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Continuum Limit of the Hierarchical $O(N)$ Non-Linear σ-Model

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Abstract. We construct rigorously the continuum limit of the $O(N)$ non-linear σ model in two euclidean dimensions with a hierarchical kinetic term. Asymptotic freedom and weak coupling Wilson renormalization group flow are established.

1. Introduction

Non-linear σ-models in two space-time dimensions have proved to be useful in several areas of physics: introduced originally as continuum versions of the classical Heisenberg spin systems, they were subsequently used as toy models for the study of asymptotic freedom and dynamical mass generation [1], as effective theories of the quantum Hall effect [2] and, last but not least, as prototypes of first quantized string theories in non-trivial backgrounds [3]. The simplest $O(N)$ version of the σ-model exhibits many properties expected for QCD and serves as a good playground for the Monte-Carlo (renormalization group) simulations of non-linear theories with symmetries, see e.g. [4]. Its S-matrix is known [5] as well as its spectrum for $N=3$ [6]. Here we start a rigorous analysis of such systems in the spirit of the constructive quantum field theory [7]. This should lead to the construction of the Green functions of the model and to establishment of their properties without recurring to their (as yet unknown) analytic form.

The action of the $O(N)$ non-linear σ-model on a two-dimensional euclidean lattice of spacing a, $a\mathbb{Z}^2$, is

$$S(\varphi)=\tfrac{1}{2}\beta(a)\sum_{|x-y|=a} a^2\left(\frac{\varphi(x)-\varphi(y)}{a}\right)^2 \equiv \tfrac{1}{2}\beta(a)\,\langle\varphi|-\Delta|\varphi\rangle \tag{1}$$

with $\varphi(x)\in\mathbb{R}^N$, $\varphi(x)^2\equiv 1$. In the present paper, to avoid the infrared problem, we shall consider the theory in a finite volume, say the unit box $|x^\mu|\leqq\frac{1}{2}$. Alternatively we could add to S a magnetic field term $h(a)\cdot\sum_a a^2\varphi(x)$ which explicitly generates mass and study also the thermodynamical limit. The problem of the continuum limit is to find the "inverse temperature" $\beta(a)$ and the field strength renormalization $Z(a)$ such that the Green functions

$$G^k(x_1, \ldots, x_k) = Z(a)^{-k/2} \frac{1}{\mathcal{N}} \int \prod_{i=1}^{k} \varphi(x_i) e^{-S(\varphi)} \prod_x \delta(\varphi(x)^2 - 1) d\varphi(x) \tag{2}$$

have a (non-trivial) $a \to 0$ limit for non-coinciding points. The natural approach to this question on the heuristic level uses the renormalization group [8]. In our rigorous analysis, we plan to use a block spin technique. In this paper, we shall examine a simplified version of the σ-model with (1) replaced by its *hierarchical approximation* which will make the study of the renormalization group flow of the model an entirely local problem. For the complete model one expects this to be true only up to exponential tails which have to be additionally controlled. Although largely simplified, the hierarchical model provides a very useful toy-case of the real problem (see e.g. the work on φ_4^4 [9]). We consider its study as an important step towards the construction of the complete model, shedding light on the new features brought about by the non-linear nature of (1) as compared to such models as φ_4^4. The version of the hierarchical model that we shall use has been advocated in [10] and differs somewhat from the Dyson model [11]. It replaces the action S on the lattice $a\mathbb{Z}^2$ for $a = L^{-M_0}$ (L an odd integer) by

$$S_h(\varphi) = \tfrac{1}{2}\beta(a) \langle \varphi | G_a^{-1} | \varphi \rangle, \qquad \varphi(x)^2 \equiv 1, \tag{3}$$

where the lattice covariance [whose inverse appears in (3)]

$$G_a(x, y) = \gamma M(x, y). \tag{4}$$

$M(x, y)$ is the largest integer $M \leqq M_0$ such that the integral parts of $L^M x$ and $L^M y$ coincide. G_a mimicks the inverse Laplacian in the unit square of $a\mathbb{Z}^2$. The point of (4) is that G_a may be naturally split into the long distance part and the local short distance one:

$$G_a(x, y) = G_{La}(\bar{x}, \bar{y}) + \gamma \delta_{x,y}, \tag{5}$$

where $x(\bar{y})$ denotes the point of $La\mathbb{Z}^2$ closest to $x(y)$.
Note that the Gibbs state generated by (3) may be written as

$$\frac{1}{\mathcal{N}} \prod_{x \in a\mathbb{Z}^2} e^{-a^2 v_a(\varphi(x))} d\mu_{G_a/\beta(a)}(\varphi), \tag{6}$$

where $d\mu_G$ stands for the Gaussian measure with mean zero and covariance G, and for more generality, we have inserted $e^{-a^2 v_a(\varphi(x))}$ instead of $\delta(\varphi(x)^2 - 1)$, the latter being a limiting case of the previous one with $v_a(\varphi(x)) = \frac{1}{8} a^{-2} \beta(a) \lambda (\varphi(x)^2 - 1)^2$ when $\lambda \to +\infty$. Now, (5) implies the factorization of the Gaussian measure $d\mu_{G_a/\beta(a)}$ into the long distance and local short distance parts so that with the replacement $\varphi(x) \to \varphi'(\bar{x}) + \zeta(x)$, (6) becomes

$$\frac{1}{\mathcal{N}} d\mu_{G_{La}/\beta(a)}(\varphi') \prod_{x \in a\mathbb{Z}^2} e^{-a^2 v_a(\varphi'(\bar{x}) + \zeta(x)) - \frac{1}{2}\beta(a)\gamma^{-1}\zeta(x)^2} d\zeta(x). \tag{7}$$

From (7) one can immediately read off the effective distribution of the long distance component φ' of the field φ. It is obtained by integrating out the short distance component ζ in (7) which gives

$$\frac{1}{\mathcal{N}} \prod_{x \in La\mathbb{Z}^2} e^{-(La)^2 v_{La}(\varphi'(x), a)} d\mu_{G_{La}/\beta(a)}(\varphi') \tag{8}$$

with

$$e^{-a^2 v_{La}(\varphi', a)} = \text{const} \int e^{-a^2 v_a(\varphi' + \zeta) - \frac{1}{2}\beta(a)\gamma^{-1}\zeta^2} d\zeta \,. \tag{9}$$

As mentioned before, for the hierarchical model the renormalization group transformation reduces to the simple recursion for the single spin potential. Notice that our recursion does not preserve the $\varphi^2 \equiv 1$ condition, smoothing the singularity of the initial potential.

Equation (9) may be repeatedly applied producing a sequence of longer and longer distance effective potentials $v_{L^n a}$. To control the continuum limit one has to show that for suitable choices of $\beta(a)$ and $Z(a)$,

$$\lim_{a \to 0} v_{\tilde{a}}(Z(a)^{1/2}\varphi, a) \quad \text{exists} \tag{10}$$

for any fixed $\tilde{a}$. We shall establish (10) first and then study the Green functions (2).

It is convenient to scale (9) a little. As there is no genuine field strength renormalization in the hierarchical model we may take $Z(a) = \beta(a)^{-1}$. Denoting

$$\tilde{a}^2 v_{\tilde{a}}(\beta(a)^{-1/2}\varphi, a) \equiv u_{\tilde{a}}(\varphi, a)\,,$$

we obtain from (9) (dropping the a dependence) the recursion

$$e^{-u_{L\tilde{a}}(\varphi)} = \left[\text{const} \int e^{-u_{\tilde{a}}(\varphi + \zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta\right]^{L^2} \tag{11}$$

with the starting point

$$u_a(\varphi) = \tfrac{1}{8} \frac{\lambda}{\beta(a)} (\varphi(x)^2 - \beta(a))^2 \tag{12}$$

and λ arbitrarily large. The constant in (11) will be chosen so that $u_{\tilde{a}}(\varphi) = 0$ at its minimum.

2. Flow of the Effective Actions

The main content of the analysis to follow is that the transformation (11) upon iteration drives $u_{L^n a}$ exponentially fast (in n) to a fixed form u^*

$$u^*_{\tilde{a}}(\varphi) = \beta_{\tilde{a}}\left[\tfrac{1}{2}\lambda^*(\beta_{\tilde{a}}^{-1/2}|\varphi| - 1)^2 + \ldots\right], \tag{13}$$

with $\beta_{\tilde{a}}$ flowing slowly down ($\beta_{\tilde{a}}$ playing the role of the effective inverse temperature, is defined as the value of φ^2 at the minimum of $u_{\tilde{a}}$). The non-linear σ-model is thus driven fast to a more $(\varphi^2)^2$ like model. In fact (13) is reached almost whatever model with the minimum at $\varphi^2 = \beta(a)$ we start with as can be seen from the analysis below. Let us formulate the main result of the analysis of $u_{L^n a}$. We assume $L > L_0(\gamma)$, $(L^2 - 1)\gamma^{-1} \equiv \lambda^* \leqq \lambda \leqq \infty$ [see (12)].

Theorem. *Choose*

$$\beta(a) = \beta + b_2 \log a + \frac{b_3}{b_2} \log(1 + \beta^{-1} b_2 \log a) \tag{14}$$

with $\beta \geqq \beta_0(L,\gamma)$, b_2, b_3 *being* λ *independent computable constants* ($b_2<0$). *Then for* $\tilde{a}=L^{-m}$,

$$e^{-u_{\tilde{a}}(\varphi,a)} \quad \textit{converges when} \quad a\to 0 \tag{15}$$

uniformly on compacts in $\mathbb{C}^N$. *The limiting* $e^{-u_{\tilde{a}}(\varphi)}$ *is entire in* φ, $O(N)$ *invariant and satisfies the stability bound*

$$|e^{-u_{\tilde{a}}(\varphi)}| \leqq \exp[-\tfrac{1}{8}\lambda^*(|\mathrm{Re}\,\varphi|-\beta(\tilde{a})^{1/2})^2+\tfrac{1}{2}L^2\gamma^{-1}(\mathrm{Im}\,\varphi)^2+1]\,. \tag{16}$$

Moreover, for φ *real*, $||\varphi|-\beta(\tilde{a})^{1/2}|<\beta(\tilde{a})^\alpha$ ($\alpha=10^{-2}$, *say*),

$$u_{\tilde{a}}(\varphi)=\sum_{n=2}^{3} g_{\tilde{a},n}(|\varphi|-\beta_{\tilde{a}}^{1/2})^n+\tilde{w}_{\tilde{a}}(|\varphi|-\beta_{\tilde{a}}^{1/2}) \tag{17}$$

with

$$g_{\tilde{a},2}=(L^2-1)\gamma^{-1}+O(\beta(\tilde{a})^{-1})\,,\qquad g_{\tilde{a},3}=O(\beta(\tilde{a})^{-3/2})\,.$$

$w_{\tilde{a}}(z)$ *is analytic for* $|z|<2\beta(\tilde{a})^\alpha$ *with* $|\tilde{w}_{\tilde{a}}|<O(\beta(\tilde{a})^{-7/4})$ *there and with three derivatives at zero vanishing. The value of* φ^2 *at the minimum of* $u_{\tilde{a}}(\varphi)$ $\beta_{\tilde{a}}=\beta(\tilde{a})+O(1)$.

Remark. Equation (17) reflects essentially the behavior (13) around the minimum of $u_{\tilde{a}}$. Equation (16) shows that $e^{-u_{\tilde{a}}}$ falls off fast away from the maximum.

Before proving these claims, let us roughly explain the main idea. A priori, the control of the flow seems difficult since all terms φ^{2n} are dimensionless and thus naively marginal under the renormalization group recursion (11). However, once we perform in (11) a shift of ζ to the location of approximate classical minimum, a single marginal variable remains corresponding to the location of the minimum $\beta_{\tilde{a}}$, the others contracting to the fixed point values exponentially fast (in the number of iteration steps). To see this, consider (11) with $u_{\tilde{a}}$ as in (12) [with $a\to\tilde{a}$, $\beta(a)\to\beta_{\tilde{a}}$]. Here and below we shall often use the shorthand notation: $f=\beta_{\tilde{a}}^{1/2}$, $\beta=\beta(\tilde{a})$. Take $\varphi=(f+\sigma)\hat{\varphi}\left(\hat{\varphi}\equiv\dfrac{\varphi}{|\varphi|}\right)$. Write $\zeta=\tilde{\sigma}\hat{\varphi}+\tilde{\pi}$ with $\tilde{\pi}\cdot\varphi=0$. Then

$$e^{-u_{L\tilde{a}}(\varphi)}=\left[\mathrm{const}\int d\tilde{\sigma}d\tilde{\pi}\exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)-\frac{\lambda}{2}(\sigma+\tilde{\sigma})^2-\tilde{u}_{\tilde{a}}(\sigma+\tilde{\sigma},\tilde{\pi})\right]\right]^{L^2}, \tag{18}$$

where

$$\tilde{u}_{\tilde{a}}(\sigma,\pi)=u_{\tilde{a}}(f+\sigma,\pi)-\frac{\lambda}{2}\sigma^2\,. \tag{19}$$

Notice that $u_{\tilde{a}}(\sigma+\tilde{\sigma},\tilde{\pi})=O(f^{6\alpha-1})$ if σ, $\tilde{\sigma}$ and $\tilde{\pi}$ are $O(\beta^\alpha)$. Of course, $\tilde{\sigma}$ and $\tilde{\pi}$ are not bounded and we shall have to work more in the large field region exploiting the small weight in the integral coming from $\exp[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)]$ and the stability properties of $e^{-\tilde{u}_{\tilde{a}}}$. Now shift in (18) $\tilde{\sigma}\to\tilde{\sigma}-\lambda\sigma(\gamma^{-1}+\lambda)^{-1}$ to get

$$e^{-u_{L\tilde{a}}(\varphi)}=e^{-\frac{1}{2}\lambda'\sigma^2}\left[\mathrm{const}\int d\tilde{\sigma}d\tilde{\pi}\exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)-\frac{\lambda}{2}\tilde{\sigma}^2-\tilde{u}_{\tilde{a}}(\mathscr{L}\sigma+\tilde{\sigma},\tilde{\pi})\right]\right]^{L^2}, \tag{20}$$

where
$$\lambda' = \frac{L^2\lambda}{1+\gamma\lambda}, \qquad \mathscr{L} = (1+\gamma\lambda)^{-1}. \tag{21}$$

The structure of (20)–(21) is clear: λ seems to have a fixed point $\lambda^* = (L^2-1)\gamma^{-1}$ where $\mathscr{L}$ takes value $\mathscr{L}^* = L^{-2}$. This fixed point is approached exponentially fast $\left(\frac{1}{\lambda'} - \frac{1}{\lambda^*} = L^{-2}\left(\frac{1}{\lambda} - \frac{1}{\lambda^*}\right)\right)$ no matter how large the initial λ is: $\lambda'|_{\lambda=\infty} = L^2\gamma^{-1} = \lambda^* + \gamma^{-1}$, the non-linear σ-model is driven to the linear regime by the renormalization group; $\tilde{u}$ will stabilize exponentially fast too since now all powers of σ are irrelevant except the first one which stays marginal and causes the slow flow of $\beta_{\tilde{a}}$ determined from the higher-order weak coupling analysis in powers of $\beta_{\tilde{a}}^{-1}$.

The stability of $e^{-u_{\tilde{a}}}$ needed to substantiate these claims is expressed by (16). To see why this property iterates, put
$$\mathscr{F}_g(\varphi) = e^{-\frac{g}{2}(|\varphi| - \beta^{1/2})^2} \tag{22}$$
for $g \geqq \frac{1}{4}\lambda^*$, and compute for real φ
$$\begin{aligned}
&[\int \mathscr{F}_g(\varphi+\zeta)e^{-\frac{1}{2}\gamma^{-1}\zeta^2}d\zeta]^{L^2} \\
&\quad = [\int \mathscr{F}_g(\zeta)e^{-\frac{1}{2}\gamma^{-1}(\zeta-\varphi)^2}d\zeta]^{L^2} = e^{-\frac{1}{2}L^2\gamma^{-1}\varphi^2 - \frac{1}{2}L^2 g\beta}[\int e^{\gamma^{-1}\zeta\cdot\varphi + g\beta^{1/2}|\zeta| - \frac{1}{2}(\gamma^{-1}+g)\zeta^2}d\zeta]^{L^2} \\
&\quad \leqq \mathrm{const}\, e^{-\frac{1}{2}L^2\gamma^{-1}\varphi^2 - \frac{1}{2}L^2 g\beta}\left[\int_0^\infty e^{(\gamma^{-1}|\varphi| + g\beta^{1/2})r - \frac{1}{2}(\gamma^{-1}+g)r^2} r^{N-1}dr\right]^{L^2} \\
&\quad \leqq \mathrm{const}\left[\left(\frac{\gamma^{-1}|\varphi| + g\beta^{1/2}}{\gamma^{-1}+g}\right)^{N-1}(\gamma^{-1}+g)^{-1/2}\right]^{L^2}\mathscr{F}_{g'}(\varphi)
\end{aligned} \tag{23}$$
with $g' = \frac{L^2 g}{1+\gamma g}$ attracted again to λ^*. Clearly, for $(|\varphi| - \beta^{1/2}) \geqq \frac{1}{4}\beta^\alpha$, we may use a fraction of g' to kill the const$[-]^{(N-1)L^2}$ factor by the $e^{-O(\beta^{2\alpha})}$ one and to accomodate for the change $\beta \to \beta' = \beta(L\tilde{a}) = \beta - O(1)$. Thus
$$\mathrm{const}[\int \mathscr{F}_g(\varphi+\zeta)e^{-\frac{1}{2}\gamma^{-1}\zeta^2}d\zeta]^{L^2} \leqq g^{-L^2/2}\mathscr{F}'_{1/4\lambda^*}(\varphi), \tag{24}$$
where $\mathscr{F}'_g$ is given by (22) with $\beta \to \beta'$. Equation (24) will imply the iteration of the stability bound for large real φ.

We shall pass now to the actual proof starting with the first step of iteration where λ may be arbitrarily large, then discussing the flow of $u_{\tilde{a}}$ in detail and finally showing that the $a \to 0$ limit may be taken. The first step of the iteration gets a special treatment since it requires estimates uniform in λ.

3. From the Non-Linear to the Linear σ-Model

Let us compute $u_{La}(\varphi, a)$ with u_a given by (12). Notice that for real φ, $e^{-u_a(\varphi)} \leqq \exp\left[-\frac{\lambda}{8}(|\varphi| - \beta^{1/2})^2\right] = \mathscr{F}_{\lambda/4}(\varphi)$. Thus (24) implies (16) for $e^{-u_{La}(\varphi)}$ with φ real, $||\varphi| - \beta'| \geqq \frac{1}{2}\beta'^\alpha$, if the constant in (11) is $\leqq O(\lambda^{1/2})$. This will follow from the small field analysis to which we turn now. Take $|\sigma| < 3\beta^\alpha$.

Consider

$$I(\sigma)=\int d\tilde{\sigma}d\tilde{\pi}\exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)-\frac{\lambda}{2}\tilde{\sigma}^2-\tilde{u}_a(\mathscr{L}\sigma+\tilde{\sigma},\tilde{\pi})\right], \tag{25}$$

see (20) and (19). Write

$$I(\sigma)=I_0(\sigma)+I_c(\sigma)$$

by inserting under the integral $1=\chi+\chi^c$ with $\chi(\tilde{\sigma},\tilde{\pi})$ being the characteristic function of $|\tilde{\sigma}|<\frac{1}{4}\beta^\alpha$, $|\tilde{\pi}|<\frac{1}{2}\beta^\alpha$. First we estimate I_c. This provides the large field contribution to the small field $u_{La}(\varphi)$ and we wish to show that it is non-perturbatively small for $|\sigma|<3\beta^\alpha$, say. Write ($f=\beta^{1/2}$ at the first step)

$$\frac{\lambda}{2}\tilde{\sigma}^2+\tilde{u}_a(\mathscr{L}\sigma+\tilde{\sigma},\tilde{\pi})=\frac{\lambda}{2}\tilde{\sigma}^2+\frac{\lambda}{8f^2}((f+\mathscr{L}\sigma+\tilde{\sigma})^2+\tilde{\pi}^2-f^2)^2,$$

$$-\frac{\lambda}{2}(\mathscr{L}\sigma+\tilde{\sigma})^2\equiv\frac{\lambda}{8f^2}((f+\tilde{\sigma})^2+\tilde{\pi}^2-f^2)^2+\tilde{v}_a(\sigma,\tilde{\sigma},\tilde{\pi}). \tag{26}$$

Straightforward algebra using boundedness of $\lambda\mathscr{L}=\frac{\lambda}{1+\gamma\lambda}$ and of σ [see also (33) below] gives

$$|\tilde{v}_a|\leqq O(f^{6\alpha-1})+O(f^{4\alpha-1})\tilde{\sigma}+O(f^{2\alpha-1})\tilde{\sigma}^2+O(f^{2\alpha-1})\tilde{\pi}^2+\left|\frac{\lambda}{2f^2}\mathscr{L}\sigma\tilde{\sigma}(\tilde{\sigma}^2+\tilde{\pi}^2)\right| \tag{27}$$

with $O(\cdot)$ uniform in λ. The last term in (27) can be estimated by

$$\begin{aligned}\left|\frac{\lambda}{2f^2}\mathscr{L}\sigma\tilde{\sigma}(\tilde{\sigma}^2+\tilde{\pi}^2)\right| &\leqq \left|\frac{\lambda}{2f^2}\mathscr{L}\sigma\tilde{\sigma}(2f\tilde{\sigma}+\tilde{\sigma}^2+\tilde{\pi}^2)\right|+O(f^{2\alpha-1})\tilde{\sigma}^2\\ &\leqq\frac{\lambda}{64f^2}(2f\tilde{\sigma}+\tilde{\sigma}^2+\tilde{\pi}^2)^2+O(f^{2\alpha-1})\tilde{\sigma}^2,\end{aligned} \tag{28}$$

$$\mathrm{Re}(26)\geqq-O(f^{6\alpha-1})-O(f^{2\alpha-1})(\tilde{\sigma}^2+\tilde{\pi}^2)+\frac{\lambda}{10f^2}((f+\tilde{\sigma})^2+\tilde{\pi}^2-f^2)^2. \tag{29}$$

Using (29), we obtain the desired estimate

$$|I_c(\sigma)|\leqq e^{-O(f^{4\alpha})}\int e^{-\frac{\lambda}{10f^2}(\varphi^2-f^2)^2}d\varphi=O(\lambda^{-1/2})e^{-O(f^{4\alpha})} \tag{30}$$

exhibiting the non-perturbative character of $I_c(\sigma)$. Thus, provided that $I_0(\sigma)=O(\lambda^{-1/2})$, $I_c(\sigma)$ is negligible.

To control $I_0(\sigma)$, we shall employ the weak coupling expansion in powers of $\frac{1}{f}$ to the order $\frac{1}{f^3}$ with simple estimates of the remainder. To this end, by Taylor expanding, we obtain from (25) and (26):

$$\log I_0(\sigma)=\log I_0(0)+\sum_{p=1}^{3}\frac{(-1)^p}{p!}\langle\tilde{v}_a^p\rangle_0^T+\frac{1}{3!}\int_0^1 dt(1-t)^3\langle\tilde{v}_a^4\rangle_t^T, \tag{31}$$

where

$$\langle - \rangle_t = \frac{1}{\mathcal{N}} \int -\chi \exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2) - \frac{\lambda}{8f^2}((f+\tilde{\sigma})^2+\tilde{\pi}^2-f^2)^2 - t\tilde{v}_a \right] d\tilde{\sigma} d\tilde{\pi} \tag{32}$$

and $\langle - \rangle_t^T$ denotes the truncated expectation. The explicit form of $\tilde{v}_a$ is

$$\begin{aligned}\tilde{v}_a = \frac{\lambda}{2f}\Bigg[&\left(3\tilde{\sigma}^2+\tilde{\pi}^2+\frac{\tilde{\sigma}^3}{f}+\frac{\tilde{\sigma}\tilde{\pi}^2}{f}\right)\mathcal{L}\sigma + \left(3\tilde{\sigma}+\frac{3}{2f}\tilde{\sigma}^2+\frac{1}{2f}\tilde{\pi}^2\right)(\mathcal{L}\sigma)^2 \\ &+\left(1+\frac{1}{f}\tilde{\sigma}\right)(\mathcal{L}\sigma)^3+\frac{1}{4f}(\mathcal{L}\sigma)^4\Bigg]. \end{aligned} \tag{33}$$

To easily obtain estimates uniform in λ, we change the variable $\tilde{\sigma}$ to $\tilde{s} = \frac{1}{2f}((f+\tilde{\sigma})^2 + \tilde{\pi}^2 - f^2)$,

$$\tilde{\sigma} = f\left[\left(1+\frac{2\tilde{s}}{f}-\frac{\tilde{\pi}^2}{f^2}\right)^{1/2}-1\right] = \tilde{s} - \frac{1}{2}\frac{\tilde{s}^2+\tilde{\pi}^2}{f} + \frac{1}{2}\frac{\tilde{s}(\tilde{s}^2+\tilde{\pi}^2)}{f^2} + O(f^{-3+r\alpha}), \tag{34}$$

where r is a small integer. Note that due to the change of the variable, λ will appear only in the propagator $\frac{1}{\gamma^{-1}+\lambda}$ of $\tilde{s}$. Now we are ready to perform the explicit $\frac{1}{f}$ analysis of the expectations appearing in (31). The calculation gives

$$\log I_0(\sigma) = \log I_0(0) - \sum_{n=1}^{5} \tilde{g}_n \sigma^n - \tilde{g}_6(\sigma), \tag{35}$$

where

$$\begin{aligned} \tilde{g}_1 &= \frac{\lambda\mathcal{L}}{2f}(3(\gamma^{-1}+\lambda)^{-1}+(N-1)\gamma) + \frac{\tilde{c}_1}{f^3}, \\ \tilde{g}_2 &= \frac{\tilde{c}^2}{f^2}, \quad \tilde{g}_3 = \frac{\lambda\mathcal{L}^3}{2f} + \frac{\tilde{c}_3}{f^3}, \\ \tilde{g}_4 &= \frac{\tilde{c}_4}{f^2}, \quad \tilde{g}_5 = \frac{\tilde{c}_5}{f^3}, \end{aligned} \tag{36}$$

with $\tilde{c}_i$ uniformly bounded in λ,

$$|\tilde{g}_6(\sigma)| \leqq O(f^{-4+r\alpha}). \tag{37}$$

Also

$$|I_0(0)| = O(\lambda^{-1/2}) \tag{38}$$

as mentioned above and so, due to (30), the representation (35) also holds for full $I(\sigma)$.

Now from (20) and the definition (25) of $I(\sigma)$ we obtain

$$u_{La}(f+\sigma, 0) = \text{const}\left[+ \frac{1}{2}\lambda'\sigma^2 + L^2 \sum_{n=1}^{5} \tilde{g}_n \sigma^n + L^2 \tilde{g}_6(\sigma) \right]. \tag{39}$$

To find the inverse effective temperature $\beta_{La}=f'^2$, let us search for the minimum of u_{La}. From (35)–(39) it follows that the equation

$$\frac{\partial u_{La}}{\partial\sigma}(f+\sigma_0,\sigma)=0 \tag{40}$$

is solved by

$$\sigma_0=-\frac{1}{2f}((N-1)\gamma+3(\gamma^{-1}+\lambda)^{-1})+\frac{\tilde{c}_6}{f^3}+O(f^{-4+r\alpha}). \tag{41}$$

Since f' is the value of $|\varphi|$ at the minimum of u_{La},

$$\beta_L=f'^2=(f+\sigma_0)^2=\beta+b_{0,2}\log L+b_{0,3}f^{-2}\log L+O(f^{-3+r\alpha}) \tag{42}$$

with

$$b_{0,2}=-(N-1)\gamma+3(\gamma^{-1}+\lambda)^{-1})/\log L. \tag{43}$$

We shall shift σ by σ_0 in (39) obtaining the ultimate small field representation for u_{La}:

$$u_{La}(\varphi)=\sum_{n=2}^{5}g_{La,n}(|\varphi|-f')^n+\tilde{w}_{La}(|\varphi|-f'), \tag{44}$$

where

$$\begin{aligned}
g_{La,2}&=\frac{1}{2}\frac{L^2\lambda}{1+\gamma\lambda}+\frac{c_2}{f'^2}+O(f'^{-4+r\alpha}),\\
g_{La,3}&=\frac{L^2\lambda}{2(1+\gamma\lambda)^3f'}+\frac{c_3}{f'^3}+O(f'^{-4+r\alpha}),\\
g_{La,4}&=\frac{c_4}{f'^3}+O(f'^{-4+r\alpha}),\\
g_{La,5}&=\frac{c_5}{f'^3}+O(f'^{-4+r\alpha}).
\end{aligned} \tag{45}$$

$\tilde{w}_{La}(\sigma)$ is analytic in σ, $|\sigma|<2\beta'^{\alpha}$, with first five derivations vanishing at zero and

$$|\tilde{w}_{La}(\sigma)|\leqq Af'^{-4+r\alpha} \tag{46}$$

there.

It is straightforward now that the stability bound (16) holds also for small real φ, $||\varphi|-\beta'|<\beta'^{\alpha}$ and thus for all real φ. The behavior for the complex fields is then implied by the translation $\zeta\to\zeta-i\,\mathrm{Im}\,\varphi$ in (11) which yields

$$|e^{-u_{L\tilde{a}}(\varphi)}|\leqq e^{1/2\,L^2\gamma^{-1}(\mathrm{Im}\,\varphi)^2}\,e^{-u_{L\tilde{a}}(\mathrm{Re}\,\varphi)}. \tag{47}$$

This establishes the properties of the first effective potential.

4. Towards the Renormalized Trajectory

We shall iterate now the above analysis for a general step (11). So assume the stability bound (16) for $e^{-u_{\tilde{a}}(\varphi)}$ and the small field representation as in (44):

$$u_{\tilde{a}}(\varphi)=\sum_{n=2}^{5}g_{\tilde{a},n}(|\varphi|-f)^n+\tilde{w}_{\tilde{a}}(|\varphi|-f) \tag{48}$$

for $||\varphi|-\beta^{1/2}|<\beta^\alpha$ with $f^2=\beta+O(1)$. Essentially the whole work has been done already. For example, the stability bound (16) will iterate as before. Only the small field perturbative analysis has to be carried out again, this time without having to take care about big λ.

To this end proceed through (20) (with $\lambda=2g_{\tilde{a},2}$) and introduce $I(\sigma)=I_0(\sigma)+I_c(\sigma)$ as in (25). For $|\sigma|<3\beta^\alpha$, using the stability bound, we obtain

$$|I_c(\sigma)|\leqq\int d\tilde{\sigma}d\tilde{\pi}\chi^c\exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)-\frac{\lambda}{2}\tilde{\sigma}^2+\frac{\lambda}{2}(\mathcal{L}\sigma+\tilde{\sigma})^2\right.$$
$$\left.+\frac{1}{2}L^2\gamma^{-1}(\mathrm{Im}\,\mathcal{L}\sigma)^2-\frac{1}{8}\lambda^*(|\mathrm{Re}\,\varphi|-\beta^{1/2})^2+1\right],\tag{49}$$

where $\varphi=(f+\mathcal{L}\sigma+\tilde{\sigma},\tilde{\pi})$. After the first step (and, as we shall see also after the next ones) we have $L^2\gamma^{-1}+O(f^{-2})\geqq\lambda+O(f^{-2})\geqq\lambda^*=(L^2-1)\gamma^{-1}$, hence $\mathcal{L}=(1+\gamma\lambda)^{-1}\leqq L^{-2}+O(f^{-2})$. Also $\lambda\gamma\mathcal{L}<1$. Let first $|\tilde{\sigma}|$ or $|\tilde{\pi}|\geqq L\beta^\alpha$. Use

$$\frac{1}{2}L^2\gamma^{-1}(\mathrm{Im}\,\mathcal{L}\sigma)^2+\frac{\lambda}{2}|\mathcal{L}\sigma|^2<8L^{-2}\gamma^{-1}\beta^{2\alpha}\tag{50}$$

and

$$|\lambda\mathcal{L}\sigma\tilde{\sigma}|\leqq\tfrac{1}{2}(L\gamma(\lambda\mathcal{L})^2|\sigma|^2+L^{-1}\gamma^{-1}\tilde{\sigma}^2)\leqq\tfrac{1}{2}L\gamma^{-1}|\sigma|^2+\tfrac{1}{2}L^{-1}\gamma^{-1}\tilde{\sigma}^2\,.\tag{51}$$

Equations (50) and (51) may be dominated by $\frac{1}{4}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)>\frac{1}{4}\gamma^{-1}L^2\beta^2$ and thus this contribution to $I_c(\sigma)$ is bounded by $e^{-O(f^{4\alpha})}$. If $|\tilde{\sigma}|$ and $|\tilde{\pi}|<L\beta^\alpha$, then

$$\tfrac{1}{8}\lambda^*(|\mathrm{Re}\,\varphi|-\beta^{1/2})^2\geqq\tfrac{1}{8}\lambda^*(\mathrm{Re}\,\mathcal{L}\sigma+\tilde{\sigma})^2-O(\beta^{3\alpha-1/2})$$

and

$$+\frac{1}{2}\lambda\tilde{\sigma}^2+\frac{1}{2}L^2\gamma^{-1}(\mathrm{Im}\,\mathcal{L}\sigma)^2+\frac{1}{2}\gamma\,\mathrm{Re}(\mathcal{L}\sigma+\tilde{\sigma})^2-\frac{1}{8}\lambda^*(\mathrm{Re}\,\mathcal{L}\sigma+\tilde{\sigma})^2$$
$$\leqq\frac{1}{2}L^2\gamma^{-1}|\mathcal{L}\sigma|^2-\frac{1}{8}\lambda^*\tilde{\sigma}^2+\frac{1}{4}(4\lambda-\lambda^*)(\mathrm{Re}\,\mathcal{L}\sigma)\tilde{\sigma}$$
$$\leqq 2L^{-2}\gamma^{-1}\beta^{2\alpha}+\frac{1}{8}\frac{(4\lambda-\lambda^*)^2}{\lambda^*}(\mathrm{Re}\,\mathcal{L}\sigma)^2\leqq 15L^{-2}\gamma^{-1}\beta^{2\alpha}\tag{52}$$

which can be again dominated by $\frac{1}{4}\gamma^{-1}(\tilde{\sigma}^2+\tilde{\pi}^2)\geqq\frac{1}{64}\gamma^{-1}\beta^{2\alpha}$, and so

$$|I_c(\sigma)|\leqq e^{-O(f^{4\alpha})}.\tag{53}$$

Thus we turn into the perturbative regime, i.e. to $I_0(\sigma)$, where

$$\tilde{u}_{\tilde{a}}(\mathcal{L}\sigma+\tilde{\sigma},\tilde{\pi})=\frac{1}{2}\lambda[(|\varphi|-f)^2-(\mathcal{L}\sigma+\tilde{\sigma})^2]+\sum_{n=3}^{5}g_{n,\tilde{a}}(|\varphi|-f)^n+\tilde{w}_{\tilde{a}}(|\varphi|-f),\tag{54}$$

and $\varphi=(f+\mathcal{L}\sigma+\tilde{\sigma},\tilde{\pi})$. Using

$$|\varphi|-f=\tilde{\sigma}+\frac{1}{2f}\tilde{\pi}^2-\frac{1}{2f^2}\tilde{\sigma}\tilde{\pi}^2+\frac{1}{2f^3}\tilde{\sigma}^2\tilde{\pi}^2-\frac{1}{8f^3}\tilde{\pi}^4+O(f^{-4+r\alpha}),\tag{55}$$

where $\bar{\sigma} = \mathscr{L}\sigma + \tilde{\sigma}$ and anticipating that $g_3 = O(f^{-1})$, $g_4 = O(f^{-2})$, $g_5 = O(f^3)$, we rewirte (54) as

$$\begin{aligned}\tilde{u} = &\frac{1}{2}\lambda\left(\frac{1}{f}\bar{\sigma}\tilde{\pi}^2 - \frac{1}{f^2}\bar{\sigma}^2\tilde{\pi}^2 + \frac{1}{4}\frac{1}{f^2}\tilde{\pi}^4 + \frac{1}{f^3}\bar{\sigma}^3\tilde{\pi}^2 - \frac{3}{4}\frac{1}{f^3}\bar{\sigma}\tilde{\pi}^4\right) \\ &+ g_3\left(\bar{\sigma}^3 + \frac{3}{2}\frac{1}{f}\bar{\sigma}^2\tilde{\pi}^2 + \frac{3}{4}\frac{1}{f^2}\bar{\sigma}\tilde{\pi}^4 - \frac{3}{2}\frac{1}{f^2}\bar{\sigma}^3\tilde{\pi}^2\right) \\ &+ g_4\left(\bar{\sigma}^4 + 2\frac{1}{f}\bar{\sigma}^3\tilde{\pi}^2\right) + g_5\bar{\sigma}^5 + \tilde{v} + \tilde{w} \equiv \tilde{u}^0 + \tilde{v} + \tilde{w}\,, \end{aligned} \tag{56}$$

where

$$|\tilde{v}| \leqq Cf^{-4+r\alpha}. \tag{57}$$

Now, we compute $I_0(\sigma)$ to the third order in $\tilde{u}^0$ $\left(\text{from which we shall extract the third order in } \frac{1}{f}\right)$:

$$\begin{aligned}\log I_0(\sigma) &= \log\int d\tilde{\sigma}d\tilde{\pi}\chi\exp\left[-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2 + \tilde{\pi}^2) - \frac{\lambda}{2}\bar{\sigma}^2 + \tilde{\mathrm{u}}^0\right] - \int_0^1 dt\langle\tilde{v} + \tilde{w}\rangle_t \\ &= \mathrm{const} + \sum_{p=1}^{3}\frac{(-1)^p}{p!}\langle(\tilde{u}^0)^p\rangle_0^T + \frac{1}{3!}\int_0^1 ds(1-s)^3\langle(\tilde{u}^0)^4\rangle_s^T - \int_0^1 dt\langle\tilde{v} + \tilde{w}\rangle_t \end{aligned} \tag{58}$$

in the obvious (abusive) notation.

Let us extract from (58) the terms up to $\frac{1}{f^3}$ order by writing

$$\sum_{p=1}^{3}\frac{(-1)^p}{p!}\langle(\tilde{u}^0)^p\rangle_0^T = \mathrm{const} + \sum_{n=1}^{5}\tilde{g}_n\sigma^n + \tilde{g}_6(\sigma) \tag{59}$$

with

$$\begin{aligned}\tilde{g}_1 &= \frac{\lambda\mathscr{L}}{2f}(N-1)\gamma + 3\mathscr{L}(\gamma^{-1} + \lambda)^{-1}g_3 + \tilde{P}_1^3(x)\,, \\ \tilde{g}_2 &= \tilde{P}_2^2(x)\,, \qquad \tilde{g}_3 = \mathscr{L}^3 g_3 + \tilde{P}_3^3(x)\,, \\ \tilde{g}_4 &= \mathscr{L}^4 g_4 + d_4 g_3^2\,, \qquad \tilde{g}_5 = \mathscr{L}^5 g_5 + d_5 g_3^2 + d_5' g_3 g_4\,. \end{aligned} \tag{60}$$

Above, $\tilde{P}_i^k(x)$ are polynomials in $x = (f^{-1}, g_3, g_4, g_5)$ of order k where the order of f^{-1} and g_3 is defined as 1, of g_4 as 2 and of g_5 as 3. The coefficients of $\tilde{P}_i^k$ and d_i depend on λ.

$$|\tilde{g}_6(\sigma)| \leqq O(f^{-4+r\alpha}) \tag{61}$$

for $|\sigma| < \frac{1}{4}\mathscr{L}^{-1}\beta^\alpha$, say. All other terms in (58) are also analytic in σ for $|\sigma| < \frac{1}{4}\mathscr{L}^{-1}\beta^\alpha$ (as then $\varphi = (f + \mathscr{L}\sigma + \tilde{\sigma}, \tilde{\pi})$ stays in the small field region). Moreover

$$|\langle(\tilde{u}_0)^4\rangle_s^T|\,, \qquad |\langle\tilde{v}\rangle_t| \leqq O(f^{-4+r\alpha}) \tag{62}$$

there. Let us assume inductively that

$$|\tilde{w}(\sigma)| \leqq A f^{-4+r_0\alpha} \tag{63}$$

for $|\sigma| < 2\beta^\alpha$. Then in (58),

$$|\langle \tilde{w} \rangle_t| \leqq 2A f^{-4+r_0\alpha} \tag{64}$$

for $|\sigma| < \frac{1}{4}\mathscr{L}^{-1}\beta^\alpha$. Thus we may rewrite (58) as

$$\log I_0(\sigma) = \text{const} + \sum_{n=1}^{5} \tilde{g}_n \sigma^n + \gamma_6(\sigma), \tag{65}$$

where, by choosing r_0 big enough, we may achieve that

$$|\gamma_6(\sigma)| \leqq 3A f^{-4+r_0\alpha} \tag{66}$$

for $|\sigma| < \frac{1}{4}\mathscr{L}^{-1}\beta^\alpha$. Now, Taylor expand γ_6 to the fifth order. This gives

$$\log I_0(\sigma) = \text{const} + \sum_{n=1}^{5} (\tilde{g}_n + O(f^{-4+r\alpha}))\sigma^n + \tilde{\gamma}_6(\sigma), \tag{67}$$

where $\tilde{\gamma}_6$ vanishes to the fifth order at zero. With the use of the maximum principle for $\tilde{\gamma}_6(\sigma)/\sigma^6$, we infer that for $|\sigma| < 3\beta^\alpha$,

$$|\tilde{\gamma}_6(\sigma)| \leqq 3A f^{-4+r_0\alpha} \left(\frac{3\beta^\alpha}{\frac{1}{4}\mathscr{L}^{-1}\beta^\alpha} \right)^6 \leqq L^{-10} A f^{-4+r_0\alpha}$$

(recall that $\mathscr{L} \simeq L^{-2}$). This contraction will lead to the iteration of the bound (63).

Equations (67) and (53) immediately imply [see (20)] that

$$\begin{aligned} u_{L\tilde{a}}(f+\sigma, 0) = \text{const} &+ \frac{1}{2}\lambda'\sigma^2 + \sum_{n=1}^{5} (L^2\tilde{g}_n + O(f^{-4+r\alpha}))\sigma^n \\ &+ L^2\tilde{\gamma}_6(\sigma) + O(e^{-O(f^{4\alpha})}). \end{aligned} \tag{68}$$

The minimum of $u_{L\tilde{a}}$ is attained at

$$\sigma_0 = -\frac{1}{2f}(N-1)\gamma - \frac{3}{\lambda}(\lambda + \gamma^{-1})^{-1} g_3 + \tilde{P}_4^3(x) + O(f^{-4+r\alpha}), \tag{69}$$

which gives for the new effective temperature

$$\beta_{L\tilde{a}} = f'^2 = (f+\sigma_0)^2 = \beta_{\tilde{a}} - (N-1)\gamma - \frac{6}{\lambda}(\lambda + \gamma^{-1})^{-1} f g_3 + \tilde{P}_5^2(x) + O(f^{-3+r\alpha}). \tag{70}$$

Shifting σ by σ_0 gives finally the desired small field representation of the new effective potentials:

$$u_{L\tilde{a}}(\varphi) = \sum_{n=2}^{5} g_n'(|\varphi| - f')^n + \tilde{w}'(|\varphi| - f') \tag{71}$$

for $||\varphi| - \beta'| < \beta'^\alpha$ with

$$\begin{aligned} g_2' &= \tfrac{1}{2}\lambda' + P_2^2(x) + \mathcal{O}, \qquad g_3' = L^2\mathscr{L}^3 g_3 + P_3^3(x) + \mathcal{O}, \\ g_4' &= L^2\mathscr{L}^4 g_4 + L^2 d_4 g_3^2 + \mathcal{O}, \qquad g_5' = L^2\mathscr{L}^5 g_5 + L^2 d_5 g_3^2 + L^2 d_5' g_3 g_4 + \mathcal{O}, \end{aligned} \tag{72}$$

with $\mathcal{O} = O(f^{-4+r\alpha})$. Due to (68), $\tilde{w}'$ satisfies (63) with $f \to f'$ for $|\sigma| < 2\beta'^\alpha$.

We still have to solve (72) to the leading order in g_i's. To this end put for $a=L^{-M}$ and $\tilde{a}=L^{-m}$,

$$\begin{aligned}
g_{\tilde{a},2} &\equiv g_{m,2} = \frac{1}{2}\lambda_{M-m} + \frac{\xi_{M-m,2}}{f_m^2} + \bar{g}_{m,2}\,,\\
g_{\tilde{a},3} &\equiv g_{m,3} = \frac{\gamma_{M-m,3}}{f_m} + \frac{\xi_{M-m,3}}{f_m^3} + \bar{g}_{m,3}\,,\\
g_{\tilde{a},4} &\equiv g_{m,4} = \frac{\gamma_{M-m,4}}{f_m^2} + \bar{g}_{m,4}\,,\\
g_{\tilde{a},5} &\equiv g_{m,5} = \frac{\gamma_{M-m,5}}{f_m^3} + \bar{g}_{m,5}\,,
\end{aligned} \tag{73}$$

where λ_k is the k^{th} iterate of $\lambda \to \lambda' = \dfrac{L^2\lambda}{1+\gamma\lambda}$ and

$$|\bar{g}_{m5}| < Bf_m^{-4+r\alpha}\,. \tag{75}$$

A straightforward analysis shows now that

$$\lambda_{M-m} \xrightarrow[M-m\to\infty]{} \lambda^* = (L^2-1)\gamma^{-1}\,, \qquad \gamma_{M-m} \xrightarrow[M-m\to\infty]{} 0\,,$$

and ξ_{M-m} stabilize, all three exponentially fast and that (75) iterates.

The crucial role in the stabilization of the leading order solutions is played by the contractive factors in front of the leading terms in (72), and by the contractivity to λ^* of $\lambda \to \lambda'$. For example, for γ_k's we obtain

$$\begin{aligned}
\gamma_{k+1,3} &= L^2\mathscr{L}(\lambda_k)^3\gamma_{k,3}\,,\\
\gamma_{k+1,4} &= L^2\mathscr{L}(\lambda_k)^4\gamma_{k,4} + L^2 d_4(\lambda_k)\gamma_{k,3}^2\,,\\
\gamma_{k+1,5} &= L^2\mathscr{L}(\lambda_k)^5\gamma_{k,5} + L^2 d_5(\lambda_k)\gamma_{k,3}^2 + L^2 d_5'(\lambda_k)\gamma_{k,3}\gamma_{k,4}
\end{aligned} \tag{76}$$

from which the behavior of γ's follows.

Going back to the changes in the effective temperature, we may rewrite (70) as $(\beta_{\tilde{a}} \equiv \beta_m)$,

$$\beta_{m-1} = \beta_m + b_{M-m,2}\log L + b_{M-m,3}\beta_m^{-1}\log L + O(\beta_m^{-3/2+r\alpha})\,, \tag{77}$$

where b_{k2} and b_{k3} stabilize exponentially fast to the values b_2 and b_3, $b_2 = -(N-1)\gamma/\log L$, appearing in (14). From this it follows by a standard induction that $\beta_{\tilde{a}} = \beta(\tilde{a}) + O(1)$ on all scales, as anticipated (see e.g. [12]).

This establishes the boundedness of the effective potentials on all scales. Note that the third order perturbative analysis in $\frac{1}{f}$ was necessary to arrive at this result (the b_3 term in (14) is indispensable for boundedness of $\beta_{\tilde{a}}$). Note also the crucial role played in the analysis by the asymptotic freedom of the model, that is the fact that $b_2 < 0$. Thanks to that, $\beta_{\tilde{a}}$ increased for small $\tilde{a}$ and the weak coupling expansion technique worked consistently on all scales. It may seem astonishing that our hierarchical model is asymptotically free also in the planar $N=2$ case, unlike the complete model exhibiting quite different behavior for two components (the Kosterlitz-Thouless phenomenon [13]). This is due to the absence of the wave function renormalization which in the real model contributes additional $\gamma/\log L$ to b_2, and hence the asymptotic freedom appears there only in the non-abelian theory.

It is worthwhile to compare the effective potentials of (48) with the initial $(\varphi^2)^2$ type theory. Writing

$$\frac{\lambda}{8\beta}(\varphi^2-\beta)^2=\frac{\lambda}{2}(|\varphi|-f)^2+\frac{\lambda}{2f}(|\varphi|-f)^3+\frac{\lambda}{8f^2}(|\varphi|-f)^4\,, \tag{78}$$

we see that (78) is far from the renormalized trajectory approached by the effective potentials $u_{\tilde{a}}$ as $a\to 0$ (see the next section). From the above analysis it follows that on the renormalized trajectory $\lambda=\lambda^*+O(f^{-2})$, $g_3=O(f^{-3})$, $|g_4|\leqq O(f^{-4+r\alpha})$. To improve the convergence to the continuum limit we could just replace the bare potential of (12) by

$$u_a'(\varphi)=\tfrac{1}{2}\lambda^*(|\varphi|-\beta(a)^{1/2})^2\,. \tag{79}$$

This would realize in the leading order of weak coupling expansion Symanzik's improvement program [14] for the hierarchical σ model.

5. Continuum Limit: The Effective Potentials

We still need to show the actual convergence of $u_{\tilde{a}}(\varphi, a)$ as $a\to 0$. So far we have seen that they stay bounded. In fact, we have already proven that the β-function stabilizes to the first two non-trivial orders. Only little work remains. We need to vary M in $a=L^{-M}$, thus denote for any function $F_{\tilde{a}}(a)\equiv F_m(M)$, $\delta F_m(M)=F_m(M+1)-F_m(M)$. We can show the following:

$$\delta\beta_m=(b_{\cdot\,-m,2}-b_2)\log L+(b_{\cdot\,-m,3}-b_3)\beta_m^{-1}\log L+\overline{\delta\beta_m}\,, \tag{80}$$

$$|\overline{\delta\beta_m}|\leqq C_1\beta(\cdot)^{-3/2+\varepsilon}\,, \tag{81}$$

$$\delta g_{m,2}=\tfrac{1}{2}\delta\lambda_{\cdot\,-m}+\delta\xi_{\cdot\,-m,2}f_m^{-2}+\overline{\delta g_{m,2}}\,, \tag{82}$$

$$\delta(g_{m,3}f_m)=\delta\gamma_{\cdot\,-m,3}+\delta\xi_{\cdot\,-m}f_m^{-2}+\overline{\delta(g_{m,3}f_m)}\,, \tag{83}$$

$$|\overline{\delta(g_{m,3}f_m)}|\leqq C_2\delta_m f_m^{-3+\varepsilon}\,,\qquad \delta_m\equiv L^{-(\cdot\,-m)}+\beta(\cdot)^{-3/2+\varepsilon}\,, \tag{84}$$

$$\delta g_{m,4}=\delta\gamma_{\cdot\,-m,4}f_m^{-2}+\overline{\delta g_{m,4}}\,, \tag{85}$$

$$\delta g_{m,5}=\delta\gamma_{\cdot\,-m,5}f_m^{-3}+\overline{\delta g_{m,5}} \tag{86}$$

$$|\overline{\delta g_{m,i}}|\leqq C_3\delta_m f_m^{-4+\varepsilon}\,,\qquad i=2,4,5\,, \tag{87}$$

$$|\delta\tilde{w}_m(\sigma)|\leqq C_4\delta_m f_m^{-4+\varepsilon}\quad\text{for}\quad|\sigma|<2\beta(m)^{\alpha} \tag{88}$$

and

$$|\delta e^{-u_m(\varphi)}|\leqq C_5\delta_m e^{-1/4\lambda^*(|\mathrm{Re}\,\varphi|-\beta(m)^{1/2})^2+L^2\gamma^{-1}(\mathrm{Im}\,\varphi)^2+1}\,, \tag{89}$$

where the dot stands for the M-dependence. From (80)–(89), our claims about the continuum limit follow immediately, as we may write

$$F_m(\infty)=F_m(m+1)+\sum_{M=m+1}^{\infty}\delta F_m(M) \tag{90}$$

and

$$\sum_{M=m+1}^{\infty} O(\delta_m(M)) \leqq \sum_{M=m+1}^{\infty} (O(L^{-(M-m)}) + O(M^{-\frac{3+\varepsilon}{2}})) < \infty . \tag{91}$$

The relations (80)–(89) are established by induction in m, starting from $m = M-1$. For the latter value they follow by inspection.

We start from (89):

$$\begin{aligned} \delta e^{-u_{m-1}(\varphi)} &= \delta[C \int e^{-u_m(\varphi+\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta]^{L^2} \\ &= (C')^{-L^2} \delta C^{L^2} e^{-u'_{m-1}(\varphi)} - C^{L^2} \delta [\int e^{-u_m(\varphi+\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta]^{L^2} , \end{aligned} \tag{92}$$

where the prime refers to $M+1$. Below we show that

$$\delta C^{L^2} = O(\delta_m) . \tag{93}$$

As for the second term on the right-hand side of (92), we write

$$\begin{aligned} \delta [\int e^{-u_m(\varphi+\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta]^{L^2} &= \int (\delta e^{-u_m(\varphi+\zeta)}) e^{-\frac{1}{2}\gamma^{-1}\zeta^2} d\zeta \\ &\times \sum_{p=1}^{L^2} [\int e^{-u'_m(\varphi+\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta]^{p-1} [\int e^{-u_m(\varphi+\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta]^{L^2-p} , \end{aligned} \tag{94}$$

translate $\zeta \to \zeta - i \operatorname{Im}\varphi$ and proceed as in (23) and (24). As for $||\operatorname{Re}\varphi| - \beta(m-1)^{1/2}| > \frac{1}{2}\beta(m-1)^\alpha$ or $|\operatorname{Im}\varphi| > \beta(m-1)^\alpha$ we may extract an additional $e^{-O(\beta^{2\alpha})}$ factor, the bound (89) iterates in this region. For other φ, which satisfy $||\varphi| - \beta(m-1)^{1/2}| < \beta(m-1)^\alpha$, it will follow from (80) to (88).

We still have to show (93). But

$$\delta C^{-1} = \int \delta(e^{-u_m(f_{m-1} + \tilde{\sigma}, \tilde{\pi})}) e^{-\frac{1}{2}\gamma^{-1}(\tilde{\sigma}^2 + \tilde{\pi}^2)} d\tilde{\sigma} d\tilde{\pi} . \tag{95}$$

Insert again $1 = \chi + \chi^c$. For the χ^c term, we obtain from (89) the bound $\delta_m e^{-O(\beta(m)^{2\alpha})}$. For the χ term, (82)–(88) gives $O(\delta_m)$ estimate, both provided (80), (81) hold for $m-1$.

Hence δu_m for small fields remains. As in the preceding sections, we start from $I(\sigma)$ estimating

$$\delta \log I(\sigma) = \delta \log I_0(\sigma) + \delta \log\left(1 + \frac{I_c(\sigma)}{I_0(\sigma)}\right) . \tag{96}$$

Again, only the first term needs some care. It is [see (58)]:

$$\begin{aligned} \delta \log I_0(\sigma) = \delta \Bigg[\text{const} &+ \sum_{p=1}^{3} \frac{(-1)^p}{p!} \langle (\tilde{u}^0)^p \rangle_0^T \\ &+ \frac{1}{3!} \int_0^1 ds (1-s)^3 \langle (\tilde{u}^0)^4 \rangle_s^T - \int_0^1 dt \langle \tilde{v} + \tilde{w} \rangle_t \Bigg] . \end{aligned} \tag{97}$$

Consider the higher order terms first. So, for example, writing $\tilde{u}^{0,r} \equiv \tilde{u}^0 + r\delta\tilde{u}^0$, we obtain

$$\begin{aligned} \delta \langle (\tilde{u}^0)^4 \rangle_s^T &= \int_0^1 dr \frac{d}{dr} \langle (\tilde{u}^{0,r})^4 \rangle_{s,r} \\ &= \int_0^1 dr [4 \langle \delta\tilde{u}^{0,r} ; (\tilde{u}^{0,r})^3 \rangle_{s,r}^T - \langle (s\delta\tilde{u}^0 + \tfrac{1}{2}\delta\lambda\tilde{\sigma}^2) ; (\tilde{u}^{0,r})^4 \rangle_{s,r}^T] , \end{aligned} \tag{98}$$

which is easily bounded by $O(\delta_m f_m^{-4+\varepsilon})$. The same holds for $\delta\langle\tilde{v}\rangle_t$ and $\delta\langle\tilde{w}\rangle_t$, and we again get the contraction of the bounds once five Taylor series terms are extracted.

The analysis of $\delta\left(\sum_{p=1}^{3}\frac{(-1)^p}{p!}\langle(\tilde{u}^0)^p\rangle_0^T\right)$ is the one of the variation δ of the relations (59), (60), (68)–(70), and (72) and is straightforward. Notice that we iterate the properties of $\delta(g_3 f)$ rather than of δg_3 to avoid the cumbersome contribution of $\gamma_3\delta\left(\frac{1}{f}\right)$ to the latter. It is just $\delta(g_3 f)$ which contributes at low orders to $\delta\beta_{m-1}$, see (70). The iteration of the bounds on $\delta g_{m,i}$ is due to the contractive properties of (72) and hence of its variation. For $\delta\beta_{m-1}$, we get

$$\delta\beta_{m-1}=\delta\beta_m+\delta b_{\cdot-m,2}\log L+\delta b_{\cdot-m,3}\beta_m^{-1}\log L+O(\delta_m f_m^{-3+\varepsilon}), \tag{99}$$

from which it follows that

$$|\overline{\delta\beta}_{m-1}|\leqq|\overline{\delta\beta}_m|+O(\delta_m f_m^{-3+\varepsilon}) \tag{100}$$

so that (81) iterates with $C_1\to C_1(1+O(L^{-(\cdot-m)/2}+f_m^{-3+\varepsilon}))$. Thus (81) holds for all m for sufficiently big C_1. This ends the proof of the convergence of the effective potentials to the continuum limit and establishes the theorem formulated in Sect. 2.

6. The Green Functions

Let us now study the correlations. We shall show that the Green functions at non-coinciding points have continuum limit. Rewrite (2) in the rescaled form ($Z(a)=\beta(a)^{-1}$):

$$G^k(x_1,\ldots,x_k)=\frac{1}{\mathcal{N}}\int\prod_{i=1}^{k}\varphi(x_i)\prod_{x\in a\mathbb{Z}^2}e^{-u_a(\varphi(x))}d\mu_{G_a}(\varphi). \tag{101}$$

Decomposing the Gaussian measure in (101) as in (7) and integrating out the fluctuation field, we obtain

$$G^k(x_1,\ldots,x_k)=\frac{1}{\mathcal{N}}\int\prod_{i=1}^{k}F_{La}(\varphi(\bar{x}_i))\prod_{x\in La\mathbb{Z}^2}e^{-u_{La}(\varphi(x))}d\mu_{G_{La}}(\varphi), \tag{102}$$

where $\bar{x}_i$ are the points in $La\mathbb{Z}^2$ closest to x_i. We shall iterate this, as long as the blocked points stay non-coinciding. This leads to the expression

$$G^k(x_1,\ldots,x_k)=\frac{1}{2}\int\prod_{i=1}^{k}F_{\tilde{a}}(\varphi(\bar{x}_i))\prod_{x\in\tilde{a}\mathbb{Z}^2}e^{-u_{\tilde{a}}(\varphi(x))}d\mu_{G_{\tilde{a}}}(\varphi), \tag{103}$$

where $\bar{x}_i$ are in $\tilde{a}\mathbb{Z}^2$. The $F_{\tilde{a}}$'s are given by the following recursion

$$F_{L\tilde{a}}(\varphi)=\frac{\int F_{\tilde{a}}(\varphi+\zeta)\,e^{-u_{\tilde{a}}(\varphi+\zeta)-\frac{1}{2}\gamma^{-1}\zeta^2}d\zeta}{\int e^{-u_{\tilde{a}}(\varphi+\zeta)-\frac{1}{2}\gamma^{-1}\zeta^2}d\zeta} \tag{104}$$

with the initial condition

$$F_a(\varphi)=\varphi\,. \tag{105}$$

To prove the existence of the continuum limit of the Green functions for non-coinciding points, it is enough to prove that $F_{\tilde{a}}(\varphi)$ have a limit when $a \to 0$ so that the integral (103) also converges. Then its limit will give the Green functions at non-coinciding continuum points x_i in the unit cube in $\mathbb{R}^2$, provided that $\tilde{a}$ is small enough so that the $\tilde{a}\mathbb{Z}^2$ approximants $\bar{x}_i$ still do not coincide.

First note that ($f = \beta_{La}^{1/2}$, $\beta = \beta(La)$) for $||\varphi| - \beta^{1/2}| < \beta^\alpha$,

$$u_{La}(\varphi) = \sum_{i=2}^{5} g_{La,i}(|\varphi| - f)^i + \tilde{w}_{La}(|\varphi| - f) = -L^2 \log \int e^{-u_a(\zeta) - \frac{1}{2}\gamma^{-1}(\zeta - \varphi)^2} d\zeta + \text{const.} \tag{106}$$

Differentiating both sides over φ, we get

$$\begin{aligned} &\sum_{i=2}^{5} i g_{La,i}(|\varphi| - f)^{i-1} \frac{\varphi}{|\varphi|} + \tilde{w}'_{La}(|\varphi| - f) \frac{\varphi}{|\varphi|} \\ &= -L^2 \gamma^{-1} \frac{\int \zeta e^{-u_a(\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta}{\int e^{-u_a(\zeta) - \frac{1}{2}\gamma^{-1}\zeta^2} d\zeta} = -L^2 \gamma^{-1}(F_{La}(\varphi) - \varphi), \end{aligned} \tag{107}$$

where we have used (104) and (105) in the last step. From (107) we obtain the following representation

$$F_{\tilde{a}}(\varphi) = b_{\tilde{a},0}\varphi[1 + b_{\tilde{a},1}(|\varphi| - f) + b_{\tilde{a},2}(|\varphi| - f)^2 + \tilde{h}_{\tilde{a}}(|\varphi| - f)], \tag{108}$$

where $\tilde{a} = La$ and

$$\begin{aligned} b_{La,0} &= 1, \quad b_{La,1} = -2L^{-2}\gamma g_{La,2} f^{-1}, \\ b_{La,2} &= -L^{-2}\gamma(3g_3 - 2g_{La,2} f^{-1}) f^{-1}. \end{aligned} \tag{109}$$

$\tilde{h}_{La}(\sigma)$ is analytic for $|\sigma| < \frac{3}{2}\beta^\alpha$ with two derivatives at zero vanishing and a bound

$$|\tilde{h}_{\tilde{a}}(\sigma)| \leqq D_1 f^{-3+\varepsilon}. \tag{110}$$

Note that F_{La} has estimates uniform λ.

Large field contributions to F_{La} will be controlled by the following stability bound with $\tilde{a} = La$,

$$|F_{\tilde{a}}(\varphi) e^{-u_{\tilde{a}}(\varphi)}| \leqq (1 + |\varphi|^2) \exp[-\tfrac{1}{8}\lambda^*(|\mathrm{Re}\,\varphi| - \beta(\tilde{a})^{1/2})^2 + \tfrac{1}{2}L^2\gamma^{-1}(\mathrm{Im}\,\varphi)^2 + 1]. \tag{111}$$

Our notation is in fact somewhat abusive, because for large fields we shall only control the product $F_{\tilde{a}} e^{-u_{\tilde{a}}}$ which is an entire function.

Now (111) iterates under (104) as (16) before, provided the small field representation (109) does with appropriate bounds. For small fields take $\varphi = (f + \sigma, 0)$ with $f = \beta_{\tilde{a}}^{1/2}$ now and write (104) using (55) as

$$\frac{F_{L\tilde{a}}(f + \sigma, 0)}{(f + \sigma, 0)} = \frac{b_0}{1 + \frac{\sigma}{f}} \left\langle \left(1 + \frac{\bar{\sigma}}{f}\right)\left[1 + b_1\left(\bar{\sigma} + \frac{1}{2f}\bar{\pi}^2\right)\right] + b_2\bar{\sigma}^2 + k + \tilde{h} \right\rangle + e^{-O(f^{4\alpha})} \tag{112}$$

with $|k| \leqq O(f^{-3+\varepsilon})$ and the expectation is with respect to measure

$$\frac{1}{\mathcal{N}} e^{-\tilde{u}^0 - \tilde{v} - \tilde{w} - \frac{1}{2}\lambda\bar{\sigma}^2 - \frac{1}{2}\gamma^{-1}(\bar{\sigma}^2 + \bar{\pi}^2)} \chi\, d\bar{\sigma} d\bar{\pi}$$

in the notation of (56). A straightforward perturbative analysis supplemented by a shift $\sigma \to \sigma + \sigma_0$, where σ_0 is given by (69) produces

$$b_0' = b_0\left(1 + \frac{1}{2g_2 + \gamma^{-1}}\left[b_2 + b_1\left(f^{-1} - 3\frac{g_3}{2g_2}\right)\right] + \mathcal{O}\right),$$
$$b_1' = \mathcal{L}\left(b_1 + \frac{1}{f}\right) - \frac{1}{f} + \mathcal{O}, \tag{113}$$
$$b_2' + \frac{b_1'}{f'} = \mathcal{L}^2\left(b_2 + \frac{b_1}{f}\right) - 3\mathcal{L}^3\gamma\left(b_1 + \frac{1}{f}\right)g_3 + \mathcal{O},$$

where $\mathcal{O} = O(f^{-3+\varepsilon})$. Moreover (110) iterates due to the contracting $\mathcal{L}$ factor in front of σ. Equation (113) implies that

$$b_{m,1} = -\frac{1 + \eta_{m,1}}{f_m} + \overline{b_{m,1}}, \qquad b_{m,2} = \frac{1 + \eta_{m,2}}{f_m^2} + \overline{b_{m,2}} \tag{114}$$

with $\eta_{m,i}$ converging exponentially fast to zero and

$$|\overline{b_{m,i}}| \leqq D_2 f_m^{-3+\varepsilon}. \tag{115}$$

Moreover, since $b_0' = b_0(1 + O(L^{-(M-m)} + f_m^{-3+\varepsilon}))$ also $b_{m,0}$ stays finite when the ultraviolet cutoff is removed.

This proves uniform boundedness of $F_{\tilde{a}}$. To show that they converge when $a \to \infty$, we study δF_a which leads to the variation δ of (113) and (114) and is standard by now, so that we leave it as an exercise. Notice again that on the renormalized trajectory b_1 and b_2 pick up fixed $O\left(\frac{1}{f}\right)$ and $O\left(\frac{1}{f^2}\right)$ terms respectively so that in order to improve the convergence to the continuum limit by implementing the Symanzik program to the leading orders, we should consider the cutoff Green functions

$$G^k_{\mathrm{imp}}(x_1, \ldots, x_k) = Z(a)^{-k/2}\frac{1}{\mathcal{N}}\int \prod_{i=1}^{k}\left[\varphi(x_i)\,(3 - 3|\varphi(x_i)| + |\varphi(x_i)|^2)\right]$$
$$\times \prod_{x \in a\mathbb{Z}^2} e^{-\frac{1}{2}\beta(a)\lambda^*(|\varphi| - 1)^2} d\mu_{G_a/\beta(a)}(\varphi) \tag{116}$$

instead of those of (2).

The renormalization group approach, especially simple for the hierarchical model, allows studying also short distance asymptotics of the continuum Green functions, especially the logarithmic corrections to scaling, see [15]. This requires a study of the composite correlations, and although straightforward, will not be pursued in this paper.

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