Highest Weight Representations of the Neveu-Schwarz and Ramond Algebras

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Abstract. We construct a family of representations $\mathscr{H}^{\xi,w}$ of the Neveu-Schwarz and Ramond algebras, which generalize the Fock representations of the Virasoro algebra. We show that the representations $\mathscr{H}^{\xi,w}$ are intertwined by a vertex operator.

The above results are used to give the proof of the conjectured formulas for the determinant of the contravariant form on the highest weight representations of the Neveu-Schwarz and Ramond algebras. Further results on the representation theory of the latter are derived from the determinant formulas.

1. Introduction

In Superstring theory physicists consider two supersymmetric extensions of the Lie algebra of vector fields on the circle $(\mathrm{Vect}(S^1))$ called the Neveu-Schwarz [19] and Ramond [20] algebras. The Neveu-Schwarz algebra has basis $\{L'_0, L_i, G_j\}$ $(i \in \mathbb{Z}, j \in \frac{1}{2} + \mathbb{Z})$, where L'_0 is central, and the bracket of two noncentral generators is given by the relations

$$[L_i, L_j] = (j-i)L_{i+j} + \delta_{i,-j}\left(\frac{i^3-i}{8}\right)L'_0,$$

$$[G_i, G_j] = -2L_{i+j} + \delta_{i,-j}\tfrac{1}{2}(i^2 - \tfrac{1}{4})L'_0,$$

$$[L_i, G_j] = (j - \tfrac{1}{2}i)G_{i+j}.$$

The Ramond algebra has the same relations, but the G_j are indexed by $\mathbb{Z}$. These algebras are "$\mathbb{Z}_2$-graded Lie algebras," i.e., $\mathbb{Z}_2$-graded vector spaces with a grading

preserving bracket operation that satisfies $\mathbb{Z}_2$-graded versions of anticommutativity and Jacobi's identity.

In [13, 14] Kac gave a formula for the determinant of the Shapovalov form for (a central extension of) Vect(S^1). Kac's formula was later proved by Feigin and Fuchs ([2])[1]. In [14] Kac also gave a brief introduction to the highest weight theory of the Neveu-Schwarz algebra, and stated (without proof) a formula for its Shapovalov determinant. An interesting application of this formula is found in the recent work of Friedan et al. ([8]) where the formula is used to obtain a series of (conjectured) unitary representations. In [8] the authors also considered the Ramond algebra, for which they conjectured a determinant formula. In the present paper we prove the determinant formulas for both the Neveu-Schwarz and Ramond algebras[2] (Sect. 4). We then use the determinant formulas to prove that the highest weights of a Verma module and that of an irreducible subquotient are "linked" (Theorem 5.1). For the Neveu-Schwarz algebra this result was first stated by Kac [14].

Our proof of the determinant formulas follows the general lines of that of Feigin and Fuchs for Vect(S^1) and is, like theirs, based on the construction of a concrete family of representations. Rather than generalizing the construction of [2], however, we generalize the one obtained by Goodman and Wallach [11]. One can show that these constructions are isomorphic via the boson-fermion correspondence [see Frenkel, I.B.: J. Funct. Anal. **44**, 259–327 (1981)]. In this approach, the operator that was used in [2] to obtain highest weight vectors becomes a vertex operator.

We would like to point out that the theory of Verma modules for the Ramond algebra, developed in this paper, is quite a bit different from that of the Neveu-Schwarz algebra. For example, the Ramond algebra admits "half" Verma modules as well as Verma modules. The former explain the extra factor in the determinant formula of this algebra.

Finally, we hope that our findings on the Ramond algebra, as well as our detailed follow-up and proofs of the results announced in [14] can serve as a basis for further investigations on the subject.

In Sect. 2 we review the background material on $\mathbb{Z}_2$-graded Lie algebras and their representations that will be needed in the paper. In Sect. 3 we present some basic results on the Neveu-Schwarz and Ramond algebras and their highest weight representations. The determinant formulas are stated at the end of this section. In Sect. 4 we construct the representations $\mathscr{H}^{\xi,w}$ (Theorem 4.2) and prove that they are intertwined by a vertex operator (Theorem 4.4). We then use the latter to obtain maximal weight vectors in $\mathscr{H}^{\xi,w}$ (Corollary 4.8). We conclude this section with the proof of the determinant formulas which consists of two parts: The degree computation (Proposition 4.10) and the proof of the vanishing of the determinant on the variety of its irreducible factors (Proposition 4.13). In Sect. 5 we construct Jantzen type filtrations following the general lines of [23], and then use them to obtain character sum formulas (Proposition 5.2). The latter imply Theorem 5.1

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[1] After this work was completed we received reference [27]

[2] After this work was completed, D. Friedan informed us that Curtright and Thorn had also succeeded to prove these formulas

mentioned above. Finally, in the Appendix, we extend to $\mathbb{Z}_2$-graded Lie algebras, some definitions and classic results about central extensions of Lie algebras.

We denote by $\mathbb{N}$ the set of positive integers, $\mathbb{Z}_+$ the set of nonnegative integers, and set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

2. Preliminaries

First we recall the necessary definitions pertaining to $\mathbb{Z}_2$-graded algebras ([1, 15, 16, 18]). All algebras and vector spaces are defined over $\mathbb{C}$.

A $\mathbb{Z}_2$-*graded algebra* is an algebra $\mathfrak{a}$ with a vector space decomposition $\mathfrak{a} = \mathfrak{a}_{\bar{0}} + \mathfrak{a}_{\bar{1}}$ such that $\mathfrak{a}_\alpha \cdot \mathfrak{a}_\beta \subset \mathfrak{a}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. The elements of $\mathfrak{a}_{\bar{0}}$ are called *even* and those of $\mathfrak{a}_{\bar{1}}$ are called *odd*. By a subalgebra of a $\mathbb{Z}_2$-graded algebra we will always mean a $\mathbb{Z}_2$-graded subalgebra. Also, by a homomorphism $\Phi: \mathfrak{a} \to \mathfrak{a}'$ of $\mathbb{Z}_2$-graded algebras we will always mean one that preserves the $\mathbb{Z}_2$-grading, i.e., such that $\Phi(\mathfrak{a}_\alpha) \subset \mathfrak{a}'_\alpha$ for all $\alpha \in \mathbb{Z}_2$.

A $\mathbb{Z}_2$-*graded Lie algebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{l} = \mathfrak{l}_{\bar{0}} \oplus \mathfrak{l}_{\bar{1}}$ equipped with a bilinear map $[\,,\,]$ such that $[\mathfrak{l}_\alpha, \mathfrak{l}_\beta] \subset \mathfrak{l}_{\alpha+\beta}$, $\alpha, \beta \in \mathbb{Z}_2$, and satisfying the following conditions:

(L1) $[X, Y] = -(-1)^{\alpha\beta}[Y, X]$, $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$,

(L2) $(-1)^{\alpha\gamma}[X, [Y, Z]] + (-1)^{\beta\alpha}[Y, [Z, X]] + (-1)^{\gamma\beta}[X, [Z, Y]] = 0$,
$X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $Z \in \mathfrak{l}_\gamma$, $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

The subalgebra $\mathfrak{l}_{\bar{0}}$ of $\mathfrak{l}$ is clearly a Lie algebra in the usual sense.

Note. We warn the reader that the terminology for the objects just defined is varied. [15, 16] use the term "Lie superalgebras." In [18] the authors consider what we would call "$\mathbb{Z}_+$-graded Lie algebras" and refer to them simply as "Lie algebras."

If $\mathfrak{a}$ is an associative $\mathbb{Z}_2$-graded algebra, set $[X, Y] = XY - (-1)^{\alpha\beta} YX$, $X \in \mathfrak{a}_\alpha$, $Y \in \mathfrak{a}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$. We denote by $\mathfrak{a}_L$ the resulting $\mathbb{Z}_2$-graded Lie algebra.

The *universal enveloping* algebra of a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{l}$ is a pair $(U(\mathfrak{l}), \sigma)$ where $U(\mathfrak{l})$ is an associative $\mathbb{Z}_2$-graded algebra and $\sigma: \mathfrak{l} \to U(\mathfrak{l})$ is a linear map preserving the $\mathbb{Z}_2$-grading, with the following universal property: $\sigma[X, Y] = \sigma(X)\sigma(Y) - (-1)^{\alpha\beta}\sigma(Y)\sigma(X)$ for all $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, and given an associative $\mathbb{Z}_2$-graded algebra $\mathfrak{a}$ and a linear map $\tau: \mathfrak{l} \to \mathfrak{a}$ preserving the $\mathbb{Z}_2$-grading, and such that $\tau[X, Y] = \tau(X)\tau(Y) - (-1)^{\alpha\beta}\tau(Y)\tau(X)$ for $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, then there is a unique homomorphism $\psi: U(\mathfrak{l}) \to \mathfrak{a}$ such that $\tau = \psi \circ \sigma$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. Let, for $p \geq 1$, $(\otimes^p V)_\alpha = \sum_{\beta_1 + \ldots + \beta_p = \alpha} V_{\beta_1} \otimes \ldots \otimes V_{\beta_p}$, $\alpha, \beta_i \in \mathbb{Z}_2$, $i = 1, \ldots, p$, and set $T^p(V) = \otimes^p V$, $p \geq 1$, $T^0(V) = \mathbb{C}$. The tensor algebra $T(V) = \bigoplus_{p=0}^{\infty} T^p(V)$ is clearly a $\mathbb{Z}_2$-graded associative algebra.

Concretely, $U(\mathfrak{l}) = T(\mathfrak{l})/I$, where I is the two-sided ($\mathbb{Z}_2$-graded) ideal in $T(\mathfrak{l})$ generated by the elements of the form $X \otimes Y - (-1)^{\alpha\beta} Y \otimes X - [X, Y]$, $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, and $\sigma: \mathfrak{l} \to U(\mathfrak{l})$ is induced by the restriction to $\mathfrak{l}$ of the canonical map $\pi: T(\mathfrak{l}) \to U(\mathfrak{l})$.

Let $\{x_i\}$ $(i \in \Lambda)$ be a homogeneous basis of $\mathfrak{l}$, where Λ is a totally order set. Let $p \in \mathbb{Z}_+$. By a p-tuple of Λ we mean a sequence of p elements of Λ. We write $\emptyset$ for the 0-tuple. If $\mathbf{I}$ is a p-tuple we set $y_{\mathbf{I}} = y_{\emptyset} = 1$ in $U(\mathfrak{l})$, for $p = 0$, and $y_{\mathbf{I}} = y_{i_1} \ldots y_{i_p}$ in $U(\mathfrak{l})$ if $\mathbf{I} = (i_1, \ldots, i_p)$, $i_j \in \Lambda, j = 1, \ldots, p$, for $p \geq 1$, where $y_i = \sigma(x_i)$ for all $i \in \Lambda$. A p-tuple of Λ is said to be *admissible* if its elements are in nondecreasing order and if no index of an odd basis element appears more than once in the sequence.

Lemma 2.1 [1]. *Let* $\{x_i\}$ $(i \in \Lambda)$ *be an ordered basis of* $\mathfrak{l}$. *Then the monomials* $y_{\mathbf{I}}$, *for all the admissible p-tuples* $\mathbf{I}$ *of* Λ, *and all* $p \geq 0$, *form a basis of* $U(\mathfrak{l})$.

From Lemma 2.1 we immediately obtain the following:

Proposition 2.2. *The canonical map* $\sigma : \mathfrak{g} \to U(\mathfrak{g})$ *is injective.*

Proposition 2.2 allows us to identify x_i and y_i. Now set $x_{\mathbf{I}} = x_{i_1} \ldots x_{i_j}$ for $\mathbf{I} = (i_1, \ldots, i_j)$, $j \geq 1$, $x_{\emptyset} = 1$. By Lemma 2.1 we have

Theorem 2.3 [1]. *Let* $\{x_i\}$ $(i \in \Lambda)$ *be an ordered basis of* $\mathfrak{l}$. *The monomials* $x_{\mathbf{I}}$ *for all admissible p-tuples* $\mathbf{I}$ *of* Λ, *and all* $p \geq 0$, *form a basis of* $U(\mathfrak{l})$.

Next, we will state the Poincaré-Birkhoff-Witt theorem for $\mathbb{Z}_2$-graded Lie algebras. First, we need a few more definitions.

An associative $\mathbb{Z}_2$-graded algebra $\mathfrak{a}$ is said to be *commutative* if $[\mathfrak{a}, \mathfrak{a}] = 0$. The *symmetric algebra* of a $\mathbb{Z}_2$-graded algebra $\mathfrak{l}$ is a pair $(S(\mathfrak{l}), \varrho)$, where $S(\mathfrak{l})$ is a commutative associative $\mathbb{Z}_2$-graded algebra and $\varrho : \mathfrak{l} \to S(\mathfrak{l})$ is a linear map preserving the $\mathbb{Z}_2$-grading, with the following property: Given any pair $(\mathfrak{a}, \tau)$ where $\mathfrak{a}$ is a commutative associative $\mathbb{Z}_2$-graded algebra and $\tau : \mathfrak{l} \to \mathfrak{a}$ is a linear map preserving the $\mathbb{Z}_2$-grading, there is a unique homomorphism $\psi : S(\mathfrak{l}) \to \mathfrak{a}$ such that $\tau = \psi \circ \varrho$. Concretely, $S(\mathfrak{l}) = T(\mathfrak{l})/J$, where J is the two-sided ($\mathbb{Z}_2$-graded) ideal in $T(\mathfrak{l})$ generated by the elements of the form $X \otimes Y - (-1)^{\alpha\beta} Y \otimes X$, $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, and ϱ is induced by the restriction to $\mathfrak{l}$ of the canonical map $\varepsilon : U(\mathfrak{l}) \to S(\mathfrak{l})$. It is clear that ε maps $\mathfrak{l}$ injectively. For this reason we will identify $\varepsilon(\mathfrak{l})$ with $\mathfrak{l}$.

We define a filtration of $T(\mathfrak{l})$ by $T_p(\mathfrak{l}) = T^0(\mathfrak{l}) \oplus \ldots \oplus T^p(\mathfrak{l})$, and let $U_p(\mathfrak{l}) = \pi(T_p(\mathfrak{l}))$ [respectively $S_p(\mathfrak{l}) = \varepsilon(T_p(\mathfrak{l}))$] be the corresponding filtration of $U(\mathfrak{l})$ [respectively of $S(\mathfrak{l})$]. Consider the $\mathbb{Z}_2$-graded vector spaces $G^p(\mathfrak{l}) = U_p(\mathfrak{l})/U_{p-1}(\mathfrak{l})$ and $G(\mathfrak{l}) = \bigoplus_{i=0}^{\infty} G^i(\mathfrak{l})$, where we have set $U_{-1}(\mathfrak{l}) = (0)$. Since $U_p(\mathfrak{l}) U_q(\mathfrak{l}) \subset U_{p+q}(\mathfrak{l})$, the multiplication in $U(\mathfrak{l})$ induces a multiplication in $G(\mathfrak{l})$. The resulting associative $\mathbb{Z}_2$-graded algebra is called the graded associative $\mathbb{Z}_2$-graded algebra associated to the filtered $\mathbb{Z}_2$-graded algebra $U(\mathfrak{l})$.

Clearly, $G^0(\mathfrak{l}) = U^0(\mathfrak{l}) = \mathbb{C} \cdot 1$ and $\mathfrak{l}$ is canonically identified with $G^1(\mathfrak{l})$ thanks to Proposition 2.2. Since

$$X Y = (-1)^{\alpha\beta} YX + [X, Y]$$

in $U_2(\mathfrak{l})$, $X \in \mathfrak{l}_\alpha$, $Y \in \mathfrak{l}_\beta$, $\alpha, \beta \in \mathbb{Z}_2$, we conclude that there is a unique homomorphism $\psi : S(\mathfrak{l}) \to G(\mathfrak{l})$ such that $\psi|_{\mathfrak{l}} = $ identity. [Here, we have identified $\mathfrak{l}$ with its image in $S(\mathfrak{l})$.] ψ is called the canonical homomorphism of $S(\mathfrak{l})$ in $G(\mathfrak{l})$. The next theorem is readily obtained from Theorem 2.3.

Theorem 2.4 (Poincaré-Birkhoff-Witt) [1]. *The canonical homomorphism of $S(\mathfrak{l})$ into $G(\mathfrak{l})$ is an isomorphism.*

Remark. The Poincaré-Birkhoff-Witt Theorem for $\mathbb{Z}$-graded Lie algebras was first proved in [18].

Corollary 2.5. *Let $\mathfrak{k} \subset \mathfrak{l}$ be a subalgebra of $\mathfrak{l}$ with an ordered basis $\{x_i\}_{i \in M}$. Let $\{x_i\}_{i \in M \cup N}$ be an ordered basis of $\mathfrak{l}$ extending $\{x_i\}_{i \in M}$, and such that $i < j$ for $i \in M$, $j \in N$. Then the inclusion $\mathfrak{k} \subset \mathfrak{l}$ induces an injective homomorphism $U(\mathfrak{k}) \to U(\mathfrak{l})$, and $U(\mathfrak{l})$ is a free $U(\mathfrak{k})$-module with basis $\{x_I\}$ for all admissible p-tuples I of N, all $p \geq 0$.*

For a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ we let $\mathrm{End}(V)$ be the associative $\mathbb{Z}_2$-graded algebra of (graded) endomorphisms, $\mathrm{End}(V) = (\mathrm{End}(V))_{\bar{0}} \oplus (\mathrm{End}(V))_{\bar{1}}$, where

$$(\mathrm{End}(V))_\alpha = \{T \in \mathrm{End}(V) \mid T(V_\gamma) \subset V_{\alpha + \beta} \text{ for all } \gamma \in \mathbb{Z}_2\},$$

$\alpha \in \mathbb{Z}_2$. We set $\mathfrak{l}(V) = (\mathrm{End}(V))_L$. A *representation* π of a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{l}$ is a homomorphism $\pi : \mathfrak{l} \to \mathfrak{l}(V)$. We set $Xv = \pi(X)v$, for $X \in \mathfrak{l}$, $v \in V$. It is clear from the definition that

$$X(V_\beta) \subset V_{\alpha + \beta} \quad \text{for all} \quad X \in \mathfrak{l}_\alpha, \alpha \in \mathbb{Z}_2,$$

and that

$$[X, Y]v = XYv - (-1)^{\alpha \beta} YXv, \quad \text{for all} \quad X \in \mathfrak{l}_\alpha, Y \in \mathfrak{l}_\beta, \alpha, \beta \in \mathbb{Z}_2.$$

We will also use the terminologies of "$U(\mathfrak{l})$-module" and "$\mathfrak{l}$-module" in place of "representation of $\mathfrak{l}$." A submodule of an $\mathfrak{l}$-module is always meant to be $\mathbb{Z}_2$-graded, and a homomorphism $\Phi : M \to M'$ of $\mathfrak{l}$-modules, means one such that $\Phi(M_\alpha) \subset M'_{\psi(\alpha)}$, where $\psi : \mathbb{Z}_2 \to \mathbb{Z}_2$ is a bijection.

Let $\mathfrak{k} \subset \mathfrak{l}$ be a subalgebra of $\mathfrak{l}$, W a $\mathfrak{k}$-module. Thanks to Corollary 2.5 we can view $U(\mathfrak{k})$ as a subalgebra of $U(\mathfrak{l})$. One can consider $U(\mathfrak{l})$ as a right $U(\mathfrak{k})$-module and therefore may form the left $U(\mathfrak{l})$-module $V = U(\mathfrak{l}) \otimes_{U(\mathfrak{k})} W$. I.e., let M be the $\mathbb{Z}_2$-graded subspace of $U(\mathfrak{l}) \otimes W$ generated by the elements of form $XY \otimes v - X \otimes Yv$ for all $X \in U(\mathfrak{l})$, $Y \in U(\mathfrak{k})$ and $v \in W$. Then V is the $\mathbb{Z}_2$-graded space $(U(\mathfrak{l}) \otimes W)/M$. V is made into a $U(\mathfrak{l})$-module by setting

$$X(Y \otimes v) = XY \otimes v, X, Y \in U(\mathfrak{l}), v \in W.$$

V is called the $U(\mathfrak{l})$-*module induced from the* $U(\mathfrak{k})$-*module* W and denoted $\mathrm{Ind}_\mathfrak{k}^\mathfrak{l}(W)$. By Corollary 2.5 the homomorphism $w \mapsto 1 \otimes w$ from W to V is injective. We identify W with $1 \otimes W$ by this map.

Proposition 2.6. *Let $\mathfrak{k} \subset \mathfrak{l}$ be a subalgebra of $\mathfrak{l}$, W a $\mathfrak{k}$-module, $V = \mathrm{Ind}_\mathfrak{k}^\mathfrak{l}(W)$ and V' and $\mathfrak{l}$-module. Any $\mathfrak{k}$-homomorphism $\psi : W \to V'$ can be uniquely extended to an $\mathfrak{l}$-homomorphism $\phi : V \to V'$. Furthermore, $\psi \mapsto \phi$ is a bijection from $\mathrm{Hom}_\mathfrak{k}(W, V')$ to $\mathrm{Hom}_\mathfrak{l}(V, V')$.*

Proof. Let $\phi(X \otimes w) = X\psi(w)$. If $X \in U(\mathfrak{l})$, $Y \in U(\mathfrak{k})$, then $\phi(XY \otimes w - X \otimes Yw) = XY\psi(w) - X\psi(Yw) = 0$, so ϕ is a homomorphism from V to V'. The restriction of ϕ to $1 \otimes W$ gives a map from $\mathrm{Hom}_\mathfrak{l}(V, V')$ to $\mathrm{Hom}_\mathfrak{k}(W, V')$ inverse to $\psi \mapsto \phi$. Q.E.D.

3. Highest Weight Representations

Let $\mathfrak{ns}$ (respectively $\mathfrak{r}$) denote the complex vector space with basis $\{L_i, G_j\}$ ($i \in \mathbb{Z}$, $j \in \frac{1}{2} + \mathbb{Z}$) (respectively $\{L_i, G_j\}$ ($i, j \in \mathbb{Z}$)), and set

$$[L_i, L_j] = (j - i) L_{i+j},$$

$$[G_i, G_j] = -2L_{i+j},$$

$$[L_i, G_j] = (j - \tfrac{1}{2} i) G_{i+j}.$$

Let $\mathfrak{v} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} L_i$, and set $\widehat{\mathfrak{ns}}_{\bar{0}} = r_{\bar{0}} = \mathfrak{v}$, $\widehat{\mathfrak{ns}}_{\bar{1}} = \bigoplus_{i \in \frac{1}{2} + \mathbb{Z}} \mathbb{C} G_i$. $\mathfrak{r}_{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} G_i$. It is easy to see that $\mathfrak{ns}$ and $\mathfrak{r}$ are $\mathbb{Z}_2$-graded Lie algebras and that $\mathfrak{v}$ is a subalgebra of $\mathfrak{ns}$ and $\mathfrak{r}$. $\mathfrak{v}$ is a Lie algebra in the usual sense. Now let

$$C(L_i, L_j) = \tfrac{1}{8} \delta_{i, -j}(i^3 - i),$$

$$C(G_i, G_j) = \tfrac{1}{2} \delta_{i, -j}(i^2 - \tfrac{1}{4}),$$

$$C(L_i, G_j) = 0.$$

It is also easy to see that C satisfies the cocycle condition for $\mathbb{Z}_2$-graded Lie algebras. The central extension [3] of $\mathfrak{ns}$ (respectively $\mathfrak{r}$) corresponding to the cocycle C is called the *Neveu-Schwarz* (respectively *Ramond*) *algebra* and is denoted by $\widehat{\mathfrak{ns}}$ (respectively $\hat{\mathfrak{r}}$) (See Appendix for the definitions of cocycle condition and central extension corresponding to a cocycle, for $\mathbb{Z}_2$-graded Lie algebras.) The central extension of $\mathfrak{v}$ corresponding to $C|_{\mathfrak{v} \times \mathfrak{v}}$ is called the *Virasoro algebra* and is denoted by $\hat{\mathfrak{v}}$.

From now on, in this section, we let $\mathfrak{g}$ denote either one of the $\mathbb{Z}_2$-graded Lie algebras $\widehat{\mathfrak{ns}}$ and $\hat{\mathfrak{r}}$. The $\mathbb{Z}_2$-grading is given by

$$g_{\bar{0}} = \left(\bigoplus_{i \in \mathbb{Z}} \mathbb{C} e_i^+ \right) \oplus \mathbb{C} e_0', \quad \mathfrak{ns}_{\bar{1}} = \bigoplus_{i \in \frac{1}{2} + \mathbb{Z}} \mathbb{C} e_i^-, \quad \mathfrak{r}_{\bar{1}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} e_i^-,$$

where $e_i^+ = (L_i, 0)$, $e_i^- = (G_i, 0)$, and $e_0' = (0, 1)$ in the notation of the Appendix. (We sometimes drop the superscripts of e_i^+, e_i^- if $\mathfrak{g} = \widehat{\mathfrak{ns}}$.)

If $\mathfrak{l}$ is a Lie algebra in the usual sense, then $U(\mathfrak{l})$ is isomorphic to a polynomial ring. Therefore, by the P-B-W theorem, $U(\mathfrak{l})$ is an integral domain. If $\mathfrak{l}$ is a $\mathbb{Z}_2$-graded Lie algebra, however, $S(\mathfrak{l})$ is not in general isomorphic with a polynomial ring. Nevertheless one has the following

Proposition 3.1. $U(\mathfrak{g})$ *is an integral domain.*

Proof. We let $B = \{e_n | n \in \mathbb{Z}\} \cup \{e_m | m \in \mathbb{Z} + \frac{1}{2}\} \cup \{e_0'\}$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, and $B = \{e_n^+ | n \in \mathbb{Z}\}$ $\cup \{e_m^- | m \in \mathbb{Z}\} \cup \{e_0'\}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. We enumerate the elements of B by $X : \mathbb{Z} \to B$, where $X(0) = e_0'$, $X(2n-1) = e_{(2n-1)/2}$, $X(2n) = e_n$, $X(-2n+1) = e_{1-n}$, $X(-2n)$ $= e_{-(2n-1)/2}$, $n \in \mathbb{N}$, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, and $X(0) = e_0'$ $X(2n-1) = e_n^-$, $X(2n) = e_{n-1}^+$, $X(-2n+1) = e_{-n}^+$, $X_{(-2n)} = e_{1-n}^-$, $n \in \mathbb{N}$, if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $x_m = X(m)$, $m \in \mathbb{Z}$. By Theorem 2.3, a basis of $U(\mathfrak{g})$ consists of the monomials $x_{\mathbf{I}} = x_{i_1} \dots x_{i_m}$, where $\mathbf{I} = (i_1, \dots, i_m)$ is an admissible m-tuple of $\mathbb{Z}$. If $x_j \in g_{\bar{0}}$ and $\mathbf{I} = (i_1, \dots, i_m)$ is an

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[3] By Proposition A.2 and the results of [21] one sees that these are universal central extensions

admissible m-tuple of $\mathbb{Z}$, let $Z(j, \mathbf{I})$ denote the set of all i_k, $k = 1, \ldots, m$, such that either $i_k = j$ or $x_{i_k} \in \mathfrak{g}_{\bar{1}}$ and $[x_{i_k}, x_{i_k}] \in -2x_j + \mathbb{C}x_0$. For $i \in \mathbb{Z}$ set $a_i = 2$ if $x_i \in \mathfrak{g}_{\bar{0}}$ and $a_i = 1$ if $x_i \in \mathfrak{g}_{\bar{1}}$. We define the j-degree of $x_{\mathbf{I}}$ as $j - \deg(x_{\mathbf{I}}) = 0$ if $Z(j, \mathbf{I}) = \emptyset$ and $j - \deg(x_{\mathbf{I}}) = \sum\limits_{i_k \in \mathbb{Z}(j, \mathbf{I})} a_{i_k}$, otherwise. We define the total degree of $x_{\mathbf{I}}$ as $t - \deg(x_{\mathbf{I}})$ $= \sum\limits_{k=1}^{m} a_{i_k}$. Note that $t - \deg(x_{\mathbf{I}}) = \sum\limits_{j \mid x_j \in \mathfrak{g}_{\bar{0}}} j - \deg(x_{\mathbf{I}})$. Given $\mathbf{I} = (i_1, \ldots, i_m)$ and $\mathbf{J} = (j_1, \ldots, j_n)$, $\mathbf{I} \neq \mathbf{J}$ such that $t - \deg(x_{\mathbf{I}}) = t - \deg(x_{\mathbf{J}})$, we choose the leading monomial between $x_{\mathbf{I}}$ and $x_{\mathbf{J}}$ as follows: Let j_0 be the largest even index (i.e. $x_{j_0} \in \mathfrak{g}_{\bar{0}}$) such that $j_0 - \deg(x_{\mathbf{I}}) \neq j_0 - \deg(x_{\mathbf{J}})$. The leading monomial is the one with greatest j_0-degree. If $x = \sum\limits_{\mathbf{I}} a_{\mathbf{I}} x_{\mathbf{I}}$ we set $t - \deg(x) = \max\limits_{\mathbf{I}} \{t - \deg(x_{\mathbf{I}})\}$. If $t - \deg(x) = p$, the leading monomial of x is, by definition, the leading monomial of total degree p. We note: 1) $t - \deg([x_i, x_j]) < t - \deg(x_i x_j)$ if x_i or $x_j \in \mathfrak{g}_{\bar{0}}$. 2) Let $x_i, x_j \in \mathfrak{g}_{\bar{1}}$, $i \neq j$, and say $i < j$. There are $k, k' \in \mathbb{Z}$ such that $[x_j, x_j] = -2x_k + c_j x_0$ and $[x_i, x_i] = -2x_{k'}$ $+ c_{ij} x_0$. Clearly $k > k'$. If $c_j \neq 0$, then $c_{ij} = 0$, and if $c_{ij} \neq 0$, then $c_j = 0$ and $k > 0$. Now, $k - \deg(x_i x_j) = 1$. This implies that $x_i x_j$ leads the terms of $[x_i, x_j]$ if $i \neq j$. 3) x_i^2 $= \frac{1}{2}[x_i, x_i] = -x_j + c_i x_0$, where $x_j \in \mathfrak{g}_{\bar{0}}$, if $x_i \in \mathfrak{g}_{\bar{1}}$. Let $a_{\mathbf{K}} x_{\mathbf{K}}$ be the leading monomial among the monomials of $x_{\mathbf{I}} x_{\mathbf{J}}$ obtained by bracketing only an odd element with itself. By 1)–3) $t - \deg(x_{\mathbf{I}} x_{\mathbf{J}}) = t - \deg(a_{\mathbf{K}} x_{\mathbf{K}}) = t - \deg(x_{\mathbf{I}}) + t - \deg(x_{\mathbf{J}})$, $j - \deg(a_{\mathbf{K}} x_{\mathbf{K}}) = j - \deg(x_{\mathbf{I}}) + j - \deg(x_{\mathbf{J}})$ for all j such that $x_j \in \mathfrak{g}_{\bar{0}}$, so that $a_{\mathbf{K}} x_{\mathbf{K}}$ is the leading term of $x_{\mathbf{I}} x_{\mathbf{J}}$. Now, if $x_{\mathbf{I}}$ is the leading monomial of x and $y_{\mathbf{J}}$ is the leading monomial of y, then the leading monomial of $x_{\mathbf{I}} x_{\mathbf{J}}$ constructed above is the leading term of xy. Hence $xy \neq 0$ if $x \neq 0$, $y \neq 0$.   Q.E.D.

Let $\mathfrak{h} = \mathbb{C}e_0^+ \oplus \mathbb{C}e_0' \subset \mathfrak{g}$. We define $\delta \in \mathfrak{h}^*$ by requiring that $\delta(e_0) = 1$ and $\delta(e_0') = 0$, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, and that $\delta(e_0^+) = 1$ and $\delta(e_0') = 0$, if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $Q = \frac{1}{2}\mathbb{Z}\delta$, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, and $Q = \mathbb{Z}\delta$, if $\mathfrak{g} = \mathfrak{r}$. For $\eta \in Q$, let

$$\mathfrak{g}^\eta = \{X \in \mathfrak{g} \mid [H, X] = \eta(H)X \text{ for all } H \in \mathfrak{h}\}.$$

We then have $\widehat{\mathfrak{ns}}^0 = \mathbb{C}e_0 \oplus \mathbb{C}e_0'$ and $\mathfrak{ns}^{i\delta} = \mathbb{C}e_i$ for all $i \in \mathbb{Z}\backslash\{0\}$, $\hat{\mathfrak{r}}^0 = \mathbb{C}e_0^+ \oplus \mathbb{C}e_0^- \oplus \mathbb{C}e_0'$ and $\hat{\mathfrak{r}}^{i\delta} = \mathbb{C}e_i^+ \oplus \mathbb{C}e_i^-$, for all $i \in \mathbb{Z}\backslash\{0\}$. Hence $\mathfrak{g} = \bigoplus\limits_{\eta \in Q} \mathfrak{g}^\eta$ and $[\mathfrak{g}^{\eta_1}, \mathfrak{g}^{\eta_2}] \subset \mathfrak{g}^{\eta_1 + \eta_2}$ for all $\eta_1, \eta_2 \in Q$. We write $i\delta > 0$ (respectively $i\delta < 0$) if $i > 0$ (respectively $i < 0$), where $i \in \frac{1}{2}\mathbb{Z}$. Let Q^+ denote the set of all $\eta \in Q$ such that $\eta > 0$. We define the subalgebras $\mathfrak{n}$ and $\mathfrak{n}^-$ of $\mathfrak{g}$ as $\mathfrak{n} = \bigoplus\limits_{\eta \in Q^+} \mathfrak{g}^\eta$ and $\mathfrak{n}^- = \bigoplus\limits_{-\eta \in Q^+} \mathfrak{g}^\eta$. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{n}$. Let $\mathfrak{b} = \mathfrak{g}^0 \oplus \mathfrak{n}$. If M is an $\mathfrak{h}$-module, and $\mu \in \mathfrak{h}^*$, let

$$M^\mu = \{v \in M \mid Hv = \mu(H)v \text{ for all } H \in \mathfrak{h}\}.$$

Definition 3.2. A $\mathfrak{g}$-module M is said to be a *weight module* if $M = \bigoplus\limits_{\mu \in \mathfrak{h}^*} M^\mu$, with $\dim M^\mu < \infty$ for all $\lambda \in \mathfrak{h}^*$. M^μ is called the μ-weightspace of M.

Definition 3.3. Let M be a $\mathfrak{g}$-module, $\mu \in \mathfrak{h}^*$, $v \in M_{\bar{0}}\backslash\{0\}$ such that $Hv = \lambda(H)v$ for all $H \in \mathfrak{h}$, $\mathfrak{n}v = 0$ and $M = U(\mathfrak{g})v$. Then M is said to be a *highest weight module with highest weight* λ.

Remark. Any highest weight module is a weight module with weightspace decomposition $M = \bigoplus\limits_{\mu \in \lambda - Q^+ \cup \{0\}} M^\mu$, where λ is the highest weight of M, and

$M^\lambda = \mathbb{C}v$, $\quad M^{\lambda-\eta} = U(\mathfrak{n}^-)^{-\eta}v$, $\quad$ if $\quad \mathfrak{g} = \widehat{\mathfrak{ns}}$, $\quad$ and $\quad M^\lambda = \mathbb{C}v \oplus \mathbb{C}e_0^- v$, $\quad M^{\lambda-\eta}$ $= (U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-)e_0^-)^{-\eta}v$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Here $U(\mathfrak{n}^-)^{-\eta}$ and $(U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-)e_0^-)^{-\eta}$ are the weightspaces of $U(\mathfrak{n}^-)$ and $U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-)e_0^-$, respectively, viewed as $\mathfrak{h}$-modules, and $M = U(\mathfrak{g})v$. Any nonzero homogeneous element of M^λ is called a *highest weight vector*.

Let $\lambda \in \mathfrak{h}^*$ and let $\lambda(e_0^+) = h$, $\lambda(e_0') = c$. If $\mathfrak{g} = \widehat{\mathfrak{ns}}$, let $\mathbb{C}(\lambda)$ denote the one-dimensional even b-module $\mathbb{C}$ defined by $e_0 1 = h1$, $e_0' 1 = c1$, $\mathfrak{n}1 = 0$, $\deg 1 = \bar{0}$. If $\mathfrak{g} = \hat{\mathfrak{r}}$ we denote by $R(\lambda)$ the two-dimensional $\mathbb{Z}_2$-graded b-module generated by v_λ, where $\mathfrak{n} \cdot v_\lambda = 0$, $e_0^+ v_\lambda = h v_\lambda$, $e_0' v_\lambda = c v_\lambda$, $\deg(v_\lambda) = \bar{0}$. If $\mathfrak{g} = \hat{\mathfrak{r}}$, and λ is such that $h = -\dfrac{c}{16}$, we denote by $S(\lambda)$ the one-dimensional even b-module $\mathbb{C}$ defined by $\mathfrak{n}1 = 0$, $e_0^+ 1 = h1$, $e_0' 1 = c1$, $e_0^- 1 = 0$, $\deg 1 = \bar{0}$. If $h \neq -\dfrac{c}{16}$ thus $R(\lambda)$ is irreducible (as a $\mathbb{Z}_2$-graded b-module). If $h = -\dfrac{c}{16}$, then $e_0^- v_\lambda$ generates a proper $U(\mathfrak{b})$-submodule of $R(\lambda)$ isomorphic to $S(\lambda)$.

Definition 3.4. Let $\lambda \in \mathfrak{h}^*$, and let $V(\lambda)$ be one of the b-modules $\mathbb{C}(\lambda)$, $R(\lambda)$, or $S(\lambda)$ if $\lambda(e_0^+) = -\lambda(e_0')/16$. Set $M(V(\lambda)) = \mathrm{Ind}_\mathfrak{b}^\mathfrak{g}(V(\lambda))$. If $V(\lambda) = \mathbb{C}(\lambda)$ or $R(\lambda)$, $M(V(\lambda))$ is called the *Verma module* associated with $\mathfrak{g}$, $\mathfrak{h}$, and λ. If $\lambda(e_0^+) = -\lambda(e_0')/16$, $M(S(\lambda))$ is called the *half Verma module* associated with $\mathfrak{g}$, $\mathfrak{h}$, and λ. (This terminology will be justified by Proposition 3.12.)

Remark. $M(V(\lambda)) = U(\mathfrak{n}^-) \otimes_\mathbb{C} V(\lambda)$ by Corollary 2.5. We have $M(V(\lambda))^{\lambda-\eta}$ $= U(\mathfrak{n}^-)^{-\eta} \otimes V(\lambda)$ and $M(V(\lambda)) = \bigoplus\limits_{\eta \in Q^+ \cup \{0\}} M(V(\lambda))^{\lambda-\eta}$.

The following is an immediate consequence of the universal property of the tensor product.

Proposition 3.5. *If* $\lambda(e_0^+) = -\lambda(e_0')/16$, *then* $M(S(\lambda))$ *is a submodule of* $M(R(\lambda))$.

We denote by $p(n)$ the number of partitions of n and by $p(S, n)$ the number of partitions of n that belong to the subset S of the set of all partitions. For a subset H of the set of positive integers, we denote by "H" the set of all partitions with parts in H, and by "H" $(\leq d)$ the set of all partitions with parts in H, in which no part appears more than d times. Let H_0 (respectively H_1) denote the even (respectively odd) positive integers. Finally, let $p_e(\mathscr{D}, n)$ [respectively $p_o(\mathscr{D}, n)$) be the number of partitions of n into an even (respectively odd) number of distinct parts, and let $p(\mathscr{D}, n)$ be the number of partitions of n into distinct parts.

If M is a weight module we set $M_\gamma^\eta = M^\mu \cap M_\gamma$, $\mu \in \mathfrak{h}^*$, $\gamma \in \mathbb{Z}_2$.

Proposition 3.6. *Let* $\lambda \in \mathfrak{h}^*$,

$$\dim M(\mathbb{C}(\lambda))^{\lambda-n\delta} = \hat{p}(n) = \tilde{p}(2n), \tag{1}$$

where

$$\tilde{p}(0) = 1, \tilde{p}(s) = \sum_{i=0}^{s} p(\text{``}H_0\text{''}, i)\, p(\text{``}H_1\text{''}(\leq 1), s-i), \ s > 0;$$

$$\dim M(R(\lambda))_\gamma^{\lambda-n\delta} = \hat{p}(n), \tag{2}$$

where

$$\hat{p}(0) = 1, \hat{p}(n) = \sum_{i=0}^{n} p(i)\, p(\mathscr{D}, n-i), \ n > 0, \gamma \in \mathbb{Z}_2;$$

If $\lambda(e_0) = -\lambda(e_0^+)/16$, *then*

$$\dim M(S(\lambda))_\gamma^{\lambda-n\delta} = \tfrac{1}{2}\hat{p}(n)) \quad for \quad n > 0, \hat{p}(n) \quad as \ in \ (2). \tag{3}$$

Proof. By the remark that follows Definition 3.4, $M(R(\lambda))_0^{\lambda-n\delta}$ is a linear combination of elements

$$e_{-i_1}^+ \dots e_{-i_r}^+ e_{-j_1}^- \dots e_{-j_s}^- \otimes v_\lambda, \ e_{-k_1}^+ \dots e_{-k_t}^+ e_{-l_1}^- \dots e_{-l_u}^- \otimes e_0^- v_\lambda,$$

where

$$i_1 + \dots + i_r + j_1 + \dots + j_s = k_1 + \dots + k_t + l_1 + \dots + l_u = n,$$

$$i_1 \geq \dots \geq i_r, j_1 > \dots > j_s, k_1 \geq \dots \geq k_t, l_1 > \dots > l_u,$$

s even, u odd. Hence,

$$\dim M(R(\lambda))_0^{\lambda-n\delta} = \sum_{i=0}^n p(i)\, p_e(\mathscr{D}, n-i) + \sum_{i=0}^n p(i)\, p_o(\mathscr{D}, n-i)$$

$$= \sum_{i=0}^n p(i)\, p(\mathscr{D}, n-i).$$

Similarly, for $M(R(\lambda))_1^{\lambda-n\delta}$. This proves (2).

To prove (3) we use a similar argument and note that

$$\sum_{i=0}^n p(i)\, p_e(\mathscr{D}, n-i) = \sum_{i=0}^n p(i)\, p_o(\mathscr{D}, n-i), \quad \text{for all} \quad n > 0,$$

since

$$\left(\sum_{n=0}^\infty p(n) q^n\right)\left(\sum_{n=0}^\infty (p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n)) q^n\right) = 1$$

(see, e.g. [10]). (1) is even easier.   Q.E.D.

Let $v_\lambda = 1 \otimes v_\lambda$ in $M(R(\lambda))$ and $v_\lambda = 1 \otimes 1$ in $M(\mathbb{C}(\lambda))$ or in $M(S(\lambda))$.

Proposition 3.7. *Let M be a highest weight module with highest weight λ and let v be a generating highest weight vector of M. If $\mathfrak{g} = \hat{\mathfrak{ns}}$, we let $V(\lambda) = \mathbb{C}(\lambda)$; if $\mathfrak{g} = \hat{\mathfrak{r}}$ we let $V(\lambda) = R(\lambda)$ if $e_0^- v \neq 0$, and $V(\lambda) = S(\lambda)$ if $e_0^- v = 0$ (in which case $\lambda(e_0^+) = -\lambda(e_0')/16$). Then there is a unique surjective homomorphism $\phi : M(V(\lambda)) \to M$ such that $\phi(v_\lambda) = v$.*

Proof. This is immediate from Proposition 2.6.   Q.E.D.

Proposition 3.8. *Let M be a highest weight module. Then M has a unique maximal proper submodule.*

Proof. Let λ be the highest weight of M. If $g = \hat{\mathfrak{ns}}$, or if $\mathfrak{g} = \hat{\mathfrak{r}}$ and $\lambda(e_0^+) \neq \lambda(e_0')/16$, then every proper submodule of M is contained in $\bigoplus_{n>0} M^{\lambda-n\delta}$. If $\mathfrak{g} = \hat{\mathfrak{r}}$ and $\lambda(e_0^+) = -\lambda(e_0')/16$, then every proper submodule of M is contained in $\bigoplus_{n>0} M^{\lambda-n\delta} \oplus M_1^\lambda$. In either case, the sum of all proper submodules is itself a proper submodule.   Q.E.D.

Definition 3.9. $L(V(\lambda)) =$ unique irreducible quotient of $M(V(\lambda))$, $V(\lambda) = \mathbb{C}(\lambda)$, $R(\lambda)$ or $S(\lambda)$.

Proposition 3.10. *Let* $\lambda(e_0^+) = -\lambda(e_0')/16$. *Then* $L(R(\lambda)) \simeq L(S(\lambda))$.

Proof. Let $M(R(\lambda))_{(1)}$ [respectively $M(S(\lambda))_{(1)}$] denote the unique maximal proper submodule of $M(R(\lambda))$ [respectively $M(S(\lambda))$]. By Proposition 3.5, $e_0^- (1 \otimes v_\lambda) \in M(R(\lambda))_{(1)}$. Therefore the image $\overline{1 \otimes v_\lambda}$ of $1 \otimes v_\lambda$ by the quotient map is annihilated by e_0^-. By Proposition 3.7, there is a unique surjective homomorphism $\phi: M(S(\lambda)) \to L(R(\lambda))$ such that $\phi(1 \otimes 1) = \overline{1 \otimes v_\lambda}$. This implies that

$$L(R(\lambda)) \simeq M(S(\lambda))/\ker \phi = L(S(\lambda)). \quad \text{Q.E.D.}$$

Definition 3.11. If M is a weight module, we define the character of M as

$$\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} (\dim M^\mu) q^\mu.$$

From now on we abbreviate $q^{n\delta}$ by q^n, $n\delta \in Q$.

Proposition 3.12. (1) $\operatorname{ch} M(\mathbb{C}(\lambda)) = q^\lambda \dfrac{\prod\limits_{n \in \mathbb{N} - 1/2} (1 + q^{-n})}{\prod\limits_{n \in \mathbb{N}} (1 - q^{-n})};$

(2) $\operatorname{ch} M(R(\lambda)) = 2q^\lambda \prod\limits_{n \in \mathbb{N}} \dfrac{(1 + q^{-n})}{(1 - q^{-n})};$

(3) $\operatorname{ch} M(S(\lambda)) = q^\lambda \prod\limits_{n \in \mathbb{N}} \dfrac{(1 + q^{-n})}{(1 - q^{-n})},$ *if* $\lambda(e_0^+) = -\lambda(e_0')/16$.

Proof.

$$\sum_{n \in \mathbb{Z}_+} \tilde{p}(n) q^{-n} = \prod_{n \text{ even}} (1 + q^{-n} + q^{-2n} + \ldots) \prod_{n \text{ odd}} (1 + q^{-n}) = \dfrac{\prod\limits_{n \text{ odd}} (1 + q^{-n})}{\prod\limits_{n \text{ even}} (1 - q^{-n})},$$

$\tilde{p}(n)$ as in Proposition 3.6. By Proposition 3.6,

$$\operatorname{ch} M(\mathbb{C}(\lambda)) = \sum_{n \in \frac{1}{2}\mathbb{Z}_+} \tilde{p}(2n) q^{\lambda - n\delta} = q^\lambda \sum_{n \in \mathbb{Z}} \tilde{p}(n) q^{-n/2} = q^\lambda \dfrac{\prod\limits_{n \in \mathbb{N} - 1/2} (1 + q^{-n})}{\prod\limits_{n \in \mathbb{N}} (1 - q^{-n})}.$$

By Proposition 3.6,

$$\operatorname{ch} M(R(\lambda)) = \sum_{n \in \mathbb{Z}_+} \dim (M(R(\lambda))^{\lambda - n\delta}) q^{\lambda - n\delta} = 2q^\lambda \sum_{n \in \mathbb{Z}_+} \hat{p}(n) q^{-n}$$

$$= 2q^\lambda \prod_{n \in \mathbb{N}} (1 + q^{-n} + q^{-2n} + \ldots) \prod_{n \in \mathbb{N}} (1 + q^{-n}) = 2q^\lambda \prod_{n \in \mathbb{N}} \dfrac{(1 + q^{-n})}{(1 - q^{-n})}.$$

This proves (1) and (2). (3) follows from (2) and Proposition 3.6. Q.E.D.

Proposition 3.13. *Let* $V(\lambda) = \mathbb{C}(\lambda)$, $V(\mu) = \mathbb{C}(\mu)$, *or* $V(\lambda) = R(\lambda)$, $V(\mu) = R(\mu)$, *or* $V(\lambda) = S(\lambda)$, $V(\mu) = S(\mu)$, *if* $\lambda(e_0^+) = -\lambda(e_0')/16$ *and* $\mu(e_0^+) = -\mu(e_0')/16$. *Every homomorphism* $\phi: M(V(\lambda)) \to M(V(\mu))$ *is injective or zero.*

Proof. Let $\phi: M(V(\lambda)) \to M(V(\mu))$ be a nonzero homomorphism. Then $\phi(v_\lambda) = Y v_\mu$, $Y \ne 0$, $Y \in U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-) e_0^-$ if $V(\lambda) = R(\lambda)$, $V(\mu) = R(\mu)$ [respectively $Y \in U(\mathfrak{n}^-)$ if $V(\lambda) = \mathbb{C}(\lambda)$, $V(\mu) = \mathbb{C}(\mu)$, or $V(\lambda) = S(\lambda)$, $V(\mu) = S(\mu)$]. If $v \in M(V(\lambda)) \backslash \{0\}$, say $v = X v_\lambda$, $X \ne 0$, $X \in U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-) e_0^-$ if $V(\lambda) = R(\lambda)$, $V(\mu) = R(\mu)$ [respectively $X \in U(\mathfrak{n}^-)$ if $V(\lambda) = \mathbb{C}(\lambda)$, $V(\mu) = \mathbb{C}(\mu)$, or $V(\lambda) = S(\lambda)$, $V(\mu) = S(\mu)$], then $XY \ne 0$ by Proposition 3.1. Now, $M(R(\mu)) \simeq (U(\mathfrak{n}^-) \oplus U(\mathfrak{n}^-) e_0^-) \otimes_{\mathbb{C}} \mathbb{C} v_\mu$, $M(\mathbb{C}(\mu)) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}$ and $M(S(\mu)) = U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}$, hence $XY v \ne 0$. Q.E.D.

We now define the contravariant form on $M(V(\lambda))$ for $V(\lambda) = \mathbb{C}(\lambda)$, $R(\lambda)$, or $S(\lambda)$ if $\lambda(e_0^+) = -\lambda(e_0')/16$.

We set $\sigma(e_i^+) = e_{-i}^+$, $\sigma(e_i^-) = e_{-i}^-$ and $\sigma(e_0') = e_0'$. Clearly,

$$\sigma([e_i^+, e_j^+]) = [\sigma(e_j^+), \sigma(e_i^+)], \tag{1}$$

$$\sigma([e_i^-, e_j^-]) = [\sigma(e_j^-), \sigma(e_i^-)], \tag{2}$$

$$\sigma([e_i^+, e_j^-]) = [\sigma(e_j^-), \sigma(e_i^+)], \tag{3}$$

$$\sigma([\mathfrak{g}, e_0']) = 0 = [\sigma(e_0'), \sigma(\mathfrak{g})]. \tag{4}$$

This implies that σ extends to a $\mathbb{Z}_2$-graded linear bijection from $\mathfrak{g}$ to $\mathfrak{q}$ such that $\sigma^2 = $ identity and $\sigma([X, Y]) = [\sigma(Y), \sigma(X)]$, for all $X, Y \in \mathfrak{q}$. By the universal property of $U(\mathfrak{g})$, σ therefore extends to a $\mathbb{Z}_2$-graded linear bijection from $U(\mathfrak{g})$ to $U(\mathfrak{q})$ such that $\sigma^2 = $ identity and $\sigma(XY) = \sigma(Y)\sigma(X)$, for all $X, Y \in U(\mathfrak{g})$. From the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{n}$ and Corollary 2.5, one sees that $U(\mathfrak{g}) = U(\mathfrak{g}^0) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n})$. Let $\phi: U(\mathfrak{g}) \in U(\mathfrak{g}^0)$ be the projection on the first factor, and let $\Pi_{\bar{0}}: U(\mathfrak{g}^0) \to (U(\mathfrak{g}^0))_{\bar{0}}$ be the projection on the even component of $U(\mathfrak{g}^0)$. Let $\beta = \Pi_{\bar{0}} \circ \phi: U(\mathfrak{g}) \to U(\mathfrak{g}^0)_0$. For $X, Y \in U(\mathfrak{g})$, $\lambda \in \mathfrak{h}^*$, we set

$$(X v_\lambda, Y v_\lambda)_\lambda = (\lambda \circ \beta)(\sigma(X) Y), \tag{5}$$

where we have extended $\lambda \in \mathfrak{h}^*$ canonically to an algebra homomorphism $\lambda: U(\mathfrak{g}^0)_{\bar{0}} \to \mathbb{C}$, and $v_\lambda \in M(V(\lambda))$, $V(\lambda) = \mathbb{C}(\lambda)$, $R(\lambda)$, or $S(\lambda)$, if $\lambda(e_0^+) = -\lambda(e_0')/16$.

Proposition 3.14. (a) $(,)_\lambda$ *defines a symmetric, bilinear form on* $M(V(\lambda))$ *such that*

$$(X v, w)_\lambda = (v, \sigma(X) w)_\lambda \quad \text{for all} \quad X \in U(\mathfrak{g}), v, w \in M(V(\lambda)), \tag{6}$$

(b) $(M(V(\lambda))_{\gamma_1}^{\mu_1}, M(V(\lambda))_{\gamma_2}^{\mu_2})_\lambda = 0$ *if* $\mu_1 \ne \mu_2$ *or* $\gamma_2 \ne \gamma_2$, $\mu_1, \mu_2 \in \mathfrak{h}^*$, $\gamma_1, \gamma_2 \in \mathbb{Z}_2$;

(c) *The radical of* $(,)_\lambda$, $\mathrm{Rad}(,)_\lambda$, *is the maximal proper submodule of* $M(V(\lambda))$;

(d) $(,)_\lambda$ *is the unique symmetric bilinear form on* $M(V(\lambda))$ *such that* $(M(V(\lambda))_{\bar{0}}, M(V(\lambda))_{\bar{1}})_\lambda = 0$, $(v_\lambda, v_\lambda)_\lambda = 1$ *and such that* (6) *is satisfied.*

Proof. (a) If $X v_\lambda = 0$, then X is in the left ideal of $U(\mathfrak{g})$ generated by $\mathfrak{n}$ and $\{H - \lambda(H) | H \in \mathfrak{h}\}$, if $V(\lambda) = \mathbb{C}(\lambda)$ or $R(\lambda)$ [respectively generated by $\mathfrak{n}$, $\mathbb{C} e_0^-$ and $\{H - \lambda(H) | H \in \mathfrak{h}\}$ if $V(\lambda) = S(\lambda)$, $\lambda(e_0^+) = -\lambda(e_0')/16$]. Therefore $(X v_\lambda, U(\mathfrak{g}) v_\lambda)_\lambda = 0$. Now,

$$(X v_\lambda, Y v_\lambda)_\lambda = (\lambda \circ \beta)(\sigma(X) Y) = (\lambda \circ \beta)(\sigma(\sigma(X) Y)) = (\lambda \circ \beta)(\sigma(Y) X) = (Y v_\lambda, X v_\lambda)_\lambda,$$

hence $(,)_\lambda$ is symmetric. Also,

$$(ZX v_\lambda, Y v_\lambda)_\lambda = (\lambda \circ \beta)(\sigma(ZX) Y) = (\lambda \circ \beta)(\sigma(X) \sigma(Z) Y) = (X v_\lambda, \sigma(Z) Y v_\lambda)_\lambda.$$

(b) $U(\mathfrak{g})_{\bar{i}} U(\mathfrak{g})_{\bar{j}} \subset U(\mathfrak{g})_{\overline{i+j}}$, $\sigma(U(\mathfrak{g})_{\bar{i}}) \subset U(\mathfrak{g})_{\bar{i}}$, $i = 0, 1$, and $\beta(U(\mathfrak{g})_{\bar{1}}) = (0)$. Hence, $\beta(\sigma(X) Y) = 0$ if $X \in U(\mathfrak{g})_{\bar{i}}$, $Y \in U(\mathfrak{g})_{\overline{1-i}}$, $i = 0, 1$. Let now $v_1 \in M(V(\lambda))^{\mu_1}$, $v_2 \in M(V(\lambda))^{\mu_2}$ and $\mu_1 \neq \mu_2$. Then $(Hv_1, v_2)_\lambda = \mu_1(H)(v_1, v_2)_\lambda$ and $(v_1, Hv_2) = \mu_2(H)(v_1, v_2)$ for all $H \in \mathfrak{h}$. By (6) $(\mu_1(H) - \mu_2(H))(v_1, v_2)_\lambda = 0$, hence $(v_1, v_2)_\lambda = 0$.

(c) Let $v \in \mathrm{Rad}(\ ,\ )_\lambda$. Then $(Xv, U(\mathfrak{g})v_\lambda)_\lambda = (v, \sigma(X) U(\mathfrak{g})v_\lambda)_\lambda = 0$ for all $X \in U(\mathfrak{g})$. Hence, $\mathrm{Rad}(\ ,\ )_\lambda$ is a proper submodule of $M(V(\lambda))$, since $(v_\lambda, v_\lambda)_\lambda = 1$. Therefore $\mathrm{Rad}(\ ,\ )_\lambda \subset$ largest proper submodule of $M(V(\lambda))$. Conversely, let v be in the largest proper submodule of $M(V(\lambda))$. Then $v \in \left(\bigoplus_{n > 0} M(V(\lambda))^{\lambda - n\delta}\right) \oplus M(V(\lambda))_{\bar{1}}^\lambda$, and so, by (b), $(v, v_\lambda)_\lambda = 0$. Now, $(v, U(\mathfrak{g})v_\lambda) = (\sigma(U(\mathfrak{g}))v, v_\lambda)_\lambda = 0$ and $v \in \mathrm{Rad}(\ ,\ )_\lambda$.

(d) Let $\langle\ ,\ \rangle_\lambda$ be a symmetric bilinear form on $M(V(\lambda))$ such that $(M(V(\lambda))_{\bar{0}}, M(V(\lambda))_{\bar{1}}) = 0$ and such that $\langle Xv, w\rangle_\lambda = \langle v, \sigma(X)w\rangle_\lambda$ for all $X \in U(\mathfrak{g})$, $v, w \in M(V(\lambda))$. Then $\langle Xv_\lambda, Yv_\lambda\rangle_\lambda = \langle v_\lambda, \sigma(X) Yv_\lambda\rangle_\lambda$. By (b), $\langle v_\lambda, \sigma(X) Yv_\lambda\rangle_\lambda = (\lambda \circ \beta)(\sigma(X) Y)\langle v_\lambda, v_\lambda\rangle_\lambda$.   Q.E.D.

Let $\lambda \in \mathfrak{h}^*$ and set $h = \lambda(e_0^+)$, $c = \lambda(e_0')$. For $n \in \frac{1}{2}\mathbb{Z}_+$ and $\gamma \in \mathbb{Z}_2$, we denote by $(\ ,\ )_{h,c,n}$ [respectively $(\ ,\ )_{h,c,n,\gamma}$] or by $(\ ,\ )_{\lambda,n}$ [respectively $(\ ,\ )_{\lambda,n,\gamma}$] the restriction of $(\ ,\ )_\lambda$ to $M(V(\lambda))^{\lambda - n\delta}$ [respectively $M(V(\lambda))^{\lambda - n\delta}$], where $V(\lambda) = \mathbb{C}(\lambda)$ or $R(\lambda)$. We now state the formulas for the determinant of $(\ ,\ )_{h,c,n,\gamma}$ (we omit the subscript γ if $\mathfrak{g} = \hat{ns}$). The formula for the Neveu-Schwarz algebra was conjectured by Kac [14], and the one for the Ramond algebra was conjectured by D. Friedan, Z. Qiu, and S. H. Shenker [7, 8]. Let $r, s \in \frac{1}{2}\mathbb{N}$, $r \leq s$. Set, for $\mathfrak{g} = \hat{ns}$

$$h_{r,s}^\pm(c) = -\tfrac{1}{8}\left[(r^2 + s^2)(5 - c) \pm \sqrt{c^2 - 10c + 9}\,(r^2 - s^2) - 8rs - \tfrac{1}{2} + \tfrac{1}{2}c\right], \quad (7)$$

and, for $\mathfrak{g} = \hat{r}$

$$h_{r,s}^\pm(c) = -\tfrac{1}{8}\left[(r^2 + s^2)(5 - c) \pm \sqrt{c^2 - 10c + 9}\,(r^2 - s^2) - 8rs - \tfrac{1}{2} + \tfrac{1}{2}c\right] - \tfrac{1}{16}. \quad (7')$$

Also, set, for $r \neq s$, $\psi_{r,s}(h, c) = (h - h_{r,s}^+(c))(h - h_{r,s}^-(c))$, and, for $\mathfrak{g} = \hat{ns}$, $\psi_{r,r}(h, c) = h + \tfrac{1}{16}(1 - c)(4r^2 - 1)$. Let $\hat{p}(i)$ be as in Proposition 3.6. Then, if $\mathfrak{g} = \hat{ns}$,

$$\det(\ ,\ )_{h,c,n} = \prod_{2rs \leq n, r - s \in \mathbb{Z}} \psi_{r,s}(h, c)^{\hat{p}(n - 2rs)}, \ [14] \quad (8)$$

and, for $\mathfrak{g} = \hat{r}$,

$$\det(\ ,\ )_{h,c,n,\gamma} = \left(h + \frac{c}{16}\right)^{\frac{1}{2}\hat{p}(n)} \prod_{2rs \leq n, r - s \in \frac{1}{2} + \mathbb{Z}} \psi_{r,s}(h, c)^{\hat{p}(n - 2rs)} \quad (9)$$

if $n \geq 0$, and $\det(\ ,\ )_{h,c,0,\bar{0}} = 1$, $\det(\ ,\ )_{h,c,0,\bar{1}} = h + \dfrac{c}{16}$ [7, 8].

Note. The nonzero constant depending on the choice of a basis of $M(V(\lambda))$ has been omitted in (8), (9).

4. The Representations $\mathcal{K}^\xi$
and the Proof of the Determinant Formulas

We retain the notation of Sect. 3. In this section we prove the determinant formulas for the restriction of $(\ ,\ )_\lambda$ to $M(V(\lambda))_\gamma^\mu$ (see Sect. 3). The main ingredients are the

construction of the representations $\mathscr{H}^{\xi,\omega}$ of $\mathfrak{g}$ (Theorem 4.2), which generalize the construction of [11], and the vertex operator.

Following [6] let $\mathfrak{t}$ be a one-dimensional space, $\mathfrak{t} = \mathbb{C}T$, and let $\langle\,,\,\rangle$ be a symmetric, bilinear form on $\mathfrak{t}$ such that $\langle T, T \rangle = 1$. Let $\hat{\mathfrak{t}} = \mathfrak{t} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}z$, and let

$$[X(m), Y(n)] = \langle X, Y \rangle m\delta_{m+n,0}z, \tag{1}$$

$$[\hat{\mathfrak{t}}, z] = 0, \tag{2}$$

where we have set $X(m) = X \otimes t^m$ for all $x \in \mathfrak{t}$, $m \in \mathbb{Z}$. We let

$$D(X \otimes t^n) = n, \quad X \in \mathfrak{t}, n \in \mathbb{Z}, \tag{3}$$

$$D(z) = 0. \tag{4}$$

We note that $\hat{\mathfrak{t}}_* = \left(\bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathfrak{t} \otimes \mathbb{C}t^n \right) \oplus \mathbb{C}z$ is a $\mathbb{Z}$-graded Heisenberg algebra with the $\mathbb{Z}$-grading given by the degree map D. Let $\hat{\mathfrak{t}}^- = \bigoplus_{n<0} \mathfrak{t} \otimes \mathbb{C}t^n \subset \hat{\mathfrak{t}}_*$, and $\mathfrak{u} = \left(\bigoplus_{n>0} \mathfrak{t} \otimes \mathbb{C}t^n \right) \oplus \mathbb{C}z \subset \hat{\mathfrak{t}}_*$. Let $\mathbb{C}(1)$ denote the one-dimensional $\mathfrak{u}$-module such that $(X \otimes t^n)1 = 0$ if $X \in \mathfrak{t}$, $n > 0$, and $z1 = 1$, and form the left $U(\hat{\mathfrak{t}}_*)$-module: $\mathrm{Ind}_{\mathfrak{u}}^{\hat{\mathfrak{t}}_*} \mathbb{C}(1)$. Note that $S(\hat{\mathfrak{t}}^-)$ is linearly isomorphic with $\mathrm{Ind}_{\mathfrak{u}}^{\hat{\mathfrak{t}}_*} \mathbb{C}(1)$. Using this isomorphism we can define a $\hat{\mathfrak{t}}_*$-module action on $S(\hat{\mathfrak{t}}^-)$ and identify $S(\hat{\mathfrak{t}}^-)$ with $\mathrm{Ind}_{\mathfrak{u}}^{\hat{\mathfrak{t}}_*} \mathbb{C}(1)$. Let $\mathbb{C}[\mathfrak{t}]$ denote the group algebra of $\mathfrak{t}$, $\mathbb{C}[\mathfrak{t}] = \bigoplus_{\omega \in \mathbb{C}} e^{\omega T}$, where $e^\alpha e^\beta = e^{\alpha+\beta}$, $\alpha, \beta \in \mathfrak{t}$, and let

$$De^\alpha = -\frac{\langle \alpha, \alpha \rangle}{2}, \quad \alpha \in \mathfrak{t}. \tag{5}$$

Let $V = S(\hat{\mathfrak{t}}^-) \otimes \mathbb{C}[\mathfrak{t}]$. We define a representation (π_1, V) of $\hat{\mathfrak{t}}$ as follows:

$$\pi_1(X) = X \otimes 1, \quad X \in \hat{\mathfrak{t}}_*, \tag{6}$$

$$\pi_1(X)(v \otimes e^\alpha) = v \otimes \langle X, \alpha \rangle e^\alpha, X \in \mathfrak{t}, v \in S(\hat{\mathfrak{t}}^-), \quad \alpha \in \mathfrak{t}, \tag{7}$$

$$\pi_1(z) = \text{identity}. \tag{8}$$

Let

$$\pi_2(e^\alpha)(v \otimes e^\beta) = v \otimes e^{\alpha+\beta}, \quad \alpha, \beta \in \mathfrak{t}, v \in S(\hat{\mathfrak{t}}^-). \tag{9}$$

The pair $(\hat{\mathfrak{t}}, \mathbb{C}[\mathfrak{t}])$ is called a Heisenberg system ([6]) and (π_1, π_2) is an irreducible representation of $(\hat{\mathfrak{t}}, \mathbb{C}[\mathfrak{t}])$ ([6]).

Next we describe the spin representation of [4, 17]. We denote by Z the set $\mathbb{Z} + \frac{1}{2}$ or the set $\mathbb{Z}$. Let $U = \bigoplus_{m \in Z} \mathbb{C}a(m)$, $U^+ = \bigoplus_{m>0, m \in Z} \mathbb{C}a(m)$, and $U^- = \bigoplus_{m \leq 0, m \in Z} \mathbb{C}a(m)$. Let $\langle\,,\,\rangle : U \times U \to \mathbb{C}$ be the symmetric bilinear form satisfying

$$\langle a(m), a(n) \rangle = \delta_{m+n,0}, \tag{10}$$

and let $\mathscr{C}\ell(U)$ be the corresponding Clifford algebra. That is, $\mathscr{C}\ell(U) = T(U)/J$, where $T(U)$ is the tensor algebra of U and J is the two-sided ideal of $T(U)$

generated by the elements of the form: $v \otimes w + w \otimes v - \langle v, w \rangle$. Let

$$Da(m) = m. \tag{11}$$

We define, for $X \in U^-$, $Y \in U^+$,

$$L_X(v_1 \wedge \ldots \wedge v_n) = X \wedge v_1 \wedge \ldots \wedge v_n, \tag{12}$$

$$\iota(Y)(v_1 \wedge \ldots \wedge v_n) = \sum_{j=1}^{n} (-1)^{j-1} \langle Y, v_j \rangle v_1 \wedge \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_n, \tag{13}$$

$$\varrho(X + Y) = L_X + \iota(Y), \tag{14}$$

$v_i \in U^-$, $i = 1, \ldots, n$. ϱ extends to an irreducible representation of $\mathscr{Cl}(U)$ on $\mathscr{Cl}(U^-)$ $= \Lambda(U^-)$, also denoted by ϱ, called the spin representation corresponding to the polarization $U = U^+ \oplus U^-$ ([4, 17]). Let $W = V \otimes \Lambda(U^-)$. We let $(\hat{t}, \mathbb{C}[t])$ act on W by letting it act trivially on $\Lambda(U^-)$, and let $\mathscr{Cl}(U)$ act on W by letting it act trivially on V. We make W into a $\mathbb{Z}_2$-graded vector space $W = \bigoplus_{\gamma \in \mathbb{Z}_2} W_\gamma$ by setting $\deg v = \bar{0}$ for all $v \in V$ and $\deg u = \bar{1}$ for all $u \in U$.

From now on we omit the representation symbols and indicate the action of an element of $(\hat{t}, \mathbb{C}[t])$ or $\mathscr{Cl}(U)$ on W by the element itself.

The ordered products: $:T(m) T(n):$ and $:a(m) a(n):$ of the operators $T(m)$ and $T(n)$, and $a(m)$ and $a(n)$, respectively, are defined as follows:

$$:T(m) T(n): = \begin{cases} T(m) T(n) & \text{if } m < n, \\ \frac{1}{2}[T(m) T(n) + T(n) T(m)] & \text{if } m = n, \\ T(n) T(m) & \text{if } m > n, \end{cases} \tag{15}$$

$$:a(m) a(n): = \begin{cases} a(m) a(n) & \text{if } m < n, \\ \frac{1}{2}[a(m) a(n) - a(n) a(m)] & \text{if } m = n, \\ -a(n) a(m) & \text{if } m > n. \end{cases} \tag{16}$$

We now define the operators that give a special case of the representations of $\mathfrak{g}$ that will be constructed. From now on, $Z = \mathbb{Z} + \frac{1}{2}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$ and $Z = \mathbb{Z}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$.

Let, for $n \in \mathbb{Z}$,

$$d_n^+ = -\frac{1}{2} \sum_{k \in \mathbb{Z}} :T(k) T(n-k): + \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} k :a(k) a(n-k): \tag{17}$$

if $\mathfrak{g} = \hat{\mathfrak{ns}}$,

$$d_n^+ = -\frac{1}{2} \sum_{k \in \mathbb{Z}} :T(k) T(n-k): + \frac{1}{2} \sum_{k \in \mathbb{Z}} k :a(k) a(n-k): - \frac{1}{16} \delta_{n,0} \tag{18}$$

if $\mathfrak{g} = \hat{\mathfrak{r}}$, and, for $n \in Z$,

$$d_n^- = \sum_{k \in \mathbb{Z}} T(k) \otimes a(n-k). \tag{19}$$

(We sometimes drop the superscripts if $\mathfrak{g} = \hat{\mathfrak{ns}}$.)

The degree map D given by (3), (4), (5), and (11) extends uniquely to a degree derivation on W which we also denote by D.

If A, B, C are operators we have the following obvious identities: $[AB, C]$ $= [A, C]B + A[B, C]$, $[AB, C] = -[A, C]_+ B + A[B, C]_+$ and $[AB, C]_+$ $= -[A, C]B + A[B, C]_+$, where $[A, B]_+$ is the anticommutator $AB + BA$.

Let $d_n^{+,0} = -\frac{1}{2} \sum_{k \in \mathbb{Z}} :T(k) T(n-k):$, $n \in \mathbb{Z}$. Then one has:

$$[d_m^{+,0}, T(n)] = nT(m+n), \quad m, n \in \mathbb{Z} \qquad ([6], \text{ see also } [12]) \tag{20}$$

$$[d_m^{+,0}, d_n^{+,0}] = (n, m)d_{m+n}^{+,0} + \frac{m^3 - m}{12}\delta_{m+n, 0}, \quad m, n \in \mathbb{Z} \tag{21}$$

([6], see also [12]). The relations of the next lemma were also known ([5]).[4] We give their proof for completeness.

Lemma 4.1. *Let* $\quad d_n^{+,1} = \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} k :a(k) a(n-k):$ *if* $\quad \mathfrak{g} = \hat{\mathfrak{ns}}$ *and* $\quad d_n^{+,1}$

$= \frac{1}{2} \sum_{k \in \mathbb{Z}} k : a(k) a(n-k): -\frac{1}{16}\delta_n$ *if* $\mathfrak{g} = \hat{\mathfrak{r}}$. *Then*

(i) $[d_m^{+,1}, a(n)] = (n + \frac{1}{2}m) a(m+n), \quad m \in \mathbb{Z}, n \in Z$;

(ii) $[d_m^{+,1}, d_n^{+,1}] = (n-m)d_{m+n}^{+,1} + \dfrac{m^3 - m}{24}\delta_{m+n, 0}, \quad m, n \in \mathbb{Z}$.

Proof. (i) $[d_m^{+,1}, a(n)] = \frac{1}{2} \sum_{k \in Z} k[a(k) a(m-k), a(n)]$, by (16). Assume that $m \neq -2n$. Then $[d_m^{+,1}, a(n)] = \frac{1}{2}[-na(-n) a(m+n) + (m+n) a(m+n) a(-n), a(n)]$ $= \frac{1}{2}(m+2n) a(m+n)$, by (10). If $m = -2n$, then, $[d_m^{+,2}, a(n)]$ $= \frac{1}{2}[-na(-n) a(-n), a(n)] = 0$;

(ii) $[[d_m^{+,1}, d_n^{+,1}], a(k)] = [d_m^{+,1}, (k + \frac{1}{2}n) a(k+n)] + [(k + \frac{1}{2}m) a(k+m), d_n^{+,1}]$ $= (k + \frac{1}{2}n)(k + n + \frac{1}{2}m) a(k+m+n) - (k + \frac{1}{2}m)(k + m + \frac{1}{2}n) a(k+m+n) = (n-m)$ $(k + \frac{1}{2}(m+n)) a(m+n+k) = [(n-m)d_{m+n}^{+,1}, a(k)]$ for all $k \in Z$, by (i). Therefore, $[d_m^{+,1}, d_n^{+,1}] - (n-m)d_{m+n}^{+,1}$ is constant, by the irreducibility of $\Lambda(U^-)$. Since $[d_m^{+,1}, d_n^{+,1}] - (n-m)d_{m+1}^{+,1}$ is homogeneous of degree $m+n$ relative to D, we conclude that this constant is zero unless $m+n = 0$. Now, if $\mathfrak{g} = \hat{\mathfrak{ns}}$ $[d_1^{+,1}, d_{-1}^{+,1}] (1 \otimes 1 \otimes 1) = d_1^{+,1}, d_{-1}^{+,1}(1 \otimes 1 \otimes 1) = (\frac{1}{4} a(\frac{1}{2})^2)(-\frac{1}{4} a(-\frac{1}{2})^2)(1 \otimes 1 \otimes 1)$ $= 0 = -2d_0^{+,1}(1 \otimes 1 \otimes 1)$. Similarly,

$$[d_2^{+,1}, d_{-2}^{+,1}](1 \otimes 1 \otimes 1) = \frac{1}{4}(1 \otimes 1 \otimes 1) = -4d_0^{+,1}(1 \otimes 1 \otimes 1)$$

and

$$[d_1^{+,1}, d_{-1}^{+,1}](1 \otimes 1 \otimes 1) = \frac{1}{8}(1 \otimes 1 \otimes 1) = -2d_0^{+,1}(1 \otimes 1 \otimes 1)$$

and

$$[d_2^{+,1}, d_{-2}^{+,1}](1 \otimes 1 \otimes 1) = \frac{1}{2}(1 \otimes 1 \otimes 1) = (-4d_0^{+,1} + \frac{1}{4})(1 \otimes 1 \otimes 1) \quad \text{if} \quad \mathfrak{g} = \hat{\mathfrak{r}}.$$

Proceeding by induction, we assume that (ii) has been proved for $n = -m$ and $m \geq 2$. Now, write $d_{m+1}^{+,1} = \dfrac{1}{m-1}[d_1^{+,1}, d_m^{+,1}]$ and $d_{-m-1}^{+,1} = \dfrac{-1}{m-1}[d_{-1}^{+,1}, d_{-m}^{+,1}]$. It is now a simple exercise to see that $[d_{m+1}^{+,1}, d_{-m-1}^{+,1}] = -2(m+1) d_0^{+,1}$ $+ \dfrac{(m+1)^3 - (m+1)}{24}$, which proves (ii). Q.E.D.

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[4] Similar calculations were also carried out in [3]

Lemma 4.1, (20) and (21) imply that the formulas (17)–(19) give a representation of $\mathfrak{g}$ on W where the central element acts as the identity operator. To obtain arbitrary central action, we let, for $\xi \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$e_n^+(\xi) = d_n^+ + \xi n T(n) + \frac{\xi^2}{2} \delta_{n,0}$$

and, for $n \in Z$,

$$e_n^-(\xi) = d_n^- - 2\xi n a(n).$$

(We sometimes drop the superscripts if $\mathfrak{g} = \hat{\mathfrak{ns}}$.)

We note that $[A \otimes B, \ C \otimes D] = [A, C] \otimes BD + CA \otimes [B, D]$ and $[A \otimes B, C \otimes D]_+ = [A, C] \otimes BD + CA \otimes [B, D]_+$.

Theorem 4.2. *Let $\xi \in \mathbb{C}$ and set $\pi_\xi(e_n^+) = e_n^+(\xi)$, $n \in \mathbb{Z}$, $\pi_\xi(e_n^-) = e_n^-(\xi)$, $n \in Z$, and $\pi_\xi(e_0') = 1 - 8\xi^2$. Then π_ξ extends to a representation of $\mathfrak{g}$ on W.*

Proof. (a) $[e_m^+(\xi), e_n^+(\xi)] = [d_m^{+,0}, d_n^{+,0}] + \xi n[d_m^{+,0}, T(n)] + [d_m^{+,1}, d_n^{+,1}] + \xi m$

$\cdot [T(m), d_n^{+,0}] + \xi^2 mn[T(m), T(n)] = (n-m) d_{m+n}^{+,0} + \dfrac{m^3 - m}{12} \delta_{m+n,0} + \xi n^2 T(m+n)$

$+ (n-m) d_{m+n}^{+,1} + \dfrac{m^3 - m}{24} \delta_{m+n,0} - \xi m^2 T(m+n) + \xi^2 m^2 n \delta_{m+n,0}$ by (20), (21), and Lemma 4.1.

Hence, $[e_m^+(\xi), e_n^+(\xi)] = (n-m) \left[d_{m+n}^+ + \xi(m+n) T(m+n) + \dfrac{\xi^2}{2} \delta_{m+n,0} \right]$

$+ \dfrac{m^3 - m}{8} \delta_{m+n,0} - \xi^2 m^3 \delta_{m+n,0} + m\xi^2 \delta_{m+n,0} = (n-m) e_{m+n}^+(\xi)$

$+ \dfrac{m^3 - m}{8} \delta_{m+n,0} (1 - 8\xi^2);$

(b) $[e_m^+(\xi), e_n^-(\xi)] = [d_m^{+,0}, d_n^-] + [d_m^{+,1}, d_n^-] + \xi m[T(m), d_n^-] - 2\xi n[d_m^{+,1}, a(n)]$

$= \sum_{k \in \mathbb{Z}} \{[d_m^{+,0}, T(k)] \otimes a(n-k) + T(k)[d_m^{+,1}, a(n-k)]$

$+ \xi m[T(m), T(k)] \otimes a(n-k)\} - 2\xi n[d_m^{+,1}, a(n)] = \sum_{k \in \mathbb{Z}} \{ kT(m+k) \otimes a(n-k)$

$+ (n-k+\tfrac{1}{2}m) T(k) \otimes a(m+n-k)\} + \xi m^2 a(m+n) - 2\xi n(n+\tfrac{1}{2}m) a(m+n)$

$= \sum_{k \in \mathbb{Z}} [(k-m+n-k+\tfrac{1}{2}m) T(k) \otimes a(m+n-k)] + \xi(m^2 - 2n^2 - mn) a(m+n)$

$= \sum_{k \in \mathbb{Z}} (n-\tfrac{1}{2}m) T(k) \otimes a(m+n-k) - 2\xi(n-\tfrac{1}{2}m) (m+n) a(m+n) = (n-\tfrac{1}{2},) e_{m+n}^-(\xi),$

by (20) and Lemma 4.1.

(c) $[e_m^-(\xi), e_n^-(\xi)]_+ = \sum_{k,l \in \mathbb{Z}} [T(k) \otimes a(m-k), T(l) \otimes a(n-l)]_+$

$- 2\xi n \sum_{k \in \mathbb{Z}} [T(k) \otimes a(m-k), a(n)]_+ - 2\xi m \sum_{l \in \mathbb{Z}} [a(m), T(l) \otimes a(n-l)]_+$

$+ 4\xi^2 mn[a(m), a(n)]_+ = \sum_{k,l \in \mathbb{Z}} \{[[T(k), T(l)] \otimes a(m-k) a(n-l)$

$+ T(l) T(k) \otimes [a(m-k), a(n-l)]_+\} - 2\xi n \sum_{k \in \mathbb{Z}} T(k) \otimes [a(m-k), a(n)]_+$

$- 2\xi m \sum_{l \in \mathbb{Z}} T(l) \otimes [a(m), a(n-l)]_+ + 4\xi^2 mn[a(m), a(n)]_+ = \sum_{k \in \mathbb{Z}} [ka(m-k) a(n+k)$

$+ T(m+n-k) T(k)] - 2\xi n T(m+n) - 2\xi m T(m+n) - 4\xi^2 m^2 \delta_{m+n,0}.$

It is clear that $ka(m-k) a(n+k) - k{:}a(m-k) a(n+k){:} = 0 = T(m+n-k) T(k)$ $-{:}T(m+n-k) T(k){:}$, unless $m+n=0$. Assume that $m>0$. If $k>m$, then both

differences are zero since also $k > 0$. If $k < 0$, then $ka(m-k)\,a(n+k)$
$-k{:}a(m-k)\,a(n+k){:} = k\delta_{m+n,0}$ and $T(m+n-k)\,T(k) - {:}T(m+n-k)\,T(k){:}$
$= -k\delta_{m+n,0}$. Therefore, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, $[e_m^-(\xi), e_n^-(\xi)]_+ = \sum\limits_{k \in \mathbb{Z}} [k{:}a(m-k)\,a(n+k){:}$
$+ {:}T(m+n-k)\,T(k){:}] + \sum\limits_{k=0}^{m-\frac{1}{2}} k\delta_{m+n,0} - 2\xi(m+n)\,T(m+n) - 4\xi^2 m^2 \delta_{m+n,0}$

$= m \sum\limits_{k \in \mathbb{Z}+\frac{1}{2}} {:}a(k)\,a(m+n-k){:} - \sum\limits_{k \in \mathbb{Z}+\frac{1}{2}} k{:}a(k)\,a(m+n-k){:} + \sum\limits_{k \in \mathbb{Z}} {:}T(k)\,T(m+n-k){:}$
$+ \frac{1}{2}(m^2 - \frac{1}{4})\delta_{m+n,0} - 2\xi(m+n)\,T(m+n) - 4\xi^2 m^2 \delta_{m+n,0} = -2e_{m+n}^+(\xi)$

$+ \frac{1}{2}(m^2 - \frac{1}{4})\delta_{m+n,0}(1 - 8\xi^2)$. If $\mathfrak{g} = \hat{\mathfrak{r}}$, $[e_m^-(\xi), e_n^-(\xi)]_+ = \sum\limits_{k \in \mathbb{Z}} [k{:}a(m-k)\,a(n+k){:}$
$+ {:}T(m+n-k)\,T(k){:}] + \sum\limits_{k=0}^{m-1} k\delta_{m+n,0} + \frac{m}{2}\delta_{m+n,0} - 2\xi(m+n)\,T(m+n)$

$- 4\xi^2 m^2 \delta_{m+n,0} = m \sum\limits_{k \in \mathbb{Z}} {:}a(k)\,a(m+n-k){:} - \sum\limits_{k \in \mathbb{Z}} k{:}a(k)\,a(m+n-k){:}$

$+ \sum\limits_{k \in \mathbb{Z}} {:}T(k)\,T(m+n-k){:} + \frac{m^2}{2}\delta_{m+n,0} - 2\xi(m+n)\,T(m+n) - 4\xi^2 m^2 \delta_{m+n,0}$

$= -2e_{m+n}^+(\xi) + \frac{1}{2}(m^2 - \frac{1}{4})\delta_{m+n,0}(1 - 8\xi^2)$. Similarly for $m \leq 0$.   Q.E.D.

For $\omega \in \mathbb{C}$, let $\mathscr{K}^{\xi,\omega} = S(\hat{\mathfrak{t}}^-) \otimes \mathbb{C}e^{\omega T} \otimes \Lambda(U^-)$. Then

$$W = \bigoplus_{\omega \in \mathbb{C}} \mathscr{K}^{\xi,\omega} \tag{22}$$

as representations of $\mathfrak{g}$. Let $\mathfrak{h}, \mathfrak{n} \subset \mathfrak{g}$ be as in Sect. 3. For $\mu \in \mathfrak{h}^*$ we denote by $(\mathscr{K}^{\xi,\omega})^\mu$ the μ-weightspace of $\mathscr{K}^{\xi,\omega}$ relative to $\mathfrak{h}$. Let $(\mathscr{K}^{\xi,\omega})^\mu_\gamma = (\mathscr{K}^{\xi,\omega})^\mu \cap (\mathscr{K}^{\xi,\omega})_\gamma$, $\gamma \in \mathbb{Z}_2$.

Proposition 4.3. *Let* $\xi, \omega \in \mathbb{C}$. *Then*
   (i) $\mathscr{K}^{\xi,\omega} = \bigoplus\limits_{\mu \in \mathfrak{h}^*} (\mathscr{K}^{\mu,\omega})^\mu$;
   (ii) $(\mathscr{K}^{\xi,\omega})^\mu \neq 0$ *only if* $\mu(e_0') = 1 - 8\xi^2$ *and* $\mu(e_0^+) = \dfrac{\xi^2 - \omega^2}{2} - n$ *for some* $n \in \frac{1}{2}\mathbb{Z}_+$, *if* $\mathfrak{g} = \widehat{\mathfrak{ns}}$, *and* $\mu(e_0^+) = \dfrac{\xi^2 - \omega^2}{2} - \frac{1}{16} - n$ *for some* $n \in \mathbb{Z}_+$, *if* $\mathfrak{g} = \hat{\mathfrak{r}}$;
   (iii) $\dim(\mathscr{K}^{\xi,\omega})^\mu_\gamma = \hat{p}(n)$ *for* μ *as in* (ii) *and* $\gamma \in \mathbb{Z}_2$, *where* $\hat{p}$ *is defined in Proposition 3.5;*
   (iv) $1 \otimes e^{\omega T} \otimes 1$ *is an* $\mathfrak{n}$-*invariant vector in* $\mathscr{K}^{\xi,\omega}$.

Proof. Let $k_1, \ldots, k_r \in \mathbb{N}$, $l_1, \ldots, l_s \in \frac{1}{2}\mathbb{N}$, and let $n = \sum\limits_{i=1}^r k_i + \sum\limits_{j=1}^s l_j$.

   Using (20) and Lemma 4.1 (i) we obtain

$e_0^+(\xi)\,(T(-k_1) \ldots T(-k_r) \otimes e^{\omega T} \otimes a(-l_1) \ldots a(-l_s))$

$= T(-k_1) \ldots T(-k_r)\,a(-l_1) \ldots a(-l_s)\left(e_0^+(\xi) - \sum\limits_{i=1}^r k_i - \sum\limits_{j=1}^s l_j\right)(1 \otimes e^{\omega T} \otimes 1)$

$= \left(\dfrac{\xi^2 - \omega^2}{2} - n\right)T(-k_1) \ldots T(-k_r) \otimes e^{\omega T} \otimes a(-l_1) \ldots a(-l_s),$

if $\mathfrak{g} = \widehat{\mathfrak{ns}}$. Similarly, for $\mathfrak{g} = \hat{\mathfrak{r}}$. This proves (i) and (ii), and (iii) follows from (ii). By Theorem 4.2, e_0^+ acts on $e_m^\pm(1 \otimes e^{\omega T} \otimes 1)$ by $\dfrac{\xi^2 - \omega^2}{2} + m$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, and by $\dfrac{\xi^2 - \omega^2}{2} - \frac{1}{16} + m$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. (iv) now follows from (ii).   Q.E.D.

Let ζ be a formal variable and let $\alpha \in \mathbb{C}$. Let

$$X(\alpha, \zeta) = \exp\left(\sum_{n \in \mathbb{N}} \frac{\alpha T(-n)}{n} \zeta^{-n}\right) \exp\left(-\sum_{n \in \mathbb{N}} \frac{\alpha T(n)}{n} \zeta^n\right)$$

$$\otimes e^{\alpha T} \zeta^{-\alpha T - \frac{\alpha^2}{2}} \otimes 1, \tag{23}$$

where we set

$$\zeta^{-\alpha T - \frac{\alpha^2}{2}} (e^{\omega T}) = e^{\omega T} \zeta^{-\alpha w - \frac{\alpha^2}{2}}, \ [6]. \tag{24}$$

We define $X_\alpha(b): W \to W$ by

$$X(\alpha, \zeta) = \sum_{b \in \mathbb{C}} X_\alpha(b) \zeta^b. \tag{25}$$

Then

$$[T(m), X_\alpha(b)] = \alpha X_\alpha(m+b), \ m \in \mathbb{Z}, \ \alpha, b \in \mathbb{C} \ [6], \tag{26}$$

$$[d_m^{+,0}, X_\alpha(b)] = \left(m + b - \frac{\alpha^2}{2} m\right) X_\alpha(m+b), \ m \in \mathbb{Z}, \ \alpha, b \in \mathbb{C}, [6]. \tag{27}$$

The vertex operator is given by

$$Y_\alpha(b) = \sum_{k \in \mathbb{Z}} X_\alpha(b-k) \otimes a(k), \ [5]. \tag{28}$$

Theorem 4.4. *Let $\alpha \in \mathbb{C}\backslash\{0\}$ and let $Y_\alpha(b)$ be as in (28). $Y_\alpha(0)$ is a $\mathfrak{g}$-homomorphism of (π^ξ, W) if and only if $\xi = \dfrac{\alpha^2 - 1}{2\alpha}$.*[5]

Proof. $[e_m^-(\xi), Y_\alpha(b)]_+ = \left[\sum_{k \in \mathbb{Z}} T(k) \otimes a(m-k) - 2\xi m a(m), \sum_{l \in \mathbb{Z} + \frac{1}{2}} X_\alpha(b-l) \otimes a(l)\right]_+$

$= \sum_{k,l} [[T(k), X_\alpha(b-l)] \otimes a(m-k) a(l) + X_\alpha(b-l) T(k) \otimes [a(m-k), a(l)]]$

$-2\xi m X_\alpha(b+m) = \sum_{k,l} [\alpha X_\alpha(k+b-l) \otimes a(m-k) a(l)$

$+ X_\alpha(b+m-k) T(k) \otimes \delta_{m-k+l,0}] - 2\xi m X_\alpha(b+m)$ by (26). This implies that
$[e_m^-(\xi), Y_\alpha(b)]_+ = \sum_{k \in \mathbb{Z}} [\alpha X_\alpha(b+m) \otimes a(m-k) a(k-m) + X_\alpha(b+m-k) T(k)]$
$-2\xi m X_\alpha(b+m)$. We define $:X_\alpha(b+m-k) T(k):$ to be $X_\alpha(b+m-k) T(k)$ if $k>0$,
$\frac{1}{2}[X_\alpha(b+m-k) T(k) + T(k) X_\alpha(b+m-k)]$ if $k=0$, and $T(k) X_\alpha(b+m-k)$ if $k<0$.
Assume that $m>0$. If $k>m$, then $a(m-k) a(k-m) - :a(m-k) a(k-m): = 0$
$= X_\alpha(b+m-k) T(k) - :X_\alpha(b+m-k) T(k):$. If $k<0$, then

$$\alpha X_\alpha(b+m) \otimes [a(m-k) a(k-m) - :a(m-k) a(k-m):] = \alpha X_\alpha(b+m),$$

and $X_\alpha(b+m-k) T(k) - :X_\alpha(b+m-k) T(k): = -\alpha X_\alpha(b+m)$ by (26). Hence,
$[e_m^-(\xi), Y_\alpha(b)]_+ = \sum_{k \in \mathbb{Z}} [\alpha X_\alpha(b+m) \otimes :a(m-k) a(k-m): + :X_\alpha(b+m-k) T(k):]$
$+ \sum_{k=1}^{m-\frac{1}{2}} \alpha X_\alpha(b+m) + \frac{1}{2} \alpha X_\alpha(b+m) - 2\xi m X_\alpha(b+m) = -\dfrac{(b+m)}{\alpha} X_\alpha(b+m)$

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[5] The commutativity of $Y_1(0)$ with (π^0, W) for $\mathfrak{g} = \widehat{\mathfrak{n}\mathfrak{s}}$ was known [5]

$+ ma X_\alpha(b+m) - 2\xi m X_\alpha(b+m)$ by (27). Similarly for $m<0$. Therefore,

$$[e_m^-(\xi), Y_\alpha(b)]_+ = \left[m\left(\alpha - 2\xi - \frac{1}{\alpha}\right) - \frac{b}{\alpha}\right] X_\alpha(b+m). \text{ Now,}$$

$$X_\alpha(m) (1 \otimes e^{\beta T} \otimes 1) = 1 \otimes e^{(\alpha + \beta)T} \otimes 1 \neq 0 \quad \text{if} \quad m = -\alpha\beta - \frac{\alpha^2}{2}.$$

Furthermore the above implies that $[e_n^+(\xi), Y_\alpha(0)] = 0$ for all $n \in \mathbb{Z}$. The result now follows for $\mathfrak{g} = \widehat{\mathfrak{ns}}$. The proof for $\mathfrak{g} = \hat{\mathfrak{r}}$ is similar.   Q.E.D.

We now use the operators $Y_\alpha(0)$ to construct $\mathfrak{n}$-invariant vectors, that is, vectors that are annihilated by $\mathfrak{n}$.

Set, for $\alpha \in \mathbb{C} \backslash \{0\}$,

$$E^\pm(\alpha, \zeta) = \exp\left(- \sum_{n \in \mathbb{N}} \frac{\alpha T(\pm n)}{\pm n} \zeta^{\pm n}\right), \tag{29}$$

and let

$$E^-(\alpha, \zeta) = \sum_{n \geq 0} f_{-n} \zeta^{-n}, \tag{30}$$

where $f_{-n} \in \mathbb{C}[\alpha T(-1), \alpha T(-2), \ldots]$ is a polynomial of D-degree $-n$ [see (3)].

Proposition 4.5. (1) *The polynomials* $f_{-1}, f_{-2}, \ldots$ *are algebraically independent;*
(2) $\mathbb{C}[\alpha T(-1), \alpha T(-2), \ldots] = \mathbb{C}[f_{-1}, f_{-2}, \ldots]$.

Proof. For a partition $v = (n_1, \ldots, n_l)$ of n we write $\alpha_v = \alpha T(-n_1) \ldots \alpha T(-n_l)$ and proceed by induction on the weight $|v| = \sum_{j=1}^{l} n_j$ of v. By the induction assumption we see that $\alpha_v \in \mathbb{C}[f_{-1}, f_{-2}, \ldots]$ for all $v \neq (n)$ such that $|v| = n$. Now, $f_{-n} = \frac{\alpha T(-n)}{n} + \sum_v c_v \alpha_v$, where $c_v \in \mathbb{C}$ and the sum is over all partitions v such that $|v| = n$ but $v \neq (n)$. Therefore $\alpha_v \in \mathbb{C}[f_{-1}, f_{-2}, \ldots]$ for all v. This implies that the subspace U_{-n} of degree $-n$ in $\mathbb{C}[\alpha T(-1), \alpha T(-2), \ldots]$ is generated by the $p(n)$ monomials in the f_{-k}. Since $\dim(U_{-n}) = p(n)$, these monomials are linearly independent. This proves (1) and (2).   Q.E.D.

Let $\alpha, w \in \mathbb{C}$, $R \in \mathbb{N}$, and let $\zeta_1, \ldots, \zeta_R$ be distinct formal variables.

Lemma 4.6.

$$X(\alpha_1, \zeta_1) \ldots X(\alpha, \zeta_R) 1 \otimes e^{(\omega - R\alpha)T}$$

$$= \prod_{1 \leq i < j \leq R} (\zeta_j - \zeta_i)^{2p} \prod_{i=1}^{R} \zeta_i^{p-q} \prod_{i=1}^{R} \exp\left(\sum_{n \in \mathbb{N}} \frac{\alpha T(-n)}{n} \zeta_i^{-n}\right) 1 \otimes e^{\omega T},$$

where $\alpha^2 = 2p \in \mathbb{Z}_+$, $\alpha w = q$.

Proof. $E^+(\alpha, \zeta_1) E^-(\alpha, \zeta_2) = (1 - \zeta_1/\zeta_2)^{2p} E^-(\alpha, \zeta_2) E^+(\alpha, \zeta_1)$ by the Baker-Campbell-Hausdorff formula. Also,

$$(\zeta^{-\alpha T} e^{\alpha T}) e^X = (e^{\alpha T} \zeta^{-\alpha T - \alpha^2}) e^X \quad \text{for all} \quad \alpha \in \mathbb{C}, X \in \mathfrak{t}, \quad \text{by (24)}.$$

The result now follows by recurrence.   Q.E.D.

Let $\Delta = \prod\limits_{1 \le i < j \le R} (\zeta_j - \zeta_i)$, the Vandermonde determinant. Let $p = \dfrac{\alpha^2}{2}$, $q = \alpha w$, $\Delta^{2p} = \sum\limits_{\mu} a_\mu \zeta^\mu$, where $\mu = (m_1, \ldots, m_R)$, $\zeta^\mu = \zeta_1^{m_1} \ldots \zeta_R^{m_R}$.

Proposition 4.7. $X_\alpha(n_1) \ldots X_\alpha(n_R) 1 \otimes e^{(\omega - R\alpha)T} \ne 0$ *if and only if there exist* $s_1, \ldots, s_R \in \mathbb{Z}_+$ *such that* $\sum\limits_v a_v \ne 0$, *where* $v = (q - p + n_1 + s_{\pi_1}, \ldots, q - p + n_R + s_{\pi_R})$ *and π ranges over the set S_R of all permutations in R elements.*

Proof. $X_\alpha(n_1) \ldots X_\alpha(n_R) 1 \otimes e^{\omega - R\alpha}$ is the coefficient of $\zeta_1^{n_1} \ldots \zeta_R^{n_R}$ in $X(\alpha, \zeta_1) \ldots X(\alpha, \zeta_R) 1 \otimes e^{(\omega - R\alpha)T}$. By Lemma 4.6, the coefficient of $f_{-s_1} \ldots f_{-s_R} \otimes e^{\omega T}$ in $X_\alpha(n_1) \ldots X_\alpha(n_R) 1 \otimes e^{(\omega - R\alpha)T}$ is a non-zero multiple of $\sum\limits_v a_v$ with v as in the statement of the Proposition. The result now follows from Proposition 4.5 (1).   Q.E.D.

Corollary 4.8. *Let $R \in \mathbb{N}$, $\alpha^2 = 2p$, where $P = p - \tfrac{1}{2}$ is a prime number, and $\alpha w = q \in \mathbb{Z}$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, $\alpha w = q \in \mathbb{Z} + \tfrac{1}{2}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Then $Y_\alpha(0)^R (1 \otimes e^{(\omega - R\alpha)T} \otimes 1) \ne 0$, and $Y_\alpha(0)^R (1 \otimes e^{(\omega - R\alpha)T} \otimes a(0)) \ne 0$ if $\mathfrak{g} = \hat{\mathfrak{r}}$, provided that $p > R + \tfrac{1}{2}$, and that $Rp - q - \dfrac{R}{2} \ge 0$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, $Rp - q - \dfrac{R}{2} - \dfrac{1}{2} \ge 0$ if $\mathfrak{g} = \hat{\mathfrak{r}}$.*

Proof. Let $\mathfrak{g} = \widehat{\mathfrak{ns}}$. The coefficient of $a(-\tfrac{1}{2}) a(-\tfrac{3}{2}) \ldots a(-R + \tfrac{1}{2})$ in $Y_\alpha(0)^R 1 \otimes e^{(\omega - R\alpha)T} \otimes 1$ is $X_\alpha(\tfrac{1}{2}) X_\alpha(\tfrac{3}{2}) \ldots X_\alpha(R - \tfrac{1}{2}) (1 \otimes e^{(\omega - R\alpha)T})$. Set $S = Rp - q - \dfrac{R}{2} \in \mathbb{Z}_+$. The coefficient of $f_{-S}^R \otimes e^{\omega T}$ in $X_\alpha(\tfrac{1}{2}) \ldots X_\alpha(R - \tfrac{1}{2}) (1 \otimes e^{(\omega - R\alpha)T})$ is a_v, where the coefficient a_v is as in Proposition 4.7 and $v = (P(R-1), P(R-1) + 1, \ldots, P(R-1) + R - 1)$. From the elementary properties of determinants one sees that the coefficient of $\zeta_1^{R-1} \ldots \zeta_R^{R-1}$ in Δ^2 is $\pm R!$ Now,

$$\Delta^{2P+1} = \Delta^{2P} \sum_{\pi \in S_R} \mathrm{sgn}(\pi) \zeta_{\pi_1}^0 \zeta_{\pi_2}^1 \ldots \zeta_{\pi_R}^{R-1}, \quad \Delta^{2P} \equiv \Delta^2(\zeta_1^P, \ldots, \zeta_R^P) \pmod{P},$$

and the coefficient of $\zeta_1^{P(R-1)} \ldots \zeta_R^{P(R-1)}$ in Δ^{2P} is $\pm R! \pmod{P}$. Hence, $a_v \ne 0$, for $v = (P(R-1), P(R-1) + 1, \ldots, P(R-1) + R - 1)$ if $P > R$. For $\mathfrak{g} = \hat{\mathfrak{r}}$, the argument is the same except for the following changes: Instead of $a(-\tfrac{1}{2}) \ldots a(-R + \tfrac{1}{2})$, $X_\alpha(\tfrac{1}{2}) \ldots X_\alpha(R - \tfrac{1}{2}) (1 \otimes e^{\omega - R\alpha})$ and $S = Rp - q - \dfrac{R}{2}$, take $a(-1) \ldots a(-R)$ [respectively $a(-1) \ldots a(-R) a(0)$], $X_\alpha(1) \ldots X_\alpha(R) 1 \otimes e^{\omega - R\alpha}$ [respectively $X_\alpha(1) \ldots X_\alpha(R) 1 \otimes e^{\omega - R\alpha} \otimes a(0)$] and $S = Rp - q - \dfrac{R}{2} - \dfrac{1}{2} \in \mathbb{Z}_+$, respectively. Q.E.D.

Next, we will prove the determinant formulas 3 (8) and 3 (9). The proof will consist of two parts: the calculation of the h-degree of $\det(\, , \,)_{h,c,n,\gamma}$ and the divisibility of $\det(\, , \,)_{h,c,h,\gamma}$ by $\psi_{r,s}(h, c)^{\bar{p}(n - 2rs)}$. We note that formula 3 (9) is obvious if $n = 0$, so we assume that $n > 0$. From now on $V(\lambda) = \mathbb{C}(\lambda)$ or $V(\lambda) = R(\lambda)$.

For $n \in \tfrac{1}{2} \mathbb{N}$, let $\mathcal{P}_n$ be the set of all $(d + f)$-tuples $\mathbf{I} = (i_1, \ldots, i_d; j_1, \ldots, j_f)$ in $\mathbb{N}^d \times \mathbb{Z}^f$ such that $i_1 \ge \ldots \ge i_d$, $j_1 > \ldots > j_f$, $j_k > 0$, $k = 1, \ldots, f$, and $|\mathbf{I}| = i_1 + \ldots + i_d + j_1 + \ldots + j_f = n$. Let $\mathcal{P}_n^e$ (respectively $\mathcal{P}_n^0$) be the subset of $\mathcal{P}_n$ consisting of those

$\mathbf{I} = (i_1, \ldots, i_d; j_1, \ldots, j_f)$ with f even (respectively f odd). We say that $\mathbf{I}$ as above is equal to $\mathbf{I}' = (i'_1, \ldots, i'_{d'}; j'_1, \ldots, j'_{f'}) \in \mathscr{P}_n$ if $d = d'$, $f = f'$, and $i_1 = i'_1, \ldots, i_d = i'_d$, $j_1 = j'_1, \ldots, j_f = j'_{f'}$. Let $\lambda \in \mathfrak{h}^*$ and set $h = \lambda(e_0^+)$, $c = \lambda(e'_0)$. If $\mathbf{I} \in \mathscr{P}_n$, set $e_{\mathbf{I}} = e_{-i_1}^+ \ldots e_{-i_d}^+ e_{-j_1}^- \ldots e_{-j_f}^-$. We now consider the basis of $M(\mathbb{C}(\lambda))^{\lambda - n\delta}$ formed by the elements $e_{\mathbf{I}} v_\lambda$ and the basis of $M(R(\lambda))_0^{\lambda - n\delta}$ [respectively $M(R(\lambda))_1^{\lambda - n\delta}$] consisting of the elements $e_{\mathbf{I}} v_\lambda$, $e_{\mathbf{K}} e_0^- v_\lambda$, where $\mathbf{I} \in \mathscr{P}_n^e$, $\mathbf{K} \in \mathscr{P}_n^0$ (respectively $\mathbf{I} \in \mathscr{P}_n^0$, $\mathbf{K} \in \mathscr{P}_n^e$). [Recall from the proof of Proposition 3.6 that the number of such $e_{\mathbf{I}} v_\lambda$ is equal to the number of such $e_{\mathbf{K}} e_0^- v_\lambda$ and is equal $\frac{1}{2} \hat{p}(n)$.]

If $F \in \mathbb{C}[h, c]$ we let h-deg(F) denote the h-degree of F, that is, the degree of F regarded as a polynomial of h.

The following lemma is the analog for $\mathfrak{g}$ of [26, Lemma 4]:

Lemma 4.9. Let $\mathbf{I} = (i_1, \ldots, i_d; j_1, \ldots, j_f)$ and $\mathbf{K} = (k_1, \ldots, k_g; l_1, \ldots, l_m) \in \mathscr{P}_n$. Then

(1) h-deg $(e_{\mathbf{I}} v_\lambda, e_{\mathbf{K}} v_\lambda)_{h, c, n, \gamma} \leq \min\{d + f, g + m\}$,

h-deg $(e_{\mathbf{I}} e_0^- v_\lambda, e_{\mathbf{K}} e_0^- v_\lambda)_{h, c, n, \gamma} \leq \min\{d + f, g + m\} + 1$,

h-deg $(e_{\mathbf{I}} e_0^- v_\lambda, e_{\mathbf{K}} v_\lambda)_{h, c, n, \gamma} \leq \min\{d + f, g + m\}$.

(2) If $d + f = g + m$ and $\mathbf{I} \neq \mathbf{K}$, then h-deg $(e_{\mathbf{I}} v_\lambda, e_{\mathbf{K}} v_\lambda)_{h, c, n, \gamma} < d + f$ and h-deg $(e_{\mathbf{I}} e_0^- v_\lambda, e_{\mathbf{K}} e_0^- v_\lambda)_{h, c, n, \gamma} < d + f + 1$.

(3) h-deg $(e_{\mathbf{I}} v_\lambda, e_{\mathbf{I}} v_\lambda)_{h, c, n, \gamma} = d + f$ and

h-deg $(e_{\mathbf{I}} e_0^- v_\lambda, e_{\mathbf{I}} e_0^- v_\lambda)_{h, c, n, \gamma} = d + f + 1$.

Proof. Let $v \in \{v_\lambda, e_0^- v_\lambda\}$. Let $d > 0$. Using the bracket relations in $\mathfrak{g}$ we obtain:

$$e_{i_1}^+ e_{\mathbf{K}} v = \sum e_{\mathbf{K}'} (a_{\mathbf{K}'} x_{M(i_1, \mathbf{K}')} + b_{\mathbf{K}'} c) e_{\mathbf{K}''} v, \tag{31}$$

where the sum is over all the $\mathbf{K}' = (k'_1, \ldots, \hat{k}'_{p_1}, \ldots, \hat{k}'_{p_r}, \ldots, k'_q; l'_1, \ldots, \hat{l}'_{q_1}, \ldots, \hat{l}'_{q_t})$ such that $k'_{p_1} + \ldots + k'_{p_r} + l'_{q_1} + \ldots + l'_{q_{t-1}} < i_1$, $k'_{p_1} + \ldots + k'_{p_r} + l'_{q_1} + \ldots + l'_{q_t} \geq i_1$, $\mathbf{K}'' = (l'_{q_{t+1}}, \ldots, l'_{m'})$, and $x_{M(i_1, \mathbf{K}')} = e_M^+$ if t is even, $x_{M(i_1, \mathbf{K}')} = e_M^-$ if t is odd, where $M = i_1 - k'_{p_1} - \ldots - k'_{p_r} - l'_{q_1} - \ldots - l'_{q_t}$, $a_{\mathbf{K}'}, b_{\mathbf{K}'} \in \mathbb{C}$. Since $(,)_\lambda$ is symmetric we may assume that $k = \deg e_{\mathbf{I}} v \leq \deg e_{\mathbf{K}} w$. The lemma now follows immediately from 3 (6) and (31) by induction on k. Similarly if $d = 0$, $f > 0$.   Q.E.D.

Proposition 4.10. For $i \in \frac{1}{2} \mathbb{N}$, set $\hat{d}(i) = \#\{(r, s) | 2rs = i, r, s \in \frac{1}{2} \mathbb{N}, r - s \in \mathbb{Z}\}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, and $\hat{d}(i) = \#\{(r, s) | 2rs = i, r, s \in \frac{1}{2} \mathbb{N}, r - s \in \frac{1}{2} + \mathbb{Z}\}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Then

(1) h-deg det$(,)_{h, c, n} = \sum_{\substack{i \in \frac{1}{2} \mathbb{N} \\ i \leq n}} \hat{p}(n - i) \hat{d}(i)$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$;

(2) h-deg det$(,)_{h, c, n, \gamma} = \frac{1}{2} \hat{p}(n) + \sum_{i = 1}^{n} \hat{p}(n - i) \hat{d}(i)$ if $\mathfrak{g} = \hat{\mathfrak{r}}$.

Proof. Let $N = \frac{1}{2} \hat{p}(n)$, and let $y_1, \ldots, y_N$ be the elements $\{e_{\mathbf{I}} v_\lambda\}$, $\mathbf{I} \in \mathscr{P}_n^e$ (respectively $\mathbf{I} \in \mathscr{P}_n^0$) in some order and $y_{N+1}, \ldots, y_{2N}$ be the elements $\{e_{\mathbf{K}} e_0^- v_\lambda\}$, $\mathbf{K} \in \mathscr{P}_n^0$ (respectively $\mathbf{K} \in \mathscr{P}_n^e$) in some order, so that $y_1, \ldots, y_{2N}$ is a basis of $M(R(\lambda))_0^{\lambda - n\delta}$ [respectively $M(R(\lambda))_1^{\lambda - n\delta}$]. Now det$(,)_{h, c, n, \gamma} = \sum_{\sigma \in S_{2N}} \text{sgn}(\sigma) \prod_{i=1}^{2N} (y_i, y_{\sigma(i)})_{h, c, n, \gamma}$. Then, by Lemma 4.9 the term corresponding to the identity permutation is the only term of h-degree $\frac{1}{2} \hat{p}(n) + \sum_{k=1}^{n} k \hat{p}(n, k)$, hence, by the same lemma,

h-deg det$(,)_{h, c, n, \gamma} = \frac{1}{2} \hat{p}(n) + \sum_{k=1}^{n} k \hat{p}(n, k)$.   Here, $\hat{p}(n, k) = \#\{\mathbf{I} = (i_1, \ldots, i_d;$

$j_1, \ldots, j_f) \in \mathscr{P}_n | d + f = k\}$, for $\mathfrak{g} = \hat{\mathfrak{r}}$. Similarly, h-deg det $(\,,\,)_{h,c,n} = \sum\limits_{k=1}^{n} k\hat{p}(n,k)$, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$. Now,

$$\frac{\prod\limits_{n \in \mathbb{N}} (1 + xq^n)}{\prod\limits_{n \in \mathbb{N}} (1 - xq^n)} = 1 + \sum\limits_{n \in \mathbb{N}} \sum\limits_{k \in \mathbb{N}} \hat{p}(n,k) x^k q^n \quad \text{in} \quad \mathbb{C}[\![x, q]\!]. \tag{32}$$

Taking $\dfrac{d}{dx}$ at $x = 1$ in (32), we get

$$\left(\sum\limits_{n \in \mathbb{N}} \frac{q^n}{1 - q^n} + \sum\limits_{n \in \mathbb{N}} \frac{q^n}{1 + q^n} \right) \left(\frac{\prod\limits_{n \in \mathbb{N}} (1 + q^n)}{\prod\limits_{n \in \mathbb{N}} (1 - q^n)} \right) = \sum\limits_{n \in \mathbb{N}} \left(\sum\limits_{k=1}^{n} k\hat{p}(n,k) \right) q^n. \tag{33}$$

Let, for $i \in \mathbb{N}$, $d(i) = \#$ of divisors of i, and set $\tilde{d}(i) = d(i) + \sum\limits_{r \in \mathbb{N}, rs = i} (-1)^{s+1}$. If $i \in 2\mathbb{N} - 1$, then $\tilde{d}(i) = 2d(i)$. Noting that if $i = rs$ then $r, s \in 2\mathbb{N} - 1$ and $i = 2\dfrac{r}{2} s = 2r\dfrac{s}{2}$, we see that $\tilde{d}(i) = \hat{d}(i)$. If $i \in 2\mathbb{N}$, then $\tilde{d}(i) = 2(\#\{r \in 2\mathbb{N} | rs = i, \; s \in 2\mathbb{N} - 1\})$ $= 2(\#\{r \in \mathbb{N} | rs = i, s \in 2\mathbb{N} - 1\} = \hat{d}(i)$. From (33) we obtain

$$\sum\limits_{n \in \mathbb{N}} \left(\sum\limits_{i \in \mathbb{N}} \hat{p}(n-i) \hat{d}(i) \right) q^n = \sum\limits_{n \in \mathbb{N}} \left(\sum\limits_{k=1}^{n} \hat{p}(n,k) \right) q^n. \tag{34}$$

This proves (2). Similarly, if $\mathfrak{g} = \widehat{\mathfrak{ns}}$,

$$\frac{\prod\limits_{n \in \mathbb{N} - \frac{1}{2}} (1 + xq^n)}{\prod\limits_{n \in \mathbb{N}} (1 - xq^n)} = \sum\limits_{n \in \frac{1}{2}\mathbb{N}} \sum\limits_{k \in \mathbb{N}} \hat{p}(n,k) x^k q^n, \quad \text{where} \quad \hat{p}(n,k) \tag{35}$$

is as before, Taking $\dfrac{d}{dx}$ at $x = 1$ in (35) we obtain

$$\left(\sum\limits_{n \in \mathbb{N}} \frac{q^n}{1 - q^n} + \sum\limits_{n \in \mathbb{N} - \frac{1}{2}} \frac{q^n}{1 + q^n} \right) \frac{\prod\limits_{n \in \mathbb{N} - \frac{1}{2}} (1 + q^n)}{\prod\limits_{n \in \mathbb{N}} (1 - q^n)} = \sum\limits_{n \in \frac{1}{2}\mathbb{N}} \left(\sum\limits_{k \in \mathbb{N}} k\hat{p}(n,k) \right) q^n. \tag{36}$$

Let, for $i \in \frac{1}{2}\mathbb{N}$, $\tilde{d}(i) = \#\{(r,s) \in \mathbb{N}^2 | rs = i\} + \sum\limits_{r \in \mathbb{N} - \frac{1}{2}, rs = i, s \in \mathbb{N}} (-1)^{s+1}$. If $i \in \mathbb{N}$, then $\tilde{d}(i) = \#$ of even divisors of $i = \hat{d}(i)$. If $i \in \mathbb{N} - \frac{1}{2}$, then $\tilde{d}(i) = d(2i) = \hat{d}(i)$. From (36) we obtain

$$\sum\limits_{n \in \frac{1}{2}\mathbb{N}} \left(\sum\limits_{i \in \frac{1}{2}\mathbb{N}} \hat{p}(n-i) \hat{d}(i) \right) q^n = \sum\limits_{n \in \frac{1}{2}\mathbb{N}} \left(\sum\limits_{k \in \mathbb{N}} k\hat{p}(n,k) \right) q^n. \tag{37}$$

Hence $\sum\limits_{i \in \frac{1}{2}\mathbb{N}} \hat{p}(n-i) \hat{d}(i) = \sum\limits_{k \in \mathbb{N}} k\hat{p}(n,k)$. This proves (1).   Q.E.D.

Lemma 4.11. *Let* $r, s \in \frac{1}{2}\mathbb{N}$ *be such that* $s - r \in \mathbb{Z}_+$ *if* $\mathfrak{g} = \widehat{\mathfrak{ns}}$ *and* $s - r \in \frac{1}{2} + \mathbb{Z}_+$ *if* $\mathfrak{g} = \hat{\mathfrak{r}}$. *Let* $p \in \frac{1}{2} + \mathbb{N}$ *be such that* $p - \frac{1}{2}$ *is prime and* $p > 2r + \frac{1}{2}$. *Let* $c = 1 - \dfrac{(2p-1)^2}{p}$,

$\gamma \in \mathbb{Z}_2$, and $h = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p}$ if $\mathfrak{g} = \mathfrak{ns}$, $h = \dfrac{2(p-1)^2 - 4(2rp-s)^2}{16p} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Then $\det(\ , \ )_{h,c,2rs,\gamma} = 0$.

Proof. Let $\xi = \dfrac{2p-1}{2\sqrt{2p}}$, $\omega = \dfrac{2rp-s}{\sqrt{2p}}$, $\alpha = \sqrt{2p}$ and $R = 2r$. Then $\xi = \dfrac{\alpha^2 - 1}{2\alpha}$, and so $Y_\alpha(0)^R$ intertwines the action of $\mathfrak{g}$ on $\mathscr{K}^{\xi, \omega - R\alpha}$ and $\mathscr{K}^{\xi, \omega}$ by Theorem 4.4. Therefore, by Theorems 4.2 and 4.4, and by Proposition 4.3, $u_{\omega, \alpha, R} = Y_\alpha(0)^R (1 \otimes e^{(\omega - R\alpha)T} \otimes 1)$ and $e_0^- u_{\omega, \alpha, R}$ are $\mathfrak{n}$-invariant vectors of $(\mathscr{K}^{\xi, \omega})^\mu$, where $\mu(e_0') = 1 - 8\xi^2$ and $\mu(e_0^+) = \dfrac{\xi^2 - \omega^2}{2} - 2rs$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, $\mu(e_0^+) = \dfrac{\xi^2 - \omega^2}{2} - \dfrac{1}{16} - 2rs$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $q = 2rp - s$. Then $q \in \mathbb{Z}$ and $Rp - q - \dfrac{R}{2} \geqq 0$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, and $q \in \mathbb{Z} + \frac{1}{2}$ and $Rp - q - \dfrac{R}{2} - \dfrac{1}{2} \geqq 0$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Also note that $\alpha^2 = 2p$ and $\alpha w = q$. By Corollary 4.8, $u_{\omega, \alpha, R} = Y_\alpha(0)^R (1 \otimes e^{(\omega - R\alpha)T} \otimes 1)$ and $e_0^- u_{\omega, \alpha, R}$ are both nonzero. Let $\mathscr{U}$ denote the $\mathfrak{g}$-submodule of $\mathscr{K}^{\xi, \omega}$ generated by $1 \otimes e^{\omega T} \otimes 1$. By the above and Proposition 3.7 there is a unique surjective homomorphism $\phi : M(V(\lambda)) \to \mathscr{U}$ such that $\phi(v_\lambda) = 1 \otimes e^{\omega T} \otimes 1$, since $c = 1 - 8\xi^2$, and $h = \dfrac{\xi^2 - \omega^2}{2}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, $h = \dfrac{\xi^2 - \omega^2}{2} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $\psi = i \circ \phi : M(V(\lambda)) \to \mathscr{K}^{\xi, \omega}$, where $i : \mathscr{U} \to \mathscr{K}^{\xi, \omega}$ is the inclusion map. If $u_{\omega, \alpha, R} \in \operatorname{Im} \psi = \mathscr{U}$ and if $e_0^- u_{\omega, \alpha, R} \in \operatorname{Im} \psi$, then the assertion follows from Proposition 3.14. If $u_{\omega, \alpha, R} \notin \operatorname{Im} \psi$, then $\dim(\operatorname{Im} \psi)_0^{\lambda - 2rs\delta} < \hat{p}(2rs) = \dim M(V(\lambda))_0^{\lambda - 2rs\delta}$. Therefore, $(\operatorname{Ker} \psi)_0^{\lambda - 2rs\delta} \neq (0)$, and so, $\det(\ , \ )_{\lambda, 2rs, \bar{0}} = 0$, by Proposition 3.14. Similarly, if $e_0^- u_{\omega, \alpha, R} \notin \operatorname{Im} \psi$.   Q.E.D.

Let $\psi_{r,s}$ be as in Sect. 3 if $r, s \in \frac{1}{2}\mathbb{N}$, and let $\psi_{0,0}(h, c) = h + \dfrac{c}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $\mathscr{V}_{r,s}$ denote the variety in $\mathbb{C}^2$ consisting of all points (h, c) such that $\psi_{r,s}(h, c) = 0$, $(r, s) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N} \cup \{(0, 0)\}$.

Lemma 4.12. *Let $r, s \in \frac{1}{2}\mathbb{N}$, $r \leqq s$. Then*

(1) $\mathscr{V}_{r,s} = \left\{ \left(\dfrac{(2p-1)^2 - 4(2pr-s)^2}{16p}, 1 - \dfrac{(2p-1)^2}{p} \right) \Big| p \in \mathbb{C} \backslash \{0\} \right\}$ *if* $\mathfrak{g} = \hat{\mathfrak{ns}}$;

(2) $\mathscr{V}_{r,s} = \left\{ \left(\dfrac{(2p-1)^2 - 4(2pr-s)^2}{16p} - \dfrac{1}{16}, 1 - \dfrac{(2p-1)^2}{p} \right) | p \in \mathbb{C} \backslash \{0\} \right\}$ *if* $\mathfrak{g} = \mathfrak{r}$.

Proof. Let $p^\pm = \dfrac{5 - c \pm \sqrt{c^2 - 10c + 9}}{8}$. Then $(2p^\pm - 1)^2 = p^\pm (1 - c)$, i.e. $c = 1 - \dfrac{(2p^\pm - 1)^2}{p^\pm}$. Now, $h = h_{r,s}^\pm(c) \Leftrightarrow h = \dfrac{1-c}{16} - r^2 p^\pm + rs - \dfrac{s^2}{4p^\pm}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, and $h = \dfrac{1-c}{16} - r^2 p^\pm + rs - \dfrac{s^2}{4p^\pm} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Hence $h = h_{r,s}^\pm(c) \Leftrightarrow h = \dfrac{1}{16p^\pm} [(1-c)p^\pm - 4(2rp^\pm - s)^2]$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, $h = \dfrac{1}{16p^\pm} [(1-c)p^\pm - 4(2rp^\pm - s)^2] - \frac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Let $\psi_{r,s}(h, c) = 0$.   Then $h = h_{r,s}^\pm(c)$.   Therefore, $c = 1 - \dfrac{(2p-1)^2}{2}$ and

$h = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, $h = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$, where

$p = p^{\pm}$. Conversely, let $c = 1 - \dfrac{(2p-1)^2}{2}$ and $h = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$,

$h = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. Then $p = p^{\pm}$ and $h = h_{r,s}^{\pm}$, i.e. $\psi_{r,s}(h,c) = 0$.
Q.E.D.

Proposition 4.13. *Let* $h, c \in \mathbb{C}$, $\gamma \in \mathbb{Z}_2$. *Let* $r, s \in \frac{1}{2}\mathbb{N}$, $r \leq s$, *and* $r - s \in \mathbb{Z}$ *if* $\mathfrak{g} = \hat{\mathfrak{ns}}$, $r - s \in \frac{1}{2} + \mathbb{Z}$ *if* $\mathfrak{g} = \hat{\mathfrak{r}}$. *Then* $\det(\,,\,)_{h,c,2rs,\gamma}$ *vanishes on* $\mathscr{V}_{r,s}$. *Furthermore, if* $\mathfrak{g} = \hat{\mathfrak{r}}$, $n \in \mathbb{Z}_+$, *then* $\det(\,,\,)_{h,c,n,\gamma}$ *vanishes on* $\mathscr{V}_{0,0}$.

Proof. Assume $(r,s) \neq (0,0)$. Let $\theta : \mathbb{C}[h,c] \to \mathbb{C}[p,p^{-1}]$ be the homomorphism defined by $\theta(c) = 1 - \dfrac{(2p-1)^2}{p}$ and $\theta(h) = \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p}$ if $\mathfrak{g} = \hat{\mathfrak{ns}}$, $\theta(h)$

$= \dfrac{(2p-1)^2 - 4(2rp-s)^2}{16p} - \dfrac{1}{16}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. By Lemma 4.11, $\theta(\det(\,,\,)_{h,c,2rs,\gamma})$ vanishes at infinitely many values of p. Therefore, $\theta(\det(\,,\,)_{h,c,2rs,\gamma}) \equiv 0$. This implies, by Lemma 4.12, that $\det(\,,\,)_{h,c,2rs,\gamma}$ vanishes on $\mathscr{V}_{r,s}$. Also, if $\mathfrak{g} = \hat{\mathfrak{r}}$, we saw that $M(R(\lambda)) \supset M(S(\lambda))$ if $h = -\dfrac{c}{16}$ (Proposition 3.5). Hence, for each $n \in \mathbb{Z}_+$, $\det(\,,\,)_{h,c,n,\gamma}$ vanishes on $\mathscr{V}_{0,0}$.   Q.E.D.

Proof of the Determinant Formulas. We proceed by induction on n. Let r, s be as in Proposition 4.13, $n \in \mathbb{Z}_+$, $2rs \leq n$. Let $u = \varphi_{r,s}(h,c)$, $v = c$. Then $\left|\dfrac{\partial(u,v)}{\partial(h,c)}\right| = \dfrac{\partial}{\partial h}$

$\cdot \psi_{r,s}(h,c)$. Hence, if $r = s$, $\dfrac{\partial(u,v)}{\partial(h,c)} = 1 \neq 0$, and if $r \neq s$, $\left|\dfrac{\partial(u,v)}{\partial(h,c)}\right| = 0 \Leftrightarrow h$

$= \dfrac{h_{r,s}^+(c) + h_{r,s}^-(c)}{2}$. Thus, at $h = h_{r,s}^{\pm}(c)$, $\left|\dfrac{\partial(u,v)}{\partial(h,c)}\right| = 0 \Leftrightarrow c = 1$ or $c = 9$. This implies that $\{u, v\}$ is a local system of coordinates near all $(h,c) \in \mathscr{V}_{r,s} \setminus \{(h_{r,s}^{\pm}(1), 1), (h_{r,s}^{\pm}(9), 9)\}$. By letting $\mathfrak{n}$ act on the basis of $M(V(\lambda))_\gamma^{\lambda - m\delta}$ described in the proof of Proposition 4.10, we see that $\det(\,,\,)_{h,c,m,\gamma} = 0$ if and only if there is a non-zero $\mathfrak{n}$-invariant vector in $M(V(\lambda))_\gamma^{\lambda - q\delta}$, $q \leq m$. Therefore, if $(h,c) \in \mathscr{V}_{r,s}$, then there is a non-zero $\mathfrak{n}$-invariant vector in $M(V(\lambda))_\gamma^{\lambda - q\delta}$, $q \leq 2rs$, by Proposition 4.13. Let

$$\mathscr{V}_{r,s}' = \mathscr{V}_{r,s} \setminus \left(\{(h_{r,s}^{\pm}(1),1), (h_{r,s}^{\pm}(9),9)\} \bigcup_{\substack{(r',s') \neq (r,s) \\ r's' \leq rs}} \mathscr{V}_{r',s'} \right)$$

if $(r,s) \neq (0,0)$, $\mathscr{V}_{0,0}' = \mathscr{V}_{0,0}$. Arguing as in [23, Lemma 3.2], we see that $\mathscr{V}_{r,s}'$ contains all but a finite number of points of $\mathscr{V}_{r,s}$. Let $\lambda_0 \in \mathscr{V}_{r,s}'$. By the induction hypothesis there is then a non-zero $\mathfrak{n}$-invariant vector in $M(V(\lambda_0))_\gamma^{\lambda_0 - 2rs\delta}$. Let $\mathscr{U}$ be the submodule of $M(V(\lambda_0))$ generated by this vector. Then $\dim(M(V(\lambda_0))_\gamma^{\lambda_0 - n\delta} \cap \mathscr{U}) = \breve{p}(n - 2rs)$, where $\breve{p}(n - 2rs) = \hat{p}(n - 2rs)$ if $(r,s) \neq (0,0)$, $\breve{p}(n) = \frac{1}{2}\hat{p}(n)$, by Propositions 3.1 and 3.6. Therefore,

$$\det{}_n(\,,\,)_{h(u,v),c(u,v),n,\gamma} = u^{\breve{p}(n - 2rs)} f(u,v), \qquad (38)$$

where f is analytic near (u_0, v_0) and u_0, v_0 are the (u, v)-coordinates of λ_0. Let $g(h, c)$ be an irreducible divisor of $\det(\ ,\ )_{h,c,n,\gamma}$ in $\mathbb{C}[h, c]$. If $g(h(u, v), c(u, v)) = uk(u, v)$, with k analytic near (u_0, v_0), then $g(h, c) = 0$ for infinitely many (h, c) in $\mathscr{V}_{r,s}$. Hence $\theta(g(h, c)) = 0$ for all (h, c) and $g(h, c) = 0$ for all (h, c) on $\mathscr{V}_{r,s}$ as in the proof of Proposition 4.13. Thus $g(h, c)$ is a nonzero multiple of $\psi_{r,s}(h, c)$. This, (38) and Proposition 13 show that $\psi_{r,s}(h, c)^{\tilde{p}(n-2rs)}$ divides $\det(\ ,\ )_{h,c,n,\gamma}$. The degree computation of Proposition 4.10 gives (8) and (9).   Q.E.D.

5. The Irreducible Subquotients of a Verma Module

We retain the notation of Sect. 3.

For $n \in \frac{1}{2}\mathbb{N}$, let $\phi_n(h, c) = \prod_{2rs=n} \psi_{r,s}(h, c)$, where the product is over all $r, s \in \frac{1}{2}\mathbb{N}$ such that $s - r \in \mathbb{N}$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$, $s - r \in \mathbb{N} - \frac{1}{2}$ if $\mathfrak{g} = \hat{\mathfrak{r}}$. As in [22], we set, for $\lambda \in \mathfrak{h}^*$, $(\frac{1}{2}\mathbb{N})_+(\lambda) = \{n \in \frac{1}{2}\mathbb{N} | \phi_n(h, c) = 0\}$. Let $\mu, \lambda \in \mathfrak{h}^*$. We write $\mu \uparrow \lambda$ if $\mu = \lambda$ or if $\mu = \lambda - \left(\sum_{i=1}^{r} m_i\right)\delta$ for some $m_1, \ldots, m_r \in \frac{1}{2}\mathbb{N}$ such that

$$m_1 \in (\tfrac{1}{2}\mathbb{N})_+(\lambda), \qquad m_i \in (\tfrac{1}{2}\mathbb{N})_+\left(\lambda - \sum_{j=1}^{i-1} m_j\delta\right), \qquad 2 \le i \le r.$$

A subquotient of a module M is a quotient N/U where $U \subset N \subset M$, U, N submodules of M. From now on, unless otherwise stated, $V(\lambda) = \mathbb{C}(\lambda)$ if $\mathfrak{g} = \widehat{\mathfrak{ns}}$ and $V(\lambda) = R(\lambda)$ if $\mathfrak{g} = \hat{\mathfrak{r}}$.

Theorem 5.1. *Let $\lambda, \mu \in \mathfrak{h}^*$. If $L(V(\mu))$ is a subquotient of $M(V(\lambda))$, then $\mu \uparrow \lambda$.*[6]

In this section we prove Theorem 5.1. The construction of filtrations of Verma modules given below is a straight-forward generalization of the one given in [23].

Let $V = U(\mathfrak{n}^-)$, $\mu \in \mathfrak{h}^*$. Let $T_\mu : V \to M(V(\mu))$ be the linear ($\mathbb{Z}_2$-graded) isomorphism defined by $T_\mu(X) = Xv_\mu$, $X \in V$. We set $\tau_\mu(Y)v = T_\mu^{-1}(YT_\mu(v))$ for $Y \in \mathfrak{g}$, $v \in V$. Then (V, τ_μ) is a representation of $\mathfrak{g}$. On $V \times V$ we define $A_\mu(v, w) = (T_\mu(v), T_\mu(w))_\mu$. It is automatic to see that A_μ defines a symmetric bilinear form on $V \times V$ such that

$$A_\mu(\tau_\mu(Y)v, w) = A_\mu(v, \tau_\mu(\sigma(Y))w) \tag{1}$$

for all $Y \in U(\mathfrak{g})$, $v, w \in V$.

We now let λ be fixed and for each $t \in \mathbb{C}$, we set

$$(\pi_t, V) = (\tau_{\lambda+t\delta}, V), \tag{2}$$

$$B_t(\ ,\ ) = A_{\lambda+t\delta}(\ ,\ ). \tag{3}$$

Now, $V = \bigoplus_{\eta \in Q^+ \cup \{0\}} V^{\lambda-\eta}$.

Let $\mathcal{O}(\mathbb{C})$ denote the space of germs $\mathbf{f}$ of complex valued holomorphic functions f at 0. Let $\mathcal{O}(V)^F$ denote the space of germs $\mathbf{f}$ of holomorphic functions at 0 with values in a finite sum of spaces $V^{\lambda-\eta}$. For $X \in \mathfrak{g}$, t near 0, $\mathbf{f} \in \mathcal{O}(V)^F$, set

$$(\pi(X)\mathbf{f})(t) = \pi_t(X)(f(t)). \tag{4}$$

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[6] For $\mathfrak{g} = \widehat{\mathfrak{ns}}$ this result was first stated in [14]

For $k \in \mathbb{Z}_+$, set

$$\mathcal{O}(V)_{(k)}^F = \{ \mathbf{f} \in \mathcal{O}(V)^F | B_t(f(t), w) \in t^k \mathcal{O}(\mathbb{C}) \, \forall w \in V \} \, . \tag{5}$$

Now, set

$$V_{(k)}^F = \{ f(0) | \mathbf{f} \in \mathcal{O}^F(V)_{(k)} \} \, , \tag{6}$$

and

$$M(V(\lambda))_{(k)} = T_\lambda(V_{(k)}^F) \, . \tag{7}$$

Clearly, $M(V(\lambda)) = M(V(\lambda))_{(0)} \supset M(V(\lambda))_{(1)} \supset \dots$. By (1), (3), $B_t(\pi_t(X)f(t), w)$ $= B_t(f(t), \pi(\sigma(X))w) \in t^k \mathcal{O}(\mathbb{C})$ for all $\mathbf{f} \in \mathcal{O}(V)_{(k)}^F$, i.e., $\pi(X)\mathcal{O}(V)_{(k)} \subset \mathcal{O}(V)_{(k)}$, by (4). This implies that $\pi_0(X) V_{(k)}^F \subset V_{(k)}^F$. Therefore, $X M(V(\lambda))_{(k)} = X T_\lambda(V_{(k)}^F)$ $= T_\lambda(\tau_\lambda(X) V_{(k)}^F) \subset T_\lambda(V_{(k)}^F) = M(V(\lambda))_{(k)}$. This shows that $(M(V(\lambda))_{(k)})_{k \in \mathbb{Z}_+}$ is a $\mathfrak{g}$-module filtration of $M(V(\lambda))$. We now show that this filtration is $\mathbb{Z}_2$-graded. We note that $B_t(V_{\gamma_1}^{\eta_1}, V_{\gamma_2}^{\eta_2}) = 0$ if $\gamma_1 \neq \gamma_2$ or $\eta_1 \neq \eta_2$. Let $P_\gamma : V \to V_\gamma$ be the obvious surjection. Let $v \in V_{(k)}^F$ and $\mathbf{f} \in \mathcal{O}^F(V)_{(k)}$ be such that $v = f(0)$. Now, $v = v_{\bar{0}} + v_{\bar{1}}$, $v_\gamma \in V_\gamma$ and $v_\gamma = f_\gamma(0)$, where $\mathbf{f}_\gamma$ is the germ of $P_\gamma \circ f$. Let $w \in V_\gamma$. Then $B_t(f(t), w) \in t^k \mathcal{O}(\mathbb{C})$. But $B_t(f(t), w) = B_t(f_\gamma(t), w)$ by Proposition 3.14 (b). Applying Proposition 3.14 (b) again we see that $B_t(f_\gamma(t), w) \in t^k \mathcal{O}(\mathbb{C})$ for all $w \in V$. Therefore, $f_\gamma \in \mathcal{O}^F(V)_{(k)}$ and so $v_\gamma \in V_{(k)}^F$ for all $\gamma \in \mathbb{Z}_2$. This proves that $(M(V(\lambda))_{(k)})_{k \in \mathbb{Z}_+}$ is $\mathbb{Z}_2$-graded. For any finite dimensional vector space M we let $\mathscr{L}_s^2(M, \mathbb{C})$ (respectively $\mathrm{Sym}(M)$) denote the space of all symmetric bilinear forms on M (respectively symmetric endomorphisms of M). Let $\mathcal{O}(M)$ denote the space of germs of M-valued holomorphic functions at 0. Let $B_t^\eta = B_{t|V^{\lambda - \eta} \times V^{\lambda - \eta}}$, and let $(\; | \;)_\eta$ be any symmetric bilinear non-singular form on $V^{\lambda - \eta}$. Set $B^\eta(t) = B_t^\eta$, $t \sim 0$. Then $\mathbf{B}^\eta \in \mathcal{O}(\mathscr{L}_s^2(V^{\lambda - \eta}, \mathbb{C}))$ and $\exists \mathbf{A}^\eta \in \mathcal{O}(\mathrm{Sym}(V^{\lambda - \eta}))$ such that $B_t^\eta(v, w) = (A_t^\eta v | w)$ for all $v, w \in V^{\lambda - \eta}$. Here $\mathbf{A}^\eta(t) = A_t^\eta$, $t \sim 0$. Set

$$\mathcal{O}(V^{\lambda - \eta})_{(k)} = \{ \mathbf{f} \in \mathcal{O}(V^{\lambda - \eta}) | A_t^\eta f(t) = t^k g(t) \text{ for some } \mathbf{g} \in \mathcal{O}(V^{\lambda - \eta}), \text{ all } t \sim 0 \}, \tag{8}$$

$$(V^{\lambda - \eta})_{(k)} = \{ f(0) | \mathbf{f} \in \mathcal{O}(V^{\lambda - \eta})_{(k)} \} \, . \tag{9}$$

Then $(V^{\lambda - \eta})_{(k)} = V_{(k)}^F \cap V^{\lambda - \eta}$. Also,

$$\mathrm{Ch} \, M(V(\lambda))_{(k)} = \sum_\eta \dim(V^{\lambda - \eta})_{(k)} q^{\lambda - \eta} \, , \tag{10}$$

where $\eta \in Q^+ \cup \{0\}$.

    If $\mathbf{f} \in \mathcal{O}(\mathbb{C})$, let $f(t) = \sum_{i \geq 0} a_i t^i$, $t \sim 0$. If $a_i \neq 0$ for at least one i, we can define the number $\mathrm{ord}_0 f(t)$ as the smallest i with this property. Applying Lemma 1.2 of [23] to $V^{\lambda - \eta}$ and $\mathbf{A}^\eta$, we obtain

$$\sum_{k \geq 1} (\dim V^{\lambda - \eta})_{(k)} = \mathrm{ord}_0 \det B_t^\eta \, . \tag{11}$$

Combining (10) and (11) we obtain

$$\sum_{k \geq 1} \mathrm{ch} \, M(V(\lambda))_{(k)} = \sum_\eta q^{\lambda - \eta} \mathrm{ord}_0 \det B_t^\eta \, . \tag{12}$$

Now, $\mathrm{ord}_0 \psi_{r,s}(h+t,c)$ is one or zero according to whether $\psi_{r,s}(h,c)=0$ or $\psi_{r,s}(h,c)\neq 0$. Let $a(i)=\{(r,s)\,|\,2rs=i\}$. 3(8) and 3(9) imply

$$\sum_{k\geq 1}\mathrm{ch}\,M(\mathbb{C}(\lambda))_{(k)}=\sum_{i\in\frac{1}{2}\mathbb{N}_+(\lambda)}a(i)\sum_{\eta=n\delta}\hat{p}(n-i)q^{\lambda-\eta}, \tag{13}$$

$$\sum_{k\geq 1}\mathrm{ch}\,M(R(\lambda))_{(k)}=\delta_{h,-\frac{c}{16}}\sum_{\eta=n\delta}\tfrac{1}{2}\hat{p}(n)q^{\lambda-\eta}+\sum_{i\in\frac{1}{2}\mathbb{N}_+(\lambda)}a(i)\sum_{\eta=n\delta}\hat{p}(n-i)q^{\lambda-\eta}. \tag{13$'$}$$

Therefore, we have

Proposition 5.2. *The following are valid:*

(1) $\displaystyle\sum_{k\geq 1}\mathrm{ch}\,M(\mathbb{C}(\lambda))_{(k)}=\sum_{i\in\frac{1}{2}\mathbb{N}_+(\lambda)}a(i)\,\mathrm{ch}\,M(\mathbb{C}(\lambda-i\delta))$,

(1$'$) $\displaystyle\sum_{k\geq 1}\mathrm{ch}\,M(R(\lambda))_{(k)}=\delta_{h,-\frac{c}{16}}\,\mathrm{ch}\,M(S(\lambda))+\sum_{i\in\frac{1}{2}\mathbb{N}_+(\lambda)}a(i)\,\mathrm{ch}\,M(R(\lambda-i\delta))$.

If $\mathbf{f},\mathbf{g}\in\mathcal{O}(V^{\lambda-\eta})$, set $\langle f,g\rangle(t)=B_t^\eta(f(t),g(t))$. Now, let $v,w\in V^{\lambda-\eta}$, $v=f(0)$, $w=g(0)$, $\mathbf{f},\mathbf{g}\in\mathcal{O}(V^{\lambda-\eta})$. The argument in [23, Sect. 1] shows that $\langle v,w\rangle=\langle f,g\rangle(0)$ defines a symmetric, bilinear form on $V^{\lambda-\eta}$ which factors to a symmetric, bilinear, nondegenerate form on $V^{\lambda-\eta}/(V^{\lambda-\eta})_{(1)}$ ([23, Lemma 1.3]). This implies that $M(V(\lambda))_{(1)}$ is the maximal proper submodule of $M(V(\lambda))$.

If M is a weight module, $M=\bigoplus_\mu M^\mu$, we set $P(M)=\{\mu\in\mathfrak{h}^*|M^\mu\neq(0)\}$. Let $\mathscr{C}$ be the full subcategory of the category of all $\mathfrak{g}$-modules M such that

$$M=\bigoplus_{\mu\in\mathfrak{h}^*}M^\mu,\ \dim M^\mu<\infty, \tag{14}$$

$$P(M)\subset\{v-(Q^+\cup\{0\})\}\quad\text{for some}\quad v\in\mathfrak{h}^*. \tag{15}$$

If $\mu_1,\mu_2\in\mathfrak{h}^*$ we set $\mu_1<\mu_2\Leftrightarrow\mu_2-\mu_1>0$. Arguing as in [9, Lemma 4.4] (see also [24, Lemma 5.1]) we obtain

Lemma 5.3. *Let $M\in\mathscr{C}$. For each $i\in\mathbb{Z}_+$ there is a submodule $_iM$ of M such that*

$$(0)=\,_0M\subset\,_1M\subset\dots,\ \bigcup_{i=0}^{\infty}\,_iM=M,\ _iM/_{i-1}M\text{ is a highest weight module with highest}$$

weight $v_i\in\mathfrak{h}^*$. Furthermore, $\mu_i<\mu_j\Rightarrow i>j$, and if $v_i=v_{i+1}$, then $\dim(_iM/_{i-1}M)^{v_i}>\dim(_{i+1}M/_iM)^{v_i}$.

Using Lemma 5.3 we can define for each $M\in\mathscr{C}$ the multiplicity $(M:L(V(\mu)))$ as in [24, Sect. 5] by first defining $(M:L(V(\mu)))$ for a highest weight module M of $\mathscr{C}$. The only distinction here is in the case $\mathfrak{g}=\hat{\mathfrak{r}}$: If M is a highest weight module with highest weight $\lambda=(h,c)$ such that $\dim M^\lambda=2$ and $\lambda(e_0^+)+\dfrac{\lambda(e_0')}{16}=0$, then we set $(M:L(V(\lambda)))=2$. Otherwise, $(M:L(V(\lambda)))=1$, for a highest weight module of highest weight λ.

Proposition 5.4. *Let $M\in\mathscr{C}$. Then*

$$\mathrm{ch}\,M=\sum_\mu(M:L(V(\mu)))\,\mathrm{ch}\,L(V(\mu)). \tag{16}$$

Proof. We will show that

$$\dim M^\xi=\sum_\mu(M:L(V(\mu)))\dim L(V(\mu))^\xi\quad\text{for all}\quad\xi\in\mathfrak{h}^*. \tag{17}$$

Let M be a highest weight module with highest weight v. If $M = L(V(v))$, then $(M : L(V(\mu))) = 1$ if $\mu = v$ and $(M : L(V(\mu))) = 0$, otherwise. So, we may assume that M is reducible. Now, $\dim L(V(v-\eta'))^v \neq 0$ only if $\eta' = 0$. Furthermore, if $\dim M^v = 2$, then either $v(e_0^+) + \frac{1}{16} v(e_0') = 0$ and $\dim L(R(v))^v = \dim L(S(v))^v = 1$ (see Proposition 3.10), or $v(e_0^+) + \frac{1}{16} v(e_0') \neq 0$ and $\dim L(R(v))^v = 2$.

In any case, for $\xi = v$, (17) follows from the definition of $(M : L(V(\mu)))$. Assume that (17) was proved for all $\xi = v - \eta'$, with $\eta' < \eta$. There are two cases to consider:

1) $\dim M^v = 1$. Let $\{_i M_{(1)}\}_{i=0}^\infty$ be a filtration of $M_{(1)}$ as in Lemma 3.5, and let s be the largest index such that $(_s M_{(1)}/_{s+1} M_{(1)})^{v-\eta} \neq (0)$. Now, $\dim M^{v-\eta} = \dim L V(v))^{v-\eta} + \dim M_{(1)}^{v-\eta}$ and $\dim M_{(1)}^{v-\eta} = \dim(_1 M_{(1)}^{v-\eta})$ $+ \dim((_2 M_{(1)}/_1 M_{(1)})^{v-\eta}) + \ldots + \dim((_s M_{(s)}/_{s-1} M_{(s)})^{v-\eta})$, where each $(_i M_{(1)}/_{i-1} M_{(1)})$ is a highest weight module of highest weight $v' < v$, hence $v - \eta = v' - \eta'$, $\eta' < \eta$. Equation (17) now follows from the induction hypothesis;

2) $\dim M^v = 2$. This case is similar but we need to apply the argument of 1) twice. Finally, for general $M \in \mathcal{C}$, we apply Lemma 5.3 and the proof for highest weight modules.   Q.E.D.

Remark. Proposition 5.4 implies that the numbers $(M : L(V(\mu)))$ are independent of the choices apparently made in their definition.

We denote by $\aleph(\mathcal{C})$ the set of all characters of elements of $\mathcal{C}$.

Proposition 5.5. *Let* $\mathbf{a} \in \aleph(\mathcal{C})$. *If* $\mathbf{a} = \sum_\mu a_\mu \operatorname{ch} L(V(\mu)) = \sum_\mu b_\mu \operatorname{ch} L(V(\mu))$, *then* $a_\mu = b_\mu$ *for all* $\mu \in \mathfrak{h}^*$.

Proof. Let μ_1 be maximal so that $a_\mu \neq 0$. Then $a_{\mu_1} \dim L(V(\mu_1))^{\mu_1} q^{\mu_1} = b_{\mu_1} \dim L(V(\mu_1))^{\mu_1} q^{\mu_1}$. Hence $a_{\mu_1} = b_{\mu_2}$. Suppose we have showed that $a_\mu = b_\mu$ for all μ such that $\mu_1 - \mu < \eta \in Q^+ \cup \{0\}$, and let μ be so that $\mu_1 - \mu = \eta \in Q^+ \cup \{0\}$. Then

$$\sum_{\mu \leq \mu_1 + \eta} a_\mu \operatorname{ch} L(V(\mu)) = \sum_{\mu \leq \mu_1 + \eta} b_\mu \operatorname{ch} L(V(\mu)).$$

Then $a_{\mu_1 + \eta} \dim L(V(\mu_1 + \eta))^{\mu_1 + \eta} q^{\mu_1 + \eta} = b_{\mu_1 + \eta} \dim L(V(\mu_1 + \eta))^{\mu_1 + \eta} q^{\mu_1 + \eta}$, i.e. $a_{\mu_1 + \eta} = b_{\mu_1 + \eta}$. The result now follows by induction.   Q.E.D.

Proposition 5.6. *Let* $M \in \mathcal{C}$. *Then* $L(V(\mu))$ *is a subquotient of* M *if and only if* $(M : L(V(\mu))) \neq 0$.

Proof. If $(M : L(V(\mu))) \neq 0$, then $L(V(\mu))$ is a subquotient of M, by the definition of $(M : L(V(\mu)))$. Conversely, let $M \supset N \supset P$, N, $P \in \mathcal{C}$ such that $N/P \simeq L(V(\mu))$. Then $\operatorname{ch} N = \operatorname{ch} P + \operatorname{ch} L(V(\mu))$. Also $\operatorname{ch} M = \operatorname{ch} N + \operatorname{ch} M/N$. By Proposition 5.4 $(M : L(V(\mu))) \geq 1$.   Q.E.D.

If $\mathbf{a} = \operatorname{ch} M \in \aleph(\mathcal{C})$ we define $(\mathbf{a} : \operatorname{ch} L(V(\mu)))$ as $(M : L(V(\mu)))$.

Proof of Theorem 5.1. Let $L(V(\mu))$ be a subquotient of $M(V(\lambda))$. We saw earlier that the maximal proper submodule of $M(V(\lambda))$ is $M(V(\lambda))_{(1)}$. Therefore, $\mu = \lambda$ or $L(V(\mu))$ is a subquotient of $M(V(\lambda))_{(1)}$. By Proposition 5.6, either $\mu = \lambda$ or $(M(V(\lambda))_{(1)} : L(V(\mu))) \neq 0$. By Proposition 5.2, $\mu = \lambda$ or $\sum_{i \in \frac{1}{2} \mathbb{N}_+ (\lambda)} a(i)(\operatorname{ch} M(V(\lambda - i\delta)) : \operatorname{ch} L(V(\mu))) \neq 0$. Hence, $\mu = \lambda$ or $L(V(\mu))$ is a subquotient of $M(V(\lambda - i\delta))$ for some $i \in \frac{1}{2} \mathbb{N}_+ (\lambda)$, by Proposition 5.6. By induction, we obtain $\mu \uparrow \lambda$.   Q.E.D.

Appendix. Central Extensions of $\mathbb{Z}_2$-Graded Lie Algebras

In this appendix we extend some definitions and classic results on central extensions of Lie algebras to $\mathbb{Z}_2$-graded Lie algebras.

As in Sect. 2, all homomorphisms of $\mathbb{Z}_2$-graded Lie algebras preserve the $\mathbb{Z}_2$-grading. Furthermore, all short exact sequences

$$0 \to \mathfrak{b} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{a} \to 0 \tag{1}$$

of $\mathbb{Z}_2$-graded Lie algebras are $\mathbb{Z}_2$-split, i.e., are such that π has a linear right inverse that preserves the $\mathbb{Z}_2$-grading (or equivalently, ι has a linear left inverse that preserves the $\mathbb{Z}_2$-grading). (1) is called an *extension* of $\mathfrak{a}$ by $\mathfrak{b}$. A *morphism* of (1) to another extension

$$0 \to \mathfrak{b}' \xrightarrow{\iota'} \mathfrak{e}' \xrightarrow{\pi'} \mathfrak{a}' \to 0 \tag{2}$$

is a triplet (ψ, ϕ, ϱ) of homomorphisms: $\psi: \mathfrak{b} \to \mathfrak{b}'$, $\phi: \mathfrak{e} \to \mathfrak{e}'$, and $\varrho: \mathfrak{a} \to \mathfrak{a}'$ such that $\phi \circ \iota = \iota' \circ \psi$ and $\pi' \circ \phi = \varrho \circ \pi$.

For any $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{a}$ we denote by $\mathfrak{z}(\mathfrak{a})$ the center of $\mathfrak{a}$, i.e., $\mathfrak{z}(\mathfrak{a}) = \{X \in \mathfrak{a} | [x, \mathfrak{a}] = 0\}$. If $\mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{e})$ in (1), then (1) is said to be a *central extension* of $\mathfrak{a}$. A morphism of the central extension (1) to a central extension

$$0 \to \mathfrak{b}' \xrightarrow{\iota'} \mathfrak{e}' \xrightarrow{\pi'} \mathfrak{a} \to 0 \tag{3}$$

is a morphism (ψ, ϕ, ϱ) of extensions from (1) to (3) such that $\varrho = \mathrm{id}|_\mathfrak{a}$. (Here, $\mathrm{id}|_\mathfrak{a}$ denotes the identity homomorphism of $\mathfrak{a}$, for any $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{a}$.) The extension (1) is said to be *equivalent* to

$$0 \to \mathfrak{b} \xrightarrow{\iota'} \mathfrak{e}' \xrightarrow{\pi'} \mathfrak{a} \to 0 \tag{4}$$

if there is a morphism (ψ, ϕ, ϱ) of extensions from (1) to (4) such that $\psi = \mathrm{id}|_\mathfrak{b}$ and $\varrho = \mathrm{id}|_\mathfrak{a}$. ϕ is necessarily an isomorphism, i.e., ϕ has a two-sided inverse homomorphism which preserves the $\mathbb{Z}_2$-grading. Therefore, equivalence of extensions of $\mathfrak{a}$ by $\mathfrak{b}$ is an equivalence relation.

Let V be a $\mathbb{Z}_2$-graded vector space: $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A bilinear map $\alpha: \mathfrak{a} \times \mathfrak{a} \to V$ will always mean one that preserves the $\mathbb{Z}_2$-grading, in the sense that $\alpha(\mathfrak{a}_{\gamma_1} \times \mathfrak{a}_{\gamma_2}) \subset V_{\gamma_1 + \gamma_2}$, $\gamma_1, \gamma_2 \in \mathbb{Z}_2$. A bilinear map $\alpha: \mathfrak{a} \times \mathfrak{a} \to V$ is *skew-symmetric* if

$$\alpha(X, Y) = -(-1)^{\gamma_1 \gamma_2} \alpha(Y, X), \quad X \in \mathfrak{a}_{\gamma_1}, Y \in \mathfrak{a}_{\gamma_2}, \gamma_1, \gamma_2 \in \mathbb{Z}_2. \tag{5}$$

We denote by $Z^2(\mathfrak{a}, V)$ the space of all skew-symmetric bilinear maps $\alpha: \mathfrak{a} \times \mathfrak{a} \to V$ such that the following condition holds:

$$(-1)^{\gamma_1 \gamma_3} \alpha(X, [Y, Z]) + (-1)^{\gamma_2 \gamma_1} \alpha(Y, [Z, X]) + (-1)^{\gamma_3 \gamma_2} \alpha(Z, [X, Y]) = 0, \tag{6}$$

where $X \in \mathfrak{a}_{\gamma_1}$, $Y \in \mathfrak{a}_{\gamma_2}$, $Z \in \mathfrak{a}_{\gamma_3}$, $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}_2$. The elements of $Z^2(\mathfrak{a}, V)$ are called *cocycles* and (6) is the *cocycle condition* for $\mathbb{Z}_2$-graded Lie algebras. Let $B^2(\mathfrak{a}, V)$ be the space of all skew-symmetric bilinear maps $\alpha: \mathfrak{a} \times \mathfrak{a} \to V$ such that

$\alpha(X, Y) = g([X, Y])$ for all $X, Y \in \mathfrak{a}$ and some linear map $g: \mathfrak{a} \to V$ such that $g(\mathfrak{a}_\gamma) \subset V_\gamma$, for all $\gamma \in \mathbb{Z}_2$. The elements of $B^2(\mathfrak{a}, V)$ are the *coboundaries*. We note that condition $(L2)$ of Sect. 2 implies that $B^2(\mathfrak{a}, V) \subset Z^2(\mathfrak{a}, V)$. Set $H^2(\mathfrak{a}, V) = Z^2(\mathfrak{a}, V)/B^2(\mathfrak{a}, V)$. $H^2(\mathfrak{a}, V)$ is the *second cohomology* group of $\mathfrak{a}$ with coefficients in V.

Next, we show how to associate to each element α in $Z^2(\mathfrak{a}, \mathfrak{b})$, $\mathfrak{b}$ abelian, a central extension of $\mathfrak{a}$ by $\mathfrak{b}$. Let $\mathfrak{a}_\alpha$ denote the $\mathbb{Z}_2$-graded vector space $\mathfrak{a} \oplus \mathfrak{b}$, where $(\mathfrak{a} \oplus \mathfrak{b})_\gamma = \mathfrak{a}_\gamma \oplus \mathfrak{b}_\gamma$, $\gamma \in \mathbb{Z}_2$. For $(X, v), (Y, w) \in (\mathfrak{a}_\alpha)_\gamma$, set

$$[(X, v), (Y, w)]_\alpha = ([X, Y], \alpha[X, Y]). \tag{7}$$

Clearly, $[\ ,\ ]_\alpha$ extends to a bilinear map from $\mathfrak{a}_\alpha \times \mathfrak{a}_\alpha$ to $\mathfrak{a}_\alpha$ and $[(\mathfrak{a}_\alpha)_{\gamma_1}, (\mathfrak{a}_\alpha)_{\gamma_2}] \subset (\mathfrak{a}_\alpha)_{\gamma_1 + \gamma_2}$, $\gamma_1, \gamma_2 \in \mathbb{Z}_2$. Condition $(L1)$ of Sect. 2 follows from (5) and condition $(L2)$ of Sect. 2 follows from (6), so that $(\mathfrak{a}_\alpha, [\ ,\ ]_\alpha)$ is a $\mathbb{Z}_2$-graded Lie algebra. We associate to $\alpha \in Z^2(\mathfrak{a}, \mathfrak{b})$ the central extension

$$0 \to \mathfrak{b} \xrightarrow{\iota_\alpha} \mathfrak{a}_\alpha \xrightarrow{\pi_\alpha} \mathfrak{a} \to 0, \tag{8}$$

where ι_α is the obvious inclusion and π_α is the projection onto $\mathfrak{a}$. Furthermore, let $\alpha, \alpha' \in Z^2(\mathfrak{a}, \mathfrak{b})$ and $\alpha' - \alpha \in B^2(\mathfrak{a}, \mathfrak{b})$, i.e., there is $\mathfrak{g}: \mathfrak{a} \to \mathfrak{b}$ linear, preserving the $\mathbb{Z}_2$-grading, such that $(\alpha' - \alpha)(X, Y) = g([X, Y])$, for all $X, Y \in \mathfrak{a}$. If we define $\phi: (\mathfrak{a}_\alpha)_\gamma \to (\mathfrak{a}_{\alpha'})_\gamma$ by $\phi((X, v)) = (X, v + g(X))$, $X \in \mathfrak{a}_\gamma$, $v \in \mathfrak{b}_\gamma$, then $(\mathrm{id}|_\mathfrak{b}, \phi, \mathrm{id}|_\mathfrak{a})$ is a morphism from (8) to

$$0 \to \mathfrak{b} \xrightarrow{\iota_{\alpha'}} \mathfrak{a}_{\alpha'} \xrightarrow{\pi_{\alpha'}} \mathfrak{a} \to 0. \tag{8'}$$

Conversely, if (1) is a given central extension of $\mathfrak{a}$, i.e., $\mathfrak{b} \in \mathfrak{z}_\mathfrak{e}$, let $\beta: \mathfrak{a} \to \mathfrak{e}$ be a linear map which preserves the $\mathbb{Z}_2$-grading and is such that $\pi \circ \beta = \mathrm{id}|_\mathfrak{a}$. Set $\alpha_0(X, Y) = [\beta(X), \beta(Y)] - \beta([X, Y])$, $X, Y \in \mathfrak{a}$. Clearly, $\pi(\alpha_0(X, Y)) = 0$, so that $\alpha_0 = \iota \circ \alpha$, where α is a skew-symmetric, bilinear map from $\mathfrak{a} \times \mathfrak{a}$ to $\mathfrak{b}$. Now, let $X \in \mathfrak{a}_{\gamma_1}$, $Y \in \mathfrak{a}_{\gamma_2}$, $Z \in \mathfrak{a}_{\gamma_3}$. Then

$$(-1)^{\gamma_1 \gamma_3} \alpha_0(X, [Y, Z]) + (-1)^{\gamma_2 \gamma_1} \alpha_0(Y, [Z, X]) + (-1)^{\gamma_3 \gamma_2} \alpha_0(Z, [X, Y])$$

$$= (-1)^{\gamma_1 \gamma_3}([\beta(x), \beta([Y, Z])] - \beta([X, [Y, Z]]))$$

$$+ (-1)^{\gamma_2 \gamma_1}([\beta(Y), \beta([Z, X])] - \beta([Y, [Z, X]]))$$

$$+ (-1)^{\gamma_3 \gamma_2}([\beta(Z), \beta([X, Y])] - \beta([Z, [X, Y]]))$$

$$= (-1)^{\gamma_1 \gamma_3}([\beta(X), \beta([Y, Z])] - [\beta(X), [\beta(Y), \beta(Z)]])$$

$$+ (-1)^{\gamma_2 \gamma_1}([\beta(Y), \beta([Z, X])] - [\beta(Y), [\beta(Z), \beta(X)]])$$

$$+ (-1)^{\gamma_3 \gamma_2}([\beta(Z), \beta([X, Y])] - [\beta(Z), [\beta(X), \beta(Y)]])$$

$$= (-1)^{\gamma_1 \gamma_3}[\beta(X), \beta([Y, Z]) - [\beta(Y), \beta(Z)]]$$

$$+ (-1)^{\gamma_2 \gamma_1}[\beta(Y), \beta([Z, X]) - [\beta(Z), \beta(X)]]$$

$$+ (-1)^{\gamma_3 \gamma_2}[\beta(Z), \beta([X, Y]) - [\beta(X), \beta(Y)]]$$

$$= (-1)^{\gamma_1 \gamma_3}[\beta(X), -\alpha_0(Y, Z)] + (-1)^{\gamma_2 \gamma_1}[\beta(Y), -\alpha_0(Z, X)]$$

$$+ (-1)^{\gamma_3 \gamma_2}[\beta(Z), -\alpha_0(X, Y)] = 0,$$

by Sect. 2 (*L2*) and since $(\iota \circ \alpha)(\mathfrak{a} \times \mathfrak{a}) \subset \iota(\mathfrak{b}) \subset \mathfrak{z}_{\mathfrak{e}}$. Therefore, $\alpha \in Z^2(\mathfrak{a}, \mathfrak{b})$ and we have associated to each central extension of $\mathfrak{a}$ by $\mathfrak{b}$, an element of $Z^2(\mathfrak{a}, \mathfrak{b})$. It is immediate to check that the equivalence class of the latter in $H^2(\mathfrak{a}, \mathfrak{b})$ does not depend on the choice of the map β used above. Now, suppose that there is a morphism $(\mathrm{id}|_{\mathfrak{b}}, \phi, \mathrm{id}|_{\mathfrak{a}})$ from the central extension (1) to the central extension (4). Let β be a right inverse of π that preserves the $\mathbb{Z}_2$-grading, as before. Let α be the element of $Z^2(\mathfrak{a}, \mathfrak{b})$ corresponding to (1) and constructed via β. Since $\phi \circ \beta$ is a right inverse of π that preserves the $\mathbb{Z}_2$-grading, it is easy to see that (4) corresponds to the equivalence class of α in $H^2(\mathfrak{a}, \mathfrak{b})$, by the previous observation. We have, thus, associated to each element $[\alpha]$ of $H^2(\mathfrak{a}, \mathfrak{b})$, an equivalence class $\Phi[\alpha]$ of central extensions of $\mathfrak{a}$ by $\mathfrak{b}$, and, to each equivalence class $[(\iota, \pi)]$ of central extensions of $\mathfrak{a}$ by $\mathfrak{b}$, an element $\Psi([(\iota, \pi)])$ of $H^2(\mathfrak{a}, \mathfrak{b})$. Here (ι, π) denotes the central extension (1). Let $CE(\mathfrak{a}, \mathfrak{b})$ denote the set of all equivalence classes of central extensions of $\mathfrak{a}$ by $\mathfrak{b}$.

Theorem 1. *The map* $\Phi : H^2(\mathfrak{a}, \mathfrak{b}) \to CE(\mathfrak{a}, \mathfrak{b})$ *is a bijection with inverse* $\Psi : CE(\mathfrak{a}, \mathfrak{b}) \to H^2(\mathfrak{a}, \mathfrak{b})$.

Proof. Let (ι, π) denote the element (1) of $CE(\mathfrak{a}, \mathfrak{b})$. Let $\Psi([(\iota, \pi)]) = [\alpha]$ as constructed above. Let (i_α, π_α) be the central extension (8). It is easy to see that $\phi : \mathfrak{a}_\alpha \to \mathfrak{e}$ defined by $\phi((X, v)) = \beta(X) + \iota(v)$, $(X, v) \in \mathfrak{a}_\alpha$, where β is the right ($\mathbb{Z}_2$-graded) inverse of π, is a morphism from (i_α, π_α) to (ι, π), so that $\Phi \circ \Psi$ is the identity map on $CE(\mathfrak{a}, \mathfrak{b})$. It is immediate from the above constructions that $\Psi \circ \Phi$ is the identity on $H^2(\mathfrak{a}, \mathfrak{b})$. Q.E.D.

As in Sect. 2, we consider, for any $\mathbb{Z}_2$-graded vector space V, the tensor algebra $T(V) = \bigoplus\limits_{p=0}^{\infty} T^P(V)$. Let K be the two-sided ($\mathbb{Z}_2$-graded) ideal in $T(V)$ generated by the elements of the form $X \otimes Y + (-1)^{\gamma_1 \gamma_2} Y \otimes X$, $X \in V_{\gamma_1}$, $Y \in V_{\gamma_2}$, $\gamma_1, \gamma_2 \in \mathbb{Z}_2$. The *exterior algebra* of V is $\Lambda(V) = T(V)/K = \bigoplus\limits_{p=0}^{\infty} \Lambda^P(V)$, where $\Lambda^P(V)$ is the image of $T^P(V)$ in $\Lambda(V)$ by the quotient map $\delta : T(V) \to \Lambda(V)$.

For any $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{a}$, let L be the subspace of $\Lambda^2(\mathfrak{a})$ generated by the elements of the form

$$(-1)^{\gamma_1 \gamma_3} X \wedge [Y, Z] + (-1)^{\gamma_2 \gamma_1} Y \wedge [Z, X] + (-1)^{\gamma_3 \gamma_2} Z \wedge [X, Y]. \tag{9}$$

The *second homology* group of $\mathfrak{a}$ with trivial coefficients is $H_2(\mathfrak{a}, \mathbb{C}) = \Lambda^2(\mathfrak{a})/L$. Following [10, Lemma I.10], we set $\alpha(X, Y) = \delta(X \wedge Y)$. By (9) one sees that $\alpha \in Z^2(\mathfrak{a}, H_2(\mathfrak{a}, \mathbb{C}))$. Let $\Phi([\alpha])$ be the central extension

$$0 \to H_2(\mathfrak{a}, \mathbb{C}) \xrightarrow{\iota_\alpha} \mathfrak{a}_\alpha \xrightarrow{\pi_\alpha} \mathfrak{a} \to 0, \tag{10}$$

as constructed prior to Theorem 1.

A central extension (1) of $\mathfrak{a}$ is said to be *universal* if, given any central extension (3) of $\mathfrak{a}$, there is a *unique* morphism from the central extension (1) to the central extension (3). Proceeding as in the proof of Lemma I.10 of [10] we obtain the $\mathbb{Z}_2$-graded analog of that lemma.

Proposition 2. *If* $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$ *then the central extension* (10) *is universal.*

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