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The $U(1)$ Higgs Model

II. The Infinite Volume Limit

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Abstract. We construct the infinite volume limit of the $U(1)$ Higgs model in two and three dimensions, and verify some of the Osterwalder-Schrader axioms. The proof uses a combination of renormalization group techniques and correlation inequalities.

1. Introduction

In a previous paper [K 1], we considered the lattice $U(1)$ Higgs model in a finite volume with periodic boundary conditions in dimensions $d=2,3$. We proved the convergence of this model to a finite limit as the lattice spacing approaches zero. The method of proof was based on the renormalization transformations introduced in [Ba 1–4]. In this paper, we continue our analysis of the model by constructing the infinite volume theory and verifying some of the Osterwalder-Schrader axioms.

This model was previously constructed in two dimensions in the papers [BFS 1–3]. In particular, a collection of useful correlation inequalities was established in [BFS 1]. We use these inequalities to define the infinite volume limit of the continuum theory derived in [K 1]. The limit is shown to satisfy reflection positivity, translation invariance, and a suitable analyticity property.

Since we approach the continuum limit through a sequence of models defined on lattices with a fixed orientation, it is not obvious that the continuum theory satisfies rotation invariance. We prove a result which, when combined with a construction of the infinite volume limit for periodic boundary conditions, implies the rotation invariance of that limit. (Although the correlation inequalities of [BSF 1] fail for periodic boundary conditions, other methods such as the cluster expansion could be used.) Specifically, we prove that two lattices whose orientations differ by a special angle θ_0 (incommensurate with 2π) give the same continuum theory in a finite volume.

The paper is organized as follows. Section 2 contains a statement of the results proved in this paper. In Sect. 3, we construct the infinite volume limit and verify the properties mentioned before. Section 4 contains the results on rotation invariance. Finally, in Sect. 5 we extend the ultra-violet stability results of [Ba 1–4] to include the different boundary conditions used in this paper.

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We recommend that the reader study [K 1] before reading this paper.

2. Statement of Results

The lattice $U(1)$ Higgs model was defined in [K 1]. In order to apply the results obtained in [BFS 1–3] concerning correlation inequalities, we restrict attention to the case of a two-component scalar field, though the ultra-violet estimates do not depend on this restriction. For convenience, the field is written

$$\phi(x) = r(x) e^{i\theta(x)}, \tag{2.1}$$

where $r \in [0, \infty)$ and $\theta \in [-\pi, \pi)$. The model is defined on a finite lattice $\Omega \subset \varepsilon \mathbb{Z}^d$, where the scalar field is a map $\phi : \Omega \to \mathbb{C}$. The vector field is considered either as a map $A : \Omega \to \mathbb{R}^d$, or as a map $A : \Omega^* \to \mathbb{R}$, where Ω^* is the set of oriented bonds on Ω. These meanings are identified by

$$A_\mu(x) = A_{\langle x, x+\varepsilon e_\mu \rangle}, \tag{2.2}$$

where e_μ is the unit lattice vector in the μ^{th} direction. The covariant derivative of the scalar field is

$$(D_\mu^\varepsilon \phi)(x) = \varepsilon^{-1}(\exp[ie\varepsilon A_\mu(x)]\phi(x+\varepsilon e_\mu) - \phi(x)), \tag{2.3}$$

where e is the electric charge. Under a local gauge transformation χ, these fields transform as

$$\begin{aligned} \phi(x) &\to \exp[ie\chi(x)]\phi(x), \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu \chi(x), \\ (D_\mu^\varepsilon \phi)(x) &\to \exp[ie\chi(x)](D_\mu^\varepsilon \phi)(x). \end{aligned} \tag{2.4}$$

Denoting by ∂^ε the usual lattice derivative on Ω, our lattice action is

$$\begin{aligned} &S^\varepsilon(\Omega, A, \phi) \\ &= \frac{1}{2} \sum_{\mu=1}^{d} \sum_{b \in \Omega^*} \varepsilon^d |\partial^\varepsilon A_\mu(b)|^2 + \frac{1}{2} \mu_0^2 \sum_{\mu=1}^{d} \sum_{x \in \Omega} \varepsilon^d |A_\mu(x)|^2 \\ &\quad + \frac{1}{2} \sum_{\langle x, y \rangle \in \Omega^*} \varepsilon^{d-2} [r(x)^2 + r(y)^2 - 2r(x)r(y)\cos(\theta(x) - e\varepsilon A_{\langle x,y \rangle} - \theta(y))] \\ &\quad + \sum_{x \in \Omega} \varepsilon^d \left[\frac{1}{2} m^2 r(x)^2 + \lambda r(x)^4 + \frac{1}{2} \delta m^2(x) r(x)^2 \right] + E_0 + E_1. \end{aligned} \tag{2.5}$$

The quadratic part of the action for the scalar field comes from expanding the gauge-invariant term

$$\frac{1}{2} \sum_{\mu=1}^{d} \sum_{x \in \Omega} \varepsilon^d |(D_\mu^\varepsilon \phi)(x)|^2.$$

We have introduced a vector field mass μ_0^2 for technical reasons connected with ultra-violet stability. The constant E_0 is a normalization for the Gaussian parts of the action (2.5):

$$\exp(E_0)=\int(dA)(d\phi)\exp\left[-\frac{1}{2}\sum_{\mu=1}^{d}\sum_{b\in\Omega^*}\varepsilon^d|\partial^\varepsilon A_\mu(b)|^2-\frac{1}{2}\mu_0^2\sum_{\mu=1}^{d}\sum_{x\in\Omega}\varepsilon^d|A_\mu(x)|^2\right.$$
$$\left.-\frac{1}{2}\sum_{b\in\Omega^*}\varepsilon^d|\partial^\varepsilon\phi(b)|^2-\frac{1}{2}m^2\sum_{x\in\Omega}\varepsilon^d|\phi(x)|^2\right]. \tag{2.6}$$

The mass and vacuum-energy counterterms δm^2 and E_1 are introduced to cancel the usual divergences of perturbation theory, and they are defined in [K 1]. For definiteness, we take Ω to be a rectangular parallelepiped with equal sides. Define for integer J,

$$C_J=\{x\in\mathbb{R}^d:\ -MJ\leqq x_\mu<MJ,\ \mu=1,\dots,d\}, \tag{2.7}$$

where M is the large integer used in Sect. 2.3 of [K 1]. As before, we define lattice spacings $\{\varepsilon_K\}$, $K=0,\dots,\infty$, by

$$\varepsilon_K=L^{-K}, \tag{2.8}$$

where L is the small integer which determines the size of the blocks used in the renormalization transformation.

In the action (2.5), we will use Neumann boundary conditions (N b.c.) for the vector field A_μ on ∂C_J. We will use Dirichlet boundary conditions (D b.c.) for the scalar field ϕ on ∂C_J (this choice of boundary conditions for ϕ is gauge-invariant). With these choices, the partition function in C_J is written

$$Z^\varepsilon_{N,D}(C_J)=\int(dA)(d\phi)\exp[-S^\varepsilon(C_J,A,\phi)], \tag{2.9}$$

and the expectation of an operator $\mathcal{O}$ is defined by

$$\langle\mathcal{O}\rangle^{\varepsilon,C_J}_{N,D}=Z^\varepsilon_{N,D}(C_J)^{-1}\int(dA)(d\phi)\mathcal{O}(A,\phi)\exp[-S^\varepsilon(C_J,A,\phi)]. \tag{2.10}$$

We are primarily interested in gauge invariant operators $\mathcal{O}$. Denote by Ω^{**} the set of plaquettes on Ω, and define the field strength tensor $F:\Omega^{**}\to\mathbb{R}$ by

$$F(p)=\sum_{b\in\partial p}\varepsilon^{-1}A_b. \tag{2.11}$$

Then for a function $g:\Omega^{**}\to\mathbb{R}$, we have

$$F(g)=\sum_{p\in\Omega^{**}}\varepsilon^dF(p)g(p). \tag{2.12}$$

For a function $h:\Omega\to\mathbb{R}$, the gauge-invariant operator for the scalar field is

$$:\phi^2:h=\sum_{x\in\Omega}\varepsilon^dh(x)[r(x)^2-2(-\Delta^{\varepsilon,\Omega}+m^2)^{-1}(x,x)], \tag{2.13}$$

where $-\Delta^{\varepsilon,\Omega}$ is the Laplacian on Ω with D b.c. The general operator we consider is

$$\mathcal{O}(zg,wh)=\exp[zF(g)+w:\phi^2:(h)], \tag{2.14}$$

where $g,h\in C_0^\infty(\mathbb{R}^d)$ and $z,w\in\mathbb{C}$.

It will be understood below that the results stated hold for dimensions $d=2,3$. The first result extends Theorem 2.1 of [K 1] to include the boundary conditions used in this paper, and the second result gives the existence of the infinite volume limit.

Theorem 2.1. *Let C_J be defined by* (2.7), *and let g, h be supported in C_J. Then for all $z, w \in \mathbb{C}$,*

$$\exists \lim_{K \to \infty} \langle \mathcal{O}(zg, wh) \rangle_{N,D}^{\varepsilon_K, C_J} = \langle \mathcal{O}(zg, wh) \rangle_{N,D}^{C_J}. \tag{2.15}$$

Theorem 2.2. *Let g, h be C^∞ functions with compact support, and $z, w \in \mathbb{C}$. Then*

$$\exists \lim_{J \to \infty} \langle \mathcal{O}(zg, wh) \rangle_{N,D}^{C_J} = \langle \mathcal{O}(zg, wh) \rangle. \tag{2.16}$$

We also establish some properties of the limit (2.16). We denote by Tf the function obtained from f by some translation T in the Euclidean group on $\mathbb{R}^d$. Also, if $\operatorname{supp} f \subset \{x \in \mathbb{R}^d\colon x_\nu > 0\}$, we denote by $\theta_\nu f$ the function obtained from f by reflection in $x_\nu = 0$.

Theorem 2.3. *The infinite volume limit* (2.16) *satisfies the following properties:*

(i)
$$\langle \mathcal{O}(Tg, Th) \rangle = \langle \mathcal{O}(g, h) \rangle; \tag{2.17}$$

(ii) *if* $\operatorname{supp} g \cup \operatorname{supp} h \subset \{x \in \mathbb{R}^d\colon x_\nu > 0\}$, *some ν, then*

$$\langle \bar{\mathcal{O}}(\theta_\nu g, \theta_\nu h) \mathcal{O}(g, h) \rangle \geqq 0, \tag{2.18}$$

where the bar denotes complex conjugation;

(iii) *if the $2N$ functions $\{g_i\}$, $\{h_j\}$, $i,j = 1, \ldots, N$, are in $C_0^\infty(\mathbb{R}^d)$, and $\{z_i\}$, $\{w_j\}$ are in $\mathbb{C}$, the function*

$$\left\langle \mathcal{O}\left(\sum_{i=1}^N z_i g_i, \sum_{j=1}^N w_j h_j \right) \right\rangle \tag{2.19}$$

is entire on $\mathbb{C}^N \times \mathbb{C}^N$.

The properties we have listed in Theorem 2.3 are a subset of the Osterwalder-Schrader axioms presented in [GJ 1]. To complete the list, we would need to prove ergodicity, rotation invariance and regularity. Ergodicity corresponds to uniqueness of the vacuum, and has been proved for other models using cluster expansion techniques. To establish regularity we would need to keep more careful track of the source functions g, h when proving ultra-violet stability using the renormalization transformation method of [Ba 1–2].

In order to prove rotation invariance, it is sufficient to show that the limit $\langle \mathcal{O}(zg, wh) \rangle$ is the same when constructed from two sequences of lattices whose orientations differ by an angle θ_0 incommensurate with 2π. The two lattices we use are illustrated in Fig. 1. One lattice has spacing ε, the other has spacing 5ε. The angle between the lattices is $\theta_0 = \tan^{-1}(3/4)$. For $d > 2$, the lattices have the same orientation in the other $d-2$ coordinate directions not shown in Fig. 1.

The methods we use to construct the continuum limit require quite explicit expressions for lattice operators with periodic, Neumann and Dirichlet boundary conditions. These hold only when the boundary is a lattice hyperplane orthogonal to a lattice coordinate direction. Clearly the two lattices in Fig. 1 never share such a boundary. Our solution of this problem is to use periodic boundary conditions. We define a torus D_J by (2.7) with M replaced by $25M$. For $L = 5$, we can "fit" D_J by

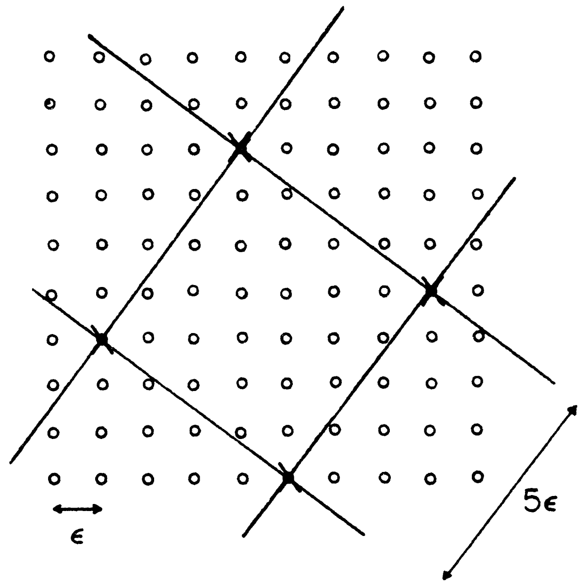


Fig. 1. Two lattices with different orientations

lattices with the same orientation as the sides of D_J and with lattice spacings $\varepsilon_K = L^{-K}$. The expectation defined by this lattice is written

$$\langle \mathcal{O} \rangle_p^{\varepsilon_K, D_J} . \tag{2.20}$$

From Fig. 1, and since the length of each side of D_J is a multiple of 25, we see that D_J is also fitted by the rotated lattice. Again taking $L=5$, and $\varepsilon_K = L^{-K}$, the expectation defined by this rotated lattice is written

$$\langle \mathcal{O} \rangle_p^{R(\varepsilon_K), D_J} . \tag{2.21}$$

We can now state our result concerning rotation invariance.

Theorem 2.4. *For g, h supported in D_J, and $z, w \in \mathbb{C}$,*

$$\lim_{K \to \infty} \langle \mathcal{O}(zg, wh) \rangle_p^{\varepsilon_K, D_J} = \lim_{K \to \infty} \langle \mathcal{O}(zg, wh) \rangle_p^{R(\varepsilon_K), D_J} . \tag{2.22}$$

Also, $\theta_0 / 2\pi$ is irrational. (2.23)

Because the expectations in (2.22) are defined with P b.c., we cannot use correlation inequalities to take the infinite volume limit. However, we can expect that a cluster expansion would give the following result:

$$\langle \mathcal{O}(zg, wh) \rangle = \lim_{\Lambda \nearrow \mathbb{R}^d} \lim_{K \to \infty} \langle \mathcal{O}(zg, wh) \rangle_p^{\varepsilon_K, \Lambda} , \tag{2.24}$$

where the limit $\Lambda \nearrow \mathbb{R}^d$ is taken through an arbitrary sequence of rectangular parallelepipeds with P b.c. Therefore, if we denote by $R_{\mu\nu} f$ the function obtained from f by rotation through an angle θ_0 in the $\mu\nu$-coordinate plane, we have

$$\begin{aligned}\langle \mathcal{O}(zR_{\mu\nu}^{-1}g, wR_{\mu\nu}^{-1}h)\rangle &= \lim_{\Lambda \nearrow \mathbb{R}^d} \lim_{K\to\infty} \langle \mathcal{O}(zR_{\mu\nu}^{-1}g, wR_{\mu\nu}^{-1}h)\rangle_p^{\varepsilon_K,\Lambda} \\ &= \lim_{\Lambda \nearrow \mathbb{R}^d} \lim_{K\to\infty} \langle \mathcal{O}(zg, wh)\rangle_p^{R_{\mu\nu}(\varepsilon_K), R_{\mu\nu}(\Lambda)} \\ &= \lim_{\Lambda \nearrow \mathbb{R}^d} \lim_{K\to\infty} \langle \mathcal{O}(zg, wh)\rangle_p^{\varepsilon_K, R_{\mu\nu}(\Lambda)}, \end{aligned} \tag{2.25}$$

where we used Theorem 2.4. Since (2.24) holds for an arbitrary sequence of regions Λ, the limit (2.25) equals (2.24). By iterating the rotation $R_{\mu\nu}^{-1}$ and using the fact that $\theta_0/2\pi$ is irrational, we generate all rotations in the $\mu\nu$-plane. Repeating the procedure for each μ, ν gives invariance under the full rotation group $SO(d)$.

3. The Infinite Volume Limit

3.1. The Continuum Limit

The renormalization transformation method was introduced in Sect. 2.2 of [K 1]. The space of field configurations is divided into cells, each corresponding to fluctuations of the field on a certain length scale. Beginning with the shortest length scales, the integral over fields in the partition function is successively analyzed in each cell. This method of phase space localization was first introduced in [GJ 2]. It was developed for the $U(1)$ Higgs model in [Ba 1–4], using a block spin transformation to separate fields into average and fluctuation parts.

In order to apply the renormalization transformation to the case when the scalar fields ϕ has D b.c., we modify the averaging operators defined in [K 1]. Given a block $B(y)$ containing L^d sites, we consider a bounded function $f: B(y) \to \mathbb{R}$ such that

$$\sum_{x \in B(y)} L^{-d} f(x) = 1 . \tag{3.1}$$

For each $y \in L\varepsilon\mathbb{Z}^d$, we introduce f on the block $B(y)$, giving a function f on $\varepsilon\mathbb{Z}^d$. We now define the averaging operator for the scalar field by

$$(Q_k(A)\phi)(y) = L^{-kd} \sum_{x \in B^k(y)} f(x) \exp[ieA(\Gamma_{y,x}^{(k)})]\phi(x) , \tag{3.2}$$

where $y \in L^k\varepsilon\mathbb{Z}^d$ and $A(\Gamma_{y,z}^{(k)})$ is defined in [K 1]. The operators and notation used in [K 1] will be taken over unchanged, with the understanding that $Q_k(A)$ is replaced by (3.2). We leave unchanged the averaging operator for the vector field. The results stated in Sect. 2.2 of [K 1] still hold, in particular the representation (2.17) of the propagator.

Using the method of multiple reflections, we can write an operator with P, N, or D b.c. on a lattice hyperplane as a sum of operators with free boundary conditions. Suppose $\Omega \subset \varepsilon\mathbb{Z}^d$ is a rectangular parallelepiped given by

$$\Omega = \{x \in \varepsilon\mathbb{Z}^d : 0 < x_\mu \leqq L_\mu, \ \mu = 1, \dots, d\} , \tag{3.3}$$

and the dimensions L_μ satisfy

$$(L^k\varepsilon)^{-1} L_\mu \in \mathbb{N}, \quad \text{each} \quad \mu = 1, \dots, d . \tag{3.4}$$

Recall that the renormalization transformation produces a sequence of covariances defined by

$$G_k^\varepsilon(\Omega) = (-\Delta^{\varepsilon,\Omega} + m^2 + a_k(L^k\varepsilon)^{-2}P_k(0))^{-1}, \tag{3.5}$$

where $P_k(0) = Q_k^*(0)Q_k(0)$ and $-\Delta^{\varepsilon,\Omega}$ is the Laplacian on Ω with some boundary conditions on $\partial\Omega$. The operator G_k^ε with free boundary conditions on $\varepsilon\mathbb{Z}^d$ is given by (3.5) with $-\Delta^{\varepsilon,\Omega}$ replaced by $-\Delta^\varepsilon$. We have an explicit Fourier representation for G_k^ε and so we want to write covariances with other boundary conditions in terms of G_k^ε. For P b.c. on Ω, we take $f=1$ in (3.1) and then

$$G_k^\varepsilon(\Omega)(x,y) = \sum_{n\in\mathbb{Z}^d} G_k^\varepsilon\left(x, y + \sum_{\mu=1}^d n_\mu L_\mu e_\mu\right). \tag{3.6}$$

For N b.c. we also take $f=1$ and then

$$G_k^\varepsilon(\Omega)(x,y) = \sum_{n\in\mathbb{Z}^d} \sum_{m\in\{0,1\}^d} G_k^\varepsilon\left(x, y + \sum_{\mu=1}^d 2n_\mu L_\mu e_\mu - \sum_{\nu=1}^d m_\nu(2y_\nu - \varepsilon)e_\nu\right). \tag{3.7}$$

Finally, for D b.c. we use a different function f. For $x\in B(y)$, define

$$f(x) = \begin{cases} \left(\dfrac{L}{L-1}\right)^d & \text{if} \quad x_\mu - y_\mu \leqq (L-2)\varepsilon \quad \text{each} \quad \mu = 1, \ldots, d, \\ 0 & \text{otherwise}. \end{cases} \tag{3.8}$$

We take $-\Delta^{\varepsilon,\Omega}$ to have D b.c. on $\{x\in\varepsilon\mathbb{Z}^d : x_\mu = 0 \text{ or } x_\mu = L_\mu, \text{ some } \mu = 1, \ldots, d\}$. Then we can write

$$G_k^\varepsilon(\Omega)(x,y) = \sum_{n\in\mathbb{Z}^d} \sum_{m\in\{0,1\}^d} (-1)^{\Sigma_\nu m_\nu} G_k^\varepsilon\left(x, y + \sum_{\mu=1}^d 2n_\mu L_\mu e_\mu - \sum_{\nu=1}^d 2m_\nu y_\nu e_\nu\right). \tag{3.9}$$

If we had taken $f=1$, the representation (3.9) would not hold.

We now wish to prove Theorem 2.1, giving the continuum limit in a region C_J. In Sect. 4, Theorems 3.1 and 3.3 of [K 1] are extended to include the case when the vector field has N b.c. and the scalar field has D b.c. Therefore, the existence of the continuum limit in C_J will follow from the convergence of the effective action, which was proved for P b.c. in [K 1]. Recall that Theorem 3.1 of [K 1] states that the generating functional $Z_{N,D}^\varepsilon(C_J)\langle\mathcal{O}(zg, wh)\rangle_{N,D}^{\varepsilon,C_J}$ is well approximated by the integral of an effective action $S^{(k),L^k\varepsilon}$ on the lattice with spacing $L^k\varepsilon$ in C_J, when z, w are real. By proving convergence of $S^{(k),L^k\varepsilon}$ as $\varepsilon\to 0$, we can deduce convergence of the generating functional, and hence of $\langle\mathcal{O}(zg, wh)\rangle_{N,D}^{\varepsilon,C_J}$, for z, w real. Furthermore,

$$|\langle\mathcal{O}(zg, wh)\rangle_{N,D}^{\varepsilon,C_J}| \leqq \langle\mathcal{O}(\mathrm{Re}\, zg, \mathrm{Re}\, wh)\rangle_{N,D}^{\varepsilon,C_J}. \tag{3.10}$$

Therefore, using Vitali's theorem, it follows that $\langle\mathcal{O}(zg, wh)\rangle_{N,D}^{\varepsilon,C_J}$ converges to an entire function in $\mathbb{C}\times\mathbb{C}$ as $\varepsilon\to 0$.

In the remainder of this section we will prove convergence of the effective action $S^{(k),L^k\varepsilon}$ in C_J for z, w real. For convenience, we adopt the following notation:

$$T_{L^k\varepsilon_K}^{(k)} = C_J \cap L^k\varepsilon_K\mathbb{Z}^d, \qquad k = 0, 1, \ldots, K. \tag{3.11}$$

When $k=0$, we write (3.11) as T_ε. Under a re-scaling $\varepsilon\to\eta$, (3.11) becomes $T^{(k)}_{L^k\eta}$. We will henceforth reserve η to mean L^{-k}. Recall that for an integer $n\geqq 1$, $\eta'=L^{-n}\eta=L^{-(k+n)}$.

The effective action $S^{(k),L^k\varepsilon}$ is defined on the lattice $T^{(k)}_{L^k\varepsilon}$. For convenience, we re-scale $L^k\varepsilon\to 1$ and then we have the following formula for $S^{(k),1}$ (see [K 1, Sect. 3.6]):

$$\begin{aligned} S^{(k),1}(T^{(k)},A_k,\phi_k,g,h)=&\tfrac{1}{2}\langle A_k,\Delta^{(k)}A_k\rangle+\ln N_k\\ &+\tfrac{1}{2}\langle\phi_k,\Delta^{(k)}(A^{(k)})\phi_k\rangle+\ln N_k(0)\\ &-\ln\left[\frac{Z_k(A^{(k)})}{Z_k(0)}\right]+P^{(k),1}(T^{(k)},A_k,\phi_k,g,h). \end{aligned} \tag{3.12}$$

The operators $\Delta^{(k)},\Delta^{(k)}(A^{(k)})$ are effective Laplacians for the fields A_k and ϕ_k. The interaction term $P^{(k),1}$ is represented by a sum of connected graphs with vertices on the lattice T_η. These graphs were described in detail in [K 1] and in [Ba 3]. The only way that the boundary conditions enter this description is through the propagators $G^\eta_k(A^{(k)})$ and G^η_k connecting the vertices. In our case, $G^\eta_k(A^{(k)})$ has D b.c. and G^η_k has N b.c. on ∂T_η. We shall specify below how the proof of convergence in Sect. 3.4 of [K 1] must be modified to take account of this.

When we localize the vertices of a graph in a cube $\square$, it may happen that $\partial\square\cap\partial T^{(k)}$ is non-empty. In this case the vertices will see a "sharp boundary;" this is the main difference between periodic boundary conditions and those used here. The following remarks refer to Sect. 3.4 of [K 1].

Having localized the vertices of a graph H in a block $\square$, we obtained the expression $E^{(k)}(H,A^{(k)};\{y_i\},\{z_q\})$. We now replace each propagator $G_k(A^{(k)})$ by the difference $G_k(\square',A^{(k)})-\delta G_k(\square',A^{(k)})$, where we take $G_k(\square',A^{(k)})$ to have D b.c. on $\partial\square'$. In the expansions (3.47)–(3.53), we take $B=0$ and $A'=\tilde{A}^{(k)}$. We omit the replacement (3.54), so every scalar field propagator in our graph is $G_k(\square',0)$. We shall prove Propositions 3.8 and 3.9 for $G_k(\square',0)$ and G_k in Appendix A. There is a minor technical point connected with the operator $Q_{k+n}(0)$. After all propagators and sources have been replaced in a graph in $E^{(k+n)}(H)$, we have (for some function K)

$$\begin{aligned} (\eta')^d\sum_{z\in B^k(y)}\sum_{x\in B^n(z)}f(x)K(z)&=\eta^d\sum_{z\in B^k(y)}K(z)\\ &=(Q_kK)(y)+\eta^d\sum_{z\in B^{k-1}(y)}\sum_{x\in B(z)}(1-f(x))|x-z|^\alpha\left[\frac{K(x)-K(z)}{|x-z|^\alpha}\right]. \end{aligned} \tag{3.13}$$

The second term in (3.13) gives a factor $L^{-\alpha k}$, since $|x-z|\leqq\sqrt{d}L^{-k+1}$ for all x,z.

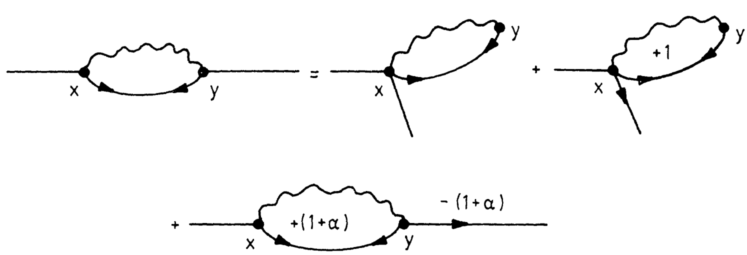
There are other modifications required when we examine subgraphs with non-positive degree. First, note that it is sufficient to assume that $\partial\square'\cap\partial T_\eta$ is non-empty; otherwise, the vertices of the graph never see a sharp boundary, and the methods of [K 1] go straight through.

We shall define the mass counterterm δm^2 using free boundary conditions; that is the propagators used to define the graphical expansion of δm^2 (see Sect. 3.3 of [K 1]) have free boundary conditions on ∂T_η. This is necessary in order to take the infinite volume limit (see the next section). At first glance, this appears to cause problems since the divergent diagrams produced by perturbation theory have propagators with N and D b.c. on ∂T_η. However, because we use D b.c. for the scalar field, the propagators entering a mass vertex go to zero near the boundary, and this is sufficient to keep the graph finite, once the most divergent part has been removed.

As a specific example, consider the following linearly divergent graph for the self-energy of the scalar field:

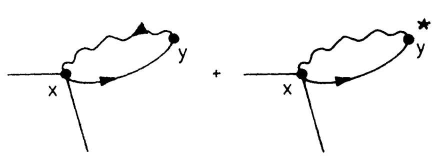
(3.14)

As explained in Sect. 3.5 of [K 1], this graph may be written as



(3.15)

The third graph on the right-hand side of (3.15) is convergent. In the first graph on the right-hand side, we integrate by parts at y to give



(3.16)

where the asterisk denotes differentiation of the localization function $\chi(y)$. Using the representations (3.7) and (3.9) we now write (3.16) and the second graph on the right-hand side of (3.15) as a sum of graphs with free propagators. The graph in δm^2 corresponding to (3.14) has an expansion similar to (3.16), and we may cancel some of the terms produced. The remaining graphs have one or more propagators connecting x to some reflection of y in ∂T_η, or they have y summed over $\eta\mathbb{Z}^d \backslash T_\eta$.

Let H represent one of these remaining graphs. Suppose that for some line l in H, there is a propagator connecting x to a reflection of y in ∂T_η. Then we see that the graph H with l omitted has positive degree [(3.16) ensured this], and the degree of H is greater than or equal to -1. We can analyse H in the manner described in Sect. 3.4 of [K 1]. The internal propagators are decomposed with the relation $G_k^\eta = \sum_{j=1}^{k-1} G_{(j)}^\eta$, and we can restrict attention to one particular ordering of the internal lines. After inserting the appropriate bounds, we get the following factor for the line l:

$$\exp[-\delta_0(L^j\eta)^{-1}\{|x-y|+\mathrm{dist}(\{x,y\},\partial\square')\}]. \tag{3.17}$$

Following the methods of [K 1], we shrink the lines in H to x and the final sum of integers is

$$\begin{aligned}&\sum_{j=0}^{j_0}(L^j\eta)^{D(H)}\exp[-\delta(L^j\eta)^{-1}\mathrm{dist}(\{x\},\partial\square')]\\ &\quad\leqq C\exp[-\delta(L^{j_0}\eta)^{-1}\mathrm{dist}(\{x\},\partial\square')]\\ &\qquad\cdot\begin{cases}(L^{j_0}\eta)^{D(H)}, & D(H)>0,\\ |\ln[\mathrm{dist}(\{x\},\partial\square')]|, & D(H)=0,\\ |\mathrm{dist}(\{x\},\partial\square')|^{D(H)}, & D(H)<0,\end{cases}\end{aligned} \tag{3.18}$$

where j_0 is the smallest integer on an external line. Suppose now that x is connected to another vertex z by a scalar field line with integer j_1; then for all $0\leqq\alpha<1$,

$$\begin{aligned}&|\mathrm{dist}(\{x\},\partial\square')|^{-\alpha}|G^{\eta}_{(j_1)}(\square';z,x)|\\ &\quad\leqq C(L^{j_1}\eta)^{2-d-\alpha}\exp[-\delta_0(L^{j_1}\eta)^{-1}|z-x|].\end{aligned} \tag{3.19}$$

This bound follows from the representation (3.9) and Proposition 3.7 of [K 1]. Therefore, we have the additional factor $|\mathrm{dist}(\{x\},\partial\square')|^{\alpha}$ available when we sum over x, and so we get

$$\begin{aligned}&\sum_{x\in\square'}\eta^d\chi(x)|\mathrm{dist}(\{x\},\partial\square')|^{D(H)+\alpha}\\ &\qquad\cdot\exp[-\delta(L^{j_1}\eta)^{-1}\{|z-x|+\mathrm{dist}(\{x\},\partial\square')\}]\\ &\quad\leqq C(L^{j_1}\eta)^{D(H)+\alpha}\exp[-\delta(L^{j_1}\eta)^{-1}\mathrm{dist}(\{z\},\partial\square')].\end{aligned} \tag{3.20}$$

Since $D(H)\geqq -1$, we can use any positive α. This extra degree of convergence has been borrowed from a line external to H; as usual, we have transferred the divergence to a bigger graph. This is exactly how divergent diagrams are renormalized in Sect. 3.5 of [K 1], so we see that H produces no divergence. The same reasoning holds when one vertex of H is restricted to the region $\eta\mathbb{Z}^d/T_\eta$. Furthermore, we may extract a convergence factor $L^{-\gamma k}$ and so prove convergence of the graph as $k\to\infty$.

The other graphs in δm^2 can be treated similarly; in fact, the cancellation of the logarithmically divergent graphs does not require D b.c. The Ward-Takahashi identities hold for our choice of boundary conditions, since this choice is gauge-invariant. We will prove in Appendix A that the operators $\Delta^{(k)}$ and $\Delta^{(k)}(0)$ converge as $k\to\infty$. Therefore, all the results of [K 1] go through, and we deduce Theorem 2.1.

3.2. Correlation Inequalities and Monotonicity in the Volume

By using an appropriate collection of correlation inequalities, we can prove that $\langle\mathcal{O}(g,h)\rangle^{C_J}_{N,D}$ is monotonic in J, for suitable g,h. The reader is referred to [BFS 1] for a derivation of the inequalities, which are quoted below.

For two operators $\mathcal{O}$ and $\mathcal{O}'$, and $0 \leqq s \leqq 2$, define the ratio of expectations

$$\mathscr{E}^J(\mathcal{O};\mathcal{O}';s)=\frac{\langle \exp[\mathcal{O}+(s-1)\mathcal{O}']\rangle_{N,D}^{C_J}}{\langle \exp[(s-1)\mathcal{O}']\rangle_{N,D}^{C_J}}. \tag{3.21}$$

We are concerned with operators of the following form:

$$A(g)=\sum_{b\in C_J^*}\varepsilon^d A(b)g(b), \tag{3.22}$$

$$r^2(h)=\sum_{x\in C_J}\varepsilon^d r^2(x)h(x). \tag{3.23}$$

The following proposition is proved in [BFS 1] using standard methods.

Proposition 3.1. *For real functions g, h with $h \geqq 0$, for any complex function u, and for $\varepsilon>0$, $0\leqq s\leqq 2$,*

$$\frac{d}{ds}\mathscr{E}^J(r^2(h)+iA(g);\,|A(u)|^2;\,s)\leqq 0, \tag{3.24}$$

$$\frac{d}{ds}\mathscr{E}^J(r^2(h)+iA(g);\,r^2(x);\,s)\geqq 0, \tag{3.25}$$

$$\frac{d}{ds}\mathscr{E}^J(r^2(h)+iA(g);\,r(x)r(y)\cos(\theta(x)-e\varepsilon A_{\langle x,y\rangle}-\theta(y));\,s)\geqq 0, \tag{3.26}$$

$$\frac{d}{ds}\mathscr{E}^J(-r^2(h)+A(g);\,|A(u)|^2;\,s)\geqq 0, \tag{3.27}$$

$$\frac{d}{ds}\mathscr{E}^J(-r^2(h)+A(g);\,r^2(x);\,s)\leqq 0, \tag{3.28}$$

$$\frac{d}{ds}\mathscr{E}^J(-r^2(h)+A(g);\,r(x)r(y)\cos(\theta(x)-e\varepsilon A_{\langle x,y\rangle}-\theta(y));\,s)\leqq 0. \tag{3.29}$$

Using Proposition 3.1, we can deduce the following theorem.

Theorem 3.2. *For g real and $h\geqq 0$, both with compact support in $\mathbb{R}^d$, $\langle \mathcal{O}(ig,h)\rangle_{N,D}^{C_J}$ and $\langle \mathcal{O}(g,-h)\rangle_{N,D}^{C_J}$ are, respectively, monotonic increasing and decreasing in J.*

Proof of Theorem 3.2. We establish monotonicity for a non-zero lattice spacing ε; continuity gives the same result for the continuum limit. For given sources g and h, we take J large enough so that $\operatorname{supp} g$ and $\operatorname{supp} h$ are bounded away from ∂C_J. Recall that

$$:\phi^2:(h)=r^2(h)-2\sum_{x\in C_J}\varepsilon^d C^\varepsilon(x,x)h(x), \tag{3.30}$$

where $C^\varepsilon=(-\Delta^\varepsilon+m^2)^{-1}$ is computed using D b.c. on ∂C_J. When x is a distance $R>1$ from C_J, the difference between $C^\varepsilon(x,x)$ computed with free and D b.c. is less than $C\exp[-mR]$. So taking $\operatorname{dist}(\operatorname{supp} h, \partial C_J)>1$, we may define the subtraction needed to renormalize $\phi^2(h)$ using free boundary conditions, and the results of Sect. 3.1 still hold. The advantage of doing this is that $:\phi^2:(h)$ is then independent of J.

Consider the expression

$$\mathscr{E}^{J+1}(r^2(h)+iF(g);\ \tfrac{1}{2}\varepsilon^{d-2}|A_\mu(b_+)-A_\mu(b_-)|^2;\ s), \tag{3.31}$$

where b is a bond in C^*_{J+1}. By (3.24), the value of (3.31) at $s=2$ is less than its value at $s=1$. Examining the definition (3.21), this means that removing the term $\frac{1}{2}\varepsilon^{d-2}|A_\mu(b_+)-A_\mu(b_-)|^2$ from the action decreases $\langle\mathcal{O}(ig,h)\rangle^{C_{J+1}}_{N,D}$. Similarly, (3.26) shows that removing a term $-\varepsilon^{d-2}r(x)r(y)\cos(\theta(x)-e\varepsilon A_{\langle x,y\rangle}-\theta(y))$ from the action also decreases this expectation. If we remove all such terms coupling sites in C_J to sites in $C_{J+1}\backslash C_J$, then the resulting expectation factorizes, and the integral over fields in $C_{J+1}\backslash C_J$ cancels.

The remaining expression is exactly $\langle\mathcal{O}(ig,h)\rangle^{C_J}_{N,D}$, since we defined δm^2 using free boundary conditions. Therefore, we deduce the inequality

$$\langle\mathcal{O}(ig,h)\rangle^{C_J}_{N,D} \leqq \langle\mathcal{O}(ig,h)\rangle^{C_{J+1}}_{N,D}. \tag{3.32}$$

A similar proof shows that

$$\langle\mathcal{O}(g,-h)\rangle^{C_J}_{N,D} \geqq \langle\mathcal{O}(g,-h)\rangle^{C_{J+1}}_{N,D}. \tag{3.33}$$

3.3. Reflection Positivity

Reflection positivity is a crucial ingredient in the reconstruction theorem of Osterwalder and Schrader [OS 1]. It was proved for the $U(1)$ Higgs model in [BFS 1], and we state their results below. Given a rectangular parallelepiped $\Omega\subset\varepsilon Z^d$, consider a hyperplane π which is perpendicular to the ν^{th} coordinate direction, and which lies midway between neighbouring sites of Ω. Define $\Omega_\pm$ to be the sites in Ω lying on opposite sides of π, and let ϱ denote reflection in π. Then we shall assume that

$$\varrho(\Omega_+)=\Omega_-. \tag{3.34}$$

We now introduce a map θ taking functionals with support in Ω_+ into functionals supported in Ω_- by

$$\theta(\mathscr{F})(A(b),\phi(x))=\bar{\mathscr{F}}(A(\varrho b),\phi(\varrho x)). \tag{3.35}$$

The bar denotes complex conjugation and $\varrho b=\varrho\langle b_-b_+\rangle=\langle\varrho b_-,\varrho b_+\rangle$. We will choose boundary conditions for the fields on Ω which are invariant under reflection in π. It is convenient to consider the vector field A as defined on oriented bonds, and to assume that

$$A(-b)=A(\langle b_+,b_-\rangle)=-A(b). \tag{3.36}$$

We define a characteristic function on Ω by

$$\Omega(x)=\begin{cases}1 & \text{if } x\in\Omega,\\ 0 & \text{otherwise}.\end{cases} \tag{3.37}$$

The action that we consider on Ω is $\left(\text{recall that } F(p)=\varepsilon^{-1}\sum_{b\in\partial p}A(b)\right)$,

$$\begin{aligned}S^{\varepsilon}_{\Omega}(A,\phi)&=\tfrac{1}{2}\sum_{p\in\Omega^{**}}\varepsilon^{d}|F(p)|^{2}+\tfrac{1}{2}\mu_0^{2}\sum_{b\in\Omega^{*}}\varepsilon^{d}|A(b)|^{2}\\&\quad+\tfrac{1}{2}\sum_{x\in\Omega}\varepsilon^{d}\left|\sum_{y\in x^{*}}\varepsilon^{-1}\Omega(y)A(\langle x,y\rangle)\right|^{2}\\&\quad+\tfrac{1}{2}\sum_{\langle x,y\rangle}\varepsilon^{d-2}[(\Omega r)(x)^{2}+(\Omega r)(y)^{2}-2(\Omega r)(x)(\Omega r)(y)\\&\quad\cdot\cos(\theta(x)-e\varepsilon A(\langle x,y\rangle)-\theta(y))]\\&\quad+\sum_{x\in\Omega}\varepsilon^{d}[\tfrac{1}{2}m^{2}r(x)^{2}+\lambda r(x)^{4}+\tfrac{1}{2}\delta m^{2}(x)r(x)^{2}].\end{aligned}\tag{3.38}$$

We have written x^{*} for the set of nearest neighbour sites of x on $\varepsilon\mathbb{Z}^{d}$. We write $\langle\mathscr{F}\rangle^{\varepsilon}_{\Omega}$ for the expectation of a functional $\mathscr{F}$ in the measure defined by (3.38). Since the quadratic form for the vector field in the action (3.38) differs from the "unitary" form $\frac{1}{2}\langle A,(d^{*}d+\mu_0^{2})A\rangle$ by a gauge-fixing term only (d is the lattice exterior derivative), we may apply Corollary 5.4 of [BFS 1] to deduce the following proposition.

Proposition 3.3. *For any gauge-invariant functional $\mathscr{F}$ supported in Ω_{+},*

$$\langle\theta(\mathscr{F})\mathscr{F}\rangle^{\varepsilon}_{\Omega}\geqq 0.\tag{3.39}$$

The action (3.38) differs from (2.5) by the omission of some terms of the form $\varepsilon^{d-2}|A(b)-A(b')|^{2}$, where b and b' intersect $\partial\Omega$. We shall use the correlation inequalities of the last section to insert these terms below.

We return now to the construction of the infinite volume limit. For any sources g,h and for $z,w\in\mathbb{C}$,

$$|\langle\mathscr{O}(zg,wh)\rangle^{\varepsilon,C_J}_{N,D}|\leqq\langle\mathscr{O}(\mathrm{Re}\,zg,\mathrm{Re}\,wh)\rangle^{\varepsilon,C_J}_{N,D}.\tag{3.40}$$

We can write $\mathrm{Re}\,wh=h_{+}-h_{-}$, with $h_{+},h\geqq 0$. Then using Schwarz's inequality and Theorem 3.2,

$$\begin{aligned}(3.40)&\leqq[\langle\mathscr{O}(2\mathrm{Re}\,zg,-2h_{-})\rangle^{\varepsilon,C_J}_{N,D}]^{1/2}\cdot[\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_J}_{N,D}]^{1/2}\\&\leqq[\langle\mathscr{O}(2\mathrm{Re}\,zg,-2h_{-})\rangle^{\varepsilon,C_{J_0}}_{N,D}\cdot[\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_J}_{N,D}]^{1/2}\\&\leqq e^{CJ_0^{d}}[\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_J}_{N,D}]^{1/2},\end{aligned}\tag{3.41}$$

where $J_0\leqq J$ is such that $\mathrm{supp}\,g\cup\mathrm{supp}\,h\subset C_{J_0-1}$, and the factor "$C$" in (3.41) depends on g,h,z,w (this is the ultra-violet stability bound). For each J, we consider $\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon}_{C_J}$ defined with the action $S^{\varepsilon}_{C_J}(A,\phi)$ given by (3.38). Then Theorem 3.2 gives

$$\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_J}_{N,D}\leqq\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon}_{C_{J+1}}\leqq\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_{J+1}}_{N,D}.\tag{3.42}$$

Using (3.42), reflection positivity and ultra-violet stability, there is a standard argument (see [GJ 1]) which allows us to deduce

$$\langle\mathscr{O}(0,2h_{+})\rangle^{\varepsilon,C_J}_{N,D}\leqq e^{CJ_0^{d}}\tag{3.43}$$

all $J \geqq J_0$, for some constant C, again depending on $\|h_+\|$. Combining monotonicity (Theorem 3.2) and (3.41), (3.43), we deduce

$$\exists \lim_{J\to\infty} \langle \mathcal{O}(zg, wh)\rangle_{N,D}^{C_J} = \langle \mathcal{O}(zg, wh)\rangle \tag{3.44}$$

for $\mathrm{Re}\, z = 0$ and $h \geqq 0$. (Notice that we took the continuum limit before taking the infinite volume limit; all our bounds and monotonicity results are uniform in ε.) Vitali's theorem then extends (3.44) to an entire function on $\mathbb{C}\times\mathbb{C}$. Furthermore, for each J, $\langle \mathcal{O}(zg, wh)\rangle_{C_J}^{\varepsilon K}$ converges on some subsequence as $K\to\infty$. This limit is reflection positive; also (3.42) implies that as $J\to\infty$ it converges to (3.44). Therefore, (3.44) is also reflection positive.

Translation invariance follows by uniqueness of the limit (3.44). Vitali's theorem allows us to deduce (2.19).

4. Rotation Invariance

As explained in Sect. 2, in this section we will consider the model on a torus D_J defined by

$$D_J = \{x \in \mathbb{R}^d:\ -25MJ \leqq x_\mu < 25MJ,\ \mu = 1, \ldots, d\}. \tag{4.1}$$

We denote by $Z^\varepsilon(D_J, g, h)$ the generating functional on the lattice T_ε with spacing ε and the "usual" orientation. We will denote by $Z^{R(\varepsilon)}(D_J, g, h)$ the generating functional on the lattice $T_{R(\varepsilon)}$ with spacing ε, but rotated by an angle θ_0 as shown in Fig. 1. The results of [Ba 1–4] apply to both these models, and we have

$$e^{-C_1|D_J|} \leqq Z^\varepsilon(D_J, g, h),\ Z^{R(\varepsilon)}(D_J, g, h) \leqq e^{C_2|D_J|}, \tag{4.2}$$

where C_1, C_2 depend on g, h but not on ε or J.

In order to compare these models we consider a modified block spin transformation. Each lattice site y in $T_{R(5\varepsilon)}$ defines a block of 5^d sites on T_ε, which we denote by $\hat{B}(y)$. We introduce corresponding averaging operators

$$(\hat{Q}A_\mu)(y) = 5^{-d} \sum_{x \in \hat{B}(y)} A_\mu(x), \tag{4.3}$$

$$(\hat{Q}(A)\phi)(y) = 5^{-d} \sum_{x \in \hat{B}(y)} U(A(\hat{\Gamma}_{y,x}))\phi(x), \tag{4.4}$$

where the contour $\hat{\Gamma}_{y,x}$ is chosen to lie entirely inside $\hat{B}(y)$. Using the operators (4.3), (4.4) we can generate from $Z^\varepsilon(D_J, g, h)$ an effective action on the lattice $T^{(1)}_{R(5\varepsilon)}$, which we write $\hat{S}^{(1), R(5\varepsilon)}(T^{(1)}, A_1, \phi_1, g, h)$. By applying succeeding steps of the renormalization transformation in the usual way, that is without rotating the blocks, we generate effective actions $\hat{S}^{(k), R(5^k\varepsilon)}(T^{(k)}, A_k, \phi_k, g, h)$ on the lattices $T^{(k)}_{R(5^k\varepsilon)}$. In Sect. 5 we will prove the following proposition.

Proposition 4.1.

$$\begin{aligned} Z^\varepsilon(D_J, g, h) \geqq & \int (dA)(d\phi)\chi_k(A_k)\chi_k(\phi_k) \\ & \cdot \exp[-\hat{S}^{(k), R(5^k\varepsilon)}(T^{(k)}, A_k, \phi_k, g, h) + C(L^k\varepsilon)^\sigma |D_J|] \end{aligned} \tag{4.5}$$

for $0 \leqq k \leqq K$, $\sigma > 0$ *and* χ_k *as defined in Sect. 3 of* [K 1].

We can also perform the usual renormalization transformations on $Z^{R(5\varepsilon)}(D_J, g, h)$. This produces effective actions $S^{(k-1),R(5^k\varepsilon)}(T^{(k)}, A_k, \phi_k, g, h)$ on the lattices $T^{(k)}_{R(5^k\varepsilon)}$, which we can compare with the previous actions.

Theorem 4.2.

$$\begin{aligned}&\chi_k(A_k)\chi_k(\phi_k)|\hat{S}^{(k),1}(T^{(k)}, A_k, \phi_k, g, h) - S^{(k-1),1}(T^{(k)}, A_k, \phi_k, g, h)|\\&\quad \leqq C(L^{-\gamma k}(L^k\varepsilon)^{-\beta} + (L^k\varepsilon)^\sigma)|D_J|,\end{aligned} \tag{4.6}$$

where $0<\gamma<1$, $0<\sigma, \beta$, and c depends on g, h.

Proof. The proof is almost identical to the proof of Theorem 3.4 in [K 1], assuming that we have corresponding results on convergence of propagators. These results will be established in Appendix B. The only significant difference in the rest of the proof is that we always decouple regions using propagators with periodic boundary conditions on some torus defined in the same way as D_J. This allows an explicit Fourier representation for free propagators.

We can now establish Theorem 2.4. First, using the results of [Ba 2] (as stated in Theorem 3.1 of [K 1]) and Proposition 4.1,

$$\begin{aligned}&Z^{R(5\varepsilon)}(D_J, g, h) - Z^\varepsilon(D_J, g, h)\\&\quad \leqq \int (dA_k)(d\phi_k)\chi_k(A_k)\chi_k(\phi_k)\\&\qquad \cdot \{\exp[-S^{(k-1),1}(T^{(k)}, A_k, \phi_k, g, h)] - \exp[-\hat{S}^{(k),1}(T^{(k)}, A_k, \phi_k, g, h)]\}\\&\qquad \cdot e^{C(L^k\varepsilon)^\sigma|D_J|} + e^{-p(L^k\varepsilon)^2 + C|D_J|}.\end{aligned} \tag{4.7}$$

We now use Theorem 4.2 and the bound

$$e^x - e^y \leqq |e^x - e^y| \leqq |x-y|(e^x + e^y), \tag{4.8}$$

which gives

$$\begin{aligned}(4.7) &\leqq C(L^{-\gamma k}(L^k\varepsilon)^{-\beta} + (L^k\varepsilon)^\sigma)|D_J|\{Z^{R(5\varepsilon)}(D_J, g, h) + Z^\varepsilon(D_J, g, h)\}\\&\qquad \cdot e^{C(L^k\varepsilon)^\sigma|D_J|} + e^{-p(L^k\varepsilon)^2 + C|D_J|}\\&\leqq C(L^{-\gamma k}(L^k\varepsilon)^{-\beta} + (L^k\varepsilon)^\sigma)e^{C|D_J|}.\end{aligned} \tag{4.9}$$

To complete the proof, we must prove the same bound for $\{Z^\varepsilon(D_J, g, h) - Z^{R(5\varepsilon)}(D_J, g, h)\}$. We do this by switching around the argument; rotated blocks are used to generate from $Z^{R(5\varepsilon)}(D_J, g, h)$ an effective action on the lattice $T^{(1)}_{5^2\varepsilon}$, and other actions on $T^{(k-1)}_{5^k\varepsilon}$. These are compared with the actions generated from $Z^\varepsilon(D_J, g, h)$ by the usual transformations. We can prove analogues of Proposition 4.1 and Theorem 4.2, which is all we need. Then using the bound (4.9) and choosing k suitably, Theorem 2.4 follows (see Sect. 3.2 of [K 1]).

It remains to prove that $\theta_0/2\pi$ is irrational. We know that $\cos\theta_0 = \frac{3}{5}$. So suppose that $\theta_0 = \frac{p}{q}2\pi$, where p, q are integers. Then

$$1 = \cos q\theta_0 = \begin{cases} (\cos\theta_0)^q \sum\limits_{n=0}^{1/2q} \binom{q}{2n} + P_{q-2}(\cos\theta_0) & \text{for } q \text{ even},\\ (\cos\theta_0)^q \sum\limits_{n=0}^{1/2(q-1)} \binom{q}{2n+1} + P'_{q-2}(\cos\theta_0) & \text{for } q \text{ odd},\end{cases} \tag{4.10}$$

where P_{q-2}, P'_{q-2} are polynomials of degree $q-2$. It is easy to show that

$$\sum_{n=0}^{1/2q}\binom{q}{2n} = \sum_{n=0}^{1/2(q-1)}\binom{q}{2n+1} = 2^{q-1}. \tag{4.11}$$

Therefore, for some integers $a_0, \ldots, a_{q-1}$, $\cos\theta_0$ is a solution of the equation

$$2^{q-1}x^q + \sum_{n=0}^{q-1} a_n x^n = 0. \tag{4.12}$$

We will now prove that 3/5 is not a solution of (4.12). Let $x = M/N$, where M, N are mutually prime. Then

$$2^{q-1}M^q + \sum_{n=0}^{q-1} a_n M^n N^{q-n} = 0. \tag{4.13}$$

Therefore, N divides $2^{q-1}M^q$, and so by assumption $N=2^s$ for some s. Therefore, $x=3/5$ is not a root of (4.12).

5. Ultra-violet Stability

5.1. Regularity and Decay of Operators with D b.c.

In this section we will extend Theorem 3.3 of [K 1] to include operators defined with D b.c. Recall that we modified the averaging operator in (3.2) by including the averaging function f defined by (3.8). Theorem 3.3 of [K 1] was proved in the paper [Ba 4], and we will modify suitably that proof below.

For any C_J defined by (2.7), we denote by T_ε the lattice in C_J with spacing ε. After rescaling to the η-lattice ($\eta = L^{-k}$), it is written T_η. We denote by $T^{(k)}$ the sublattice of T_η with unit spacing. Let $\Omega \subset T_\eta$ be such that $\Omega^{(k)}$ is a union of large blocks on $T^{(k)}$. We will write $\partial\Omega$ to mean that part of the boundary of Ω which does not belong to ∂T_η. Recall that $G_k(A)$ is defined with D b.c. on ∂T_η, and $G_k(\Omega, A)$ with D b.c. on $\partial\Omega \cup \partial T_\eta$. Also a vector field configuration A is regular on Ω if

$$|\partial_\mu^n A_\nu(x)| \leqq C(e(L^k\varepsilon)^{2-d/2})^{\beta-1} \tag{5.1}$$

for $\mu, \nu = 1, \ldots, d$, $\beta > 0$, and $x \in \Omega$.

Proposition 5.1. *Let A be regular on Ω, and $f: \Omega \to \mathbb{R}$. Let $\varepsilon = L^{-K}$; then for all $k \leqq K$, for $x \in \Omega$ and $\delta_0 > 0$,*

$$|G_k(\Omega, A)f(x)| \leqq C\exp[-\delta_0 \operatorname{dist}(x, \operatorname{supp} f)]\,\|f\|, \tag{5.2}$$

$$|G_k(\Omega, A)f(x) - G_k(A)f(x)|$$
$$\leqq C\exp[-\delta_0 \operatorname{dist}(x, \operatorname{supp} f) - \delta_0 \operatorname{dist}(x, \partial\Omega) - \delta_0 \operatorname{dist}(\operatorname{supp} f, \partial\Omega)]\,\|f\|. \tag{5.3}$$

Corresponding bounds hold for $D_{A,\mu}G_k(\Omega, A)$ and the α-Hölder derivatives of $G_k(\Omega, A)$, $D_{A,\mu}G_k(\Omega, A)$ for $0 < \alpha < 1$.

Proof. We will mimic the random walk representation presented in [Ba 4] in order to prove Proposition 5.1. For each $j \in \mathbb{Z}^d$ define the set

$$\square_j = \Omega \cap \{\text{the union of large blocks with } Mj \text{ at one corner}\}. \tag{5.4}$$

We introduce a partition of unity on $\mathbb{Z}^d$ as follows. For each $j \in \mathbb{Z}^d$, define

$$h_j(x) = \prod_{\mu=1}^{d} h\left(\frac{x_\mu}{M} - j_\mu\right), \tag{5.5}$$

where h is C^∞, supported on $[-2/3, 2/3]$ and $h=1$ on $[-1/3, 1/3]$. Also we choose h so that

$$\sum_{j \in \mathbb{Z}^d} h_j^2 = 1. \tag{5.6}$$

We define the operator

$$G_0 = \sum_j h_j G_k(\square_j, A) h_j; \tag{5.7}$$

this will serve as a first approximation to $G_k(\Omega, A)$. A straightforward calculation (see [Ba 4]) shows that

$$\begin{aligned}
&((-\Delta^{\eta,D}_{A,\Omega} + m^2(L^k\varepsilon)^2 + a_k P_k(A))G_0)(x, x') \\
&\quad = \sum_j [h_j(x)((-\Delta^{\eta,D}_{A,\square_j} + m^2(L^k\varepsilon)^2 + a_k P_k(A))G_k(\square_j, A))(x, x')h_j(x')] \\
&\quad - \sum_{b \in st(x)} (\partial^\eta h_j)(b)(D_A G_k(\square_j, A))(b, x')h_j(x') \\
&\quad + (-\Delta^\eta h_j)(x) G_k(\square_j, A; x, x')h_j(x') \\
&\quad + (R_k(A, \partial^\eta h_j) G_k(\square_j, A))(x, x')h_j(x'),
\end{aligned} \tag{5.8}$$

where $st(x)$ is the set of bonds connected to x, and we have included a superscript to denote D b.c. Also

$$\begin{aligned}
R_k(A, \partial^\eta h)\phi(x) = &\sum_{x' \in B^k(y^k(x))} \eta^d f(x') U(A(\Gamma^{(k)}_{x, y^k(x), x'})) \\
&\cdot \eta^{-1}(h(x) - h(x'))\phi(x').
\end{aligned} \tag{5.9}$$

If we define the operators

$$(K_j\phi)(x) = \square_j(x)\left\{\sum_{b \in st(x)} (\partial^\eta h_j)(b)(D^\eta_A \phi)(b) + (\Delta^\eta h_j)(x)\phi(x) - R_k(A, \partial^\eta h_j)\phi(x)\right\}, \tag{5.10}$$

$$R = \sum_j K_j G_k(\square_j, A) h_j, \tag{5.11}$$

then (5.8) implies that

$$G_k(\Omega, A) = G_0(I - R)^{-1} = \sum_{n=0}^{\infty} G_0 R^n. \tag{5.12}$$

We can now deduce the random walk representation. Define a path $\omega = \{\omega_0, \omega_1, \ldots, \omega_n\}$, $\omega_i \in \mathbb{Z}^d$, where ω_i, ω_{i+1} are vertices of the same unit cube. Then (5.12) can be written

$$G_k(\Omega, A) = \sum_\omega h_{\omega_0} G_k(\square_{\omega_0}, A) h_{\omega_0} K_{\omega_1} G_k(\square_{\omega_1}, A) h_{\omega_1} \ldots K_{\omega_n} G_k(\square_{\omega_n}, A) h_{\omega_n}. \tag{5.13}$$

In [Ba 4], it is shown that Proposition 5.1 follows from the following two lemmas.

Lemma 5.2. *Let $\square$ be a sum of a few large blocks. Then*

$$\begin{aligned}&\|G_k(\square, A)f\|_2, \|D^\eta_{A,\mu}G_k(\square, A)f\|_2,\\ &\quad \|G_k(\square, A)D^{\eta *}_{A,\mu}f\|_2, \|D^\eta_{A,\mu}G_k(\square, A)D^{\eta *}_{A,\nu}f\|_2 \leqq C\|f\|_2 .\end{aligned} \tag{5.14}$$

Lemma 5.3. *Let $\square$ be a rectangular parallelepiped which is a sum of a few large blocks. Then*

$$\|G_k(\square, A)f\|_\infty \leqq C\|f\|_\infty . \tag{5.15}$$

Corresponding bounds hold for $D^\eta_{A,\mu}G_k(\square, A)$ and the α-Hölder derivatives of $G_k(\square, A)$, $D^\eta_{A,\mu}G_k(\square, A)$ for $0<\alpha<1$.

Proof of Lemma 5.2. Since A is regular on $\square$, we can write

$$A = B + A', \tag{5.16}$$

where B is a constant vector field and

$$|A'|, |\partial^\eta_\mu A'| \leqq C(L^k\varepsilon)^{(2-d/2)(\beta-1)} . \tag{5.17}$$

In the paper [K 1], we deduced the following expansion for $G_k(\square, A)$:

$$G_k(\square, A) = G_k^{1/2}(\square, B)(I - G_k^{1/2}(\square, B)V_k(A', B)G_k^{1/2}(\square, B))^{-1}G_k^{1/2}(\square, B) . \tag{5.18}$$

If we assume that Lemma 5.2 holds with A replaced by B, then (5.18) implies (5.14). Furthermore, we can gauge transform from B to the zero vector field configuration. The lemma then follows by getting lower bounds on quadratic forms. We see that

$$\langle\phi, (-\Delta^{\eta, D}_\square + m^2(L^k\varepsilon)^2 + a_kP_k)\phi\rangle \geqq \langle\phi, (-\Delta^{\eta, N}_\square + m^2(L^k\varepsilon)^2 + a_kP_k)\phi\rangle , \tag{5.19}$$

where $-\Delta^{\eta, N}_\square$ has N b.c. on $\partial\square$. The proof then proceeds exactly as in [Ba 4]; we separate $-\Delta^{\eta, N}_\square$ across unit blocks using N b.c. On each block P_k projects onto the zero eigenvector of the Laplacian. The other eigenvalues of the Laplacian are greater than π^2.

Proof of Lemma 5.3. By expanding (5.18), we get

$$G_k(\square, A) = \sum_{n=0}^{\infty} G_k(\square, B)(V_k(A', B)G_k(\square, B))^n . \tag{5.20}$$

Once again, we assume that Lemma 5.3 holds with A replaced by B. By combining (5.20) with the bound

$$\|V_k(A', B)G_k(\square, B)\|_\infty \leqq Ce(L^k\varepsilon)^\beta , \tag{5.21}$$

we then deduce the required result. Recall that (see [K 1])

$$\begin{aligned}-V_k(A', B) = {}& D^{\eta *}_B F_{1,k}(A')U(B) + U^*(B)F_{1,k}(A')^* D^\eta_B\\ &+ |F_{1,k}(A')|^2 + a_kF_{2,k}(A', B)^*Q_k(B)\\ &+ a_kQ^*_k(B)F_{2,k}(A', B) + a_kF_{2,k}(A', B)^*F_{2,k}(A', B)\end{aligned} \tag{5.22}$$

(the operators $F_{1,k}$ and $F_{2,k}$ are defined in [K 1]). As mentioned in [Ba 4], $V_k(A', B)$ is a first order differential operator acting on the function to its right. The only obstacle to this interpretation is the term $D_B^{\eta *}F_{1,k}(A')$. Explicitly computing it, we get

$$\begin{aligned}
&(D_B^{\eta *}F_{1,k}(A')\phi)(x)\\
&= \sum_{\mu=1}^{d} \eta^{-1}\{U(B)\eta^{-1}(U(A'(\langle x-\eta e_\mu, x\rangle))-1)\phi(x)\chi(x)\chi(x-\eta e_\mu)\\
&\quad -\eta^{-1}(U(A'(\langle x, x+\eta e_\mu\rangle))-1)\phi(x+\eta e_\mu)\chi(x)\chi(x+\eta e_\mu)\}, \qquad (5.23)
\end{aligned}$$

where

$$\chi(x)=\begin{cases}1 & \text{if } x\in\square\backslash\partial\square,\\ 0 & \text{otherwise}.\end{cases} \qquad (5.24)$$

In Appendix A, we will show that

$$|\eta^{-1}(G_k(\square, B)\phi)(x)|\leqq C, \qquad (5.25)$$

when $x+\eta e_\mu$ or $x-\eta e_\mu$ lies on $\partial\square$ [this is a consequence of the fact that $G_k(\square, B)$ has D b.c. on $\partial\square$]. Therefore, (5.23) is a regular operator on $G_k(\square, B)$, and so (5.21) holds. It remains to prove (5.15) for a constant vector field. By gauge covariance, this is equivalent to having zero vector field. In Appendix A, we will show how these bounds follow from an explicit Fourier representation.

The next proposition gives a lower bound on the effective Laplacian which is crucial in the proof of ultra-violet stability. Again, we assume that $\Omega^{(k)}$ is a union of large blocks on $T^{(k)}$, and we denote by $\bar{\Omega}^{(k)}$ that part of $\partial\Omega^{(k)}$ which intersects $\partial T^{(k)}$.

Proposition 5.4. *When A is regular on Ω,*

$$\begin{aligned}
\langle\phi, \Delta^{(k)}(\Omega, A)\phi\rangle \geqq \gamma_0\Big\{&\sum_{x\in\bar{\Omega}^{(k)}}|\phi(x)|^2+\sum_{\langle x,x'\rangle\in\Omega^{(k)*}}|U(A(\langle x,x'\rangle))\phi(x')-\phi(x)|^2\\
&+m^2(L^k\varepsilon)^2\sum_{x\in\Omega^{(k)}}|\phi(x)|^2\Big\} - C(e(L^k\varepsilon)^{2-d/2})^{2-\alpha}\sum_{x\in\Omega^{(k)}}|\phi(x)|^2. \qquad (5.26)
\end{aligned}$$

Proof. The bound (5.26) is established in [Ba 4] for N b.c., but without the term $\gamma_0 \sum_{x\in\bar{\Omega}^{(k)}}|\phi(x)|^2$. This term is present only because we are using D b.c. Since

$$-\Delta_A^{\eta, D}\geqq -\Delta_A^{\eta, N}, \qquad (5.27)$$

the remaining terms on the right-hand side of (5.26) follow from the bound on the operator with N b.c. (It is important to notice that the proof presented in [Ba 4] requires only L_2-bounds on operators.) So we will prove the lower bound for the first term on the right-hand side of (5.26).

We may separate $\Delta^{(k)}(\Omega, A)$ by Neumann boundary conditions across the boundaries of each unit block in $\Omega^{(k)}$. On each block away from $\bar{\Omega}^{(k)}$, we estimate the operator by zero. So let $\square$ be the block on Ω defined by some $x\in\bar{\Omega}^{(k)}$. Then

$$\langle\phi, \Delta^{(k)}(\square, A)\phi\rangle = a_k|\phi(x)|^2 - a_k^2\langle\phi, Q_k(A)G_k(\square, A)Q_k^*(A)\phi\rangle. \qquad (5.28)$$

Since A is regular on Ω, we can write $A=B+A'$, with B constant and A' small. We can then gauge transform (5.28) to remove B, and expand about $A'=0$

in the usual way (see Sect. 3.4 of [K 1]). Using the L_2-bounds of Lemma 5.2, the terms in the expansion of order e^2 and higher can be bounded by $C(e(L^k\varepsilon)^{2-d/2}p(L^k\varepsilon))^2|\phi(x)|^2$. The term of order e is proportional to $\mathrm{Re}(\phi^*(x)i\phi(x))=0$. So we are left with the piece independent of A, which is

$$a_k|\phi(x)|^2 - a_k^2\langle\phi, Q_k G_k(\square)Q_k^*\phi\rangle. \tag{5.29}$$

For simplicity, we change $G_k(\square)$ so that $-\Delta_\square^\eta$ has D b.c. on one side of $\square$ only, and N b.c. on the other sides. We denote this side by B; it contains η^{1-d} sites. We will use the integral representation of (5.29):

$$\begin{aligned}&C\exp[-\tfrac{1}{2}(5.29)]\\&\quad=\int d\phi'|_\square \exp[-\tfrac{1}{2}a_k|\phi(x)-(Q_k\phi')(x)|^2-\tfrac{1}{2}\langle\phi',(-\Delta_\square^\eta+m^2(L^k\varepsilon)^2)\phi'\rangle],\end{aligned} \tag{5.30}$$

where C is some constant. Clearly, for some A_0,

$$(5.30)=C\exp[-\tfrac{1}{2}A_0|\phi(x)|^2]. \tag{5.31}$$

In order to evaluate A_0, we will Fourier transform (5.30). Since (5.30) does not mix the components of $\phi(x)$, it is sufficient to consider each component separately. Then

$$\int d\phi \exp-[i\xi\phi+\tfrac{1}{2}A_0\phi^2]=\sqrt{2\pi A_0^{-1}}\exp[-\tfrac{1}{2}A_0^{-1}\xi^2]. \tag{5.32}$$

If we change ϕ' to $\phi+\phi'$ in (5.30), we get

$$\begin{aligned}&\int d\phi'|_\square \exp\Big[-\tfrac{1}{2}a_k|Q_k\phi'(x)|^2-\tfrac{1}{2}\langle\phi',(-\Delta_\square^{\eta,N}+m^2(L^k\varepsilon)^2)\phi'\rangle\\&\quad-\tfrac{1}{2}\sum_{y\in B}\eta^{d-2}|\phi'(y)+\phi(x)|^2\Big],\end{aligned} \tag{5.33}$$

where $-\Delta_\square^{\eta,N}$ has N b.c. on every side of $\square$. We can write

$$\begin{aligned}\sum_{y\in B}\eta^{d-2}|\phi'(y)+\phi(x)|^2=&\sum_{y\in B}\eta^d\Big|\eta^{-1}\phi'(y)-\sum_{y'\in B}\eta^{d-2}\phi'(y')\Big|^2\\&+\eta^{-1}\Big|\phi(x)+\sum_{y\in B}\eta^{d-1}\phi'(y)\Big|^2.\end{aligned} \tag{5.34}$$

If we define

$$\begin{aligned}\langle\phi',A\phi'\rangle=&\,a_k|Q_k\phi'(x)|^2+\langle\phi',(-\Delta_\square^{\eta,N}+m^2(L^k\varepsilon)^2)\phi'\rangle\\&+\tfrac{1}{2}\sum_{y\in B}\eta^d\Big|\eta^{-1}\phi'(y)-\sum_{y'\in B}\eta^{d-2}\phi'(y')\Big|^2,\end{aligned} \tag{5.35}$$

then (5.32) becomes

$$\begin{aligned}&\int d\phi\exp(-i\xi\phi)\int d\phi'\exp\Big[-\tfrac{1}{2}\eta^{-1}\Big|\phi(x)+\sum_{y\in B}\eta^{d-1}\phi'(y)\Big|^2-\tfrac{1}{2}\langle\phi',A\phi'\rangle\Big]\\&\quad=\int d\phi\exp(-i\xi\phi-\tfrac{1}{2}\eta^{-1}\phi^2)\int d\phi'\exp\Big[i\xi\sum_{y\in B}\eta^{d-1}\phi'(y)-\tfrac{1}{2}\langle\phi',A\phi'\rangle\Big]\\&\quad=C\exp[-\tfrac{1}{2}\eta\xi^2-\tfrac{1}{2}\xi^2\langle f,A^{-1}f\rangle],\end{aligned} \tag{5.36}$$

where

$$f(y)=\sum_{y'\in B}\eta^{d-1}\delta^\eta(y-y'). \tag{5.37}$$

Therefore, we deduce

$$A_0^{-1}=\eta+\langle f, A^{-1}f\rangle. \tag{5.38}$$

From (5.35), we see that

$$\langle\phi, A\phi\rangle \geqq \langle\phi,(-\Delta_{\square}^{\eta, N}+m_0^2)\phi\rangle, \tag{5.39}$$

where m_0 is $O(1)$. Therefore, we get

$$A_0^{-1} \leqq O(1), \tag{5.40}$$

and hence (5.26) follows.

5.2. Upper and Lower Bounds on the Generating Functional

We will now establish Theorem 3.1 of [K 1] for the generating functional $Z_{N,D}^{\varepsilon}(C_J, g, h)$ defined with N b.c. for the vector field and D b.c. for the scalar field. Of course, the proof follows by modifying suitably the material in [Ba 1] and [Ba 2].

Because the covariant Laplacian for the scalar field has D b.c. on ∂C_J, there is an additional positive factor in each effective action $S^{(k)}$ which keeps the scalar field ϕ_k small near $\partial T^{(k)}$. Specifically, we introduce the following bounds on the scalar field at each step of the renormalization transformation:

$$\prod_{x\in\Lambda_0^{(k)}\cap\partial T^{(k)}} \chi(|\phi_k(x)| \leqq Cp(L^k\varepsilon)), \tag{5.41}$$

where $\Lambda_0^{(k)}$ is the subset of $T^{(k)}$ on which all fields are small (see [Ba 2]). If the bound in (5.41) is not satisfied at some point, then Proposition 5.4 shows that we get a small factor $\exp(-p(L^k\varepsilon)^2)$ from the integral.

The additional bounds (5.41) are enough to ensure that the proof of the upper bound goes through (there is no change in the proof of the lower bound). First, (5.41) ensures that the fluctuation fields ϕ' are always small. Recall that the field ϕ' at the k^{th} step is defined by Eq. (2.110) of [Ba 2]:

$$\phi_k=\phi'+aL^{-2}C_{\Lambda_4^{(k)}}^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\phi_{k+1}. \tag{5.42}$$

As shown in [Ba 2], when $\operatorname{dist}(x, \partial\Lambda_4^{(k)})>p(L^k\varepsilon)$, $x\in\Lambda_4^{(k)}$, we have

$$\begin{aligned} &aL^{-2}C_{\Lambda_4^{(k)}}^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})Q^*(B^{(k+1),\eta})\phi_{k+1}(x) \\ &\quad=(Q^*(B^{(k+1),\eta})\phi_{k+1})(x)+Cp(L^k\varepsilon). \end{aligned} \tag{5.43}$$

Therefore, suppose that $x\in\Lambda_4^{(k)}$, and $\operatorname{dist}(x, \partial T^{(k)})<p(L^k\varepsilon)$. Then using (5.41) and the bounds on the covariant derivative of ϕ_k,

$$\begin{aligned} |\phi_k(x)| &= |U(B^{(k+1),\eta}(\Gamma_{x,y}))\phi_k(y)+\phi_k(x)-U(B^{(k+1),\eta}(\Gamma_{x,y}))\phi_k(y)| \\ &\leqq Cp(L^k\varepsilon)^2, \end{aligned} \tag{5.44}$$

where $y\in\partial T^{(k)}$ and $\Gamma_{x,y}$ is a contour connecting x and y. A similar bound holds for $(Q^*(B^{(k+1),\eta})\phi_{k+1})(x)$. Using the exponential decay of $C_{\Lambda_4^{(k)}}^{(k)}(B^k(\Lambda_2^{(k)}), B^{(k+1),\eta})$, it follows that for $x\in\Lambda_4^{(k)}$, with $\operatorname{dist}(x, \partial T^{(k)})<p(L^k\varepsilon)$,

$$|\phi'(x)| \leqq Cp(L^k\varepsilon)^2, \tag{5.45}$$

as required.

Second, (5.41) also ensures that a large scalar field near $\partial T^{(k)}$ produces a small damping factor in the integral. Recall that the dominant term in the effective action is

$$-\lambda(L^k\varepsilon)^{4-d} \sum_{x\in B^k(\Lambda_7^{(k-1)'})} \eta^d|\phi^{(k)}(x)|^4, \tag{5.46}$$

where the field $\phi^{(k)}(x)$ is given by

$$\phi^{(k)}(x) = (a_k G_k(\Lambda_7^{(k-1)'}, A^{(k)}) Q_k^*(A^{(k)})\phi_k)(x). \tag{5.47}$$

Suppose that $x \in B^k(\Lambda_7^{(k-1)'})$ is a distance R or greater from $\partial T^{(k)}$. Then by taking R to be a small fraction of $p(L^k\varepsilon)$, we have

$$\begin{aligned}\phi^{(k)}(x) &= (a_k G_k(A^{(k)}) Q_k^*(A^{(k)})\phi_k)(x) + O(L^k\varepsilon)^\alpha \\ &= \phi_k(y) + Cp(L^k\varepsilon)|x-y| + O(L^k\varepsilon)^\alpha,\end{aligned} \tag{5.48}$$

where $G_k(A^{(k)})$ has periodic boundary conditions on $T^{(k)}$, and y is any point in $B^k(\Lambda_7^{(k-1)'})$. Therefore, by choosing R suitably, we deduce for any sufficiently large subset Ω,

$$-\lambda(L^k\varepsilon) \sum_{x\in B^k(\Omega)} \eta^d|\phi^{(k)}(x)|^4 \leqq -\tfrac{1}{2}\lambda(L^k\varepsilon)\sum_{y\in\Omega}|\phi_k(y)|^4 + C(L^k\varepsilon)^\alpha|\Omega|, \tag{5.49}$$

where $\alpha > 0$. This bound allows us to extract a small factor for each large field $\phi_k(y)$, and so the proof of the upper bound goes through.

5.3. Lower Bound for the Rotated Model

The proof of Proposition 4.1 follows by verifying the properties used in [Ba 1] to prove a lower bound. We will write $\hat{G}_k(A)$, $\hat{C}^{(k)}(A)$ etc. to denote the propagators produced by the averaging operators (4.3), (4.4). The following proposition gives the required properties of $\hat{G}_k(A)$.

Proposition 5.5. *Let A be regular on T_η, and $f: T_\eta \to \mathbb{R}^N$. Then*

$$|(\hat{G}_k(A)f)(x)| \leqq C\exp(-\delta_0 \operatorname{dist}(x, \operatorname{supp} f))\,\|f\|_\infty, \tag{5.50}$$

$$|(D_{A,\mu}\hat{G}_k(A)f)(x)| \leqq C\exp[-\delta_0 \operatorname{dist}(x, \operatorname{supp} f)]\,\|f\|_\infty, \tag{5.51}$$

$$\begin{aligned}&\frac{1}{|x-y|^\alpha}|U(A(\Gamma_{x,y}))(D^\eta_{A,\mu}\hat{G}_k(A)f)(y) - (D^\eta_{A,\mu}\hat{G}_k(A)f)(x)| \\ &\quad\leqq C\exp[-\delta_0 \operatorname{dist}(\{x,y\}, \operatorname{supp} f)]\,\|f\|_\infty,\end{aligned} \tag{5.52}$$

where $0<\alpha<1$, for all $x, y \in T_\eta$.

Proposition 5.5 guarantees that the effective action $\hat{S}^{(k),R(5^k\varepsilon)}(T^{(k)}, A_k, \phi_k, g, h)$ has the necessary properties to act as a small, local perturbation of the free field measure at each step of the renormalization transformation. In order to apply the results of [Ben 1] to perform a cluster expansion at each step of the renormalization transformation, we need the following results concerning $\hat{C}^{(k)}(A)$. For $\Omega \subset T_1^{(k)}$, define

$$\hat{C}_\Omega^{(k)}(A) = ((\hat{\Delta}^{(k)}(A) + aL^{-2}P(A))|_\Omega)^{-1}. \tag{5.53}$$

Proposition 5.6. *Let $\Omega \subset T_1^{(k)}$ be a sum of big blocks; then for $x, y \in \Omega$,*

$$|\hat{C}_\Omega^{(k)}(A; x, y)| \leqq C \exp[-\delta_0 |x-y|]. \tag{5.54}$$

Proof of Proposition 5.5. Once again we use a random walk representation. This time, we use blocks $\{\square_j\}$ which are small scale versions of D_J, that is they can be fitted by both the usual lattice and the rotated lattice. Our first approximation is the operator

$$\hat{G}_0 = \sum_j h_j \hat{G}_k(\square_j, \tilde{A}) h_j, \tag{5.55}$$

where $\hat{G}_k(\square, \tilde{A})$ has periodic boundary conditions on the block $\square$. Also, we take $\tilde{A}$ equal to A in the interior of $\square$, and changing regularly to a constant configuration near $\partial\square$. We then deduce the representation

$$\hat{G}_k(A) = \sum_\omega h_{\omega_0} \hat{G}_k(\square_{\omega_0}, \tilde{A}_{\omega_0}) h_{\omega_0} K_{\omega_1} \hat{G}_k(\square_{\omega_1}, \tilde{A}_{\omega_1}) \dots K_{\omega_n} \hat{G}_k(\square_{\omega_n}, \tilde{A}_{\omega_n}) h_{\omega_n}. \tag{5.56}$$

To deduce Proposition 5.5, we need the following lemmas.

Lemma 5.7.

$$\begin{gathered} \|\hat{G}_k(\square, \tilde{A}) f\|_2, \|D^\eta_{A,\mu} \hat{G}_k(\square, \tilde{A}) f\|_2, \\ \|\hat{G}_k(\square, \tilde{A}) D^{\eta *}_{A,\mu} f\|_2, \|D^\eta_{A,\mu} \hat{G}_k(\square, \tilde{A}) D^{\eta *}_{A,\nu} f\|_2 \leqq C \|f\|_2. \end{gathered} \tag{5.57}$$

Lemma 5.8.

$$\|\hat{G}_k(\square, \tilde{A}) f\|_\infty \leqq C \|f\|_\infty. \tag{5.58}$$

Corresponding bounds hold for $D^\eta_{A,\mu} \hat{G}_k(\square, A)$ and the α-Hölder derivatives of these operators, $0 < \alpha < 1$.

Proof. As with Lemma 5.2, Lemma 5.7 follows from the lower bound

$$\langle \phi, (-\Delta^\eta_\square + m^2 (L^k \varepsilon)^2 + a_k \hat{P}_k) \phi \rangle \geqq C \langle \phi, \phi \rangle. \tag{5.59}$$

To prove (5.59), we cover $\square$ by cubes $\{\Delta_i\}$ on the η-lattice, with sides parallel to the coordinate planes on that lattice. The size of Δ is chosen so that each cube contains fully at least one block of 5^{kd} sites defined by $\hat{P}_k$. We then estimate (5.59) from below by omitting $\hat{P}_k$ except on one block fully contained in each Δ_i, and omitting the (derivative)^2 terms on bonds connecting different Δ_i. We then get

$$\begin{aligned} (5.59) &\geqq \sum_i \langle \phi, (-\Delta^{\eta, N}_{\Delta_i} + m^2 (L^k \varepsilon)^2 + a_k \hat{P}_k(\Delta_i)) \phi \rangle \\ &\geqq \sum_i C \langle \phi|_{\Delta_i}, \phi|_{\Delta_i} \rangle, \end{aligned} \tag{5.60}$$

since $\hat{P}_k(\Delta_i)$ projects onto the constant mode on Δ_i, and $-\Delta^{\eta, N}_{\Delta_i}$ has non-zero eigenvalues greater than $(\pi/R)^2$, where R is the dimension of Δ_i. This gives (5.57) (the other bounds follow as before).

To prove Lemma 5.8, we write $\tilde{A} = B + A'$ with B constant and A' small, and expand about $A' = 0$. As in the proof of Lemma 5.3, we reduce to the case $\tilde{A} = 0$, which we prove in Appendix B.

Therefore, Proposition 5.5 holds for $\hat{G}_k(A)$ as required. To prove Proposition 5.6, we use the method presented in Chap. 3 of [Ba 4]. This method requires only

L_2-bounds on the operators $\hat{G}_k(\Omega, A)$, and these may be established in the same way as Lemma 5.7. Hence (5.54) holds for the set Ω.

Appendix A

Using the representations (3.7), (3.9), it is sufficient to consider the operators G_k, $G_k(0)$ with free boundary conditions. The only difference from the case studied in [K 1] is the averaging function $f(x)$, given by (3.8). The Fourier transform of f is equal to

$$\tilde{f}_\eta(p) = L^{-d} \sum_{x \in B(y)} f(x) e^{-ip(x-y)} = \prod_{\mu=1}^{d} \left[\frac{e^{-ip_\mu \eta (L-1)} - 1}{(L-1)(e^{-ip_\mu \eta} - 1)} \right]. \tag{A.1}$$

We can easily deduce the explicit representation corresponding to (4.2) of [K 1]:

$$(a_k G_k Q_k^*)(x, y) = (2\pi)^{-d} \int_{|p'| \leq \pi} dp' \Delta^{(k)}(p') \sum_l e^{i(p'+l)(x-y)} \cdot \frac{\tilde{f}_\eta(p'+l) u_{k-1}^{L\eta}(p'+l)}{\Delta^\eta(p'+l)}. \tag{A.2}$$

The terms in (A.2) have the same meaning as the corresponding terms in (4.2) of [K 1], with the following change:

$$\Delta^{(k)}(p') = \left(a_k^{-1} + \sum_l |\tilde{f}_\eta(p'+l)|^2 |u_{k-1}^{L\eta}(p'+l)|^2 (\Delta^\eta(p'+l))^{-1} \right)^{-1}. \tag{A.3}$$

Lemma 5.3 follows by exactly the same considerations used in [Ba 4] with $f=1$. The integrand in (A.2) may be analytically continued in a strip around the real p'_μ-axis, giving exponential decay. The sum over l is controlled by the estimates

$$\begin{aligned} |\tilde{f}_\eta(p'+l) u_{k-1}^{L\eta}(p'+l)| &\approx \prod_\mu |(p'+l)_\mu|^{-1}, \\ \Delta^\eta(p'+l)^{-1} &\approx |p'+l|^{-2}. \end{aligned} \tag{A.4}$$

Furthermore, suppose that we have D b.c. on the hyperplane $x_\mu = 0$. We wish to prove that

$$|\eta^{-1}(G_k(x, y) - G_k(x, y - 2y_\mu e_\mu))| \leq C, \tag{A.5}$$

when $y_\mu = \eta$ (this shows that the propagator with D b.c. goes to zero near the boundary; G_k in (A.5) has free boundary conditions).

It is sufficient to show that

$$|\eta^{-1}(a_k G_k Q_k^*)(x, y) - a_k G_k Q_k^*(x, y - 2y_\mu e_\mu))| \leq C. \tag{A.6}$$

From (A.2), we get the factor

$$|\eta^{-1}(e^{-i(p'+l)_\mu \eta} - e^{i(p'+l)_\mu \eta})| \leq C|p'+l|. \tag{A.7}$$

There are sufficient negative powers of $|p'+l|$ in the sum over l in (A.2) to control (A.7). Therefore, (A.6) holds and hence also (A.5).

In order to prove the necessary convergence estimates for (A.2) which were proved with $f=1$ in Sect. 4 of [K 1], we specify below the alterations needed to include f. We leave unchanged (4.6) and the conditions (4.7), (4.8). In order to reprove Lemma 4.1, we consider

$$\begin{aligned} &|\tilde{f}_\eta(p)|^2|u_{k-1}^{L\eta}(p)|^2-|u_k^\eta(p)|^2 \\ &\quad =|u_{k-1}^{L\eta}(p)|^2 \sum_{x,\omega\in B(y)} L^{-2d}[f(x)f(\omega)-1]e^{-ip(x-\omega)} \\ &\quad =2|u_{k-1}^{L\eta}(p)|^2 \sum_{x,\omega\in B(y)} L^{-2d}[1-f(x)f(\omega)]\sin^2[\tfrac{1}{2}p(x-\omega)], \end{aligned} \tag{A.8}$$

since $\tilde{f}(0)=1$. Hence

$$\begin{aligned} &||\tilde{f}_\eta(p)|^2|u_{k-1}^{L\eta}(p)|^2-|u_k^\eta(p)|^2| \\ &\quad \leqq C|u_{k-1}^{L\eta}(p)|^2 \sum_{\mu,\nu=1}^{d} p_\mu p_\nu\left\{\sum_{x,\omega\in B(y)} L^{-2d}(x-\omega)_\mu(x-\omega)_\nu\right\} \\ &\quad \leqq CL^{-2k}|p|^2|u_{k-1}^{L\eta}(p)|^2. \end{aligned} \tag{A.9}$$

Using (A.9) and the estimates (4.10), (4.11) of [K 1], and also $\sum_l |u_{k-1}^{L\eta}(p'+l)|^2=L^d$, we deduce Lemma 4.1 of [K 1]. Next, we again notice that $\Delta^{(k+n)}(p')$ can be written in the form of $\bar{\Delta}^{(k)}(p')$ with

$$D^{-1}(p)=a_n^{-1}L^{-2k}+\sum_{l'}|\tilde{f}_{\eta'}(p+l')|^2|u_{n-1}^{L\eta'}(p+l')|^2\Delta^{\eta'}(p+l')^{-1}. \tag{A.10}$$

Also, we have

$$\begin{aligned} |\tilde{f}_{\eta'}(p)|^2|u_{n-1}^{L\eta'}(p)|^2 &= \sum_{x,\omega\in B^n(y)} L^{-2nd}f(x)f(\omega)e^{-ip(x-\omega)} \\ &=1+\sum_{x,\omega\in B^n(y)} L^{-2nd}f(x)f(\omega)[e^{-ip(x-\omega)}-1]. \end{aligned} \tag{A.11}$$

Using (A.10) and (A.11), we can follow the proof of Lemma 4.2 in [K 1], and deduce convergence of $\Delta^{(k)}(p')$. The rest of Sect. 4 of [K 1] goes through with virtually no changes, and so we deduce the convergence lemmas needed for the proof of the continuum limit.

Appendix B

We denote by $\hat{G}_k$ the propagator generated by the averaging operators (4.3). Consider a region Ω given by

$$\Omega=\{x\in\mathbb{R}^d:\ 0<x_\mu\leqq 25L_\mu,\ \mu=1,\dots,d\}, \tag{B.1}$$

where $\{L_\mu\}$ are integers. We denote by T_η the torus with lattice spacing $\eta=5^{-k}$ which fits inside Ω. The orientation of this lattice is the same as that of $\partial\Omega$. The lattice operator $\hat{G}_k(\Omega)$ on the torus T_η is then defined by the equation

$$(-\Delta^\eta+m^2(L^k\varepsilon)^2+a_k\hat{Q}^*Q_{k-1}^*Q_{k-1}\hat{Q})\hat{G}_k(\Omega)f=f, \tag{B.2}$$

for any function f on T_η. We can now state the multiple reflection representation for $\hat{G}_k(\Omega)$ (the operator $\hat{G}_k$ is defined with free boundary conditions):

$$\hat{G}_k(\Omega)(x,y)=\sum_{n\in\mathbb{Z}^d}\hat{G}_k\left(x,y+\sum_{\mu=1}^{d}25n_\mu L_\mu e_\mu\right). \tag{B.3}$$

Therefore, it is sufficient to prove regularity and decay estimates and convergence properties for $\hat{G}_k$. We will denote by $\eta\mathbb{Z}^d$ the original lattice on which $\hat{G}_k$ is defined, and by $L^j\eta\hat{\mathbb{Z}}^d$ the rotated lattices defined by the renormalization transformation, for $j=1, \ldots, k$. The dual of $\eta\mathbb{Z}^d$ is P_η; the dual of $L^j\eta\hat{\mathbb{Z}}^d$ is $\hat{P}_{L^j\eta}$ (of course, $L=5$ here). We introduce the function

$$w_\eta(p)=5^{-d} \sum_{x\in\hat{B}(y)} e^{-ip\cdot(x-y)}, \tag{B.4}$$

for $p\in P_\eta$. By introducing a Fourier representation on $\eta\mathbb{Z}^d$, we deduce the following explicit formula:

$$\begin{aligned}(a_k\hat{G}_k\hat{Q}^*Q^*_{k-1})(x,y)=(2\pi)^{-d} &\int_{p'\in\hat{P}_1} dp'\hat{\Delta}^{(k)}(p')\\ &\cdot\sum_l e^{i(p'+l)\cdot(x-y)}\frac{w_\eta(p'+l)u^{L\eta}_{k-1}(p'+l)}{\Delta^\eta(p'+l)},\end{aligned} \tag{B.5}$$

where

$$\hat{\Delta}^{(k)}(p')=\left(a_k^{-1}+\sum_{l'}|w_\eta(p'+l')|^2|u^{L\eta}_{k-1}(p'+l')|^2\Delta^\eta(p'+l')^{-1}\right)^{-1}, \tag{B.6}$$

and we sum l, l' over $2\pi\hat{\mathbb{Z}}^d$, such that $p'+l$ or $p'+l'$ belongs to P_η. Using the representation (B.5), we can deduce the regularity and decay estimates for $\hat{G}_k$ in the same way as in Appendix A. Similarly, the convergence proofs go through in almost the same way. We will give the proof of convergence for $\tilde{\Delta}^{(k)}(p')-\Delta^{(k)}(p')$; the other estimates then follow straightforwardly (see Sect. 4 of [K 1]). First, we have

$$\Delta^{(k)}(p')=\left(a_k^{-1}+\sum_{l'}|u^\eta_k(p'+l')|^2\hat{\Delta}^\eta(p'+l')^{-1}\right)^{-1}, \tag{B.7}$$

where $\hat{\Delta}^\eta(p)$ is the Laplacian on the lattice $\eta\hat{\mathbb{Z}}^d$, and l' is summed over $2\pi\hat{\mathbb{Z}}^d$ such that $p'+l'\in\hat{P}_\eta$. It is clear that

$$\Delta^{(k)}(p'), \hat{\Delta}^{(k)}(p')\leqq a\,. \tag{B.8}$$

So we need only prove convergence of $(\Delta^{(k)}(p')^{-1}-\hat{\Delta}^{(k)}(p')^{-1})$. This is done by comparing similar terms in (B.6) and (B.7). We will use μ, α to denote components taken with respect to $\eta\hat{\mathbb{Z}}^d, \eta\mathbb{Z}^d$. Then

$$\begin{aligned}\hat{\Delta}^\eta(p'+l')&=4\eta^{-2}\sum_\mu \sin^2[\tfrac{1}{2}\eta(p'+l')_\mu]+m^2(L^k\varepsilon)^2\,,\\ \Delta^\eta(p'+l')&=4\eta^{-2}\sum_\alpha \sin^2[\tfrac{1}{2}\eta(p'+l')_\alpha]+m^2(L^k\varepsilon)^2\,.\end{aligned} \tag{B.9}$$

Using $x^2\geqq\sin^2x\geqq x^2-\frac{1}{3}x^4$, we get

$$|\hat{\Delta}^\eta(p'+l')-\Delta^\eta(p'+l')|\leqq CL^{-2k}[(p'+l')^2]^2\,. \tag{B.10}$$

Hence we deduce

$$|\hat{\Delta}^\eta(p+l')^{-1}-\Delta^\eta(p'+l')^{-1}|\leqq CL^{-2k}\,. \tag{B.11}$$

It is also straightforward to show that

$$|1-|w_\eta(p)|^2|\leqq CL^{-2k}|p|^2\,. \tag{B.12}$$

Combining these estimates with the results obtained in Sect. 4 of [K 1], we deduce

$$|\hat{\Delta}^{(k)}(p') - \Delta^{(k)}(p')| \leqq CL^{-2k}\Delta^{(k)}(p'), \tag{B.13}$$

which is the required result.

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