In the case of a cyclic group of order equal to the product of different prime numbers we obtain a complete description of all periodic Gibbs states at low temperatures.

1. Introduction

One of the main tasks of statistical mechanics is to describe the family of equilibrium states of the system for a given interaction, external parameters, and temperature. This problem is addressed here in the context of certain classical lattice gas models. Namely, at each site of the lattice there is a variable which can take on a finite number of values. One may think of these as different species which can occupy the lattice sites or $2l+1$ orientations of a spin l particle. The particles or spins interact through many body potentials. Central in the analysis of low temperature behavior of such systems is the notion of a ground state and the so-called Peierls condition. A ground state is a configuration of particles with a minimal potential energy per lattice site. A model satisfies the Peierls condition if the creation of an "island" of one periodic ground state in a "sea" of another periodic ground state leads to an increase of energy which is greater than some fixed positive constant times the length of the boundary of the "island". In the case of models which have a finite number of periodic ground states and satisfy the Peierls condition there is a complete theory due to Pirogov and Sinai [1, 2] (see also the review article by Slawny [3]). Namely, the phase diagram at low temperatures is obtained by perturbation of the zero

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temperature phase diagram. In particular the number of extremal periodic Gibbs states is equal to the number of periodic ground states. If the above conditions are not satisfied then no general results can be inferred. However, there is one class of models, the translation invariant ferromagnet spin 1/2 systems, where the complete description of all periodic Gibbs states is available: Holsztyński, Slawny [3–6]. In such systems, at each site of the lattice there is a spin variable which can take on just two values: 1 and -1 (spin up or spin down). Many body interactions generalize that of a ferromagnetic Ising model: all coupling constants are negative.

We generalize their results to the case of arbitrarily many (but finite) spin orientations or equivalently a finite number of different species which can occupy lattice sites. To use the algebraic techniques developed by Holsztyński and Slawny [5] for simple cubic lattices and Slawny [3, 6] for arbitrary lattices we make the set of different species into an abelian group. Although one may think of a cyclic group of finite order, to be more general we deal with an arbitrary finite abelian group. Lattice site variables become now characters of that group. Every finite product of the characters at different lattice sites can constitute a bond of the interaction. We restrict ourselves entirely to the ferromagnetic case, i.e., negative coupling constants. More precisely, the finite volume Hamiltonian as a function on the space (the product group) of particle configurations has a Fourier decomposition with non-positive coefficients.

Formally:

$$H_\Lambda = -\sum_{\hat{B}\in\hat{\chi}_\Lambda} J(\hat{B})\hat{B}; \quad J(\hat{B}) \geqq 0, \tag{1.1}$$

where the summation is over all characters of the product group of particle configurations χ_Λ in a finite volume Λ. By discussing the arbitrary finite abelian group case we enlarge the family of ferromagnetic models.

One of the models which fits into our scheme is the ferromagnetic Potts model with m components. The interaction is given by:

$$\delta(X(a), X(b)) = \begin{cases} 1 & \text{if } X(a) = X(b), \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where a and b are a pair of nearest neighbors on the cubic lattice, and $X(a)$ and $X(b)$ are the configurations at sites a and b respectively. Since

$$\delta(X(a), X(b)) = 1/m \sum_{k=0}^{m-1} \exp\{(2\pi ik/m)[X(a) - X(b)]\}, \tag{1.3}$$

the interaction is of the type (1.1) with $\mathbb{Z}_m$, the cyclic group of order m, as the group of configurations at each site of the lattice.

The phase structure of Z_m models was studied recently by Fröhlich and Spencer [7]. They have shown that if m is above some critical value, a massless soft phase, characteristic of the XY model, will appear between ordered and disordered massive phases of the Ising type. When m goes to infinity, the ordered phase disappears except at zero temperature and the remaining two phases are those of the XY model.

$\mathbb{Z}_m$ models were used by Gruber, Hintermann and Merlini [8] to investigate higher spin systems. They mapped a spin l system into a $\mathbb{Z}_m$ system with $m = 2l + 1$

by a set of local transformations from the set of spin orientations to the set of integers modulo m. They used the group structure to obtain certain results such as the analyticity of the pressure, and then they inverted the transformations returning to the original system.

Here we investigate the low temperature periodic Gibbs states in ferromagnetic abelian lattice systems described above. We generalize the algebraic methods developed by Holsztyński and Slawny [3, 5, 6] for spin 1/2 models, i.e., the case of $\mathbb{Z}_2$, the cyclic group of order two. It is known [4, 9, 10] that to describe the periodic Gibbs states at low temperatures in such systems it is enough to know the unbroken part, $\mathscr{S}^+$, of the group of all transformations, $\mathscr{S}$, which leave the interaction invariant. In particular, the number of extremal periodic Gibbs states—pure phases—is equal to $|\mathscr{S}/\mathscr{S}^+|$ at low temperatures.

$\mathscr{S}$ can be identified with the ground states of the model so we can have a finite number of pure phases even in the case of an infinite number of periodic ground states. Here and in the following paper [11] we present the explicit expression for $\mathscr{B}^+$, the group orthogonal to $\mathscr{S}^+$, in terms of the bonds of the system. The main idea is to use the Peierls argument as in [5]. Namely, in the case of a spin 1/2 on the cubic lattice, a finite volume magnetization $\rho_\Lambda^+(\sigma_a)$, where ρ_Λ^+ is a finite volume Gibbs state with " + " boundary condition (all spins up outside Λ), is studied.

$$\rho_\Lambda^+(\sigma_a) = 1 - 2\cdot\text{probability}_\Lambda\{X(a) = -1\}. \tag{1.4}$$

There is a standard Peierls estimate for the probability that the spin at the site a is oriented downward:

$$\text{probability}_\Lambda\{X(a) = -1\} \leqq \sum_E n_E e^{-\beta E}, \tag{1.5}$$

where n_E is the number of indecomposable excitations from the " + " ground state (all spins up) with energy E and the spin at site a oriented downward. If n_E can be bounded from above by some expression which grows slower than $e^{\beta E}$ as E increases, then the series in (1.5) becomes a geometric series, can be summed up and the sum approaches zero independently of Λ as β increases. It follows that:

$$\rho^+(\sigma_a) = \lim_{\Lambda \to \mathbb{L}} \rho_n^+(\sigma_a) > 0. \tag{1.6}$$

The magnetization per lattice site is non-zero in the ρ^+ state. The symmetry of the Hamiltonian—flipping all spins—is therefore broken at low temperatures.

On the other hand, if there are too many indecomposable excitations with the same energy, some symmetries can be preserved at low temperatures. It would be best to have a criterion for when the appropriate estimates can be obtained and the Peierls argument can be applied. The main goal of this paper is to find such a criterion. It is known [5] that for a spin 1/2 on the cubic lattice the Peierls argument can be applied if and only if the greatest common divisor of an ideal generated by the bonds of the system in a certain ring is a unit. If this is not the case, then one can pass to the "reduced system," where the standard argument can be used.

The case of a spin 1/2 on an arbitrary lattice is more complicated. The energy of the ground state can be invariant under a local change of the configuration. If this is the case then the system is a gauge model. Complete results as in [5] were obtained by Slawny in [3, 6]. However, more advanced mathematics is required. In place of a ring and its ideals, modules over the ring must be considered. The same technique is required for the case of an arbitrary finite abelian group on the simple cubic lattice. In fact, an arbitrary lattice case can always be reduced to the cubic lattice case. An arbitrary lattice is a finite union of simple cubic lattices. We can glue together the corresponding sites to make the simple cubic lattice with a finite abelian group being the product of the corresponding, not necessarily identical, $\mathbb{Z}_m$ groups.

In Sect. 2 we introduce the notation and describe the model. We show that in some cases the group of configurations can be reduced. Namely, if we have a group, $\mathscr{G}$ at each site of the lattice, then for some interactions a system can be constructed with group $\mathscr{G}'$ with smaller order. There is a natural one-to-one correspondence between the families of the periodic Gibbs states of these two systems. The second system can be studied by the methods described in the next sections.

Section 3 deals with the algebraic structure of the system. In particular, the contours are defined.

In Sect. 4 a condition for using the Peierls argument, the so-called decomposition property is formulated. It says simply that the excitations which are not connected (in some sense described later) can be decomposed into the sum of independent excitations. We give a criterion for this property to hold. Consider for example the group $\mathbb{Z}_{p^n}$ on $\mathbb{Z}^\nu$, the simple cubic lattice in ν dimensions, where p is a prime number. Let $\mathscr{A}$ be the ideal in $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$, the group ring of functions from $\mathbb{Z}^\nu$ to $\mathbb{Z}_{p^n}$ with finite support, generated by all bonds of the system. Then it can be shown that the decomposition property holds in the system with the ideal $\mathscr{A}$ if and only if it holds in the system with the ideal $p^{n-1}\mathscr{A}$. Because $p^{n-1}\mathscr{A}$ is a $\mathbb{Z}_p[\mathbb{Z}^\nu]$-module and $\mathbb{Z}_p$ is a field ($\mathbb{Z}_2$ in the case of a spin 1/2), an easy generalization of a spin 1/2 technique can be then applied.

In Sect. 5 we show that it is not necessary to consider gauge models on $\mathbb{Z}^\nu$ in the $\mathscr{G} = \mathbb{Z}_m$ case. Passage to the new system, which is non-gauge, can be achieved as described in Sect. 2.4.

In Sect. 6 we generalize the results of [5] to the case of $\mathbb{Z}_m$ on $\mathbb{Z}^\nu$, where m is the product of different prime numbers. Namely, we present the explicit expression for $\mathscr{B}^+$ in terms of the bonds of the system.

Systems with a finite abelian group on an arbitrary lattice are discussed in [11]. The general construction of the "reduced system" is presented there. This allows us to find $\mathscr{B}^+$ and to construct all periodic Gibbs states as before.

2. Notation and Description of the Model

2.1 Configuration Space. By the lattice $\mathbb{L}$ is meant any $\mathbb{Z}^\nu$ invariant, discrete subset of $\mathbb{R}^\nu$. A finite abelian group $\mathscr{G}$ is placed at each site of the lattice.

$$\chi = \mathop{\times}_{i \in \mathbb{L}} \mathscr{G} \tag{2.1}$$

is the configuration space of the system.

For $A \in \chi$, $pr_{\{a\}} A = A(a)$, $a \in \mathbb{L}$. If $\mathscr{G}$ is equipped with the discrete topology then χ

becomes a compact abelian group with the product topology.

$$\chi_\Lambda = \bigoplus_{i\in\Lambda} \mathscr{G} \tag{2.2}$$

is a finite volume configuration space, where Λ is any finite subset of $\mathbb{L}$ and

$$\chi_f = \bigoplus_{i\in\mathbb{L}} \mathscr{G} \tag{2.3}$$

is the group of configurations equal to the identity of the group $\mathscr{G}$ everywhere but on the finite subsets of the lattice. We provide χ_f with the discrete topology. $\mathscr{G}$ as a finite abelian group can be decomposed as follows:

$$\mathscr{G} \simeq \mathscr{G}_{p_1} \oplus \cdots \oplus \mathscr{G}_{pk}, \tag{2.4}$$

where p_i are prime numbers and $\mathscr{G}_{p_i}$ are p_i primary groups,

$$\mathscr{G}_{p_i} \simeq \mathbb{Z}_{p_i^{i_1}} \oplus \mathbb{Z}_{p_i^{i_s}}. \tag{2.5}$$

The group dual to χ is isomorphic to χ_f:

$$\hat{\chi} \simeq \chi_f. \tag{2.6}$$

If $A \in \chi_f$ then we write $\hat{A}$ for the corresponding element in $\hat{\chi}$.

2.2 Interaction. The Hamiltonian in a finite volume, H_Λ, is a real, negative definite and translation invariant function on χ_Λ. By this it follows that the Fourier decomposition of H_Λ is the following:

$$H_\Lambda = -\sum_{B\in\chi_\Lambda} J(B)\hat{B}, \tag{2.7}$$

where $J(B) \geqq 0, J(B) = J(B^{-1})$ and if B_2 can be obtained from B_1 by a translation then $J(B_1) = J(B_2)$. The family of bonds is defined as follows:

$$\mathscr{B} = \{B\in\chi_f : J(B) > 0\}. \tag{2.8}$$

We assume that there is a finite fundamental family of bonds, $\mathscr{B}_0$, such that any element of $\mathscr{B}$ can be obtained in a unique way by a translation of an element from $\mathscr{B}_0$. Let $K(B) = \beta J(B)$, where β is the inverse temperature. Sometimes we refer to K as the interaction of a system.

2.3 Gibbs States. Let $e_{\mathscr{G}}$, the identity of the group $\mathscr{G}$, be placed everywhere outside Λ. With this as a boundary condition, a finite volume Gibbs state can be constructed. It is denoted traditionally by ρ_Λ^+.

Recently Pfister [9] proved some correlation inequalities for the $\mathscr{G} = \mathbb{Z}_m$ case. They can be trivially generalized to the arbitrary finite abelian group case. In fact, they can be obtained for any compact abelian group: Fernandez [12]. Using them one can obtain the standard conclusions. In particular, the Gibbs state ρ^+ can be constructed as a limit of ρ_Λ^+ when $\Lambda \to \mathbb{L}$. ρ^+ is a translation invariant state, extremal in the set of all Gibbs states and therefore mixing.

The following definitions are standard:

$$\mathscr{A} - \text{a subgroup of } \chi_f \text{ generated by } \mathscr{B}, \tag{2.9a}$$

$$\mathscr{B}^+ = \{A\in\chi_f : \rho^+(\hat{A}) > 0\}, \tag{2.9b}$$

$$\mathscr{S} = \{A \in \chi : \hat{B}(A) = 1 \text{ for any } B \in \mathscr{B}\}, \tag{2.9c}$$

$$\mathscr{S}^+ = \{G \in \mathscr{S} : \rho_G^+(\hat{A}) \equiv \hat{A}(G) \cdot \rho^+(\hat{A}) = \rho^+(\hat{A}) \quad \text{for any} \quad A \in \chi_f\}. \tag{2.9d}$$

It is known [4, 9, 10] that if the pressure (the Gibbs free energy) is differentiable with respect to the temperature then every periodic Gibbs state ρ has the following representation:

$$\rho = \int_{\mathscr{S}/\mathscr{S}^+} \rho_G^+ \, \mu(dG), \tag{2.10}$$

where μ is a measure on $\mathscr{S}/\mathscr{S}^+$. The differentiability of the pressure at low temperatures was proven by Slawny for spin 1/2 systems in [13] and for higher spins in [4]. The finite abelian group case can be treated in the same way and the same result holds [14]. Both $\mathscr{B}^+$ and $\mathscr{S}^+$ depend upon temperature. It will be shown, however, that they are constant at low enough temperatures. The number of extremal, periodic Gibbs states—pure phases—is equal to $|\mathscr{S}/\mathscr{S}^+|$ at low temperatures. $\mathscr{S}/\mathscr{S}^+$ is isomorphic to the group dual to $\mathscr{B}^+/\mathscr{A}$, hence $|\mathscr{S}/\mathscr{S}^+| = |\mathscr{B}^+/\mathscr{A}|$.

2.4 Reduction of the Group of Configurations. In some systems the space of configurations can be reduced. Let $\chi = \times_{a \in \mathbb{L}} \mathscr{G}$ be the configuration space of the system.

Let

$$\mathscr{G}_a = \operatorname{Ker}(pr_{\{a\}}\mathscr{B}) \quad \text{and} \quad \chi' = \times_{a \in \mathbb{L}} \mathscr{G}_a.$$

Since $\chi' \subset Cl\mathscr{S}_f$, any Gibbs state of the system is χ' invariant [5].

Let

$$\chi'' = \chi/\chi' = \times_{a \in \mathbb{L}} (\mathscr{G}/\mathscr{G}_a).$$

To describe the family of Gibbs states of the system it is enough to take χ'' as the configuration space. In particular, it can be assumed without loss of generality that the least common multiple of the orders of elements of $\mathscr{B}$ is equal to the least common multiple of the orders of elements of $\mathscr{G}$. In the case of $\mathbb{L} = \mathbb{Z}^\nu$ and $\mathscr{G} = \mathbb{Z}_m$ this means that there is always a non-zero divisor in $\mathscr{B}_0$ (cf. Theorem A1).

3. Algebraic Structure of the System

We introduce here the notions of contours and cycles. These are generalizations of definitions from [5].

3.1 Contours and Cycles. Let $A \in \chi_f$. $\hat{A}(\chi)$ is a finite group with the multiplication of complex numbers as a group action.

Proposition 3.1. *$\hat{A}(\chi)$ is a cyclic group with order equal to $|A|$, the order of A. In particular, for every $A \in \chi_f$ there is $X_A \in \chi_f$ such that $\hat{A}(X_A) = \exp\{2\pi i/|A|\}$.*

Proof. $\hat{A}(\chi)$ is a subgroup of the group of $|\mathscr{G}|$-th roots of unity. Hence $\hat{A}(\chi)$ is cyclic as a subgroup of the cyclic group $Z_{|\mathscr{G}|}$. If for any $a \in \hat{A}(\chi)$ $a^k = 1$ then $A^k = 1$. This proves the equality of the orders.

Denote:

$$\mathcal{M} = \underset{B \in \mathcal{B}}{\times} \mathbb{Z}_{|B|}, \tag{3.1}$$

$$\mathcal{M}_f = \bigoplus_{B \in \mathcal{B}} \mathbb{Z}_{|B|}. \tag{3.2}$$

If each cyclic group is equipped with the discrete topology then $\mathcal{M}$ becomes a compact abelian group with the product topology. $\mathcal{M}_f$ with the discrete topology is a locally compact abelian group. Let m be the least common multiple of the orders of elements of $\mathcal{B}_0$. By remarks in 2.4 m is equal to the l.c.m. of the orders of elements of G. Both $\mathcal{M}$ and $\mathcal{M}_f$ are $\mathbb{Z}_m[\mathbb{Z}^\nu]$-modules, where $\mathbb{Z}_m[\mathbb{Z}^\nu]$ is the group ring of all functions from $\mathbb{Z}^\nu$ to $\mathbb{Z}_m$ with finite support. Moreover, $\mathcal{M}_f$ is a reflexive, finitely generated module. χ and χ_f are also $\mathbb{Z}_m[\mathbb{Z}^\nu]$-modules. Two useful module homomorphisms can be constructed. Let

$$\gamma(X) = (\hat{B}(X))_{B \in \mathcal{B}}, \quad X \in \chi. \tag{3.3}$$

Then by Prop. 3.1 it can be written: $\gamma(X) = \alpha \in M$ where $\hat{B}(X) = \exp\{2\pi i \alpha(B)/|B|\}$, so $\gamma: \chi \to \mathcal{M}$.

Let now $\alpha \in \mathcal{M}$,

$$\varepsilon(\alpha) = \sum_{B \in \mathcal{B}} \alpha(B) B, \tag{3.4}$$

so $\varepsilon: \mathcal{M} \to \chi$. The sum converges in the topology of χ because the interaction is of finite range. It is easy to see that both γ and ε are continuous module homomorphisms.

The following definitions are standard:

$$\Gamma = \mathrm{Im}(\gamma) \quad \Gamma_f = \gamma(\chi_f), \tag{3.5a}$$

$$K = \mathrm{Ker}(\varepsilon) \quad K_f = \kappa \cap \mathcal{M}_f. \tag{3.5b}$$

From the definition $S = \mathrm{Ker}(\gamma)$, $\mathcal{A} = \varepsilon(\mathcal{M}_f)$ and let $\mathcal{S}_f = \mathcal{S} \cap \chi_f$. By the continuity of ε and γ, $\mathcal{K}$ and $\mathcal{S}$ are closed subgroups of $\mathcal{M}$. Because χ is compact so is Γ and as a subset of a Hausdorff space $\mathcal{M}$, Γ is closed. By the density of χ_f in χ, Γ_f is dense in Γ. In general, $\mathcal{K}_f$ and $\mathcal{S}_f$ are not dense in $\mathcal{K}$ and $\mathcal{S}$ respectively. It is shown later that the density of $\mathcal{K}_f$ in $\mathcal{K}$ is equivalent to the so-called decomposition property and the density of $\mathcal{S}_f$ in $\mathcal{S}$ to the absense of phase transitions at low temperatures. Elements of Γ_f are called contours and elements of $\mathcal{K}_f$ are called cycles.

3.2 Bicharacters on $\chi \times \chi_f$ and $\mathcal{M} \times \mathcal{M}_f$. It is known that χ, χ_f and $\mathcal{M}, \mathcal{M}_f$ are mutually dual groups. For $X \in \chi$ and $Y \in \chi_f$ we have:

$$\langle X, Y \rangle \equiv \hat{Y}(X) = \prod_{a \in L} \exp\left\{2\pi i \left(\sum_{i=1}^{r} X_i(a) Y_i(a)\right) \Big/ |\mathcal{G}_i| \right\}, \tag{3.6}$$

where $\mathcal{G} = \bigoplus_{i=1}^{r} \mathcal{G}_i$ is the decomposition into cyclic groups,

$$\chi = \bigoplus_{i=1}^{r} \chi_i, \quad \chi_i = \underset{a \in \mathbb{L}}{\times} \mathcal{G}_i, \quad X = \sum_{i=1}^{r} X_i, \quad X_i \in \chi_i;$$

similarly for $\alpha_1 \in \mathcal{M}$ and $\alpha_2 \in \mathcal{M}_f$:

$$\langle \alpha_1, \alpha_2 \rangle = \prod_{B \in \mathcal{B}} \exp\{2\pi i \alpha_i(B)\alpha_2(B)/|B|\}. \tag{3.7}$$

Proposition 3.2.

(a) $\langle \gamma(X), \alpha \rangle = \langle X, \varepsilon(\alpha) \rangle$, *where* $X \in \chi, \alpha \in \mathcal{M}_f$,
(b) $\langle \alpha, \gamma(X) \rangle = \langle \varepsilon(\alpha), X \rangle$, *where* $X \in \chi_f, \alpha \in \mathcal{M}$.

Proof. Denote $\gamma(X) = \beta \in \mathcal{M}$.

$$\begin{aligned}\langle \gamma(X), \alpha \rangle &= \prod_{B \in \mathcal{B}} \exp\{2\pi i \beta(B)\alpha(B)/|B|\} = \prod_{B \in \mathcal{B}} B(X_B)^{\beta(B)\alpha(B)} \\ &= \prod_{B \in \mathcal{B}} \hat{B}(X_B)^{\beta(B)\alpha(B)} = \prod_{B \in \mathcal{B}} B(X)^{\alpha(B)} = \langle X, \varepsilon(\alpha) \rangle.\end{aligned}$$

The second part of the proposition can be proven in the same manner. ■

Let $\mathcal{N} \subset \mathcal{M}_f$ (χ_f) then

$$cl\mathcal{N} \equiv Cl\mathcal{N} \cap \mathcal{M}_f (Cl\mathcal{N} \cap \chi_f),$$
$$\mathcal{N}^\perp \equiv \{\alpha \in \mathcal{M}(\chi) : \langle \alpha, \beta \rangle = 1 \quad \text{for all} \quad \beta \in \mathcal{N}\}.$$

If $\mathcal{N} \in \mathcal{M}(\chi)$ then

$$\mathcal{N}^\perp \equiv \{\alpha \in \mathcal{M}_f(\chi_f) : \langle \beta, \alpha \rangle = 1 \quad \text{for all} \quad \beta \in \mathcal{N}\},$$
$$\Gamma_f^0 \equiv \{f : f : \mathcal{M}_f \to \mathbb{Z}_m[\mathbb{Z}^\nu], f(\beta) = 0 \quad \text{for every} \quad \beta \in \Gamma_f\}.$$

It can be proven [6, 15] that $cl\Gamma_f = \{\alpha \in \mathcal{M}_f : f(\alpha) = 0$ for every $f \in \Gamma_f^0\}$.

Proposition 3.3.

(a) $\Gamma_f^\perp = \mathcal{K}$, $\mathcal{K}^\perp = \Gamma_f$, $\mathcal{K}_f^\perp = \Gamma$, $\Gamma^\perp = \mathcal{K}_f$,
(b) $(Cl\mathcal{K}_f)^\perp = cl\Gamma_f$,
(c) $\mathcal{S} = \mathcal{A}^\perp$,
(d) $(Cl\mathcal{S}_f)^\perp = cl\mathcal{A}$,
(e) $\mathcal{B}^+ = (\mathcal{S}^+)^\perp$,
(f) $\mathcal{S}/\mathcal{S}^+ \simeq (\mathcal{B}^+/\mathcal{A})$.

Proof. Let $\alpha \in \mathcal{M}_f$ then by Proposition 3.2(a) $\alpha \in \Gamma^\perp$ iff $\varepsilon(\alpha) = e_\chi$, so $\alpha \in \mathcal{K}_f$, hence $\Gamma^\perp = \mathcal{K}_f$. Because Γ is closed, $\mathcal{K}_f^\perp = \Gamma^{\perp\perp} = \Gamma$. The rest of (a) follows from Proposition 3.2 in the same way.
(b) $(Cl\mathcal{K}_f)^\perp = \mathcal{K}_f^\perp \cap \mathcal{M}_f = \Gamma \cap \mathcal{M}_f = cl\Gamma_f$.
(c) follows from Proposition 3.2(a).

To prove (d) it is enough to notice that by Proposition 3.2(b) $\mathcal{S}_f = (Cl\mathcal{A})^\perp$, so $\mathcal{S}_f^\perp = Cl\mathcal{A}$.

$$(Cl\mathcal{S}_f)^\perp = \mathcal{S}_f^\perp \cap \chi_f = cl\mathcal{A}.$$

(e) follows directly from the definition of $\mathcal{S}^+, \mathcal{B}^+$ and the fact that $\mathcal{B} \subset \mathcal{B}^+$.
(f) follows from (c) and (e). ■

4. The Decomposition Property and the Peierls Argument

There is a standard Peierls estimate for the probability of occurrence of a given contour (Lemma 4.5). It vanishes exponentially as temperature approaches zero. It can be shown that in some systems (the systems which satisfy the decomposition property) contours which are not the sum of two other contours are connected (in the sense described below). If this is the case, then in the estimate of the type (1.5), only connected contours are used. The number of connected contours can be majorized (Proposition 4.4) and again the geometric series can be summed up as in (1.5).

4.1 The Decomposition Property. Let $B\in\chi_f \underline{B}=\{a\in\mathbb{L}:B(a)\neq e_{\mathscr{G}}\}$,

$$\alpha\in\mathscr{M},\underline{\alpha}=\bigcup_{B:\alpha(B)\neq 0}\underline{B},\quad \operatorname{supp}\alpha=\{B:\alpha(B)\neq 0\}.$$

For $a\in\mathbb{L}\subset\mathbb{R}^\nu$ $a=(a_1,\dots,a_\nu)$

$$|a|=\max\{|a_1|,\dots,|a_\nu|\},$$
$$\operatorname{diam}B=\max\{|a-b|:a,b\in\underline{B}\},\quad \operatorname{diam}\alpha=\operatorname{diam}\underline{\alpha}.$$

For $B_1,B_2\in\chi_f$

$$\delta(B_1,B_2)=\operatorname{dist}(\underline{B}_1,\underline{B}_2)=\inf\{|a-b|:a\in\underline{B}_1,b\in\underline{B}_2\},$$
$$\operatorname{mesh}\mathscr{B}_0=\max\{\operatorname{diam}\underline{B}:B\in\mathscr{B}_0\}.$$

Let N be an integral greater than the range of the interaction. α can be treated as a graph $(V,E)_N$, where

$$V=\{\underline{B}:\alpha(B)\neq 0\},$$
$$E=\{(U,W):U,W\in V,\quad \delta(U,W)\leqq N\}.$$

The components of this graph will be called N-components.

Definition 4.1. The interaction has the decomposition property if there exists a non-negative integer N such that the N-components of contours are again contours. In other words the decomposition property holds if and only if there exists an integer N such that for each $X\in\chi_f$ there exists an integer $n=n(X)$ and $X_1,\dots,X_n\in\chi_f$ such that:

(a) $\gamma(X_i)$ is N-connected,
(b) $\operatorname{supp}\gamma(X_i)\cap\operatorname{supp}\gamma(X_j)=\phi,\quad 1\leqq i<j\leqq n,$
(c) $\sum_{i=1}^{n}\gamma(X_i)=\gamma(X)$.

In that case $\gamma\left(\sum_{i=1}^{n}X_i\right)=\gamma(X)$ so $X=\sum_{i=1}^{n}X_i+Y$, where $Y\in\mathscr{S}_f$ and if $pr_\Lambda X\notin pr_\Lambda\mathscr{S}_f$ then there is i such that $pr_\Lambda X_i\notin pr_\Lambda\mathscr{S}_f$. If $\alpha_{1|\operatorname{supp}\alpha_1}=\alpha_{2|\operatorname{supp}\alpha_1}$ then we write $\alpha_1\subset\alpha_2$.

4.2 Estimates and the Main Theorem. For finite $\Lambda\subset\mathbb{L}$, let

$$\Gamma(N,\Lambda,l)=\gamma(\{X\in\chi_f:\gamma(X)\text{ is }N\text{-connected},\ |\gamma(X)|=l,pr_\Lambda X\notin pr_\Lambda\mathscr{S}_f\})$$

$$\Gamma(N,\Lambda)=\bigcup_{l=1}^{\infty}\Gamma(N,\Lambda,l).$$

The following lemmas and theorems are generalizations of the corresponding theorems in [6]. For proofs see [6, 14].

Lemma 4.2. *Let l be a natural number and let $B \in \mathscr{B}$. The number of α's such that $\alpha \in \mathscr{M}_f$ and is N-connected, $|\alpha| = l$, and $B \in \operatorname{supp} \alpha$ is not greater than $[(2b + 2N + 1)^\nu b]^{2l-2}$, where $b = \max(|\mathscr{B}_0||G|, \text{mesh } \mathscr{B}_0)$.*

Lemma 4.3. *For any $B \in \mathscr{B}$ the number of translates of B contained in $\{\operatorname{supp} \alpha : \alpha \in \Gamma(N, \Lambda, l)\}$ is not greater than $[2(l(b + N) + m) + \operatorname{diam} \Lambda]^\nu$.*

Now it follows immediately that:

Proposition 4.4.

$$\operatorname{Card} \Gamma(N, \Lambda, l) \leqq b[(2b + 2N + 1)^\nu b]^{2l-2} [2(l(b + N) + m) + \operatorname{diam} \Lambda]^\nu .$$

For $\Lambda \subset \Lambda'$ we have the standard Peierls estimate:

Lemma 4.5.

$$\rho^+_{\Lambda'}(\{X, \alpha \subset \gamma(X)\}) \leqq \exp\left\{ -c \sum_{B \in \operatorname{supp} \alpha} K(B) \right\},$$

where $c = 2 - 2\cos(1/|\mathscr{G}|)$, $\alpha \in \Gamma(N, \Lambda, l)$ and $K(B) > 0$ for any $B \in \mathscr{B}$.

The estimates can be applied to obtain the following theorem:

Theorem 4.6. *The decomposition property implies that for any finite $\Lambda \subset L$ and $\varepsilon > 0$, for low enough temperatures*

$$\rho^+(\{X \in \chi : pr_\Lambda X \in pr_\Lambda S_f\}) > 1 - \varepsilon .$$

Corollary 4.7. *For systems with the decomposition property $\mathscr{S}^+ = Cl \mathscr{S}_f$ at low temperatures.*

Corollary 4.8. *For systems with the decomposition property $\mathscr{B}^+ = cl \mathscr{A}$ at low temperatures.*

4.3 Criteria for the Decomposition Property to Hold. The following two theorems are due Slawny [6].

Theorem 4.9. *The following conditions are equivalent to the decomposition property:*
(a) $Cl \mathscr{K}_f = \mathscr{K}$,
(b) $cl \Gamma_f = \Gamma_f$,
(c) $\mathscr{K}_f^\perp \cap \mathscr{M}_f = \Gamma_f$.

Theorem 4.10. *If the symmetry group $\mathscr{S}$ is finite and the dimension of the lattice is greater than or equal to 2 then the decomposition property holds.*

In the case of $\mathbb{L} = \mathbb{Z}^\nu$, the so-called transitive case, a useful criterion can be obtained. Let us begin with $\mathscr{G} = \mathbb{Z}_m$, where m is a product of different prime numbers. χ_f can be identified with the group ring $\mathbb{Z}_m[\mathbb{Z}^\nu]$ of all functions from $\mathbb{Z}^\nu$ to $\mathbb{Z}_m$ with finite support. One can generalize the result of [5] to get the following theorem [14]:

Theorem 4.11. *The system has the decomposition property if and only if* g.c.d. $(\mathscr{B}_0)$ *is a unit.*

Now let $\mathscr{A}$ be an ideal in $\mathbb{Z}_m[\mathbb{Z}^\nu]$, m any natural number and let $\mathscr{B}_0 = \{B_1, \ldots, B_n\}$ be a finite family of its generators ($\mathbb{Z}_m[\mathbb{Z}^\nu]$ is a Noetherian ring so there is at least one such family). The lattice system with $\mathscr{B}_0$ as a fundamental family of bonds has the decomposition property if it has this property with respect to any other finite family of generators of $\mathscr{A}$. To see this it is enough to combine Theorem 4.9(b) with the fact that $\mathscr{S} = \mathscr{A}^\perp$. If this is the case then $\mathscr{A}$ satisfies the decomposition property or simply $\mathscr{A}$ is reduced.

Proposition 4.12. *Let $\mathscr{A}_1 \subset \mathscr{A}_2$ be the two ideals in $Z_p[Z^\nu]$ where p is a prime number then if $\mathscr{A}_1$ is reduced then $\mathscr{A}_2$ is reduced.*

Proof. If $\mathscr{A}_1$ is reduced then by Theorem 4.11 g.c.d. of its generators is a unit. Then g.c.d. of the generators of $\mathscr{A}_2$ is also a unit so $\mathscr{A}_2$ is reduced by the same theorem. ∎

Let $\mathscr{A}$ be the ideal in $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$, p-prime and $\mathscr{A}^p \equiv \{A \in \mathscr{A} : pA = 0\}$, $\mathscr{S}(\mathscr{A}^p) \equiv (\mathscr{A}^p)^\perp, \mathscr{S} \equiv \mathscr{A}^\perp$.

Proposition 4.13. $\mathscr{S}(\mathscr{A}^p) = p\chi + \mathscr{S}$, where $\chi = \underset{i \in \mathbb{Z}^\nu}{\times} \mathbb{Z}_{p^n}$.

Proof. Let $\mathscr{Y} = \chi / \mathscr{S}$ so $\hat{\mathscr{Y}} \simeq \mathscr{A}$. It will be shown that $\mathscr{A}^p = (p\mathscr{Y})^\perp$. Really for every $Y \in \mathscr{Y}$ and $A \in \mathscr{A}, \langle pA, Y \rangle = \langle A, pY \rangle$. Let $A \in (p\mathscr{Y})$ then $\langle pA, Y \rangle = 1$ for every $Y \in \mathscr{Y}$, so $pA = 0$, hence $A \in \mathscr{A}^p$. Conversely if $A \in \mathscr{A}^p$ then $\langle A, pY \rangle = 1$ for every $Y \in \mathscr{Y}$, so $A \in (p\mathscr{Y})^\perp$.

Now let $f : \chi \to \mathscr{Y}$ be a canonical homomorphism and $(\mathscr{A}^p)^\perp$ the annihilator of $\mathscr{A}^p$ in χ, then $(\mathscr{A}^p)^\perp = f^{-1}(p\mathscr{Y})$ and finally $(\mathscr{A}^p)^\perp = p\chi + \mathscr{S}$. ∎

By Theorem 4.9(b) the decomposition property holds if and only if $cl\Gamma_f = \Gamma_f$. This is equivalent to the following implication:

$$\gamma(X) \text{ is finite} \to X = Y + S \quad X \in \chi, \quad Y \in \chi_f, \quad X \in \mathscr{S}.$$

Now in the case of $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$ we prove the following lemma:

Lemma 4.14. *If $p\mathscr{A}$ is reduced then $\mathscr{A}$ is reduced.*

Proof. Let $\gamma(X)$ be finite; then obviously $\gamma^p(X)$ is finite, where $\gamma^p(X)$ is a contour for the $\mathscr{A}^p$ system for a fixed choice of generators of $\mathscr{A}^p$ as bonds. By the assumption and Lemma 4.12, $\mathscr{A}^p$ is reduced. By Proposition 4.13 $X = Y_1 + S_1 + pX_1$, where $Y_1 \in \chi_f$, $S_1 \in S, X_1 \in \chi$. By the general assumption (cf. 2.4) there is a bond in the $\mathscr{A}$ system whose order is equal to p^n. Now it is easy to see that $\gamma_p(X)$ is finite, where γ_p is a contour for the $p\mathscr{A}$ system. $X_1 = Y_2 + S_2$, where $Y_2 \in \chi_f, S_2 \in \mathscr{S}(p\mathscr{A}) \equiv (p\mathscr{A})^\perp$ and obviously $pS_2 \in \mathscr{S}$. Finally $X = Y_1 + pY_2 + \mathscr{S}_1 + pS_2$ and hence $\mathscr{A}$ is reduced. ∎

Theorem 4.15. *$\mathscr{A}$ is reduced if and only if p^{n-1} $\mathscr{A}$ is reduced.*

Proof. the if part

Let us introduce $\mathscr{A}^{p^k} \equiv \{A \in p^{k-1}\mathscr{A} : pA = 0\} \quad k = 1, \ldots, n-1.$

$$p^{n-1}\mathscr{A} \subset \mathscr{A}^{p^{n-1}} \subset \mathscr{A}^{p^{n-2}} \subset \cdots \subset \mathscr{A}^p. \tag{4.1}$$

It is easy to see that p^{n-1} $\mathscr{A}$ and $\mathscr{A}^{p^k}$ are reduced in $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$ if and only if $(1/p^{n-1})$ $[p^{n-1}\mathscr{A}]$ and $(1/p^{n-1})\mathscr{A}^{p^k}$ are reduced in $\mathbb{Z}_p[\mathbb{Z}^\nu]$.

Now if $p^{n-1}\mathscr{A}$ is reduced then by (4.1) and Proposition 4.12 $\mathscr{A}^{p^k}$ is reduced for every $k=1,\dots,n-1$.

$$p^{n-1}\mathscr{A},\mathscr{A}^{p^{n-1}}\subset p^{n-2}\mathscr{A},$$

so by Lemma 4.14 $p^{n-2}\ \mathscr{A}$ is reduced.

$$p^{n-1}\mathscr{A},\mathscr{A}^{p^{n-1}}\subset p^{n-k-1}\mathscr{A};\quad k=2,\dots,n-1,$$

so using the lemma $n-2$ times more $\mathscr{A}$ is reduced.

The only if part (by contraposition)

Assume that $p^{n-1}B=p^{n-1}(B/D)D$ for $B\in\mathscr{B}_0$-family of generators of $\mathscr{A}$, where $B/D, D\in\mathbb{Z}_p[\mathbb{Z}^\nu]$ and D is not a unit. Let

$$\gamma_0=p^{n-1}\sum_{B\in\mathscr{B}}(I(B)/I(D))\alpha_B, \tag{4.2}$$

where $\alpha_B\in\mathscr{M}_f$; $\alpha_B(B')=\delta_{B,B'}, B'\in\mathscr{B}$.

$\gamma_0\neq\Gamma_f$ because otherwise for $B\in\mathscr{B}_0$ such that $|B|=p^n$

$$p^{n-1}(I(B)/I(D))=RI(B)\quad\text{in}\quad \mathbb{Z}_{p^n}[\mathbb{Z}^\nu], \tag{4.3}$$

where $R\in\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$.

Now it is easy to see (cf. the proof of Proposition A1) that $R=p^{n-1}R'$, $R'\in\mathbb{Z}_p[\mathbb{Z}^\nu]$ so

$$p^{n-1}(I(B))=p^{n-1}R'I(B)I(D), \tag{4.4}$$

and D is a unit in $\mathbb{Z}_p[\mathbb{Z}^\nu]$.

$\gamma_0\in cl\Gamma_f$. Really if $f\in(\Gamma_f)^0$, then

$$I(D)f(\gamma_0)=f(I(D)\gamma_0)=0. \tag{4.5}$$

The last equality follows from the fact that $I(D)\gamma_0=p^{n-1}\gamma(X)$, where $X\in\chi_f$, $X(a)=\delta_{a,0}$. Now $f(\gamma_0)=0$ because $I(D)$ is not a zero divisor as an element of $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$ (cf. 2.4). By Theorem 4.9(b) $\mathscr{A}$ is not reduced. ■

It is now easy to generalize Theorem 4.15 to the case of $\mathscr{G}=\mathbb{Z}_m$ with arbitary m. Let $m=\prod_{i=1}^n p_i^{k_i}$, where p_i are prime numbers. If $X\in\chi$ then

$$X=\sum_{i=1}^n(m/p_i^{k_i})X_i,\quad X_i\in\times_{j\in\mathbb{L}}\mathbb{Z}_{p_i^{k_i}},$$

$\mathscr{B}_i=\{(m/p_i^{k_i})B, B\in\mathscr{B}\}$, where $\mathscr{B}$ is the family of bonds. If $\gamma(X)$ is a contour with respect to $\mathscr{B}$ then $\gamma_i(X)$ is a contour with respect to $\mathscr{B}_i$.

Lemma 4.16. *$\gamma(X)\in cl\Gamma_f$ iff $\gamma_i(X)\in cl\Gamma_{fi}$ for all i.*

Proof. It follows easily from the properties of prime numbers. ■

Theorem 4.17. *The system $\mathscr{B}$ has the decomposition property* iff *$\mathscr{B}_i$ has the decomposition property for every i.*

Proof. the if part

Assume that $\gamma(X)$ is finite then by Lemma 4.16 $\gamma_i(X)$ is finite for $i=1,\dots,n$.

Because $\mathscr{B}_i$ is reduced, there is $Y^i \in \chi_f$ such that $\gamma_i(Y^i) = \gamma_i(X)$ and we can choose $(Y^i)_j = 0$ if $i \neq j$.

Let

$$y = \sum_{i=1}^{n} Y^i \in \chi_f.$$

$\gamma(Y) = \gamma(X)$ hence $\mathscr{B}$ is reduced.

the only if part

Assume that $\gamma_i(X)$ is finite, then $\gamma((m/p_i^{k_i})X_i)$ is finite, so there is $Y \in \chi_f$ such that $\gamma((m/p_i^{k_i})X_i) = \gamma(Y)$ and finally $\gamma_i(X) = \gamma_i(Y)$. ∎

The $\mathscr{B}_i$ system can be investigated by Theorem 4.15. Theorem 4.17 is obviously true for any lattice. However, in the case of $\mathbb{Z}_{p^n}$ a complete characterization of the decomposition property is not available.

5. Non-Gauge Models

If one can change the configuration locally without changing its contours then the model is called a guage model. This is equivalent to non-triviality of $\mathscr{S}_f$ and non-injectivity of γ on χ_f.

Theorem 5.1. *If the symmetry group $\mathscr{S}$ of the non-gauge model is finite then there are $|\mathscr{S}|$ pure phases at low temperatures.*

Proof. $\mathscr{S}$ is finite so the decomposition property holds and $\mathscr{S}^+ = Cl\mathscr{S}_f$. If $\mathscr{S}_f$ is non-trivial then it is infinite so $\mathscr{S}_f$ is trivial because $\mathscr{S}$ is finite. Finally $\mathscr{S}^+$ is trivial or equivalently $\mathscr{B}^+ = \chi_f$ at low temperatures. ∎

It is a well known fact that in the case of spin 1/2 there are no gauge models on the $\mathbb{L} = \mathbb{Z}^\nu$ lattice. This is also true in a more general setting.

Theorem 5.2. *There are no gauge models on $\mathbb{L} = \mathbb{Z}^\nu$ in the $\mathscr{G} = \mathbb{Z}_m$ case.*

Proof. First consider the case of $\mathscr{G} = \mathbb{Z}_{p^n}$. Let $X \in \chi_f, X \neq e_\chi$ and l the largest number such that p^l divides $pr_{\{a\}} X \equiv X(a)$ for all $a \in \chi$. Let b be the last (in the sense of lexicographic order) element of χ such that p^{l+1} does not divide $X(b)$. Now take $B \in \mathscr{B}_0$ such that p does not divide $B(a)$ for some $a \in \mathscr{B}$ (such B exists by the assumption about the interaction, cf. 2.4). Let a' be the first of such a's and $B_{a'}$ the translate of B by $b - a'$. It can be shown that $\hat{B}_{a'}(X) \neq 1$.

$$\hat{B}_{a'}(X) = \exp\left\{\left(2\pi i \sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a)\right) \Big/ p^n\right\}$$

so $\hat{B}_{a'}(X) = 1$ if and only if $\sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a) = 0 \pmod{p^n}$.

$$\sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a) = B_{a'}(b) X(b) + \sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a),$$

and p^{l+1} divides $\sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a)$ and does not divide $B_{a'}(b) X(b)$ so

$$\sum_{a \in \mathbb{Z}^\nu} B_{a'}(a) X(a) \neq 0 \pmod{p^n}.$$

This finishes the proof for the $\mathscr{G} = \mathbb{Z}_{p^n}$ case.

Assume now that $\mathscr{G} = \mathbb{Z}_m$ so $\mathscr{G} = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \mathbb{Z}_{p_k^{n_n}}$, where the p_i's are different prime numbers. If $X \in \chi_f, X \neq e_\chi$ denote $X = (X_1, \ldots, X_k)$ where $X_i \in \chi_{fi}$ $\chi_{fi} = \bigoplus_{j \in \mathbb{Z}^\nu} Z_{p_i^{n_i}}$. There exists l, $1 \leqq l < k$ such that $X_l \neq e_{\chi fi}$. From the previous case we know that there is some $B \in \mathscr{B}_0$ such that $\hat{B}(e_{\chi_1}, \ldots, X_l, \ldots, e_{\chi_k}) \neq 1$. It is easy to see that $\hat{B}(X) \neq 1$. ■

6. Reduction

The results of [5] are generalized here to the case of $\mathscr{G} = \mathbb{Z}_m$ on $\mathbb{Z}^\nu$, where m is the product of different prime numbers.

Let $\mathscr{B}$ be a translation invariant family of elements of χ_f with a finite fundamental subfamily $\mathscr{B}_0$. Let $D \in \mathbb{Z}_m[\mathbb{Z}^\nu]$ and suppose that D is not a zero divisor.

Let

$$\mathscr{B}' = \{DB : B \in \mathscr{B}\}.$$

Let K be a ferromagnetic translation invariant interaction with bonds $\mathscr{B}$ and let $K'(DB) = K(B)$; $B \in \mathscr{B}$. The following theorem relates the systems with interactions K and K' respectively.

Theorem 6.1. *If ρ^+ and ρ'^+ are the equilibrium states corresponding to the interactions K and K' respectively, then*

$$\rho'^+(\widehat{DA}) = \rho^+(\hat{A}) \quad \text{for all} \quad A \in \chi_f, \tag{6.1}$$

$$\rho'^+(\hat{A}) = 0 \quad \text{if} \quad A \notin (D), \tag{6.2}$$

where $(D) = \{DA : A \in \chi_f\}$.

The proof follows exactly the proof of the corresponding theorem in [5] and is contained in [14].

Let now K be any ferromagnetic, translation invariant interaction of finite range and let $D = \text{g.c.d.}\,(\mathscr{B})$. Let K' be the interaction with bonds $\mathscr{B}'$ obtained by factoring out D from the bonds of $\mathscr{B}$: $\mathscr{B} = \{DB : B \in \mathscr{B}\}$ and such that $K'(B) = K(DB)$. Obviously, g.c.d. $(\mathscr{B}')$ is a unit. Applying Theorem 6.1 and then Theorem 4.11 and Theorem 5.1 to the system with interaction K'—the so-called reduced system—we obtain

Theorem 6.2. *For any temperature*

$$\rho^+(\widehat{DA}) = \rho'^+(\hat{A}) \quad \text{for all} \quad A \in \chi_f, \tag{6.3}$$

$$\rho^+(\hat{A}) = 0 \quad \text{if} \quad A \notin (D), \tag{6.4}$$

where ρ^+, ρ'^+ correspond to K and K' respectively. In particular, for low enough temperatures, $\mathscr{B}^+ = (D)$.

Appendix

Proposition A1. *If $D \in \mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$ is such that there exists $a \in \mathbb{Z}^\nu$, $D(a) \neq 0$ and p does not divide $D(a)$ ($D(a)$ is not a zero divisor in $\mathbb{Z}_{p^n}$) then D is not a zero divisor in $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$.*

Proof. Let A be any non-zero element from $\mathbb{Z}_{p^n}[\mathbb{Z}^\nu]$. It will be shown that $DA \neq 0$. Let z be the least (in the sense of lexicographic order) $a \in \mathbb{Z}^\nu$ such that $D(a) \neq 0$ and p does not divide $D(a)$. Let k be the least number such that there exists $b \in \mathbb{Z}^\nu$ such that $A(b) = sp^k$ and p does not divide s. Let w be the least of such b's. It will be shown that $(DA)(z+w) \neq 0$.

$$(DA)(z+w) = D(z)A(w) + \sum_{\substack{a_1, a_2 \\ a_1 + a_2 = z + w \\ a_1 \neq z, a_2 \neq w}} D(a_1)A(a_2)$$

$$= D(z)A(w) + \sum_{\substack{a_1, a_2 \\ a_1 + a_2 = z + w \\ a_1 < z, a_2 > w}} D(a_1)A(a_2) + \sum_{\substack{a_1, a_2 \\ a_1 + a_2 = z + w \\ a_1 > z, a_2 < w}} D(a_1)A(a_2).$$

Now it is easy to see that $(DA)(z+w)$ is not a multiple of p^n because it is not a multiple of p^{k+1}. ■

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