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A Supersymmetric Transfer Matrix and Differentiability of the Density of States in the One-Dimensional Anderson Model

Massimo Campanino* and Abel Klein**
Department of Mathematics, University of California, Irvine, Ca 92717, USA

Abstract. Let $H = -\Delta + V$ on $l^2(\mathbb{Z})$, where $V(x)$, $x \in \mathbb{Z}$, are i.i.d.r.v.'s with common probability distribution ν. Let $h(t) = \int e^{-itv} d\nu(v)$ and let $k(E)$ be the integrated density of states. It is proven: (i) If h is n-times differentiable with $h^{(j)}(t) = O((1+|t|)^{-\alpha})$ for some $\alpha > 0$, $j = 0, 1, \ldots, n$, then $k(E)$ is a C^n function. In particular, if ν has compact support and $h(t) = O((1+|t|)^{-\alpha})$ with $\alpha > 0$, then $k(E)$ is C^∞. This allows ν to be singular continuous. (ii) If $h(t) = O(e^{-\alpha|t|})$ for some $\alpha > 0$ then $k(E)$ is analytic in a strip about the real axis.

The proof uses the supersymmetric replica trick to rewrite the averaged Green's function as a two-point function of a one-dimensional supersymmetric field theory which is studied by the transfer matrix method.

1. Introduction

The one-dimensional Anderson model is given by the random Hamiltonian $H = H_0 + V$ on $l^2(\mathbb{Z})$, where

$$(H_0 u)(x) = \tfrac{1}{2}(u(x+1) + u(x-1))$$

and $V(x)$, $x \in \mathbb{Z}$, are independent identically distributed random variables with common probability distribution ν. We will denote by h its characteristic function, i.e., $h(t) = \int e^{-itv} d\nu(v)$.

Let Λ be an interval in $\mathbb{Z}$, we will denote by H_Λ the operator H restricted to $l^2(\Lambda)$ with boundary condition $u(x) = 0$ for x not in Λ.

The integrated density of states, $k(E)$, is defined by

$$k(E) = \lim_{|\Lambda| \to \infty} \# \{\text{eigenvalues of } H_\Lambda \leq E\}.$$

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* Permanent address: Dipartimento di Matematica, II Universita di Roma, Via Orazio Raimondo, I-00173 (La Romanina) Roma, Italy

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It is a consequence of the ergodic theorem that for almost every potential the limit exists for all E and is independent of the potential [14]; $k(E)$ is always a continuous function [5]. Under some mild condition on ν $k(E)$ was shown to be log-Holder continuous [6] and Holder continuous on compact intervals [7].

Without restrictions on ν we cannot expect too much more regularity. There is an argument of Halperin (see [8]) that shows that when $\nu = \frac{1}{2}\delta(v) + \frac{1}{2}\delta(v-a)$, given any $\alpha > 0$ one can choose a so that $k(E)$ is not Holder continuous of order α; in particular it gives examples where $k(E)$ is not C^1.

Further results have required ν to be absolutely continuous with respect to Lebesgue measure, say $d\nu(v) = F(v)dv$. If F is bounded, Wegner [9] proved that $k(E)$ is absolutely continuous with a bounded derivative. This has been extended by Maier [10] to $F \in L^p$, $p > 1$. If $\int v^2 F(v)dv < \infty$, Lacroix [11] has shown $k(E)$ is C^1.

Constantinescu, Fröhlich and Spencer [12] proved that if F is analytic in a strip of certain width, then $k(E)$ is real analytic for $|E|$ large enough; If ν is Gaussian they proved that for large disorder $k(E)$ is a real analytic function of E. Carmona [4], using an idea of Molcanov, gives a simple proof that if $|h(t)| \leqq C'e^{-C|t|}$, where $C' < C$, then $k(E)$ is analytic in a strip; this holds for ν Gaussian for large disorder. Another argument for the same result due to Simon can be found in [12].

Using the supersymmetric replica trick and a cluster expansion Klein and Perez [13] showed how to use decay properties of $h(t)$ and its derivatives to derive differentiability for $k(E)$ for either large disorder or large $|E|$; they also obtained analyticity results. Their methods have strongly influenced this article.

Recently, Simon and Taylor [8] proved the surprising (at least at first sight) result that if $d\nu(v) = F(v)dv$, where F has compact support and $F \in L^1_\alpha(\mathbb{R}) = \{f \in L^1(\mathbb{R}) |$ there exists $g \in L^1(\mathbb{R})$ such that $\hat{g}(t) = (1+t^2)^{\alpha/2}\hat{f}(t)\}$, with $\alpha > 0$, then $k(E)$ is C^∞. They also conjectured that it should be enough to require that $(1+t^2)^{\alpha/2}h(t)$ be bounded for some $\alpha > 0$, and that the hypothesis of compact support should not be essential. As they remarked, there are singular continuous ν satisfying this condition (see [27, Theorem XII.10.12] and [28]).

In this article we prove Simon and Taylor's conjecture. We also prove analyticity results for the density of states.

Our condition will be stated in terms of h, the characteristic function of ν. We will only be interested in $h(t)$ for $t \geqq 0$ (of course, $h(-t) = \overline{h(t)}$) and we will only consider the right-hand side derivatives at $t=0$.

We will now state our results.

Theorem 1.1. *Let $n \geqq 1$. If h is $(n-1)$-times differentiable for $t \geqq 0$ with $h^{(n-1)}$ absolutely continuous, and $(1+|t|)^\alpha h^{(j)}(t)$ is bounded for some $\alpha > 0$ and $j = 0, 1, 2, \ldots, n$, then $k(E)$ is a C^n function of E.*

Corollary 1.2. *Let $(1+|t|)^\alpha h(t)$ be bounded for some $\alpha > 0$. If $\int |v|^{n+\varepsilon} d\nu(v) < \infty$ for some $\varepsilon > 0$ $k(E)$ is C^n. In particular, if ν has finite moments of all orders $k(E)$ is C^∞.*

Our result on analyticity is

Theorem 1.3. *If $e^{\alpha|t|}h(t)$ is bounded for some $\alpha > 0$ then $k(E)$ is analytic in a strip $|\mathrm{Im}\, E| < \alpha_1$ for some $\alpha_1 > 0$.*

We approach the density of states thru the Green's function of H. Let $G(x,y;z) =$

$\langle x|(H-z)^{-1}|y\rangle$ where x, $y\in\mathbb{Z}$, $\operatorname{Im} z>0$. Then (e.g., [4, 14]) $G(z)=\mathbb{E}(G(0,0;z))$ is the Borel transform of $dk(E)$, i.e.,

$$G(z)=\int\frac{dk(E)}{E-z},$$

and we have:

i) $G(E+i0)=\lim_{\eta\downarrow 0} G(E+i\eta)$ exists for a.e. $E\in R$,

ii) if $dk_{\text{a.c.}}$ denotes the absolutely continuous part of the measure dk,
$$\frac{dk_{\text{a.c.}}}{dE}=\frac{1}{\pi}\operatorname{Im} G(E+i0),$$

iii) $dk_{\text{sing}}\equiv dk-dk_{\text{a.c.}}$ is supported by
$$\{E\in R|\lim_{\eta\downarrow 0}\operatorname{Im} G(E+i\eta)=\infty\}.$$

Thus Theorem 1.1 and 1.3 follow from

Theorem 1.4. *Let* $n\geqq 1$. *If* h *is* $(n-1)$*-times differentiable for* $t\geqq 0$ *with* $h^{(n-1)}$ *absolutely continuous and* $(1+|t|)^{\alpha}h^{(j)}(t)$ *is bounded for some* $\alpha>0$ *and all* $j=0, 1,\dots,n$, *then* $G(E+i0)=\lim_{\eta\downarrow 0} G(E+i\eta)$ *exists for all* $E\in R$ *and is a* C^{n-1} *function of* E.

Theorem 1.5. *If* $e^{\alpha|t|}h(t)$ *is bounded for some* $\alpha>0$ *then* $G(z)$ *has an analytic continuation to* $\operatorname{Im} z+\alpha_1>0$ *for some* $\alpha_1>0$.

We will now describe the strategy of our proof. Let $\Lambda_l=\{-l,-l+1,\dots,0,\dots,l\}$, $H_l=H_{\Lambda_l}$, and

$$G_l(z)=\mathbb{E}(\langle 0|(H_l-z)^{-1}|0\rangle),$$

so

$$G(z)=\lim_{l\to\infty} G_l(z) \text{ for } \operatorname{Im} z>0.$$

In Sect 2 we will use the supersymmetric replica trick [15–18] to rewrite $G_l(z)$ as a two-point function of a one-dimensional supersymmetric field theory. We will introduce a supersymmetric transfer matrix and do explicitly the integration over the anticommuting variables. This will give us

$$G_l(z)=2i\int_0^{\infty}\{[(TB(z))^l 1](r^2)\}^2\beta(r^2;z)r\,dr, \tag{1.1}$$

where $\beta(r;z)=h(r)e^{izr}$, $B(z)$ denotes the operator multiplication by $\beta(\cdot\,;z)$, and T is the operator given by

$$(Tf)(r^2)=-2\int_0^{\infty}J_0(rs)f'(s)s\,ds,$$

where J_0 is the Bessel function of order zero. This operator is studied in Sect. 3.

Since the proof of Theorem 1.5 is simpler, we give it first on Sect. 4. Recall $G_l(z)\to G(z)$ as $l\to\infty$ for $\operatorname{Im} z>0$. It will be easy to see that under the hypothesis of

Theorem 1.5 $G_l(z)$ can be analytically continued to $\operatorname{Im} z + \alpha > 0$ and (1.1) still holds. We show that (1.1) yields bounds on $G_l(z)$, uniformly on l, so an application of Vitali's Theorem gives Theorem 1.5.

Section 5 contains the proof of Theorem 1.4. We first show that for large $l(TB(z))^l 1$ has n derivatives with good decay properties at infinity. This uses the Calderon-Lions method of complex interpolation. The theorem is stated in Sect. 5 but proved in Sect. 6. In addition, we show that in this Sobolev-type space $TB(z)$ has 1 as an algebraically simple eigenvalue with a gap in the spectrum. If $\xi(\cdot\,;z)$ is the corresponding eigenvector, we will conclude that

$$G(z) = 2i \int_0^\infty \xi(r^2;z)^2 \beta(r^2;z) r dr.$$

Since our estimates will have uniformity properties in z, we will be able to let $\eta = \operatorname{Im} z \downarrow 0$ and obtain the conclusions of Theorem 1.4.

Corollary 1.2 is proven in Sect. 7.

Notes. 1) If dv/dv has an analytic continuation to a strip with decay at infinity, analyticity of the density of states can be derived [31] from formula (IX.5) in [32] and by the methods [29] of [8].

2) Rene Carmona has shown us a manuscript by March and Sznitman [30] with related results. In particular they obtain formula (1.1) by probabilistic methods.

2. A Supersymmetric Transfer Matrix

The supersymmetric replica trick [15–18] says that, if $x_1, x_2 \in \Lambda_l$, $\operatorname{Im} z > 0$,

$$\begin{aligned} G_l(x_1,x_2;z) &= \langle x_1 | (H_l - z)^{-1} | x_2 \rangle \\ &= i \int \psi(x_1)\bar{\psi}(x_2) \exp\left\{ -i \sum_{x=-l}^{l} \Phi(x)\cdot[(H_l - z)\Phi](x) \right\} D_l \Phi, \end{aligned}$$

where $\Phi(x) = (\phi(x), \psi(x), \bar{\psi}(x))$, $\phi(x) \in R^2$, $\psi(x)$, $\bar{\psi}(x)$ are anticommuting "variables" (i.e., elements of a Grassman algebra),

$$\Phi(x)\cdot\Phi(y) = \phi(x)\cdot\phi(y) + \tfrac{1}{2}(\bar{\psi}(x)\psi(y) + \bar{\psi}(y)\psi(x)),$$

and

$$D_l \Phi = \prod_{x=-l}^{l} d\Phi(x), \quad \text{where} \quad d\Phi(x) = \frac{1}{\pi} d\bar{\psi}(x) d\psi(x) d^2\phi(x)$$

(see [29, 18, 13, 20, 21, 22]). Notice that $\int e^{-\Phi(x)\cdot\Phi(x)} d\Phi(x) = 1$.

Since we are working with a finite lattice the above formula is fully rigorous. To compute functions of ψ, $\bar{\psi}$ we expand in power series that terminate after a finite number of terms due to the anticommutativity. All $\{\psi(x), \bar{\psi}(x);\ x = -l, \dots, l\}$ anticommute. The linear functional denoted by integration against $d\bar{\psi}(x)d\psi(x)$ (it is not an actual integral) is defined by

$$\int (a_0 + a_1\psi(x) + a_2\bar{\psi}(x) + a_3\bar{\psi}(x)\psi(x)) d\bar{\psi}(x) d\psi(x) = -a_3.$$

To simplify our notation, we will write $\Phi(x)^2 = \Phi(x)\cdot\Phi(x)$, $\phi(x)^2 = \phi(x)\cdot\phi(x)$.

Recalling the definition of H_l we have

$$G_l(x_1, x_2; z) = i\int \psi(x_1)\bar{\psi}(x_2)\exp\left\{-i\sum_{x=-l}^{l} V(x)\Phi(x)^2 + iz\sum_{x=-l}^{l}\Phi(x)^2 - i\sum_{x=-l}^{l-1}\Phi(x)\cdot\Phi(x+1)\right\}D_l\Phi. \quad (2.1)$$

Let us first assume that $\int|v|d\nu(v) < \infty$. This implies that h is continuously differentiable with a bounded derivative. Since in this case

$$\begin{aligned}\int e^{iv\Phi^2}d\nu(v) &= \int e^{-iv(\phi^2+\bar{\psi}\psi)}d\nu(v)\\ &= \int e^{-iv\phi^2}(1 - iv\bar{\psi}\psi)d\nu(v) = h(\phi^2) + h'(\phi^2)\bar{\psi}\psi = h(\phi^2+\bar{\psi}\psi) = h(\Phi^2),\end{aligned}$$

we can average over the random potential in (2.1) to obtain

$$\mathbb{E}(G_l(x_1, x_2; z)) = i\int \psi(x_1)\bar{\psi}(x_2)\prod_{x=-l}^{l}\beta(\Phi(x)^2; z)\exp\left\{-i\sum_{x=-l}^{l-1}\Phi(x)\cdot\Phi(x+1)\right\}D_l\Phi, \quad (2.2)$$

where $\beta(r; z) = h(r)e^{izr}$.

By an approximation argument we have

Theorem 2.1. *Let the characteristic function h be absolutely continuous with a bounded derivative. Then (2.2) holds for* $\operatorname{Im} z > 0$.

Since in this article we are interested in the density of states we will now take $x_1 = x_2 = 0$, but our methods work for general x_1, x_2 and give exponential decay for $\lim_{\eta\downarrow 0}\mathbb{E}(G(x_1, x_2; E + i\eta))$.

So let

$$G_l(z) = \mathbb{E}(G_l(0,0;z)) = i\int \psi(0)\bar{\psi}(0)\prod_{x=l}^{l}\beta(\Phi(x)^2; z)\cdot\exp\left\{-i\sum_{x=-l}^{l-1}\Phi(x)\cdot\Phi(x+1)\right\}D_l\Phi.$$

We now introduce a supersymmetric transfer matrix: let

$$\mathbf{T}(\Phi_1, \Phi_2) = e^{-i\Phi_1\Phi_2},$$

and let us define the operator $\mathbf{T}$ on supersymmetric functions (e.g., [20]) by

$$(\mathbf{T}F)(\Phi_1^2) = \int \mathbf{T}(\Phi_1, \Phi_2)F(\Phi_2^2)d\Phi_2.$$

Let us denote by $B(z)$ the operator multiplication by $\beta(\cdot\,; z)$, i.e.,

$$(B(z)F)(\Phi^2) = \beta(\Phi^2; z)F(\Phi^2).$$

Then (2.3) can be rewritten as

$$G_l(z) = i\int \psi(0)\bar{\psi}(0)\beta(\Phi(0)^2; z)\{[(\mathbf{T}B(z))^l 1](\Phi(0)^2)\}^2 d\Phi(0).$$

We now perform the integration over the anticommuting variables $\psi(0)$, $\bar{\psi}(0)$ and obtain

$$G_l(z) = \frac{i}{\pi} \int \beta(\phi(0)^2; z) \{ [(TB(z))^l 1](\phi(0)^2) \} d^2\phi(0), \tag{2.4}$$

where

$$(Tf)(\phi_1^2) = -\frac{1}{\pi} \int e^{-i\phi_1 \phi_2} f'(\phi_2^2) d^2\phi_2.$$

Here we used the fact that if

$$F(\Phi^2) = f(\phi^2) + f'(\phi^2)\bar{\psi}\psi,$$

then

$$(\mathbf{T}F)(\Phi^2) = (Tf)(\phi^2) + (Tf)'(\phi^2)\bar{\psi}\psi. \tag{2.5}$$

If we now change to polar coordinates (2.4) and (2.5) become

$$G_l(z) = 2i \int_0^\infty \{ [(TB(z))^l 1](r^2) \}^2 \beta(r^2; z) r dr \tag{2.6}$$

and

$$(Tf)(r^2) = -2 \int_0^\infty J_0(rs) f'(s^2) s ds, \tag{2.7}$$

where

$$J_0(s) = \frac{1}{2\pi} \int_0^{2\pi} e^{-is\cos\theta} d\theta \tag{2.8}$$

is the Bessel function of order zero.

3. Some Harmonic Analysis on $[0, \infty)$

We will now study the operator T given by (2.7). By an integration by parts,

$$(Tf)(r^2) = f(0) + (Rf)(r^2), \tag{3.1}$$

where

$$(Rf)(r^2) = r \int_0^\infty J_{-1}(rs) f(s^2) ds. \tag{3.2}$$

We recall that the Bessel functions of integral order n can be defined by

$$J_n(s) = (-1)^n s^n \left(\frac{d}{sds} \right)^n J_0(s), n = 0, 1, \ldots,$$

$$J_n(s) = (-1)^n J_{-n}(s) \quad \text{for} \quad n = -1, -2, \ldots,$$

where $J_0(s)$ is given by (2.8).

T and R can be expressed in terms of Hankel transforms, which are defined by (e.g., [23, 24])

$$H_n(g)(r) = \int_0^\infty (rs)^{1/2} J_n(rs) g(s) ds$$

for $n \in \mathbb{Z}$.

It is easy to see that $\| H_n(g) \|_\infty \leqq 2/\pi \| g \|_1$, and there is a Plancherel theorem for Hankel transforms [23] on $L^2([0, \infty), dr)$: $\| H_n(g) \|_2 = \| g \|_2$. It follows from the Riesz convexity theorem that one has a Hausdorff-Young inequality for Hankel transforms:

$$\| H_n(g) \|_{p'} \leqq \| g \|_p \quad \text{for} \quad 1 \leqq p \leqq 2, \frac{1}{p} + \frac{1}{p'} = 1.$$

Thus (2.7) and (3.2) can be rewritten as

$$r^{1/2}(Tf)(r^2) = -2H_0(s^{1/2} f'(s^2))(r), \tag{3.3}$$

$$r^{-1/2}(Rf)(r^2) = H_{-1}(s^{-1/2} f(s^2))(r). \tag{3.4}$$

We have the following general formula for derivatives of Hankel transforms [24]:

$$r^{n+1/2} \left(\frac{d}{rdr} \right)^m (r^{m-n-1/2} g(r)) = H_n((-s)^m [H_{n-m}(g(t))(s)])(r)$$

for $n = 0, 1, 2, \ldots$, and also for $n = -1$ if $g(0) = 0$. Thus

$$(-2)^m r^{m+k-1/2} (Qf)^{(m)}(r^2) = (-2)^k H_{m+k-1}(s^{m+k-1/2} f^{(k)}(s^2))(r) \tag{3.5}$$

holds with $Q = R$ for $m = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots$, and for $Q = T$ with $m = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots$, and $m + k \geqq 1$.

So we are led to define the Hilbert spaces:

$$\mathscr{H}_0 = \{ f : [0, \infty) \to \mathbb{C} \text{ measurable}; \ |||f|||_0 = \| r^{-1/2} f(r^2) \|_2 < \infty \},$$

$\mathscr{H}_n = \{ f : [0, \infty) \to \mathbb{C}$ continuous; f is $(n-1)$-times differentiable on $(0, \infty)$ with $f^{(n-1)}$ absolutely continuous with

$$|||f|||_n^2 = \sum_{m=1}^{n} \sum_{k=0}^{m} \| 2^k r^{m-1/2} f^{(k)}(r^2) \|_2^2 < \infty \}$$

for $n = 1, 2, \ldots$, and

$$\mathscr{H}_0^0 = \mathscr{H}_0, \mathscr{H}_n^0 = \{ f \in \mathscr{H}_n ; f(0) = 0 \} \quad \text{for} \quad n = 1, 2, \ldots.$$

It follows from (3.5) that T is a unitary operator on $\mathscr{H}_n$ for $n = 1, 2, \ldots$, and R is unitary on $\mathscr{H}_n^0$ for $n = 0, 1, 2, \ldots$. In addition (3.1) says that $T = R$ on $\mathscr{H}_n^0$ for $n \geqq 1$; in particular T leaves $\mathscr{H}_n^0$ invariant.

Let us now denote by B the operator multiplication by $\beta \in \mathscr{H}_1$. Then $(TB)^l 1$ is well defined. It also follows that $r^{-1/2} \beta(r^2) \in L^1$, so by (3.4) $R\beta$ is well defined and a bounded continuous functions with $(R\beta)(0) = 0$. Thus if we apply (3.1) l times we get

$$(TB)^l 1 = (I + RB + (RB)^2 + \cdots + (RB)^l) 1. \tag{3.6}$$

For later use we rewrite (3.6) as

$$(TB)^l 1 = 1 + RB + (I + RB + \cdots + (RB)^{l-2})(RB)^2 1, \tag{3.7}$$

and

$$(TB)^l 1 = (TB)1 + (I + RB + \cdots + (RB)^{l-2})(RB)^2 1. \tag{3.8}$$

If only assume that $\beta(r^2) \in L^\infty$, we still have RB as a bounded operator on $\mathscr{H}_0$. The following lemmas will be of importance.

Lemma 3.1. *Let* $\beta(r^2) \in L^p([0, \infty), dr)$, *where* $2 < p \leqq \infty$. *Then* $\|(RB)^2\|_{\mathscr{H}_0} \leqq \|\beta(r^2)\|_p^2$.

Proof. Let $f \in \mathscr{H}_0$. Then

$$\begin{aligned} |||(RB)^2 f|||_0 &= |||BRBf|||_0 = \|r^{-1/2}\beta(r^2)(RBf)(r^2)\|_2 \\ &\leqq \|\beta(r^2)\|_p \|r^{-1/2}(RBf)(r^2)\|_{(1/2-1/p)^{-1}} \\ &\leqq \|\beta(r^2)\|_p \|r^{-1/2}(Bf)(r^2)\|_{(1/2+1/p)^{-1}} \\ &= \|\beta(r^2)\|_p \|r^{-1/2}\beta(r^2)f(r^2)\|_{(1/2+1/p)^{-1}} \\ &\leqq \|\beta(r^2)\|_p^2 \|r^{-1/2}f(r^2)\|_2 = \|\beta(r^2)\|_p^2 \|f\|_0. \quad ■ \end{aligned}$$

Lemma 3.2. *Suppose* β *is a continuous function such that* $(1 + r^2)^{\gamma/2}\beta(r^2)$ *is bounded for some* $\gamma > 0$. *Then* $(RB)^2 1 \in \mathscr{H}_0$.

Proof. It follows that $r^{1/2}\beta(r^2) \in L^q$ for all $1 < q_1 < q < 2$, where q_1 depends only on γ, and $\beta(r^2) \in L^p$ for all large p. Thus $r^{-1/2}(R\beta)(r^2) \in L^{q'}$, where $1/q + 1/q' = 1$, and $r^{-1/2}\beta(r^2)(R\beta)(r^2) \in L^2$. ■

4. Proof of Theorem 1.5

We first assume that h is also absolutely continuous with h' bounded, so Theorem 2.1 applies and we have, from (2.6) and (3.6), that

$$G_l(z) = 2i \int_0^\infty \left\{\left[\sum_{k=0}^{l} [(RB(z))^k 1](r^2)\right\}^2 \beta(r^2; z) r dr \quad \text{for} \quad \operatorname{Im} z > 0. \tag{4.1}$$

By an approximation argument we can now extend (4.1) to h as in the hypothesis of Theorem 1.5.

Since $\beta(r^2; z) = h(r^2)e^{izr^2}$, and $e^{\alpha r^2}h(r^2)$ is bounded with $\alpha > 0$, we can use the right-hand side of (4.1) to analytically continue $G_l(z)$ to $\operatorname{Im} z + \alpha > 0$.

Since $|h(t)| < 1$ for all $t \neq 0$, there exists $2 < p < \infty$ such that $\int_0^\infty (h(r^2)|^p dr < 1$. Since $e^{\tau r^2}h(r^2) \in L^p$ for $\tau < \alpha$, we can select $0 < \tau < \alpha$ such that $\|e^{\tau r^2}h(r^2)\|_p < 1$.

It now follows from (4.1), (3.7), Lemmas 3.1 and 3.2 that $G_l(z)$ is uniformly bounded in l and in z for $\operatorname{Im} z + \tau > 0$. It follows from Vitali's Theorem that $G(z)$ is analytic for $\operatorname{Im} z + \tau > 0$. ■

5. Proof of Theorem 1.4

Under the hypothesis of Theorem 1.4, $\beta(r;z)=h(r)e^{izr}$ is $(n-1)$-times differentiable for $r\geqq 0$ with $\beta^{(n-1)}(r;z)$ absolutely continuous, and, if $\operatorname{Im} z\geqq 0$, $(1+r^2)^{\gamma/2}\beta^{(j)}(r^2;z)$ is bounded, $j=0,1,\dots,n$, for some $\gamma>0$. As before $B(z)$ will denote the operator multiplication by $\beta(\cdot\,;z)$. Notice that $B(z)$ is a bounded operator on $\mathscr{H}_m$, leaving $\mathscr{H}_m^0$ invariant, for $\operatorname{Im} z\geqq 0$ and $m=0,1,\dots,n$.

We will need more. We will need that applying $RB(z)$ repeatedly takes $\mathscr{H}_0$ to $\mathscr{H}_n^0$.

Theorem 5.1. *Let $\beta(r)$ be $(n-1)$-times differentiable with $\beta^{(n-1)}(r)$ absolutely continuous, such that $(1+r^2)^{\gamma/2}\beta^{(j)}(r^2)$ is bounded, $j=0,1,\dots,n$, for some $\alpha>0$. Let B be the operator multiplication by β. Then there exists k_0 depending only on γ, such that for all $k\geqq k_0$, $(RB)^k$ is a bounded operator from $\mathscr{H}_0$ to $\mathscr{H}_n^0$. Furthermore, if $\beta(r;z)=\beta(r)e^{izr}$ and $B(z)$ is the corresponding multiplication operator, the norm of $(RB(z))^k$ as an operator from $\mathscr{H}_0$ to $\mathscr{H}_n^0$ is uniformly bounded for* $\operatorname{Im} z\geqq 0$ *and bounded* $\operatorname{Re} z$.

If $\gamma>1$ (e.g., if the probability distribution v is the uniform distribution on a bounded interval) it is not hard to prove this theorem. But for small γ it requires the Calderon-Lions method of complex interpolation, so we will postpone it to the next section.

Let $g(t)$ be a real valued C^∞ function with compact support on $\mathbb{R}$ such that $g(t)=1$ for $|t|\leqq 1$. Let $h_1(t)=g(t)h(t)$, $h_2(t)=h(t)-h_1(t)$, and let $\beta_j(r;z)=h_j(r)e^{izr}$, $j=1,2$. Then

$$\beta(r;z)=\beta_1(r;z)+\beta_2(r;z)\quad\text{and}\quad \beta_1(r;z)\in\mathscr{H}_n,\ \beta_2(r;z)\in\mathscr{H}_0$$

for $\operatorname{Im} z\geqq 0$.

Recall that (3.8) holds for $\operatorname{Im} z>0$, so we have

$$(TB(z))^l 1 = T\beta_1(z)+R\beta_2(z)+(I+RB(z)+\dots+(RB(z))^{l-2}(RB(z))^2 1 \tag{5.1}$$

for $\operatorname{Im} z>0$.

By Lemma 3.2, $(RB(z))^2 1\in\mathscr{H}_0$ for $\operatorname{Im} z\geqq 0$, and the right-hand side of (5.1) is well defined for $\operatorname{Im} z\geqq 0$.

Now let us pick k_0 from Theorem 5.1. It follows that

$$\begin{aligned}(TB(z))^{l+k_0}1 &= (TB)^{k_0}T\beta_1(z)+(RB(z))^{k_0}[R\beta_2(z)\\ &\quad +(T+RB(z)+\dots+(RB(z))^{l-2})(RB(z)^2 1]\end{aligned} \tag{5.2}$$

is in $\mathscr{H}_n$ for $\operatorname{Im} z>0$ and the right-hand side is a continuous function of z, $\operatorname{Im} z\geqq 0$, with values in $\mathscr{H}_n$. We have proved the first part of

Lemma 5.2. *There exists l_0 such that for $l\geqq l_0(TB(z))^l 1\in\mathscr{H}_n$ for* $\operatorname{Im} z>0$, *is a continuous function of z with values in $\mathscr{H}_n$, and can be extended by continuity to* $\operatorname{Im} z\geqq 0$. *Furthermore $\xi(z)=\lim_{l\to\infty}(TB(z))^l 1$ exists in $\mathscr{H}_n$ for* $\operatorname{Im} z\geqq 0$, *the convergnece being uniform in* $\operatorname{Im} z\geqq 0$ *with bounded* $\operatorname{Re} z$.

Proof. The lemma follows from (5.2) and Lemma 3.1. Just notice that $\|\beta(r^2;z)\|_p\leqq\|h(r^2)\|_p$ for $\operatorname{Im} z\geqq 0$, and that $\|h(r^2)\|_p<1$ for p large enough. ■

Notice that Lemma 3.2 and (2.6) tell us that

$$G(z) = 2i \int_0^\infty \xi(r^2; z)^2 \beta(r^2; z) r dr, \tag{5.3}$$

and $G(z)$ is a continuous function of z for $\operatorname{Im} z \geqq 0$. This is Theorem 1.4 for $n = 1$.

Lemma 3.2 and its proof also tell us that $TB(z)\xi(z) = \xi(z)$ and $\xi(0; z) = 1$. In fact we have more:

Lemma 5.3. *Let* $\operatorname{Im} z \geqq 0$. *Then* 1 *is an algebraically simple eigenvalue for* $TB(z)$ *in* $\mathscr{H}_n$ *with corresponding unique eigenvector* $\xi(z)$ *normalized by* $\xi(0; z) = 1$. *Furthermore, the direct* sum $\mathscr{H}_n = \mathbb{C}\xi(z) \oplus \mathscr{H}_n^0$ *diagonalizes* $TB(z)$ *in the form* $TB(z) = \delta_0 \xi(z) \oplus RB(z)$, *where* $\delta_0(f) = f(0)$. *In addition, the operator norm of* $(RB(z))^2$ *in* $\mathscr{H}_n^0$ *is bounded by a constant* < 1 *uniformly in* $\operatorname{Im} z \geqq 0$ *and bounded* $\operatorname{Re} z$.

Proof. If $f \in \mathscr{H}_n$, then $f = f(0)\xi(z) + [f - f(0)\xi(z))]$ and $f - f(0)\xi(z) \in \mathscr{H}_n^0$. Thus $\mathscr{H}_n = \mathbb{C}\xi(z) \oplus \mathscr{H}_n^0$. The lemma now follows from Lemmas 5.2, 3.1, and Theorem 5.1. ■

To finish the proof of Theorem 1.4 for $n \geqq 2$, we must show that $G(E + i0)$ is a C^{n-1} function of E. From (5.3) we have

$$G(E + i0) = 2i \langle M\xi(E), B(E) M\xi(E) \rangle, \tag{5.4}$$

where

$$\langle u, v \rangle = \int_0^\infty u(r^2) v(r^2) r^{-1} dr$$

is a continuous bilinear form on 0 and M is the operator multiplication by the function $r^{1/2}$, i.e., $(Mu)(r^2) = ru(r^2)$.

Let us fix $E_0 \in R$, $\delta > 0$, and let

$$\tau_0^2 = \sup\{ \|(RB(E))^2\|_{\mathscr{H}_n^0}; |E - E_0| < \delta\} < 1$$

by Lemma 5.3. Let Υ denote the circle $\{z \in \mathbb{C}; |z - 1| = \frac{1}{2}(1 - \tau_0)\}$, $\xi_0 = \xi(E_0)$. Then

$$\xi(E) = \frac{1}{2\pi i} \int_\Upsilon (z - TB(E))^{-1} dz \xi_0 \tag{5.5}$$

for $|E - E_0| < \delta$.

Since $E \to TB(E)$ is a continuous function with values in $\mathscr{L}(\mathscr{H}_n)$, the space of bounded linear operators on $\mathscr{H}_n$, it follows from (5.5) that $E \to \xi(E)$ is continuous with values in $\mathscr{H}_n$.

Now, $TB(E)$ is not differentiable as a function with values in $\mathscr{L}(\mathscr{H}_n)$, but it is as a function with values in $\mathscr{L}(\mathscr{H}_n, \mathscr{H}_{n-2})$, the space of bounded linear operators from $\mathscr{H}_n$ to $\mathscr{H}_{n-2}$, as long as $n \geqq 2$, and $d/dt\ TB(E) = iTM^2 B(E)$. So it follows from (5.5) that $\xi(E)$ is continuously differentiable with $d\xi/dE(E) \in \mathscr{H}_{n-2}$.

More generally, if $2k \leqq n$, $TB(E)$ is k times continuously differentiable with

$$\frac{d^k}{(dE)^k} TB(E) = i^k T M^{2k} B(E) \in \mathscr{L}(\mathscr{H}_n, \mathscr{H}_{n-2k}),$$

and it follows from (5.5) that $\xi(E)$ is k-times differentiable with $d^k/dE^k\ \xi(E)\in\mathscr{H}_{n-2k}$.

It now follows from (5.4) that $G(E+i0)$ is a C^{2k} function of E if $2k\leqq n$.

But we can do better, in fact we will show $G(E+i0)$ is C^{n-1}.

To do so notice that R is self-transpose with respect to $\langle\,,\rangle$ on $\mathscr{H}_0$, i.e.,

$$\langle Rf,g\rangle=\langle f,Rg\rangle \quad \text{for} \quad f,g\in\mathscr{H}_0.$$

Similarly, $B(E)^t=B(E)$, the transpose being with respect to $\langle\,,\rangle$ on $\mathscr{H}_0$, so

$$(RB(E))^t=B(E)R, [(z-RB(E))^{-1}]^t=(z-B(E)R)^{-1}.$$

We also recall that $T=R$ if $f(0)=0$.

From (5.4) and (5.5) we get, for $|E-E_0|<\delta$,

$$G(E+i0)=\frac{1}{2\pi^2 i}\int_\gamma dz\int_\gamma dz'\langle MK(z,E)\xi_0,B(E)MK(z',E)\xi_0\rangle,$$

where $K(z,E)=(z-TB(E))^{-1}$.

If $n=2$, it is not hard to see that since $\xi_0\in\mathscr{H}_2$,

$$\frac{d}{dE}G(E+i0)=\frac{1}{2\pi^2 i}\int_\gamma dz\int_\gamma dz'\{2\langle K(z,E)TiM^2B(E)K(z,E)\xi_0,\ M^2B(E)K(z',E)\xi_0\rangle + \langle M^2K(z,E)\xi_0,\ iM^2B(E)K(z',E)\xi_0\rangle\},$$

a continuous function of E.

The same procedure can be used for general n. For an operator valued function $A(E)$, let $\Delta_eA(E)=1/e(A(E+e)-A(E))$.

When we compute $\lim\limits_{e\to 0}\Delta_e(d^k/dE^k)\,G(E+i0)$, we must move some operators from one side to the other of the bilinear form $\langle\,,\rangle$ using the transposed operators. We illustrate the procedure in the following term that appears in $\Delta_e\,(d/dE)\,G(E+i0)$:

$$2\langle MK(z,E+e)T(\Delta_eB(E))K(z,E)TiM^2B(E+e)K(z,E+e)\xi_0,B_{E+e}MK(z',E+e)\xi_0\rangle. \tag{5.6}$$

In this case $\xi_0\in\mathscr{H}_3$. We cannot just take the limit as $e\to 0$ for the vector appearing on the right-hand side of $\langle\,,\rangle$ because the vector to which the last operator T would be applied would not necessarily be in $\mathscr{H}_0$ since $\Delta_eB(E)\to iM^2B(E)$, and we may only have $\xi_0\in\mathscr{H}_3$. But (5.6) can be rewritten as

$$\begin{aligned}&2\langle(\Delta_eB(E))K(z,E)TiM^2B(E+e)K(z,E+e)\xi_0,\\ &TK(z,E+e)^tM^2B(E+e)K(z',E+e)\xi_0\rangle.\end{aligned} \tag{5.7}$$

The rearrangement is legitimate since all vectors are in the right spaces. We can now take the limit as $e\to 0$ and obtain

$$2\langle iMB(E)K(\xi,E)TiM^2B(E)K(z,E)\xi_0,MTK(E,z)^tM^2B(E)K(z',E)\xi_0\rangle.$$

The same procedure can be applied to all terms appearing in $\Delta_e\,(d^k/dE^k)\,G(E+i0)$, $k\leqq n-2$, to give existence and continuity of $(d^{k+1}/dE^{k+1})\,G(E+i0)$. This proves Theorem 1.4. ■

6. Proof of Theorem 5.1

The proof will proceed by induction on n.

If $n=0$, there is nothing to prove since RB is a bounded operator from $\mathscr{H}_0$ to $\mathscr{H}_0$ (notice that the theorem makes sense for $n=0$, the hypothesis being simply that $\beta(r)$ is a bounded measurable function).

So let us assume the theorem is true for $n-1$, $n \geqq 1$; we will prove the theorem is then true for n.

We are going to use repeatedly the Calderon-Lions interpolation theorem [25, 26]. We will use the notation V_t, $0 \leqq t \leqq 1$, for the interpolating spaces between V_0 and V_1. We will write $V_t^{(1)} = V_t$, $V_t^{(m)}$ = the t^{th} interpolating space between $V_t^{(m-1)}$ and V_1. In what follows S: $V \to W$ or $V \overset{S}{\to} W$ mean that S is a bounded operator from V to W. For all spaces V_0 and V_1 between which we will interpolate we will have I: $V_1 \to V_0$. We start by introducing the following spaces:

$$X_0 = Y_0 = Z_0 = \mathscr{H}_0, \quad Z_1 = \mathscr{H}_n^0,$$

and

$$X_1 = \{f:[0,\infty) \to \mathbb{C} \text{ measurable; } \|(1+r^2)^{n/2} r^{-1/2} f(r^2)\|_2 < \infty\},$$

$Y_1 = \{f:[0,\infty) \to \mathbb{C}$ $(n-1)$-times differentiable on $(0,\infty)$ with $f^{(n-1)}$ absolutely continuous; $\sum_{k=0}^{n} \|r^{k-1/2} f^{(k)}(r^2)\|_2^2 < \infty\}$.

We can identify the interpolating spaces X_t [26]:

$$X_t = \{f:[0,\infty) \to \mathbb{C} \text{ measurable; } \|(1+r^2)^{nt/2} r^{-1/2} f(r^2)\|_2 < \infty\}.$$

From (3.5) we have

$$X_0 \overset{R}{\to} Y_0 \overset{R}{\to} X_0, \quad X_1 \overset{R}{\to} Y_1 \overset{R}{\to} X_1,$$

so we conclude that $X_t \overset{R}{\to} Y_t \overset{R}{\to} X_t$ for all $t \in [0,1]$. Recall $R^2 = I$.

Let us write $\sigma = \gamma/n$ and notice that $X_0 \overset{R}{\to} X_0 \overset{B}{\to} X_\sigma \overset{R}{\to} Y_\sigma$.

We now interpolate between the Y's and the Z's. Let $S(\zeta) = e^{\zeta^2} B(1+r^2)^{(\sigma-\zeta)n/2}$ for $\operatorname{Re}\zeta \in [0,1]$. Then $S(0)$: $Y_0 \to Z_0$ and $S(1) = Y_1 \to Z_1$ by the hypothesis on β. It is easy to see that $S(\zeta)$ satisfies the hypothesis of Theorem IX. 20 in [25], so we can conclude that $S(t)$: $Y_t \to Z_t$ for $t \in [0,1]$. Taking $t=\sigma$, we get B: $Y_\sigma \to Z_\sigma$.

We have so far shown that $(RB)^2: X_0 \to Z_\sigma$. Since $(RB)^2: Z_1 \to Z_1$, we have that $(RB)^4: X_0 \to Z_\sigma^{(2)}$. Reiterating the argument, we get that $(RB)^{2m}$: $X_0 \to Z_\sigma^{(m)}$.

Now let $W_0 = \mathscr{H}_{n-1}^0$, $W_1 = \mathscr{H}_n^0$. By the induction hypothesis there exists k_1 such that (RB): $Z_0 \to W_0$ and, of course, $(RB)^{k_1}$: $Z_n \to W_n$. It follows $(RB)^{k_1+2m}$: $X_0 \to W_\sigma^{(m)}$.

Now let D be the operator defined by $(Df)(r^2) = f'(r^2)$, and let

$$V_t = \{f:[0,\infty) \to \mathbb{C} \text{ measurable; } \|r^{n-1+t-1/2} f(r^2)\|_2 < \infty\},$$

where $0 \leqq t \leqq 1$. If $k = 0, 1, \ldots, n-1$, $D^k: W_0 \to V_0$, $D^k: W_1 \to V_1$, so it follows that D^k: $W_\sigma^{(m)} \to V_\sigma^{(m)}$.

But we can identify $V_\sigma^{(m)}$ [26]: $V_\sigma^{(m)} = \{f\colon [0,\infty) \to \mathbb{C}$ measurable; $\|r^{n-(1-\sigma)^m - 1/2} f(r^2)\|_2 < \infty\}$.

So we choose m such that $(1-\sigma)^m < \gamma$. If $f \in W_\sigma^{(m)}$, $f^{(k)} \in V_\sigma^{(m)}$ for $k = 0, 1, \ldots, n-1$. Thus $(Bf)^{(k)} \in V_1$ for $k = 0, 1, \ldots, n-1$. It follows from (3.5) that $(RBf)^{(k)} \in V_1$ for $k = 1, \ldots, n$.

Now let $f \in X_0$. Then $(RB)^{k_1 + 2m} f \in W_\sigma^{(m)}$, so

$$((RB)^{k_1 + 2m + 1} f)^{(k)} \in V^1 \quad \text{for} \quad k = 1, \ldots, n. \tag{6.1}$$

On the other hand, $RB: X_0 \to X_0$, so

$$B(RB)^{k_1 + 2m + 1} f \in V_1. \tag{6.2}$$

From (6.1) and (6.2) we conclude that $B(RB)^{k_1 + 2m + 1} f \in \mathscr{H}_n$, and hence is in $\mathscr{H}_n^0$, so $(RB)^{k_1 + 2m + 2} f \in \mathscr{H}_n^0$ for all $f \in X_0 = \mathscr{H}_0$.

If $\beta(r; z) = \beta(r) e^{izr}$, $B(z)$ the corresponding multiplication operator, it is easy to check in the proof that we get the desired uniformity in z for the norm of $(RB(z))^k$. ■

7. Proof of Corollary 1.2

Corollary 1.2 follows from

Lemma 7.1. *Let $(1 + |t|^\alpha) h(t)$ be bounded for some $\alpha > 0$ and let $\int |v|^{n+\varepsilon} d\nu(v) < \infty$ for some $\varepsilon > 0$. Then h is n-times differentiable and there exists $\delta > 0$ such that $(1 + |t|)^\delta h^{(j)}(t)$ is bounded for $j = 0, 1, \ldots, n$.*

Proof. We will show that there exist $\delta > 0$ such that $(1 + |t|)^\delta h^{(n)}(t)$ is bounded. Let $\chi(v)$ be a C^∞ function such that $\chi(v) = v^n$ for $|v| \leqq 1$, $\chi(v) = 0$ for $|v| \geqq 2$, and $|\chi(v)| \leqq 2$ for all v. For $R > 0$ let $\chi_R(v) = R^n \chi(R^{-1} v)$.

For any $k \geqq 0$ there exists $C_k < \infty$ such that if $\hat{\chi}(t) = \int e^{-itv} \chi(v) dv$, $|\hat{\chi}(t)| \leqq C_k (1 + |t|^k)^{-1}$. It follows that

$$|\hat{\chi}_R(t)| \leqq C_k R^{n+1} (1 + R^k |t|^k)^{-1}. \tag{7.2}$$

Since $h^{(n)}(t) = (-i)^n \int v^n e^{-itv} d\nu(v)$, we have that for $R \geqq 2$

$$\begin{aligned} |h^{(n)}(t) - (-i)^n \int \chi_R(v) e^{-itv} d\nu(v)| &= \Big| \int_{|v| \geqq R} (v^n - \chi_R(v)) e^{-itv} d\nu(v) \Big| \\ &\leqq 2 \int_{|v| \geqq R} |v|^n d\nu(v) \leqq 2R^{-\varepsilon} \int |v|^{n+\varepsilon} d\nu(v). \end{aligned}$$

We have

$$\begin{aligned} \int \chi_R(v) e^{-itv} d\nu(v) &= (2\pi)^{-1} (\hat{\chi}_R * h)(t) = (2\pi)^{-1} \int_{|s| \leqq t/2} \hat{\chi}_R(s) h(t-s) ds \\ &\quad + (2\pi)^{-1} \int_{|s| > t/2} \hat{\chi}_R(s) h(t-s) ds. \end{aligned} \tag{7.3}$$

We now use (7.1) to estimate each term; we have

$$\left| \int_{|s| \leqq t/2} \hat{\chi}_R(s) h(t-s) ds \right| \leqq D_k R^n (1 + |t|)^{-\alpha}, \tag{7.4}$$

and

$$\left| \int_{|s| > t/2} \hat{\chi}_R(s) h(t-s) ds \right| \leqq C_k R^{n+1} \int_{|s| > t/2} (1 + R^k |s|^k)^{-1} ds \leqq D'_k R^n (R|t|)^{1-k}, \tag{7.5}$$

where D_k and D'_k are finite if we take $k > 1$.

From (7.2), (7.3), (7.4) and (7.5), we get

$$|h^{(n)}(t)| \leqq K_1 R^{-\varepsilon} + K_2 R^n ((1 + |t|)^{-\alpha} + (R|t|)^{1-k}), \tag{7.6}$$

with K_1 and K_2 finite constants depending on $k > 1$. Fix k. Then for fixed t pick $R = R(t) = |t|^{-\sigma}$. It is clear from (7.6) that we can pick an appropriate $\sigma > 0$ to get the desired result. ∎

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