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A Rigorous Upper Bound of a Scalar Self-propagator in a Random Lattice Ensemble in Higher Dimensions

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Abstract. Using an improved weight for a scalar field on a random lattice, it is rigorously proved that the self-propagator, averaged over an ensemble of random lattices with site density ρ, is bounded from above in D dimensions ($D > 2$) i.e.:

$$\Delta_0 \leqq \left(4 + \frac{D}{2(D-2)}\omega_D^{2/D-1} D!^{2/D-1} D^{D/2-1-2/D}(D+1)^{D/2-1/2-1/D}\right) \cdot D^{D-2+2/D}\omega_D^{2-2/D}\Gamma\left(D-1+\frac{2}{D}\right)\rho^{1-2/D},$$

where ω_D is the solid angle in D dimensions.

I. Introduction

One of the most intriguing problems in mathematical physics is the ultraviolet divergence. This problem arises from the uncontrollably vast fluctuation of the field configuration in the space-time continuum. By renormalization, one can get around this difficulty in a limited number of field theories. To deal with unrenormalizable theories, a formulation of discrete mechanics has been developed [1]. The basic assumption of this theory is: Within a finite space-time domain, say Ω, one can only perform a finite number, say $\mathcal{N}$, of measurements on space time coordinates and field variables $(x_n, t_n; \Phi_n)$. The weight of each measurement is $\exp(-A_L)$, where A_L is the action of the field defined on a random lattice generated by (x_n, t_n). Physical observables are obtained by averaging the ensemble of all measurements. Such a development invites a program of studying the effect in all branches of mathematical physics due to the discreteness of space-time. In this paper I pick up the simplest system of a free massless scalar field and focus on the finiteness of its ultraviolet behavior.

Originally such an attempt was made by Friedberg and Yancopoulos [2]. They succeeded in figuring out a rigorous upper bound of the scalar self-propagator in two dimensions based on the weight given in [3]. Their reasoning depends crucially upon the fact that the lattice scalar field action in two dimensions can be viewed as a continuum action with a piecewise linear field configuration plugged in. An improved definition of the weight proposed recently in [1] renders the above property true for the lattice action in all dimensions. Thus Friedberg's method can be generalized to higher dimensions. With some further technical modifications, the problem of bounding the self-propagator becomes tractable.

This paper is organized as follows: In Sect. II, properties of the improved weight for a scalar field will be reviewed and the problem will be formulated; in Sect. II, I will follow a series of steps similar to that in [2] to bound the self-propagator on an individual lattice; in Sect. IV, the ensemble average of the bound will be calculated.

II. Formulation of the Problem

Let $x_i (i = 0, \ldots, \mathcal{N} - 1)$ be a set of sites distributed arbitrarily within a D-dimensional Euclidean volume Ω. Following [4] a unique rotationally invariant simplicial structure can be linked up by requiring that the inside of the circumsphere of each simplex be free of other lattice sites. We define a scalar field Φ_i at each site and the lattice action for the scalar field is given by:

$$A_L = \tfrac{1}{2} \sum_{l_{ij}} \lambda_{ij} (\Phi_i - \Phi_j)^2, \tag{1}$$

where l_{ij} is a link and λ_{ij} is its weight. To determine λ_{ij} we fill the inside of each simplex with a linear function $\Phi(x)$ such that $\Phi(x_i) = \Phi_i$, where x_i is a vertex of the simplex. Then we define the lattice action to be the continuum action evaluated from this piecewise linear configuration, $\Phi(x)$, referred to later as the lattice configuration. The weight of the link l_{ij} can be shown to be:

$$\lambda_{ij} = \sum_S \lambda_{ij}(S)|_{l_{ij} \in S}, \tag{2}$$

and

$$\lambda_{ij}(S) = -\frac{1}{D^2} \frac{V_{i\mu} V_{j\mu}}{V}, \tag{3}$$

where V is the volume of S, a D-simplex, and $V_{i\mu}$ is the volume, projected normal to the μ-axis, of the $(D-1)$-simplex in S opposite to site i. The orientation of V_i is taken so that

$$\sum_i V_{i\mu} = 0. \tag{4}$$

It can be shown that the weight (2), (3) agrees with the weight defined in [3] in $D = 2$ only.

A physical observable is given by the following ensemble average:

$$\langle O \rangle = \frac{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{\mathcal{N}-1} d\Phi_i \exp(-A_L) O(\Phi)}{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{\mathcal{N}-1} d\Phi_i \exp(-A_L)}, \tag{5}$$

where J is a Jacobian which has no effect on the continuum limit. The self-propagator at p_0 is given by

$$\Delta_0 = \frac{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{\mathcal{N}-1} d\Phi_i \exp(-A_L)\Phi_0^2}{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{\mathcal{N}-1} d\Phi_i \exp(-A_L)}. \tag{6}$$

This expression is, however, not well defined because of the zero mode of A_L. To remove this ambiguity I specify the following boundary condition: Draw a large closed $(D-1)$-dimensional surface $\mathscr{B}$ whose shortest distance to p_0 is R and $\rho^{1/D} R \gg 1$. We call a $(D-1)$-simplex an exterior simplex if its dual is severed by $\mathscr{B}$. The union of the exterior simplices forms a piecewise flat boundary $\mathscr{B}_L$ of the network. The potentials at the sites on $\mathscr{B}_L$ and outside it are fixed to be zero[1]. With this boundary condition we have:

$$\begin{aligned}\Delta_0 &= \frac{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{N-1} d\Phi_i \exp(-A'_L)\Phi_0^2}{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J \int \Pi_{i=0}^{N-1} d\Phi_i \exp(-A'_L)} \\ &= \frac{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J (\det \mathcal{M})^{-1/2} (\mathcal{M}^{-1})_{00}}{\int \Pi_{i=0}^{\mathcal{N}-1} d^D x_i J (\det \mathcal{M})^{-1/2}},\end{aligned} \tag{7}$$

where $A'_L \equiv \frac{1}{2}\tilde{\Phi}\mathcal{M}\Phi$ is the lattice action with the boundary condition plugged in and N, determined by the distribution of sites, is the total number of the sites whose potential are not fixed by the boundary condition. For the sake of simplicity we will take

$$J = (\det \mathcal{M})^{1/2}, \tag{8}$$

Then

$$\Delta_0 = \frac{1}{\Omega^{\mathcal{N}}} \int \prod_{j=0}^{\mathcal{N}-1} d^D x_j (\mathcal{M}^{-1})_{00}. \tag{9}$$

The continuum limit of the self-propagator is:

$$\Delta(0) = \frac{1}{(2\pi)^D} \int d^D k \frac{1}{k^2}, \tag{10}$$

which is highly divergent. Intuitively we expect (9) to be finite. However the ensemble includes some lattices whose density near p_0 is very much greater than ρ and $(\mathcal{M}^{-1})_{00}$ will be correspondingly large, since the system acts like a continuum roughly down to the lattice spacing. As we shall see later the ensemble integration is indeed convergent. It follows from the positivity of $\mathcal{M}$ that $|(\mathcal{M}^{-1})_{ij}| \leqq \frac{1}{2}((\mathcal{M}^{-1})_{ii} + (\mathcal{M}^{-1})_{jj})$. Therefore, once the ensemble average of $(\mathcal{M}^{-1})_{00}$ is bounded the ensemble averages of all other $(\mathcal{M}^{-1})_{ij} (i \neq j)$ are also bounded.

III. The Bound of the Self-propagator on an Individual Lattice

The lattice scalar field system can be viewed as an electric network in multi-dimensional space. Φ_i can be identified as the voltage at each site, λ_{ij}^{-1} as the resistance of each link and the action as the total power dissipated.

Let p_0 be the site on which the potential is fixed at Φ_0 and the boundary

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1 One can, of course, choose other boundary conditions. The ultraviolet behavior is not sensitive to the boundary condition

condition be as given in the last section. By a variational principle it is easy to show that the absolute minimum of the power is given by

$$P = (A'_L)_{\min} = \frac{\Phi_0^2}{2g}, \tag{11}$$

where

$$g = (\mathcal{M}^{-1})_{00}. \tag{12}$$

Alternatively, we can also identify the power dissipated in the network as the power in a continuous conducting medium of unit conductivity evaluated from the lattice configuration and the boundary condition that $\Phi = 0$ on $\mathcal{B}$ with the potential at p_0 being held at $\Phi(p_0) = \Phi_0$. We will take this point of view in what follows.

Theorem. *For g defined above we have:*

$$g \leqq g_1 + g_2, \tag{13}$$

where

$$g_1 = \left(\frac{4R^2}{V}\right)_{\max}, \tag{14}$$

$$g_2 = \frac{D}{2(D-2)} \omega_D^{2/D-1} D!^{2/D-1} D^{D/2-1-2/D} (D+1)^{D/2-1/2-1/D} \left(\frac{R^2}{V}\right)_{\max}. \tag{15}$$

ω_D is the total solid angle in D dimensions, R is the circumradius of a simplex containing p_0 and V is its volume. The minimum is taken among all the simplices containing p_0.

Proof. To prove this theorem we subject the configuration to a series of modifications, each of which can only decrease or leave unchanged the total power dissipated. In fact most of the strategy follows [2], but some details are different because of the higher dimensionality.

We first define the intermediate polyhedron. It is the union of all the D-simplices containing p_0. The sites on this polyhedron except p_0 are called the intermediate sites and their potentials are called the intermediate potentials. The total power is then divided into two parts:

$$P = P_1 + P_2, \tag{16}$$

where P_1 is the power dissipated inside the intermediate polyhedron and P_2 is the power dissipated outside the intermediate polyhedron.

1) Let p_1 be the intermediate site with the lowest potential, $\Phi_1 = \Phi(p_1)$, V_1 be the volume of a simplex S_1 containing $\overline{p_0 p_1}$ and $P(S_1)$ the power dissipated in S_1 only. Then:

$$P_1 \geqq P(S_1) \geqq \frac{1}{2}\left(\frac{\Phi_0 - \Phi_1}{l}\right)^2 V_1 \geqq \frac{1}{2}\left(\frac{\Phi_0 - \Phi_1}{2R}\right)^2 V_1 \geqq \frac{(\Phi_0 - \Phi_1)^2}{g_1}, \tag{17}$$

where

$$g_1 = 4\left(\frac{R^2}{V}\right)_{\max}.$$

2) We study P_2 by first lowering the potential on the intermediate polyhedron to its minimum Φ_1. Then the power is decreased further.

3) Relax the constraint of the lattice configuration and remove the boundary $\mathscr{B}$. We have then:

$$P_2 \geqq P_c, \tag{18}$$

where P_c is the power dissipated in a conducting medium of an infinite size, when the intermediate polyhedron is held at constant potential Φ_1.

4) Solving the Laplacian equation for a polyhedral boundary is awkward. In [2] the authors embedded a circle inside this polyhedron. The center of this circle is p_0 and the radius of the circle is the minimum altitude dropped from p_0 among all the simplices containing p_0. Then they fixed the potential on this circle at Φ_1 and solved the Laplacian equation with the circular boundary. However this technique can not be generalized to dimensionality higher than 3, as we shall see in the next section. The alternative I use is to pick up a simplex containing p_0 and embed an ellipsoid, referred to as the inner ellipsoid in what follows, completely inside this simplex. In Appendix A, we give a definition of such an ellipsoid. The volume of it is:

$$\tau = (D-1)!\,\omega_D D^{-D/2}(D+1)^{-(D+1)/2}\,V, \tag{19}$$

where

$$\omega_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \tag{20}$$

is the total solid angle in D dimensions and V is the volume of the simplex chosen. Now the potential on the inner ellipsoid is fixed at Φ_1. The power remains unchanged. Then we relax the potentials outside the ellipsoid. The power is decreased further.

5) In Appendix B we show that the power dissipated in a continuous medium by an ellipsoidal source at the potential Φ_1 is given by:

$$P_c = \frac{\omega_D \Phi_1^2}{\kappa}, \tag{21}$$

where

$$\kappa = \int_0^\infty \frac{ds}{\sqrt{(s+a_1^2)\cdots(s+a_D^2)}}, \tag{22}$$

and $a_1,\ldots,a_D$ are the half-axes of the ellipsoid. κ is a complicated integral and is bounded from above by:

$$\kappa \leqq \frac{D}{D-2}\left(\frac{\omega_D}{D\tau}\right)^{1-2/D}, \tag{23}$$

where τ is the volume of the ellipsoid. Using (18), (21), (22) and (23), we have:

$$P_2 \geqq P_c \geqq \frac{\Phi_1^2}{2g_2}, \tag{24}$$

with g_2 given in (15). In deriving (24) we have used the inequality $\tau \leqq V \leqq V_S$, where V is the volume of the simplex considered and V_S is the volume of the circumsphere of the simplex.

Combining (17) and (24) we obtain:

$$P \geqq \frac{(\Phi_0 - \Phi_1)^2}{2g_1} + \frac{\Phi_1^2}{2g_2}. \tag{25}$$

Minimizing the right-hand side of (25) with respect to Φ_1, we have

$$P \geqq \frac{\Phi_0^2}{2(g_1 + g_2)}. \tag{26}$$

The theorem is proved.

From (12) we have the upper bound of the self-propagator for the individual lattice:

$$(\mathcal{M}^{-1})_{00} \leqq g_1 + g_2. \tag{27}$$

Therefore the ensemble average (5) is bounded by:

$$\Delta_0 \leqq \langle g_1 \rangle + \langle g_2 \rangle. \tag{28}$$

IV. The Ensemble Average of the Upper Bound

Both g_1 and g_2 are local quantities. Their ensemble averages can be easily bounded. Denote by S_a, $a = 1, \ldots, m$ the D-simplices containing p_0, and by R_a and V_a the circumradius and the volumes of S_a. The probability density of $P(g_1)$ can be written as

$$P(g_1) = \left\langle \delta\left(g_1 - \max\left(\frac{4R_1^2}{V_1}, \ldots, \frac{4R_m^2}{V_m} \right)\right)\right\rangle, \tag{29}$$

where $\langle \cdots \rangle$ means the ensemble average over random lattices. Define

$$\tilde{P}(g_1) = \left\langle \sum_{a=1}^{m} \delta\left(g_1 - \frac{4R_a^2}{V_a} \right)\right\rangle. \tag{30}$$

Clearly

$$P(g_1) \leqq \tilde{P}(g_1), \tag{31}$$

$$\int dg_1 P(g_1) = 1, \tag{32}$$

$$\int dg_1 \tilde{P}(g_1) = N_{D/0}, \tag{33}$$

where $N_{D/0}$ is the average number of D-simplices containing a common site. Therefore

$$\begin{aligned}
\langle g_1 \rangle &= \int_0^\infty g_1 P(g_1) dg_1 \leqq \int_0^\infty g_1 \tilde{P}(g_1) dg_1 \\
&= \frac{1}{\Omega^{\mathcal{N}-1}} \int \prod_{i=1}^{\mathcal{N}-1} d^D x_i \sum_{a=1}^{m} \frac{4R_a^2}{V_a} = \frac{1}{\Omega^{\mathcal{N}-1}} \int \prod_{i=1}^{\mathcal{N}-1} \\
&\quad \cdot d^D x_i \sum_{x_{i_1}, \ldots, x_{i_D}} f(x_{i_1}, \ldots, x_{i_D}) \frac{4R^2(x_{i_1}, \ldots, x_{i_D})}{V(x_{i_1}, \ldots, x_{i_D})} \\
&= \frac{1}{\Omega^{\mathcal{N}-1}} \frac{(\mathcal{N}-1)!}{D!(\mathcal{N}-D-1)!} \int \prod_{i=1}^{\mathcal{N}-1} d^D x_i f(x_1, \ldots, x_D) \frac{4R^2(x_1, \ldots, x_D)}{V(x_1, \ldots, x_D)},
\end{aligned} \tag{34}$$

where

$$f(x_1,\ldots,x_D)=\begin{cases}1 & \text{if } p_0,\ldots,p_D \text{ form a simplex}\\ 0 & \text{otherwise}\end{cases}$$

Let $p_0, p_1, \ldots, p_D$ be a D-simplex, p_C be its circumcenter, R be its circumradius and V be its volume. The volume of its circumsphere is $v=(\omega_D/D)R^D$. Equation (34) can be written as

$$\langle g_1\rangle \leqq \frac{1}{\Omega^{\mathcal{N}-1}}\frac{(\mathcal{N}-1)!}{D!(\mathcal{N}-D-1)!}\int d^Dx_1\cdots d^Dx_D(\Omega-v)^{\mathcal{N}-D-1}\frac{4R^2}{V}. \tag{35}$$

As $\mathcal{N}\to\infty$ with $\mathcal{N}/\Omega=\rho$ held fixed, we have

$$\langle g_1\rangle = \frac{\rho^D}{D!}\int d^Dx_1\cdots d^Dx_D e^{-\rho v}\frac{4R^2}{V}. \tag{36}$$

Following (4), let us introduce a dummy point x by inserting into (36) the following factor:

$$1=\int d^Dx\,\delta^D(x-x_c)=\int d^Dx\,2^DD!V\delta(R_0^2-R_1^2)\cdots\delta(R_0^2-R_D^2), \tag{37}$$

we have

$$\langle g_1\rangle = \frac{4\rho^D}{D!}\int d^Dx_1\cdots d^Dx_D e^{-(\rho\omega_D/D)R^D}\frac{R^2}{V}\int d^Dx\,\delta^D(x-x_c) \tag{38}$$

$$=2^{D+2}\rho^D\int d^Dx_0\cdots d^Dx_D e^{-(\rho\omega_D/D)R^D}R^2\delta(R_0^2-R_1^2)\cdots\delta(R_0^2-R_D^2) \tag{39}$$

where $R_i=\overline{p_ip_c}$, and we have replaced d^Dx by d^Dx_0. Introducing spherical coordinates $d^Dx_i=R_i^{D-1}dR_i d\omega_i$, we have

$$\langle g_1\rangle \leqq 4\rho^D\int_0^\infty dR\,R^{(D-2)(D+1)+3}e^{-(\rho\omega_D/D)R^D}\omega_D^{D+1}$$

$$=4D^{D-2+2/D}\omega_D^{2-2/D}\Gamma\left(D-1+\frac{2}{D}\right)\rho^{1-2/D}. \tag{40}$$

From Eq. (15) we have

$$\langle g_2\rangle = \frac{D}{D-2}\omega_D^{2/D-1}D!^{2/D-1}D^{D/2-1-2/D}(D+1)^{D/2-1/2-1/D}\tfrac{1}{8}\langle g_1\rangle. \tag{41}$$

Therefore

$$\Delta_0 \leqq \left(4+\frac{D}{2(D-2)}\omega_D^{2/D-1}D!^{2/D-1}D^{D/2-1-2/D}(D+1)^{D/2-1/2-1/D}\right)$$

$$\cdot D^{D-2+2/D}\omega_D^{2-2/D}\Gamma\left(D-1+\frac{2}{D}\right)\rho^{1-2/D}. \tag{42}$$

If we had used the method indicated in [2] to bound g_2, then $g_2\sim h^{2-D}$ with h as the height of a simplex. However the measure of the ensemble average in (39) is $\sim h$ as h small. Thus the integral would have diverged for $D>3$. This is avoided by using the inner ellipsoid.

Table I. Numerical bound in various dimensions

D	The bound
3	$1.302 \times 10^3 \rho^{1/3}$
4	$4.372 \times 10^4 \rho^{1/2}$
5	$2.249 \times 10^6 \rho^{3/5}$
6	$1.665 \times 10^8 \rho^{2/3}$

In Table I we list the numerical values of (42) in a few dimensions. It is not surprising that the bound is too high since only one of the simplices containing p_0 among many of them is picked up in bounding P_1 and P_2 (in $D = 4$, say, there are about 160 simplices on the average containing a common site).

Appendix A. The Inner Ellipsoid of a D-Simplex[2]

Let $\overline{p_0,\dots,p_D}$ form a D-simplex, S_D, in coordinate space $(x_1,\dots,x_D)$, with $\mathbb{1}_{0i}$ as the links of S_D. By a linear transformation:

$$x = \sum_{i=1}^{D} \zeta_i \mathbb{1}_{0i}, \tag{A1}$$

S_D is deformed to a simplex $\mathscr{S}_D$ in coordinate space $(\zeta_1,\dots,\zeta_D)$. The links $\overline{\mathbb{1}_{01},\dots,\mathbb{1}_{0D}}$ of $\mathscr{S}_D$ have unit lengths and are perpendicular to each other. Let $\overline{p'_0,\dots,p'_D}$ form a regular D-simplex, S'_D, in coordinate space $(x'_1,\dots,x'_D)$ with unit link length. By a linear transformation similar to (A1):

$$x' = \sum_{i=1}^{D} \zeta_i \mathbb{1}'_{0i}, \tag{A2}$$

S'_D is deformed to the same $\mathscr{S}_D$. The combination of (A1) and the inverse of (A2) deforms S_D to S'_D. Since the transformation is linear, the volume ratio of two geometrical objects is invariant under such a deformation. Draw the inscribed sphere of S'_D. The radius of this sphere is

$$r'_D = \sqrt{\frac{1}{2D(D+1)}}, \tag{A3}$$

and its volume is

$$\tau'_D = 2^{-D/2} \omega_D D^{-D/2} (D+1)^{-D/2}. \tag{A4}$$

The volume of the regular simplex is

$$V'_D = 2^{-D/2} \frac{\sqrt{D+1}}{D!}. \tag{A5}$$

The volume ratio of this sphere and this regular simplex is

$$\frac{\tau'_D}{V'_D} = (D-1)!\, \omega_D D^{-D/2} (D+1)^{-(D+1)/2}. \tag{A6}$$

2 I would like to thank R. Friedberg for suggesting this method of defining the inner ellipsoid

When S'_D is deformed back to the original simplex, S_D, the inscribed sphere is deformed to an inscribed ellipsoid which we call the inner ellipsoid. From (A6), the volume of this ellipsoid is

$$\tau_D = (D-1)!\omega_D D^{-D/2}(D+1)^{-(D+1)/2}V_D. \tag{A7}$$

This gives (19) in the text.

Appendix B. Ellipsoidal Coordinates in Arbitrary Dimensions

Consider a family of $(D-1)$-dimensional quadratic surfaces in D-dimensional Euclidean space

$$\sum_{i=1}^{D} \frac{x_i^2}{a_i^2+\theta} = 1; \quad a_1 \geqq \cdots \geqq a_D, \tag{B1}$$

where θ is a parameter. Introduce a Dth-degree polynomial [5],

$$f(\theta) = (a_1^2+\theta)\cdots(a_D^2+\theta)\left(1 - \frac{x_1^2}{a_1^2+\theta}\cdots - \frac{x_D^2}{a_D^2+\theta}\right). \tag{B2}$$

One can readily show that $f(\theta)$ has D real roots in the following regions

$$-a_i^2 \leqq \zeta_i \leqq -a_{i+1}^2 \quad (a_{D+1}^2 = \infty), \tag{B3}$$

and

$$f(\theta) = (\theta-\zeta_1)\cdots(\theta-\zeta_D). \tag{B4}$$

Equations (B1) and (B2) give the transformation rule from the cartesian coordinates $(x_1,\ldots,x_D)$ to the ellipsoidal coordinates $(\zeta_1,\ldots,\zeta_D)$, i.e.

$$x_i^2 = (-1)^D \frac{(a_i^2+\zeta_1)\cdots(a_i^2+\zeta_D)}{(a_1^2-a_i^2)\cdots(a_{i-1}^2-a_i^2)(a_{i+1}^2-a_i^2)\cdots(a_D^2-a_i^2)}. \tag{B5}$$

It can be shown that $(\zeta_1,\ldots,\zeta_D)$ form a system of orthogonal coordinates and

$$\sum_{i=1}^{D} dx_i^2 = \sum_{i=1}^{D} \lambda_i d\zeta_i^2, \tag{B6}$$

where

$$\lambda_i = \frac{1}{4}\frac{(\zeta_i-\zeta_1)\cdots(\zeta_i-\zeta_{i+1})\cdots(\zeta_i-\zeta_D)}{(a_1^2+\zeta_i)\cdots(a_D^2+\zeta_i)}. \tag{B7}$$

The Laplacian equation in ellipsoidal coordinates is

$$\sum_{i=1}^{D} \frac{1}{\sqrt{g}}\frac{\partial}{\partial\zeta_i}\left(\frac{\sqrt{g}}{\lambda_i}\frac{\partial}{\partial\zeta_i}\Phi\right) = 0, \tag{B8}$$

where

$$g = \lambda_1\cdots\lambda_D. \tag{B9}$$

Consider an ellipsoid described by

$$\sum_{i=1}^{D} \frac{x_i^2}{a_i^2} = 1 \tag{B10}$$

at constant potential Φ_1, immersed in an infinite medium of unit conductivity. The potential in the medium will be a function of $\xi = \zeta_D$ only. In this case (B8) becomes

$$\frac{d}{d\xi}\left(\sqrt{(a_1^2+\xi)\cdots(a_D^2+\xi)}\frac{d\Phi}{d\xi}\right)=0. \tag{B11}$$

The solution satisfying $\Phi(0)=\Phi_1$ is

$$\Phi(\xi)=\frac{\Phi_1}{\kappa}\int_\xi^\infty \frac{ds}{\sqrt{(a_1^2+s)\cdots(a_D^2+s)}}, \tag{B12}$$

where κ is given by

$$\kappa=\int_0^\infty \frac{ds}{\sqrt{(a_1^2+s)\cdots(a_D^2+s)}}. \tag{B13}$$

The total power dissipated in the medium is

$$P_c=\tfrac{1}{2}\int d^Dx(\nabla\Phi)^2=\frac{\omega_D\Phi_1^2}{\kappa}. \tag{B14}$$

Finally let us derive an upper bound of κ. Let δ be an arbitrary positive number. Then

$$\kappa=\int_0^\delta \frac{ds}{\sqrt{(a_1^2+s)\cdots(a_D^2+s)}}+\int_\delta^\infty \frac{ds}{\sqrt{(a_1^2+s)\cdots(a_D^2+s)}} \tag{B15}$$

$$\leqq \frac{1}{a_1\cdots a_D}\delta+\int_\delta^\infty \frac{ds}{s^{D/2}}=\frac{\omega_D\delta}{D\tau}+\frac{1}{\frac{D}{2}-1}\delta^{1-D/2}. \tag{B16}$$

Minimizing the right-hand side of (B16) we obtain

$$\kappa\leqq\frac{D}{D-2}\left(\frac{\omega_D}{D\tau}\right)^{1-2/D}. \tag{B17}$$

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