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The Loop Expansion for the Effective Potential in the $P(\phi)_2$ Quantum Field Theory*

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Abstract. We study the loop expansion for the effective potential, defined as the Fenchel transform (convex conjugate) of the pressure in an external field, in the $P(\phi)_2$ quantum field theory. For values of the classical field a for which the classical potential $U_0(a)=P(a)+\frac{1}{2}m^2a^2$ equals its convex hull and has nonvanishing curvature we prove that the 1-PI loop expansion is asymptotic as $\hbar \downarrow 0$. We also give an example of a double well classical potential for which the 1-PI loop expansion fails to be asymptotic, and find the true asymptotics.

1. Introduction

The effective potential for the $P(\phi)_2$ Euclidean quantum field theory is defined as the Fenchel transform of the pressure in an external field:

$$V(\hbar, a)=\sup_{\mu\in\mathbb{R}}[\mu a-p(\hbar,\mu)]. \tag{1.1}$$

Here the positive parameter $\hbar$ is Planck's constant divided by 2π, the classical field a is real, and $p(\hbar,\mu)$ is given by

$$p(\hbar,\mu)=\hbar \lim_{\Lambda\uparrow\mathbb{R}^2}\frac{1}{|\Lambda|}\ln\int\exp\left[\frac{-1}{\hbar}\int_\Lambda[:P(\phi(x)):-\mu\phi(x)]\,dx\right]d\mu_{\hbar C}, \tag{1.2}$$

where $C=(-\Delta+m^2)^{-1}$ for some $m^2>0$, $d\mu_{\hbar C}$ is Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with covariance $\hbar C$, the Wick order is with respect to $\hbar C$, and $\Lambda\uparrow\mathbb{R}^2$ through a sequence of rectangles. In [14] the limit (1.2) is shown to exist for a wide variety of boundary conditions on $\partial\Lambda$, in particular for periodic boundary conditions which we will use unless otherwise indicated.

The importance of the effective potential is that it characterizes the occurrence of phase transitions in the theory [2, 16]: linear portions of $V(\hbar,\cdot)$ are in a one-

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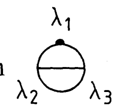
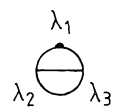
one correspondence with points of nondifferentiability of $p(\hbar, \cdot)$, and hence with discontinuities in the one point function $D_2 p(\hbar, \cdot)$ [9].

Since $p(\hbar, \cdot)$ is strictly convex [9], if the supremum in Eq. (1.1) is attained it is attained at a unique $\mu(\hbar, a)$. But by the large external field cluster expansion of [19] if $\eta(\mu)$ is the location of the unique (for large $|\mu|$) minimum of $P(x)-\mu x$ then $|D_2 p(\hbar, \mu)-\eta(\mu)|$ is bounded uniformly in large $|\mu|$, and hence, $D_2^{+} p(\hbar, \mu) \to \pm\infty$ as $\mu \to \pm\infty$. It follows that the supremum in (1.1) is attained for all $\hbar, a$, at the unique μ for which $a \in [D_2^{-} p(\hbar, \mu), D_2^{+} p(\hbar, \mu)]$, where $D_2^{(\pm)}$ denotes the right (left) derivative.

The most common method for calculation of the effective potential is to approximate it by the first few terms of the loop expansion [2, 15], which provides a power series in $\hbar$: $V(\hbar, a) \approx \sum_{n=0}^{N} v_n(a)\hbar^n$. In [2, 15] it is argued that the coefficients $v_n(a)$ are given by sums of one-particle irreducible (1-PI) n-loop Feynman diagrams. In particular, $v_0(a)$ is given by the classical potential $U_0(a)=P(a)+\frac{1}{2}m^2a^2$ so that $V(\hbar, a)$ is in some sense a quantum analogue of $U_0(a)$. The main results of this paper are a proof that for the $P(\phi)_2$ model the usual loop expansion is asymptotic as $\hbar \downarrow 0$ for those values of a at which U_0 has nonvanishing second derivative and is equal to its convex hull, and an example of a double well classical potential for which the usual expansion fails to be asymptotic. In the example it is shown that for values of the classical field lying between the minima of the classical potential the true asymptotic expansion of $V(\hbar, a)$ involves connected rather than 1-PI n-loop diagrams.

Before stating the main results we describe the graph notation used in this paper. To begin with an example and a fixed translation invariant covariance $C(x, y)=C(x-y)$, the graph 

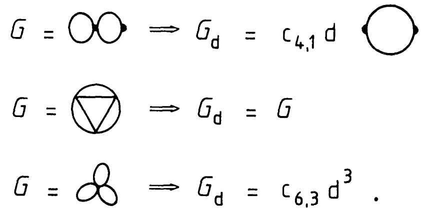
is by definition equal to



$$=\lambda_1\lambda_2\lambda_3 \int dx_1 dx_2 C(0, x_1)C(0, x_2)C(x_1, x_2)^2 . \tag{1.3}$$

The right side of (1.3) is obtained from the left side by identifying any one vertex as the origin in $\mathbb{R}^2$ and associating with the remaining vertices the variables x_1 and x_2. To every line there corresponds a factor of C evaluated at the endpoints of the line. These factors are multiplied by the vertex factors λ_i. This procedure is followed to obtain the value of any graph. Usually the vertex factors depend only on the number of lines emanating from a vertex and are understood to be part of the graph without writing them explicitly. Graphs also usually include combinatoric factors as explained below. Some standard terminology is: A *self-line* is a line connecting a vertex to itself, a *connected graph* is a graph for which any two vertices are path connected by lines, a *one-particle irreducible* (1-PI) *graph* is a connected graph such that the removal of any one line leaves a connected graph, a *one-particle reducible* (1-PR) *graph* is a graph that is not 1-PI, and a graph having L lines and V vertices is an *n-loop graph*, where $n=L-V+1$. Finally, we need the following definition.

Definition 1.1. Given a graph $\mathscr{G}$ and $d \in \mathbb{R}$, the *d-renormalized graph* $\mathscr{G}_d$ is the graph obtained by removing all self-lines from $\mathscr{G}$, introducing a vertex factor d for each removed self-line, and introducing a factor $c_{k,j}$ for every k-legged vertex of $\mathscr{G}$ having j self-lines, where $c_{k,j} = \dfrac{k!}{2^j j!\,(k-2j)!}$. For example,



The following definition introduces the set B governing the asymptotics of $V(\hbar, a)$; $\operatorname{conv} f$ denotes the convex hull of a function f.

Definition 1.2. For $U_0(a) = P(a) + \frac{1}{2}m^2 a^2$, define

$$B_1 = \{a \in \mathbb{R}\colon\ U_0(a) \neq (\operatorname{conv} U_0)(a)\}\,, \qquad B_2 = \{a \in \mathbb{R}\colon\ U_0''(a) = 0\}\,, \qquad B = B_1 \cup B_2\,.$$

In the remainder of this section we state the main results and comment on their proofs.

Theorem A. $\lim\limits_{\hbar \downarrow 0} V(\hbar, a) = (\operatorname{conv} U_0)(a)$.

This theorem is proved in Sect. 4 by first using the elementary convex analysis of Sect. 2 to show that the limit can be taken under the supremum, reducing the problem to finding $\lim\limits_{\hbar \downarrow 0} p(\hbar, \mu)$. This limit is found to be $-m(\mu) = -\min\limits_{x} U_\mu(x)$, where $U_\mu(x) = U_0(x) - \mu x$, by translating the field ϕ in the functional integral (1.2) by the location $\xi(\mu)$ of the global minimum of U_μ (which is unique for all but finitely many μ) to obtain

$$p(\hbar, \mu) = -m(\mu) + \hbar \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_\Lambda \sum_{k=2}^{n} \frac{P^{(k)}(\xi(\mu))}{k!} :\phi^k:\right] d\mu_{\hbar C}\,, \tag{1.4}$$

where we drop the variable x of integration from the interaction. For those μ such that U_μ has a uniquely attained global minimum and $U_\mu''(\xi(\mu)) \neq 0$ it will be shown in Sect. 4 using an estimate of Sect. 3 that the argument of the logarithm in (1.4) can be bounded above uniformly in Λ and small $\hbar$ by $e^{K|\Lambda|}$, using the fact that for some $\delta(\mu) > 0$,

$$\sum_{k=2}^{n} \frac{P^{(k)}(\xi(\mu))}{k!} x^k + \tfrac{1}{2}m^2 x^2 = U_\mu(x + \xi(\mu)) - U_\mu(\xi(\mu)) > \delta(\mu)(x^2 + x^n)\,,$$

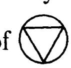
where $n = \deg P$. Since by Jensen's inequality the argument of the logarithm is bounded below by 1, $\lim\limits_{\hbar \downarrow 0} p(\hbar, \mu) = -m(\mu)$.

Theorem B. (a) *Let $a \notin B$. Then there exists a $\gamma > 0$ such that $V(\hbar, a)$ is analytic in $\hbar$ for $\hbar \in (0, \gamma)$. Moreover, $V(\hbar, a)$ is C^∞ at $\hbar = 0^+$, and so the expansion $V(\hbar, a) \sim \sum_{n=0}^{\infty} v_n(a)\hbar^n$ is asymptotic, where $v_n(a) = D_1^n V(0^+, a)/n!$.*

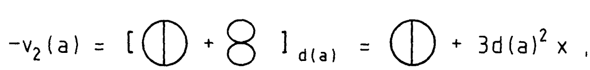
(b) *Let $a \notin B$. Then $v_0(a) = U_0(a)$ and*

$$v_1(a) = -\gamma(a) \equiv -\lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int \exp\left[-\int_\Lambda \frac{P''(a)}{2} :\phi^2: \right] d\mu_C.$$

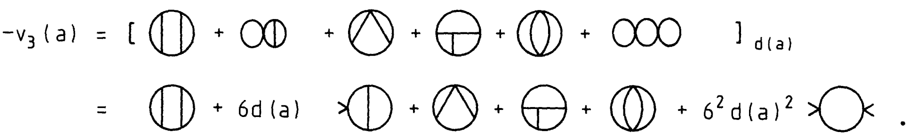
For $n \geqq 2$, $-v_n(a)$ is the (finite) sum of all $d(a)$-renormalized 1-PI *n-loop diagrams with k-legged vertices taking factors $-P^{(k)}(a)/k!$ ($3 \leqq k \leqq \deg P$) and lines corresponding to the free covariance of mass $U_0''(a)^{1/2}$, where $d(a) = -\frac{1}{4\pi}\log\frac{U_0''(a)}{m^2}$. A combinatorial factor is associated with each graph – see Remark 1 below.*

Remark 1. The renormalized graphs in $-v_n(a)$ are to be understood to include combinatorial factors. Given a renormalized graph, let V_{kj} be the number of vertices that originally had k legs and have been renormalized with the removal of $j \geqq 0$ self-lines. The combinatorial factor for the graph is the factor associated with the graph by Wick's theorem divided by $\prod_{j,k} V_{jk}!$. For example, the combinatorial factor of  is $\frac{1}{3!}1728 = 288$.

As an example of Theorem B we obtain a renormalized (and rigorous) version of a result of [15]. Let $U_0(x) = x^4 + \frac{1}{2}x^2$ and $P(x) = x^4$. Then $B = \phi$ and for $d(a) = -\frac{1}{4\pi}\log(1 + 12a^2)$,



and



Lines are $(-\Delta + 1)^{-1}$ lines and 3- and 4-legged vertices take factors $-4a$ and -1, respectively. Amputated legs have been partly drawn to keep clear what the vertex factors should be.

The proof of Theorem B(a), given in Sect. 4, involves translation of the field ϕ in $p(\hbar, \mu)$ in (1.1) by a to obtain a new pressure having vertices as in Theorem B(b), together with some elementary convex analysis (Sect. 2) which reduces the study of smoothness of $V(\hbar, a)$ in $\hbar$ to smoothness of the translated pressure in both $\hbar$ and the external field. Smoothness of the translated pressure is obtained via a high temperature cluster expansion [12] whose convergence is shown to follow in Sect. 3 from the $e^{K|\Lambda|}$ upper bound on the partition function used in the proof of Theorem A. The $a \notin B$ requirement is needed for this bound. Theorem B(b) is proved in Sect. 5 using an irreducibility analysis in the spirit of [3].

Similar methods can be used to show that for any compact $K \subset B^c$ there is a $\varrho > 0$ such that $V(\hbar, \cdot)$ is analytic in an open neighborhood of K, for all $\hbar < \varrho$.

Finally, in Sect 6 we prove the following result which gives an asymptotic expansion for $V(\hbar, a)$ when a is the bad set B, for the classical potential $U_0(a) = (a^2 - \frac{1}{8})^2$.

Theorem C. *Let $V(\hbar, a)$ denote the effective potential for $m=1$ and $P(x) = \left(x^2 - \frac{1}{8}\right)^2 - \frac{1}{2}x^2$. Then for $|a| < \frac{1}{\sqrt{8}}$, $D_1 V(0^+, a) = -\gamma\left(\frac{1}{\sqrt{8}}\right) = 0$, and for $n \geqq 2$, $-\frac{1}{n!} D_1^n V(0^+, a)$ is given by the sum of all n-loop <u>connected</u> graphs with no self-lines, with three- and four-legged vertices taking factors $\frac{-1}{3!} P^{(3)}\left(\frac{1}{\sqrt{8}}\right) = -\sqrt{2}$ and $\frac{-1}{4!} P^{(4)}\left(\frac{1}{\sqrt{8}}\right) = -1$, respectively, and lines corresponding to the free covariance of mass 1. Graphs take combinatorial factors as per Remark 1.*

A number of authors [10, 1, 4] have recently calculated the $O(\hbar)$ contribution to the effective potential corresponding to the classical potential considered in Theorem C, and find it to be the straight line interpolation of the $O(\hbar)$ approximation given for $|a| > \frac{1}{\sqrt{8}}$ by Theorem B. Theorem C gives a rigorous justification of this fact; the proof is an easy consequence of using the Fenchel transform to define $V(\hbar, a)$ and the known fact that there is a phase transition in this model if $\hbar$ is sufficiently small [13].

2. Preliminaries

In this section we prove some elementary theorems that will be used to reduce the study of smoothness of the effective potential to smoothness of the corresponding pressure, and comment on some properties of the classical potential.

For $f: \mathbb{R} \to \mathbb{R}$, its convex conjugate or Fenchel transform f^* is given by

$$f^*(a) = \sup_{\mu \in \mathbb{R}} [\mu a - f(\mu)]. \tag{2.1}$$

Denote by $\mathscr{C}_s$ the set of strictly convex functions f for which $\lim_{\mu \to \pm\infty} D^{\pm} f(\mu) = \pm\infty$. Then for $f \in \mathscr{C}_s$ the supremum in (2.1) is finite and attained at the unique μ for which $a \in [D^- f(\mu), D^+ f(\mu)]$.

Theorem 2.1. *Suppose $f(\hbar, \cdot)$ and f are in $\mathscr{C}_s$ for all $\hbar > 0$, and suppose $\lim_{\hbar \downarrow 0} f(\hbar, \mu) = f(\mu)$ for all μ. Denote by $\mu(\hbar, a)$ and $\mu(a)$ the unique values of μ where $\mu a - f(\hbar, \mu)$ and $\mu a - f(\mu)$ attain their suprema. Then $\lim_{\hbar \downarrow 0} \mu(\hbar, a) = \mu(a)$ and $\lim_{\hbar \downarrow 0} f^*(\hbar, a) = f^*(a)$.*

Proof. We first prove that $\lim_{\hbar \downarrow 0} \mu(\hbar, a) = \mu(a)$. Fix $a \in R$ and $\varepsilon > 0$. Choose $\varrho \in (0, \varepsilon)$ such that $Df(\mu(a) \pm \varrho)$ exist. Let

$$\alpha = \tfrac{1}{2} \min \{Df(\mu(a) + \varrho) - D^+ f(\mu(a)), D^- f(\mu(a)) - Df(\mu(a) - \varrho)\}.$$

Since f is strictly convex, $\alpha>0$. Then $Df(\mu(a)+\varrho)>D^+f(\mu(a))+\alpha\geqq a+\alpha$ and similarly $Df(\mu(a)-\varrho)<a-\alpha$. By convexity, $\lim_{\hbar\downarrow 0} D_2^{\pm}f(\hbar,\mu)=Df(\mu)$ if $Df(\mu)$ exists, so there is a $\delta>0$ such that

$$|D^{\pm}f(\hbar,\mu(a)\pm\varrho)-Df(\mu(a)\pm\varrho)|<\frac{\alpha}{2}\quad\text{for all}\quad \hbar<\delta\,.$$

Therefore, $D_2^-f(\hbar,\mu(a)-\varrho)<a<D_2^+f(\hbar,\mu(a)+\varrho)$ for all $\hbar<\delta$ and so $\mu(\hbar,a)\in[\mu(a)-\varrho,\mu(a)+\varrho]$ for all $\hbar<\delta$, and hence $\lim_{\hbar\downarrow 0}\mu(\hbar,a)=\mu(a)$.

Now

$$\begin{aligned}|f^*(\hbar,a)-f^*(a)|&=\left|\sup_{\mu}[\mu a-f(\hbar,\mu)]-\sup_{\mu}[\mu a-f(\mu)]\right|\\&\leqq\sup_{\mu\in[\mu(a)-\varrho,\,\mu(a)+\varrho]}|f(\hbar,\mu)-f(\mu)|\end{aligned}$$

for any $\hbar<\delta$. Since $f(\hbar,\mu)\to f(\mu)$ uniformly in compact intervals, the right side goes to zero as $\hbar\downarrow 0$.

Theorem 2.2. *Suppose $f(\hbar,\cdot)$ and f belong to the set $\mathscr{C}_s$, with $\lim_{\hbar\downarrow 0}f(\hbar,\mu)=f(\mu)$ for all $\mu\in R$. Fix a and suppose that for some $\lambda>0$ there is an open interval I containing $\mu(a)$, such that f is analytic in $(\hbar,\mu)\in(0,\lambda)\times I\subset\mathbb{C}^2$ and*

$$|D_2^2f(\hbar,\mu)|\geqq C>0\quad\textit{for every}\quad(\hbar,\mu)\in(0,\lambda)\times I\,. \tag{2.2}$$

Then for some $\gamma'>0$, $f^*(\hbar,a)$ is analytic in $\hbar\in(0,\gamma')$. If, in addition, there are constants $M_{m,n}$ such that

$$|D_1^mD_2^nf(\hbar,\mu)|\leqq M_{m,n}\quad\textit{for every}\quad(\hbar,\mu)\in(0,\gamma)\times I\,;\qquad m,n=0,1,2,\ldots, \tag{2.3}$$

then $f^*(\hbar,a)$ is C^∞ at $\hbar=0^+$.

Proof. By Theorem 2.1 we can choose $\gamma'<\gamma$ such that $\mu(\hbar,a)\in I$ if $\hbar<\gamma'$. Also, it follows from analyticity of f and the bound (2.1) that there is a neighborhood $0_\gamma\supset(0,\gamma)\times I$ on which $|D_2^2f(\hbar,\mu)|>\frac{C}{2}$. Let $g(\hbar,\mu)=\frac{\partial}{\partial\mu}[\mu a-f(\hbar,\mu)]$ $=a-D_2f(\hbar,\mu)$ for $(\hbar,\mu)\in V_{\gamma'}$ where we set $V_{\gamma'}=0_\gamma\cap\{(\hbar,\mu)\in\mathbb{C}^2:\ 0<\mathrm{Re}\hbar<\gamma'\}$. Then $\mu(\hbar,a)$ is uniquely defined by $g(\hbar,\mu(\hbar,a))=0$, for $\hbar<\gamma'$. By the fact that $D_2^2f(\hbar,\mu)\geqq\frac{C}{2}$ in 0_γ and the implicit function theorem it follows that $\mu(\hbar,a)$ is analytic in $\hbar$ in an open neighborhood $U_{\gamma'}\supset(0,\gamma')$, with $(\hbar,\mu(\hbar,a))\in V_{\gamma'}$ for all $\hbar\in U_{\gamma'}$. Therefore, $f^*(\hbar,a)=\mu(\hbar,a)a-f(\hbar,\mu(\hbar,a))$ is analytic in $\hbar\in U_{\gamma'}$.

Differentiating the equation $g(\hbar,\mu(\hbar,a))=0$ with respect to $\hbar$ gives

$$D_1\mu(\hbar,a)=\frac{-D_1D_2f(\hbar,\mu(\hbar,a))}{D_2^2f(\hbar,\mu(\hbar,a))}\,.$$

The bounds (2.2) and (2.3) imply that $|D_1\mu(\hbar,a)|$ is bounded uniformly in $\hbar\in(0,\gamma')$. Repeated differentiation shows that $|D_1^n\mu(\hbar,a)|$ is also bounded in $\hbar\in(0,\gamma')$. These bounds on $D_1^n\mu(\hbar, a)$ and the bound (2.3) imply that $|D_1^n f^*(\hbar, a)|$ is bounded uniformly in $\hbar \in (0, \gamma')$ and hence $D_1^n f^*(0^+, a)$ exists and equals $\lim_{\hbar \downarrow 0} D_1^n f^*(\hbar, a)$. □

We now turn our attention to establishing some elementary properties of the classical potential U_0. For $U_0(x) = P(x) + \frac{1}{2}m^2x^2$, let $U_\mu(x) = U_0(x) - \mu x$. Let $G_1 = \{\mu \in \mathbb{R}: U_\mu$ has a uniquely attained global minimum$\}$, and for $\mu \in G_1$ denote the location of the minimum by $\xi(\mu)$. It is not hard to see that G_1^c is finite. Define $F = \{\mu \in G_1: U_0''(\xi(\mu)) = 0\}$ and $G = G_1 \backslash F$. Since ξ is strictly increasing, F is finite, and hence G^c is finite. Let $m(\mu) = \min_x U_\mu(x)$. Then for $\mu \in G_1$, $m(\mu) = U_\mu(\xi(\mu))$.

Lemma 2.3. *The functions m and ξ are analytic on G, with $m'(\mu) = -\xi(\mu)$ and $\xi'(\mu) = \dfrac{1}{U_0''(\xi(\mu))}$. Furthermore, ξ is strictly increasing on G_1, continuous on G_1, and discontinuous on G_1^c, $\lim_{\mu \to \pm\infty} \xi(\mu) = \pm\infty$, and $-m \in \mathscr{C}_s$.*

Proof. The derivative U_0' is an entire function, and for $\mu \in G$, $U_0'(\xi(\mu)) = \mu$ and $U_0''(\xi(\mu)) > 0$. By the Inverse Function Theorem there are open neighborhoods O containing μ and V containing $\xi(\mu)$ such that $U_0'|_V$ is invertible and the inverse is analytic on O. This inverse is an extension of ξ. Since for $\mu \in G$, $m(\mu) = U_\mu(\xi(\mu)) = U_0(\xi(\mu)) - \mu\xi(\mu)$, m is also analytic on G with $m'(\mu) = -\xi(\mu)$. To calculate $\xi'(\mu)$, differentiate the equation $U_0'(\xi(\mu)) = \mu$ with respect to μ to obtain $\xi'(\mu) = \dfrac{1}{U_0''(\xi(\mu))}$.

The fact that ξ is strictly increasing and discontinuous on G_1^c is clear from the definition of ξ. It is also easy to see that ξ is continuous on F, and hence on G_1. For large μ, $\xi(\mu)$ is the unique root of $U_0'(x) = \mu$. As $\mu \to \pm\infty$ that root diverges to $\pm\infty$, so $\lim_{\mu \to \pm\infty} \xi(\mu) = \pm\infty$. This last fact, together with the strict monotonicity of ξ and the equation $-m'(\mu) = \xi(\mu)$, implies that $-m \in \mathscr{C}_s$. □

Lemma 2.4. *$B^c = \xi(G)$.*

Proof. Suppose $a \in \xi(G)$. Then there is a $\mu_a \in G$ such that $\xi(\mu_a) = a$. Since $U_0''(a) = U_0''(\xi(\mu_a)) > 0$, $a \notin B_2$. We now show $a \notin B_1$. Now $(\operatorname{conv} U_0)(a) = U_0^{**}(a) = \sup_\mu [\mu a - U_0^*(\mu)]$. Since

$$U_0^*(\mu) = \sup_x [\mu x - U_0(x)] = -\min_x U_\mu(x) = -m(\mu), \tag{2.4}$$

$(\operatorname{conv} U_0)(a) = \sup_\mu [\mu a + m(\mu)]$. But $-m$ is differentiable at μ_a and $D(-m)(\mu_a) = \xi(\mu_a) = a$. Since $-m \in \mathscr{C}_s$, this implies that

$$(\operatorname{conv} U_0)(a) = \mu_a a + m(\mu_a) = \mu_a a + U_{\mu_a}(\xi(\mu_a)) = U_0(\xi(\mu_a)) = U_0(a). \tag{2.5}$$

Since G is a union of open intervals and ξ is strictly increasing and continuous on G, $\xi(G)$ is a union of open intervals. Together with Eq. (2.5), this implies that $a \notin B_1$. Hence $\xi(G) \subset B^c$.

On the other hand, let $a \in B^c$. Suppose contrary to the statement of the lemma that $a \notin \xi(G)$, i.e., $a \in \xi(F)$ or $a \in \xi(G_1)^c$. If $a \in \xi(F)$, then $U_0''(a) = 0$ so $a \in B_2$. Therefore, $a \in \xi(G_1)^c$. By Lemma 2.3 there must be a $\mu_0 \in G_1^c$ for which

$a \in [\xi(\mu_0^-), \xi(\mu_0^+)] \subset \xi(G_1)^c$. The interval $[\xi(\mu_0^-), \xi(\mu_0^+)]$ is nontrivial since $\mu_0 \in G_1^c$ is a point where ξ undergoes a jump discontinuity. Since $\xi(\mu_0^\pm) = D^\pm(-m)(\mu_0)$ by Lemma 2.3, $a \in [D^-(-m)(\mu_0), D^+(-m)(\mu_0)] \subset \xi(G_1)^c$. It follows from the fact that $a \in B^c$ and Eq. (2.4) that

$$\begin{aligned} U_0(a) &= U_0^{**}(a) = (-m)^*(a) \\ &= \mu_0 a + m(\mu_0) \quad \text{for all} \quad a \in [D^-(-m)(\mu_0), D^+(-m)(\mu_0)]\,. \end{aligned}$$

But this is impossible because U_0 cannot have a linear segment. □

Definition 2.5. For $\delta, L > 0$ denote by $\mathcal{T}_{\delta,L}$ the set of all polynomials

$$T(x) = \sum_{k=2}^{n} t_k x^k \quad \text{with} \quad |t_k| \leqq L \quad (k = 2, \ldots, n)$$
$$\text{and} \quad T(x) \geqq \delta(x^n + x^2) \quad \text{for all } x\,.$$

Lemma 2.6. *Suppose $T(x) = \sum_{k=2}^{n} t_k x^k$ attains its global minimum at $x = 0$ only, where $t_n, t_2 > 0$. Then there exist $\delta, L > 0$ such that $T \in \mathcal{T}_{\delta,L}$.*

Proof. Let $L = \max\{|t_k| : 2 \leqq k \leqq n\}$. For large $|x|$, say $|x| > A$, there is a $\delta_1 > 0$ such that $T(x)$ is bounded below by $\delta_1 x^n$, while for small $|x|$, say $|x| < \varepsilon$, there is a $\delta_2 > 0$ such that $T(x)$ is bounded below by $\delta_2 x^2$. Let $a = \min_{\varepsilon \leqq |x| \leqq A} T(x) > 0$. Then for $|x| \in [\varepsilon, A]$, $T(x) \geqq a \geqq \frac{a}{2A^n}(x^2 + x^n)$. Let $\delta = \min\left\{\frac{1}{2}\delta_1, \frac{1}{2}\delta_2, \frac{a}{2A^n}\right\}$. □

3. The Main Estimates

To prove analyticity in $\hbar$ for the effective potential and obtain the desired form for the derivatives at $\hbar = 0$ it is convenient to perform a change of variable, so as to explicitly isolate the leading term. Let $C = (-\Delta + m^2)^{-1}$ with periodic BC on $\partial\Lambda$ and recall that $U_\mu(x) = U_0(x) - \mu x$, where $U_0(x) = P(x) + \frac{1}{2}m^2 x^2$. Fix $a \in \mathbb{R}$. Translating ϕ by a gives [11]

$$\begin{aligned} &\int \exp\left[\frac{-1}{\hbar} \int_\Lambda [:P(\phi): - \mu\phi]\right] d\mu_{\hbar C} \\ &= \exp\left[\frac{-1}{\hbar}|\Lambda| U_\mu(a)\right] \int \exp\left[\frac{-1}{\hbar} \int_\Lambda \left[\sum_{k=1}^{n} \frac{U_\mu^{(k)}(a)}{k!} :\phi^k: - \frac{1}{2}m^2 :\phi^2:\right]\right] d\mu_{\hbar C}\,. \end{aligned} \tag{3.1}$$

Here and throughout this paper the Wick dots appearing in an integrand are with respect to the covariance of the measure unless otherwise indicated. By definition of the pressure in Eq. (1.2), Eq. (3.1) implies

$$p(\hbar, \mu) = -U_\mu(a) + \hbar\sigma_1(\hbar, \mu - U_0'(a))\,, \tag{3.2}$$

where

$$\sigma_1(\hbar, j) = \lim_{\Lambda \uparrow R^2} \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_\Lambda \left[\sum_{k=2}^{n} \frac{P^{(k)}(a)}{k!} :\phi^k: - j\phi\right]\right] d\mu_{\hbar C}\,. \tag{3.3}$$

Inserting Eq. (3.2) in the definition of V in Eq. (1.1) gives

$$\begin{aligned} V(\hbar, a) &= \sup_{\mu\in R} [\mu a + U_\mu(a) - \hbar\sigma_1(\hbar, \mu - U_0'(a))] \\ &= U_0(a) + \sup_{\mu\in R} [-\hbar\sigma_1(\hbar,\mu)]. \end{aligned} \tag{3.4}$$

Next, we perform a mass shift so as to explicitly isolate the $O(\hbar)$ contribution to the effective potential. Let $m_1^2 = U_0''(a) = P''(a) + m^2$. For $a \notin B$, $m_1^2 > 0$. For the remainder of this section we assume $a \notin B$. Let $C_1 = (-\Delta + m_1^2)^{-1}$ with periodic BC on $\partial\Lambda$. By a mass shift [11] it follows from Eq. (3.3) that

$$\begin{aligned} \sigma_1(\hbar,\mu) &= \lim_\Lambda \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar}\int_\Lambda \left[\sum_{k=3}^n \frac{P^{(k)}(a)}{k!} :\phi^k:_{\hbar C} - \mu\phi\right]\right] d\mu_{\hbar C_1} \\ &\quad + \lim_\Lambda \frac{1}{|\Lambda|}\ln\int \exp\left[\frac{-1}{\hbar}\int_\Lambda \frac{P''(a)}{2} :\phi^2:\right] d\mu_{\hbar C}. \end{aligned} \tag{3.5}$$

Introducing

$$\gamma(a) = \lim_\Lambda \frac{1}{|\Lambda|}\ln\int \exp\left[\frac{-1}{\hbar}\int_\Lambda \frac{P''(a)}{2} :\phi^2:\right] d\mu_{\hbar C}, \tag{3.6}$$

it is clear by scaling $\phi \to \hbar^{1/2}\phi$ that γ is independent of $\hbar > 0$.

Let

$$\sigma_2(\hbar,\mu) = \lim_\Lambda \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar}\int_\Lambda \left[\sum_{k=3}^n \frac{P^{(k)}(a)}{k!} :\phi^k:_{\hbar C} - \mu\phi\right]\right] d\mu_{\hbar C_1}. \tag{3.7}$$

Then by Eqs. (3.4) and (3.5),

$$V(\hbar, a) = U_0(a) - \hbar\gamma(a) + \sup_{\mu\in R}[-\hbar\sigma_2\hbar,\mu)]. \tag{3.8}$$

The next step is to Wick re-order the interaction in σ_2 to match the covariance C_1. Writing $a_k = P^{(k)}(a)/k!$ $(k=3,\ldots,n)$ and using the standard Wick reordering formula [11]

$$:\phi^n(x):_{C_1} = \sum_{k=0}^{[n/2]} c_{nk}[\delta C(x)]^k :\phi^{n-2k}(x):_{C_2},$$

where $\delta C(x) = \lim_{y\to x}[C_2(x,y) - C_1(x,y)]$ and $c_{nk} = \dfrac{n!}{2^k k!(n-2k)!}$, the interaction in $\sigma_2(\hbar, 0)$ can be rewritten as

$$\sum_{k=3}^n a_k :\phi^k:_{\hbar C} = \sum_{k=0}^n q_k(\hbar) :\phi^k:_{\hbar C_1}, \tag{3.9}$$

where each q_k is a polynomial of degree $\left[\dfrac{n-1-k}{2}\right]$ in $\hbar d = \hbar\left[\left(\dfrac{-1}{4\pi}\log\dfrac{m_1^2}{m^2}\right)\right.$ plus an $\hbar$-independent term that goes to zero as $\left.\Lambda \uparrow R^2\right]$. To simplify the notation we drop the Λ-dependent term (which is insignificant for large Λ and disappears in the

$\Lambda \uparrow R^2$ limit). The q_k's obey

$$q_0(\hbar)=O(\hbar^2), \quad q_1(\hbar)=O(\hbar), \quad q_2(\hbar)=O(\hbar), \quad q_k(\hbar)=a_k+O(\hbar) \quad (3 \leqq k \leqq n). \tag{3.10}$$

Explicitly,

$$q_0(\hbar)=\sum_{k=2}^{n/2} c_{2k,k} a_{2k} (\hbar d)^k . \tag{3.11}$$

Writing μ_a for the unique element of G satisfying $\xi(\mu_a)=a$ (which exists by Lemma 2.4) we have

$$\sum_{k=2}^{n} q_k(0) x^k + \frac{1}{2} m_1^2 x^2 = U_{\mu_a}(x+\xi(\mu_a)) - U_{\mu_a}(\xi(\mu_a)),$$

so by Lemma 2.6 there exists δ, $L>0$ such that

$$\sum_{k=2}^{n} q_k(0) x^k + \frac{1}{2} m_1^2 x^2 \in \mathcal{T}_{\delta, L} .$$

Inserting Eq. (3.9) in Eq. (3.7) gives

$$\hbar \sigma_2(\hbar, \mu) = -q_0(\hbar) + \hbar \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_{\Lambda} \left[\sum_{k=2}^{n} q_k :\phi^k: -(\mu - q_1)\phi\right]\right] d\mu_{\hbar C_1} . \tag{3.12}$$

Let

$$\sigma(\hbar, j) = \lim_{\Lambda} \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_{\Lambda} \left[\sum_{k=2}^{n} q_k :\phi^k: - j\phi\right]\right] d\mu_{\hbar C_1}, \tag{3.13}$$

so that

$$\hbar \sigma_2(\hbar, \mu) = -q_0(\hbar) + \hbar \sigma(\hbar, \mu - q_1(\hbar)) . \tag{3.14}$$

Inserting Eq. (3.14) into Eq. (3.8) gives

$$V(\hbar, a) = U_0(a) - \hbar \gamma(a) + q_0(\hbar) + \sup_{\mu} [-\hbar \sigma(\hbar, \mu)], \qquad a \notin B . \tag{3.15}$$

Observe that $\dfrac{D^k q_0(0)}{k!} = c_{2k,k} a_{2k} d^k$ gives the value of the d-renormalized k loop graph with a single $2k$ legged vertex a_{2k} and legs joined up in pairs. To show that the translated effective potential

$$E(\hbar) = \sup_{\mu} [-\hbar \sigma(\hbar, \mu)] \tag{3.16}$$

is analytic in small $\hbar>0$ and C^∞ at $\hbar=0^+$, we will use Theorem 2.2 to reduce the problem to the study of $\hbar\sigma(\hbar, \mu)$. This pressure is studied using a high temperature cluster expansion.

Convergence of the cluster expansion follows from upper and lower bounds on a partition function. We now give the first steps towards obtaining these bounds. The idea for the proof of Lemma 3.1 below originated in work of Spencer [19].

After this research was completed the author learned of a paper by Eckmann [7] where an estimate very similar to Eq. (3.17) was obtained by essentially the same method.

Let $S_{\theta,\gamma}=\{z\in\mathbb{C}:\ 0<\operatorname{Re}z<\gamma,\ 0<|\operatorname{Arg}z|<\theta\}$, and denote by $d\mu_{m^2}(s)$ the Gaussian measure on $\mathscr{S}^1(\mathbb{R}^2)$ with covariance

$$\sum_{\Gamma\subset\mathscr{B}_\Lambda}\prod_{b\in\Gamma}s_b\prod_{b\in\Gamma^c}(1-s_b)(-\Delta^{\Gamma^c}+m^2)^{-1},$$

where $\mathscr{B}_\Lambda$ is the set of all bonds joining nearest neighbor sites in the periodic lattice $\Lambda\cap Z^2$ and Δ^{Γ^c} is the Laplacian with Dirichlet BC on Γ^c and PBC on $\partial\Lambda$.

Lemma 3.1. *Let* $T(\hbar,x)=\sum_{k=2}^{n}a_k(\hbar)x^k$ *and* $a_1(\hbar)=O(\hbar^{1/2})$, *where the* a_k *are continuous in* $\bar{S}_{\theta',\gamma'}$ *for some* $\theta',\gamma'>0$. *Suppose* $\operatorname{Re}T(0,\cdot)\in\mathscr{T}_{\delta,L}$ *for some* $\delta,L>0$. *Then there exist* $\theta,\gamma>0$ *such that*

$$\left|\int\exp\left[-\int_V\left[\hbar^{-1}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}(s)\right|\leqq e^{K|V|} \tag{3.17}$$

for every $\hbar\in S_{\theta,\gamma}$ *and for every* s, *and for every finite union* V *of lattice squares in* Λ. *The constant* K *depends on* δ *and* L.

Proof. It is not hard to see that without loss of generality we may take $\hbar$ and the a_k to be real. Furthermore, by conditioning [14] we may take $s\equiv1$, corresponding to the covariance $(-\Delta+m^2)^{-1}$ with PBC on $\partial\Lambda$. By performing a mass shift we obtain

$$\int\exp\left[-\int_V\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}$$

$$=\frac{\int\exp\left[-\int_V\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):_{m^2}+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:_{m^2}\right]\right]\exp\left[-\int_V\left(\frac{1}{2}m^2-\frac{\delta}{2}\right):\phi^2:^\delta\right]d\mu^\delta}{\int\exp\left[-\int_V\left(\frac{1}{2}m^2-\frac{\delta}{2}\right):\phi^2:\right]d\mu^\delta}, \tag{3.18}$$

where $d\mu^\delta$ is the Gaussian measure with periodic covariance $(-\Delta+\delta\chi_V+m^2\chi_{\Lambda\setminus V})^{-1}$. Wick order with respect to $d\mu^\delta$ is denoted $:\ :^\delta$. Applying Jensen's inequality to the denominator of the right side of Eq. (3.18) we obtain

$$\int\exp\left[-\int_V\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}\leqq\int\exp\left[-\int_V:A(\phi):^\delta\right]d\mu^\delta,$$

where

$$:A(\phi):^\delta=\hbar^{-1}:T(\hbar,\hbar^{1/2}\phi):_{m^2}+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:_{m^2}+\left(\frac{1}{2}m^2-\frac{\delta}{2}\right):\phi^2:^\delta.$$

The polynomial A has the form

$$A(x)=\hbar^{-1}T(\hbar,\hbar^{1/2}x)-\frac{\delta}{2}x^2+\sum_{k=0}^{n-2}\tilde{a}_k(\hbar)\hbar^{k/2}x^k\,, \tag{3.19}$$

with the $\tilde{a}_k$ bounded in absolute value by a constant depending only on δ and L.

We now introduce the momentum cutoff field $\phi_r(x)=\int\phi(y)\delta_{r,x}(y)dy$, where $\delta_{r,x}(y)=r^2h(r(x-y))$ with $h\in C_0^\infty(\mathbb{R}^2)$, $h\geqq 0$, $h(0)>0$ and $\int h(y)dy=1$, and obtain a lower bound on $:A(\phi_r):^\delta$ as follows. Let $\sigma_r(x)=\int\delta_{r,x}(y)C(y,z)\delta_{r,x}(z)dydz$. Then $\sigma_r(x)=O(\log r)$ as $r\to\infty$. Using Eq. (3.19) and undoing the Wick order gives

$$:A(\phi_r):^\delta=\hbar^{-1}T(\hbar,\hbar^{1/2}\phi_r)-\frac{\delta}{2}\phi_r^2+\sum_{k=0}^{n-2}c_k(\hbar,r)\hbar^{k/2}\sigma_r^{\frac{n-k}{2}}\phi_r^k$$

with $|c_k(\hbar,r)|$ uniformly bounded in small $\hbar$ and large r. Since $T(0,\cdot)\in\mathcal{T}_{\delta,L}$, it follows that for $\hbar$ sufficiently small $T(\hbar,\cdot)\in\mathcal{T}_{\frac{\delta}{2},L+\frac{\delta}{2}}$, and hence

$$:A(\phi_r):^\delta\geqq\frac{\delta}{2}\hbar^{\frac{n}{2}-1}\phi_r^n+\sum_{k=0}^{n-2}c_k\hbar^{k/2}\sigma_r^{\frac{n-k}{2}}\phi_r^k\geqq\sigma_r^{n/2}\left[\frac{\delta}{2}x^n+\sum_{k=0}^{n-2}c_kx^k\right],$$

where $x=\hbar^{1/2}\sigma_r^{-1/2}\phi_r$. Therefore, there is a constant independent of small $\hbar$ and large r such that

$$:A(\phi_r):^\delta\geqq-(\text{const})(\log r)^{n/2}\,. \tag{3.20}$$

The estimate (3.17) follows from (3.20) by a standard result [6]. □

The following theorem gives bounds which imply convergence of the cluster expansion.

Theorem 3.2. *Let* $T(\hbar,x)=\sum_{k=2}^{n}a_k(\hbar)x^k$ *and* $a_1(\hbar)=O(\hbar^{1/2})$, *where the* a_k *are continuous in* $\bar{S}_{\theta',\gamma'}$ *for some* $\theta',\gamma'>0$. *Suppose* $\operatorname{Re}T(0,x)\in\mathcal{T}_{\delta,L}$ *for some* $\delta,L>0$, *and fix* $m,\varepsilon>0$. *Then there exist* $\theta,\gamma,b>0$ *such that if* $|a_2(0)-\frac{1}{2}m^2|<b$, *then*

$$\left|\int\exp\left[-\int_V\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}(s)\right|\leqq e^{\varepsilon|V|} \tag{3.21}$$

for every $\hbar\in S_{\theta,\gamma}$, *for every* s, *and for every finite union* V *of unit lattice squares in* Λ. *Moreover,*

$$\left|\int\exp\left[-\int_\Delta\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}(s)\right|\geqq\frac{1}{2}, \tag{3.22}$$

for every $h\in S_{\theta,\gamma'}$ *for every* s, *and for every unit lattice square* Δ.

Proof. The proof follows [19].

For $\Delta\subset V$ we define

$$\psi_\Delta=\exp\left[-\int_\Delta\left[:\frac{1}{\hbar}T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]-1\,. \tag{3.23}$$

Then

$$\int \exp\left[-\int_V \left[:\frac{1}{\hbar}T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}(s)$$
$$=\int \prod_{\Delta\subset V}(\psi_\Delta+1)d\mu_{m^2}(s)=\sum_{X\subset V}\int\prod_{\Delta\subset X}\psi_\Delta d\mu_{m^2}(s)\,. \tag{3.24}$$

We claim that there is a $\gamma=\gamma(\varepsilon,\delta,L)$ such that for $\hbar<\gamma$,

$$\left|\int\prod_{\Delta\subset X}\psi_\Delta d\mu_{m^2}(s)\right|\leqq\varepsilon^{|X|}\,. \tag{3.25}$$

Given (3.25), it follows from (3.24) that

$$\left|\int \exp\left[-\int_V \left[:\frac{1}{\hbar}T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_{m^2}(s)\right|$$
$$\leqq\sum_{X\subset V}\varepsilon^{|X|}=\sum_{m=0}^{|V|}\binom{|V|}{m}\varepsilon^m\leqq(1+\varepsilon)^{|V|}\leqq e^{\varepsilon|V|},$$

which proves (3.21). The bound (3.22) follows from (3.25) with $X=\Delta$.

It remains only to prove the inequality (3.25). To simplify the notation, let

$$:S(V):=\int_V\left[\frac{1}{\hbar}:T(\hbar,\hbar^{1/2}\phi):+a_1(\hbar)\phi-\frac{1}{2}m^2:\phi^2:\right].$$

By the Fundamental Theorem of Calculus.

$$\psi_{\Delta_i}=-\int_0^1 d\lambda_i:S(\Delta_i):\exp[-\lambda_i:S(\Delta_i):]\,. \tag{3.26}$$

By Eq. (3.26) and Hölder's inequality

$$\left|\int\prod_{\Delta\subset X}\psi_\Delta d\mu_{m^2}(s)\right|\leqq\left\|\prod_i:S(\Delta_i):\right\|_{p'}\sup_{0\leqq\lambda_i\leqq1}\left\|\exp\left[-\sum_i\lambda_i:S(\Delta_i):\right]\right\|_p\,, \tag{3.27}$$

where $p>1$ will be chosen below to be near one. The norm $\|\cdot\|_p$ is the norm in $L^p(d\mu_{m^2}(s))$.

By assumption the coefficients of S are $O(\hbar^{1/2})$ or $O(b)$. For $\hbar<\gamma$, it follows from standard estimates on Gaussian integrals [11] that for given fixed p',

$$\left\|\prod_{\Delta_i\subset X}:S(\Delta_i):\right\|_{p'}\leqq(\max\{\gamma^{1/2},b\}\cdot M)^{|X|} \tag{3.28}$$

for some constant M independent of $\hbar$ and s.

To bound the other factor on the right side of Eq. (3.27), we cannot use Lemma 3.1 directly because when $\lambda_i=0$ the classical potential will not be in any $\mathscr{T}_{\delta,L}$. However, the proof of Lemma 3.1 can be modified to overcome this difficulty, as we will now show. As in the proof of Lemma 3.1 we assume that $\hbar$ and T are real and that $s=1$. Note that for $p>1$ and $\gamma\in(0,m^2)$,

$$\int\exp[-p\lambda_i:S(V):]d\mu_{m^2}=\frac{\int\exp[-p\lambda_i:S(V):_{m^2}]\exp\left[-\frac{1}{2}(m^2-\gamma)\int_V:\phi^2:\right]d\mu^\gamma}{\int\exp\left[-\frac{1}{2}(m^2-\gamma)\int_V:\phi^2:\right]d\mu^\gamma}$$
$$\leqq\int\exp\left[-p\lambda_i:S(V):_{m^2}-\frac{1}{2}(m^2-\gamma)\int_V:\phi^2:\right]d\mu^\gamma$$

by Jensen's inequality. But

$$p\lambda_i[:\hbar^{-1}T(\hbar,\hbar^{1/2}\phi_r):_{m^2}+a_1(\hbar)\phi_r-\tfrac{1}{2}m^2:\phi_r^2:_{m^2}]+\tfrac{1}{2}(m^2-\gamma):\phi_r^2:^{\gamma}$$
$$\geqq \lambda_i\left[p\hbar^{-1}:T(\hbar,\hbar^{1/2}\phi_r):_{m^2}+pa_1(\hbar)\phi_r-\frac{\delta}{2}:\phi_r^2:_{m^2}\right]$$
$$+\left[\lambda_i\left(\frac{\delta}{2}-\frac{p}{2}m^2\right)+\frac{1}{2}(m^2-\gamma)\right]:\phi_r^2:_{m^2}-C\,.$$

If $T(0,\cdot)\in\mathscr{T}_{\delta,L}$, then for $p\in(1,2)$, $pT(0,\cdot)\in\mathscr{T}_{\delta,2L}$, so the estimates of the proof of Lemma 3.1 shows that

$$ph^{-1}:T(\hbar,\hbar^{1/2}\phi_r):_{m^2}+pa_1(\hbar)\phi_r-\frac{\delta}{2}:\phi_r^2:_{m^2}\geqq -M_1(\log r)^{n/2}\,.$$

Choosing $p=1+\frac{\delta}{m^2}$ and $\gamma=\min\left\{\frac{m^2}{2},\frac{\delta}{2}\right\}$ gives $\lambda_i\left(\frac{\delta}{2}-\frac{p}{2}m^2\right)+\frac{1}{2}(m^2-\gamma)\geqq 0$ for $\lambda_i\in[0,1]$. Therefore,

$$p\lambda_i[:\hbar^{-1}T(\hbar,\hbar^{1/2}\phi_r):_{m^2}+a_1(\hbar)\phi_r-\tfrac{1}{2}m^2:\phi_r^2:_{m^2}]$$
$$+\tfrac{1}{2}(m^2-\gamma):\phi_r^2:^{\gamma}\geqq -M_2(\log r)^{n/2}\,,$$

so by [6]

$$\sup_{0\leqq\lambda_i\leqq 1}\left\|\exp\left[-\sum_i\lambda_i:S(\Delta_i):\right]\right\|_p\leqq e^{K|X|}\,. \tag{3.29}$$

Using the bounds (3.29) and (3.28), Eq. (3.25) follows from Eq. (3.27) by taking b and γ sufficiently small. □

Theorem 3.2 and standard results [12, 5], together with a standard scaling argument, imply the following corollary:

Corollary 3.3. *For an interaction T and a function $a_1(\hbar)=O(\hbar^{1/2})$ as in Theorem 3.2, there exist $\theta,\gamma,b>0$ such that the cluster expansion for the interaction $\frac{1}{\hbar}T(\hbar,\hbar^{1/2}\phi)$ $+a_1(\hbar)\phi-\frac{1}{2}m^2\phi^2$ and mass m converges with bounds depending only on m, δ, and L, independent of Λ and of $\hbar\in S_{\theta,\gamma}$. In particular, truncated expectations of the form $\frac{1}{|\Lambda|}\langle:\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\rangle_{\hbar,\Lambda}$ are bounded in absolute value uniformly in Λ and $\hbar\in S_{\theta,\gamma}$, where $\langle\cdot\rangle_{\hbar,\Lambda}$ denotes the expectation corresponding to the given interaction in a periodic volume Λ.* □

The following theorem, whose proof relies on Corollary 3.3, is the key to the proof of Theorem B(a).

Theorem 3.4. *Let $T(\hbar,x)=\sum_{k=2}^{n}a_k(\hbar)x^k$, where the a_k are analytic in an interval $(0,\varrho)$ and C^∞ at 0^+, and $T(0,\cdot)\in\mathscr{T}_{\delta,L}$. Then for $|a_2(0)-\frac{1}{2}m^2|$ sufficiently small there exist*

$\gamma>0$ and complex open neighborhoods $0_\gamma \supset (0,\gamma)$ and D containing 0 such that

$$\begin{aligned}\hbar\tau(\hbar,\mu) &= \hbar\lim_{\Lambda}\tau_\Lambda(\hbar,\mu)\\ &\equiv \hbar\lim_{\Lambda}\frac{1}{|\Lambda|}\ln\int\exp\left[\frac{-1}{\hbar}\int_\Lambda\left[:T(\hbar,\phi):-\frac{1}{2}m^2:\phi^2:-\mu\phi\right]\right]d\mu_{\hbar C}\end{aligned}$$

is jointly analytic in $(\hbar,\mu)\in 0_\gamma\times D$ and C^∞ at $\hbar=0^+$, with uniformly bounded derivatives. Moreover, there is a $c>0$ such that

$$|D_2^2\hbar\tau(\hbar,\mu)|\geqq c \quad \textit{for all} \quad (\hbar,\mu)\in 0_\gamma\times D\,. \tag{3.30}$$

Proof. Since $T(0,\cdot)\in\mathscr{T}_{\delta,L}$ there exist $\gamma',\varepsilon'>0$ such that for $\mu\in(-\varepsilon',\varepsilon')$ and $\hbar\in(0,\gamma')$, $T_\mu(\hbar,x)=T(\hbar,x)-\mu x$ has a uniquely attained global minimum, at say $\xi(\hbar,\mu)$, with

$$\begin{aligned}S(\hbar,\mu;x)=T_\mu(\hbar,x+\xi(\hbar,\mu))-T_\mu(\hbar,\xi(\hbar,\mu))\in\mathscr{T}_{\delta',L'}\\ \text{for all}\quad(\hbar,\mu)\in[0,\gamma')\times(-\varepsilon',\varepsilon')\,.\end{aligned}$$

Moreover, ξ is analytic in $V_{\gamma'}\times D_{\varepsilon'}$, where $D_{\varepsilon'}=\{z\in\mathbb{C}:|z|<\varepsilon'\}$ and $V_{\gamma'}$ is an open neighborhood of $(0,\gamma')$, and C^∞ at $\hbar=0^+$.

Translating in τ_Λ by ξ and then scaling $\phi\to\hbar^{1/2}\phi$ gives

$$\begin{aligned}\hbar\tau_\Lambda(\hbar,\mu)=&-T_\mu(\hbar,\xi(\hbar,\mu))+\frac{\hbar}{|\Lambda|}\\ &\cdot\ln\int\exp\left[-\int_\Lambda\left[\frac{1}{\hbar}:S(\hbar,\mu;\hbar^{1/2}\phi):-\frac{1}{2}m^2:\phi^2:\right]\right]d\mu_C,\end{aligned} \tag{3.31}$$

for all $(\hbar,\mu)\in V_\gamma\times D_{\varepsilon'}$. Since $\frac{1}{2}D_3^2S(0,\mu;0)=\frac{1}{2}D_2^2T(0,\xi(0,\mu))$, we can make $\frac{1}{2}D_3^2S(0,\mu;0)$ as close as desired to $\frac{1}{2}m^2$ by taking ε' and $|a_2(0)-\frac{1}{2}m^2|$ sufficiently small. Then by Corollary 3.3 expectations of the form $\frac{1}{|\Lambda|}\langle:\phi^{k_1}(\Lambda):;\ldots;:\phi^{k_r}(\Lambda):\rangle_{\tilde{S},\Lambda}$ are bounded in absolute value independent of $\Lambda,\hbar,\mu$, where

$$\tilde{S}(\hbar^{1/2},\mu;x)=\hbar^{-1}S(\hbar,\mu;\hbar^{1/2}x)-\tfrac{1}{2}m^2x^2$$

and

$$\langle\cdot\rangle_{P,\Lambda}=\frac{\int\cdot\exp\left[-\int_\Lambda:P(\phi):\right]d\mu_{m^2}}{\int\exp\left[-\int_\Lambda:P(\phi):\right]d\mu_{m^2}}.$$

The first term on the right side of Eq. (3.31) is analytic in $(\hbar,\mu)\in V_{\gamma'}\times D_{\varepsilon'}$ and C^∞ at $\hbar=0^+$, and does not depend on Λ. Its derivatives are uniformly bounded. To see that $\hbar\tau(\hbar,\mu)$ is analytic, we note that the infinite volume limit of the second term on the right side of (3.31) is analytic in small $(\hbar^{1/2},\mu)$ by Corollary 3.3 and Vitali's theorem. To see that $\hbar\tau(\hbar,\mu)$ is C^∞ at $\hbar=0^+$, we need only show that odd derivatives of

$$\zeta_\Lambda(t,\mu)\equiv\frac{1}{|\Lambda|}\ln\int\exp\left[-\int_\Lambda:\tilde{S}(t,\mu;\phi):\right]d\mu_C$$

with respect to t vanish as $t\to 0$. Now

$$D_1^{2k+1}D_2^l\zeta_\Lambda(t,\mu)=\frac{1}{|\Lambda|}\sum_{\pi\in\mathscr{P}_{2k+1}}\sum_{\sigma\in\mathscr{P}_l}c_{\pi\sigma}\left\langle\int_\Lambda:D_1^{|\pi_1|}D_2^{|\sigma_1|}\tilde{S}:;\ldots;\right.$$
$$\left.\cdot\int_\Lambda:D_1^{|\pi_{|\pi|}|}D_2^{|\sigma_{|\pi|}|}\tilde{S}:;\int_\Lambda:D_2^{|\sigma_{|\pi|+1}|}\tilde{S}:;\ldots;\int_\Lambda:D_2^{|\sigma_{|\sigma|}|}\tilde{S}:\right\rangle_{\tilde{S},\Lambda},\tag{3.32}$$

where $\mathscr{P}_n$ is the set of partitions of $\{1,\ldots,n\}$, π_i are the elements of a partition π, $c_{\pi\sigma}$ are positive integers, and $|\sigma_1|,\ldots,|\sigma_{|\pi|}|$ may be zero. Since

$$\tilde{S}(t,\mu;\phi)=\sum_{k=2}^n\frac{D_2^kT(t^2,\xi(t^2,\mu))}{k!}t^{k-2}\phi^k-\frac{1}{2}m^2\phi^2,$$

the $t=0$ contribution to $D_1^jD_2^j\tilde{S}$ is a linear combination of terms of the form $c(\mu)\phi^r(\Lambda)$, where r is odd if j is odd and r is even if j is even. By Corollary 3.3, as $t\to 0$ the right side of Eq. (3.32) approaches uniformly in Λ a sum of terms of the form

$$c(\mu)\langle:\phi^{r_1}(\Lambda):;\ldots;:\phi^{r_{|\sigma|}}(\Lambda):\rangle_{\tilde{S}(0,\mu;\cdot),\Lambda},\tag{3.33}$$

wherwhere $r_1,\ldots,r_{|\pi|}$ have the same parity as $|\pi_1|,\ldots,|\pi_{|\pi|}|$, and $r_{|\pi|+1},\ldots,r_{|\sigma|}$ all equal 2. Since $|\pi_1|+\ldots+|\pi_{|\pi|}|=2k+1$ is odd, $r_1+\ldots+r_{|\pi|}$ is also odd. The expectation in (3.33) is invariant under $\phi\to-\phi$ since $\tilde{S}(0,\mu;\cdot)$ is quadratic, and hence equals zero.

It remains to prove the lower bound (3.30). By differentiating under the integral sign, translating by $\xi(\hbar,\mu)$ and scaling $\phi\to\hbar^{1/2}\phi$ it is seen

$$\hbar D_2^2\tau(\hbar,\mu)=\lim_\Lambda\frac{1}{|\Lambda|}\langle\phi(\Lambda);\phi(\Lambda)\rangle_{\tilde{S},\Lambda},$$

which approaches $\lim_\Lambda\frac{1}{|\Lambda|}\langle\phi(\Lambda);\phi(\Lambda)\rangle_{\tilde{S}(0,\mu;\cdot),\Lambda}$ as $\hbar\downarrow 0$ by Corollary 3.3. This last quantity is continuous in μ and equals $\int_{\mathbb{R}^2}(-\Delta+2a_2(0))^{-1}(x)dx$ for $\mu=0$. Therefore, taking ε and γ smaller if necessary, the lower bound (3.30) holds. □

4. Proofs of Theorems A and B(a)

Theorem A. $\lim_{\hbar\downarrow 0}V(\hbar,a)=(\operatorname{conv}U_0)(a)$.

Proof. As was pointed out in Sect. 1, $p(\hbar,\cdot)$ is strictly convex [9] and $\lim_{\mu\to\pm\infty}D_2p(\hbar,\mu)=\pm\infty$, so $p(\hbar,\cdot)\in\mathscr{C}_s$. In Theorem 4.1 below we will show that $\lim_{\hbar\downarrow 0}p(\hbar,\mu)=-m(\mu)$ for all μ. Using this, and the fact that $-m\in\mathscr{C}_s$ by Lemma 2.3, it follows from Theorem 2.1 and Eq. (2.4) that for all $a\in\mathbb{R}$,

$$\lim_{\hbar\downarrow 0}V(\hbar,a)=-m^*(a)=U_0^{**}(a)=(\operatorname{conv}U_0)(a).$$

We now prove the promised limit, which is a Laplace's method type result for functional integrals on $\mathscr{S}'(R^2)$. For related results in the context of Gaussian integrals on $C[0,1]$, see [8, 18].

Theorem 4.1. $\lim_{\hbar \downarrow 0} p(\hbar, \mu) = -m(\mu)$, *for all* $\mu \in R$.

Proof. Let

$$p_\Lambda(\hbar, \mu) = \frac{\hbar}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_\Lambda [:P(\phi): -\mu\phi]\right] d\mu_{\hbar C},$$

and fix $\mu \in G$. Let $T(x) = U_\mu(x + \xi(\mu)) - U(\xi(\mu))$. By Lemma 2.6, $T \in \mathscr{T}_{\delta, L}$ for some $\delta, L > 0$. Translating the field by $\xi(\mu)$ gives

$$p_\Lambda(\hbar, \mu) = -U_\mu(\xi(\mu)) + \hbar \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_\Lambda \left[:T(\phi): - \frac{1}{2} m^2 :\phi^2:\right]\right] d\mu_{\hbar C}. \quad (4.1)$$

By Jensen's inequality the argument of the logarithm on the right side of Eq. (4.1) is bounded below by one, and by Lemma 3.1 it is bounded above by $e^{K|\Lambda|}$ if $\hbar$ is sufficiently small. These bounds and Eq. (4.1) show that $|p_\Lambda(\hbar, \mu) + m(\mu)| \to 0$ uniformly in Λ, as $\hbar \downarrow 0$, for $\mu \in G$. But since G^c is finite, $\lim_{\hbar \downarrow 0} p(\hbar, \mu) = -m(\mu)$ for all $\mu \in \mathbb{R}$ by convexity. □

Theorem B(a). *Let $a \notin B$. There exists a $\gamma > 0$ such that $V(\hbar, a)$ is analytic in $\hbar$ for $\hbar \in (0, \gamma)$. Moreover, $V(\hbar, a)$ is C^∞ at $\hbar = 0^+$, and so the expansion $V(\hbar, a) \sim \sum_{n=0}^{\infty} v_n(a) \hbar^n$ is asymptotic, where* $v_n(a) = \dfrac{D_1^n V(0^+, a)}{n!}$.

Proof. Recall Eq. (3.15)

$$V(\hbar, a) = U_0(a) - \hbar\gamma(a) + q_0(\hbar) + \sup_\mu [-\hbar\sigma(\hbar, \mu)], \qquad a \notin B,$$

where q_0 and σ are functions of a. Fix $a \notin B$. Since q_0 is a polynomial we need only show that $E(\hbar) = \sup_\mu [-\hbar\sigma(\hbar, \mu)]$ is analytic on $(0, \gamma)$ and C^∞ at $\hbar = 0^+$. We show this using Theorem 2.2.

Note that it suffices to show that

$$\lim_{\hbar \downarrow 0} \hbar\sigma(\hbar, \mu) = -m_0(\mu), \quad \text{for all} \quad \mu \in R, \quad (4.2)$$

where $m_0(\mu) = \min_x \left[\sum_{k=3}^{n} q_k(0) x^k + \frac{1}{2} m_1^2 x^2 - \mu x\right]$. To see that this is sufficient, note that by Lemma 2.3 the location of the supremum in $\sup_\mu [+m_0(\mu)]$ is the unique μ, say $\mu(0)$, for which $\sum_{k=3}^{n} q_k(0) x^k + \frac{1}{2} m^2 x^2 - \mu x$ attains its global minimum at zero. Since $a \notin B$, there are $\delta, L > 0$ such that

$$\sum_{k=3}^{n} q_k(0) x^k + \frac{1}{2} m_1^2 x^2 \in \mathscr{T}_{\delta, L},$$

and so $\mu(0) = 0$. Now given Eq. (4.2), it follows from Theorem 3.4 that $\hbar\sigma(\hbar, \mu)$ satisfies the analyticity requirements of Theorem 2.2, as well as the necessary bounds on the derivatives, and hence E is analytic in $(0, \gamma)$ and C^∞ at $\hbar = 0^+$. It remains to prove Eq. (4.2).

We show that (4.2) holds for $\mu \in G(0)$, where for $\lambda \geqq 0$

$$G(\lambda) = \left\{ \mu \in R: \sum_{k=3}^{n} q_k(\lambda) x^k + \frac{1}{2} m_1^2 x^2 - \mu x \right.$$

$$\text{has a uniquely attained global minimum}$$

$$\left. \text{and has positive curvature at the minimum} \right\}.$$

Since $G(0)^c$ is finite, (4.2) holds for all μ if it holds for $\mu \in G(0)$, by convexity.
Let

$$\sigma_\Lambda(\hbar, \lambda, \mu) = \frac{1}{|\Lambda|} \ln \int \exp\left[\frac{-1}{\hbar} \int_\Lambda \left[\sum_{k=2}^{n} q_k(\lambda) :\phi^k: - \mu\phi \right] \right] d\mu_{\hbar C},$$

and let $\sigma_\Lambda(\hbar, \mu) = \sigma_\Lambda(\hbar, \hbar, \mu)$, so $\sigma(\hbar, \mu) = \lim_\Lambda \sigma_\Lambda(\hbar, \mu)$. By the Fundamental Theorem of Calculus,

$$|\hbar\sigma_\Lambda(\hbar, \mu) + m_0(\mu)| \leqq |\hbar\sigma_\Lambda(\hbar, 0, \mu) + m_0(\mu)| + \hbar \int_0^\hbar |D_2\sigma_\Lambda(\hbar, \lambda, \mu) d\lambda|. \tag{4.3}$$

By Theorem 4.1, the infinite volume limit of the first term on the right side of (4.3) goes to zero as $\hbar \downarrow 0$. As for the second term, fix $\mu \in G(0)$ and $\gamma > 0$ sufficiently small that $\mu \in G(\lambda)$ for $\lambda \in (0, \gamma)$. In the expectation $\hbar D_2\sigma_\Lambda(\hbar, \lambda, \mu)$, translate the field by the location $\xi(\lambda, \mu)$ of the global minimum of $\sum_{k=3}^{n} q_3(\lambda)\phi^k + \frac{1}{2} m_1^2\phi^2 - \mu\phi$, scale the field $\phi \to \hbar^{1/2}\phi$, shift the quadratic term of the interaction over to the measure, and Wick re-order the interaction to match the new measure. Then by Corollary 3.3, $\hbar|D_2\sigma_\Lambda(\hbar, \lambda, \mu)|$ is bounded uniformly in Λ and in small $\hbar$ and λ, and therefore, the second term on the right side of (4.3) is $O(\hbar)$ uniformly in Λ. □

Note that it was also proven in Theorem 2.2 that the point $\mu(\hbar)$ at which $\sup_\mu [-\hbar\sigma(\hbar, \mu)]$ is attained is analytic and bounded on $(0, \gamma)$ and hence C^∞ at $\hbar = 0^+$. In particular,

$$\lim_{\hbar \downarrow 0} \mu(\hbar) = \mu(0) = 0. \tag{4.4}$$

5. Proof of Theorem B(b)

Theorem B(b). *Let $a \notin B$. Then $v_0(a) = U_0(a)$ and*

$$v_1(a) = -\gamma(a) \equiv -\lim_\Lambda \frac{1}{|\Lambda|} \ln \int \exp\left[-\int_\Lambda \frac{P''(a)}{2} :\phi^2: \right] d\mu_C.$$

For $N \geqq 2$, $-v_N(a) = \frac{1}{N!} D_1^N V(0^+, a)$ is the (finite) sum of all $d(a)$-renormalized 1-PI N-loop diagrams with k-legged vertices taking factors $\frac{-P_{(a)}^{(k)}}{k!}$ $(3 \leqq k \leqq \deg P)$ and lines corresponding to the free covariance of mass $(U_0''(a))^{1/2}$, where $d(a)$

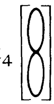
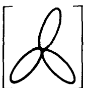
$=-\frac{1}{4\pi}\log\frac{U_0''(a)}{m^2}$. *A combinatorial factor is associated with each graph as per Remark 1 of Sect. 1.*

This section contains the proof of Theorem B(b). Fix $a \notin B$. By Eq. (3.15),

$$V(\hbar, a) = U_0(a) - \hbar\gamma(a) + q_0(\hbar) + E(\hbar), \tag{5.1}$$

where $E(\hbar) = \sup_\mu [-\hbar\sigma(\hbar, \mu)] = -\hbar\sigma(\hbar, \mu(\hbar))$ and σ is given by (3.13). By Eq. (3.11)

$$-\frac{1}{k!}D^k q_0(0) = \begin{cases} -c_{2k,k}a_{2k}d^k, & k=2, \ldots, \frac{1}{2}(\deg P), \\ 0, & \text{otherwise}, \end{cases} \tag{5.2}$$

i.e., in the notation of Definition 1.1, $-\frac{1}{2!}D^2 q_0(0) = -a_4 \Bigg[$  $\Bigg]_d$, $-\frac{1}{3!}D^3 q_0(0)$ $= -a_6 \Bigg[$  $\Bigg]_d$, etc., where $d = -\frac{1}{4\pi}\log(U_0''(a)/m^2)$. As we will now show, $E(\hbar)$ $= O(\hbar^2)$, and for $N \geqq 2$, $-D^N E(0)$ is given by a sum of graphs having the specified lines and vertices. Afterwards these graphs will be identified to be as in the statement of Theorem B(b).

Lemma 5.1. *For some $\gamma > 0$, $\sigma(\hbar, \mu(\hbar))$ is C^∞ in $\hbar \in [0, \gamma)$, with $\sigma(0, \mu(0)) = 0$.*

Proof. By Theorem B(a) and Eq. (5.1), E is C^∞ in $\hbar \in [0, \gamma)$, so it suffices to show that $\lim_{\hbar \downarrow 0} \sigma(\hbar, \mu(\hbar)) = 0$. By (4.4) and the fact that $\sum_{k=2}^{n} q_k(0)x^k + \frac{1}{2}m_1^2 x^2 \in \mathscr{T}_{\delta, L}$, if $\hbar$ is sufficiently small the polynomial $Q(\hbar, \cdot)$ given by

$$Q(\hbar, x) = \sum_{k=2}^{n} q_k(\hbar)x^k + \frac{1}{2}m_1^2 x_1^2 - \mu(\hbar)x$$

has a uniquely attained global minimum, at say $\xi(\hbar)$, with ξ smooth and $\xi(0) = 0$. Moreover, there exist δ', $L' > 0$ such that

$$T(\hbar, \cdot) \equiv Q(\hbar, \xi(\hbar) + \cdot) - Q(\hbar, \xi(\hbar)) \in \mathscr{T}_{\delta', L'},$$

if $\hbar$ is sufficiently small. Translating by $\xi(\hbar)$ and scaling $\phi \to \hbar^{1/2}\phi$ in Eq. (3.13) gives

$$\sigma(\hbar, \mu(\hbar)) = -\frac{1}{\hbar}Q(\hbar, \xi(\hbar)) + \lim_{\Lambda} \frac{1}{|\Lambda|}$$

$$\cdot \ln \int \exp\left[-\int_\Lambda \left[\hbar^{-1} : T(\hbar, \hbar^{1/2}\phi) : -\frac{1}{2}m_1^2 : \phi^2 :\right]\right] d\mu_{C_1}. \tag{5.3}$$

By (4.4) $Q(\hbar, \xi(\hbar)) = O(\hbar^2)$, so the first term on the right side of (5.3) vanishes as $\hbar \downarrow 0$. Call the second term on the right side of (5.3) $\beta(\hbar) \equiv \lim_\Lambda \beta_\Lambda(\hbar)$. By Corollary 3.3 there is a constant M such that $\left|\frac{\partial}{\partial \hbar^{1/2}}\beta_\Lambda(\hbar)\right| \leqq M$ for all Λ and small $\hbar$ and so $\beta(\hbar) \to 0$ as $\hbar \downarrow 0$. $\square$

Corollary 5.2. $E(0)=DE(0)=0$.

Writing $n=\deg P$, let

$$q_{kj}=\frac{1}{j!}D^j q_k(0) \qquad \left(j=0,\ldots,\frac{n}{2}\right), \tag{5.4}$$

so that $q_k(\hbar)=\sum_{j=0}^{n/2} q_{kj}\hbar^j$.

Theorem 5.3. *For $a\notin B$ and $N\geqq 2$, the derivative $-D^N E(0)$ is given by a linear combination of connected graphs with no self-lines, with positive or negative coefficients, made up of lines of mass m_1 and k-legged vertices taking factors $-q_{kj}$, $k=2,3,\ldots,n$; $j=0,1,\ldots,\frac{n}{2}$.*

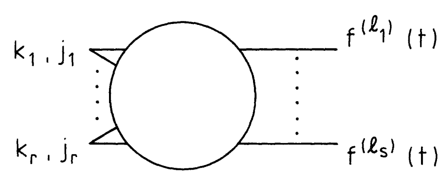
Proof. Let $f(t)=\mu(t^2)t^{-1}$. By (4.4), $f(t)=O(t)$. Let

$$\zeta_\Lambda(t,x)=\frac{1}{|\Lambda|}\ln\int\exp\left[-\int_\Lambda\left[\sum_{k=2}^{n} q_k(t^2)t^{k-2}:\phi^k:-x\phi\right]\right]d\mu_{C_1},$$
$$\text{and}\quad \zeta(t,x)=\lim_\Lambda \zeta_\Lambda(t,x).$$

Then $E(t^2)=-t^2\sigma(t^2,\mu(t^2))=-t^2\zeta(t,f(t))$, and it suffices to show that $\left.\frac{d^{2N}}{dt^{2N}}\right|_0 \zeta(t,f(t))$ is a sum of graphs as stated, for $N\geqq 2$. Now $\frac{d^k}{dt^k}\zeta_\Lambda(t,f(t))$ is a sum of positive integers multiplied by nonnegative powers of t multiplied by expressions of the form

$$\frac{1}{|\Lambda|}\langle -q_{k_1j_1}:\phi^{k_1}(\Lambda):;\ldots;-q_{k_rj_r}:\phi^{k_r}(\Lambda):;f^{(l_1)}(t)\phi(\Lambda);\ldots;f^{(l_s)}(t)\phi(\Lambda)\rangle_{t,\Lambda}, \tag{5.5}$$

where $\langle\cdot\rangle_{t,\Lambda}$ is the expectation corresponding to the measure occurring in $\zeta(t,f(t))$ and $k_i\in\{2,\ldots,n\}$, $j_i\in\left\{0,1,\ldots,\frac{n}{2}\right\}$, $r\geqq 0$, $s\geqq 0$, $l_i\geqq 1$. We denote the infinite volume limit of the expression (5.5) graphically by



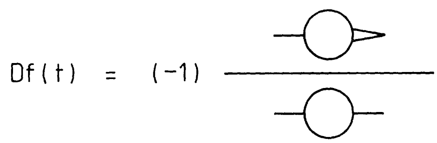
(5.6)

We now show that the vertex factors $f^{(l_i)}(0)$ are actually graphs which hook onto the corresponding legs. To simplify the notation we use  to denote a linear combination of terms of the form (5.6) with vertex factors 1 instead of $f^{(l_i)}(t)$, which linear combination will be apparent from the context. The coefficients of the linear combination will include combinatorial factors and powers of t.

Since $D_2\sigma(h, \mu(h))=0$, $D_2\zeta(t, f(t))=0$ and

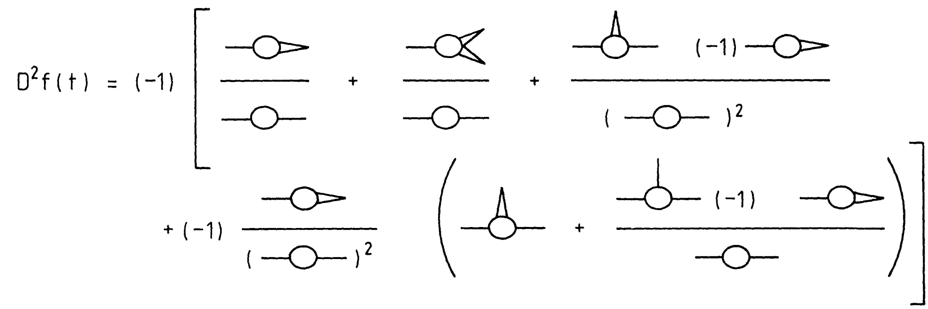
$$Df(t)=\frac{-D_1D_2\zeta(t,f(t))}{D_2^2\zeta(t,f(t))}. \tag{5.7}$$

Using the graph notation described in the last paragraph, Eq. (5.7) can be written



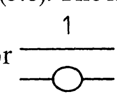
(5.8)

As explained below, differentiation of Eq. (5.8) gives

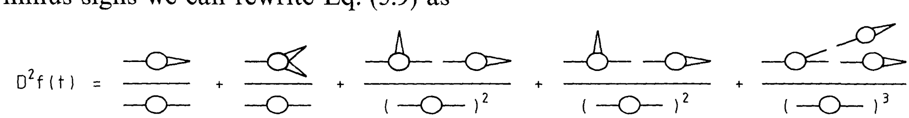


(5.9)

The terms on the right side of Eq. (5.9) arise as follows. The first three terms come from differentiating the numerator of Eq. (5.8): the first term comes from differentiating t's appearing as coefficients of ; the second term from differentiating the $\sum_{k=2}^{n} q_k(t^2)t^{k-2}:\phi^k:$ part of the interaction; the third term from differentiating the $f(t)\phi$ part of the interaction and using Eq. (5.8). The last term on the right side of Eq. (5.9) comes from differentiating the factor



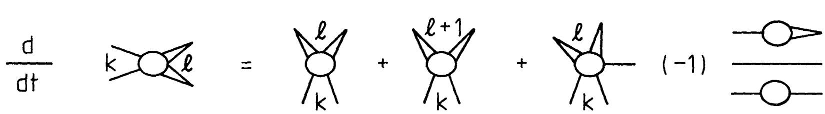
. Dropping minus signs we can rewrite Eq. (5.9) as



(5.10)

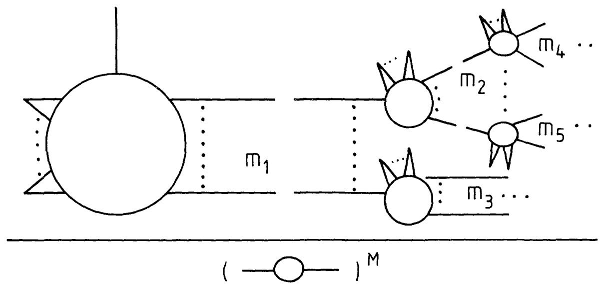
In the last three numerators of (5.10) note how all but one of the single legged vertices can be matched in pairs, and that the power of in the denominator exceeds the number of matched pairs by one.

We will now show how Eq. (5.1) generalizes to higher order derivatives. By the same reasoning used to differentiate above,

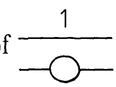
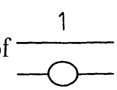
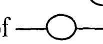


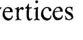
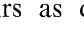
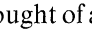
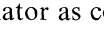
(5.11)

Using the formula (5.11) it follows from Eq. (5.8) and induction that $D^k f(t)$ is a linear combination of quotients of the form

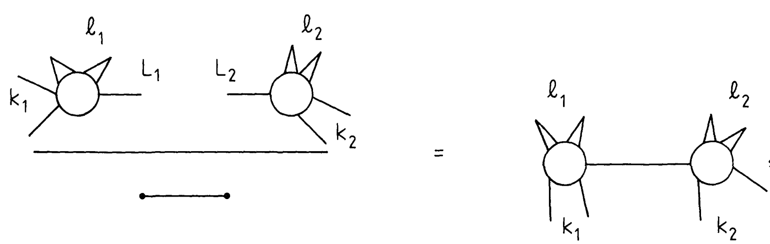


, (5.12)

where the diagram eventually terminates, $M-1$ is the total number of matched pairs of legs, i.e., $M = m_1 + m_2 + m_3 + \ldots + 1$, and there is only one unmatched leg. To see this, suppose $D^{k-1} f(t)$ is of the form (5.12) and note that differentiation of any factor of the numerator [using (5.11)] produces a sum of terms of the form (5.12). Also, considering each factor of  to be associated with a different matched pair of legs in the numerator, differentiation of  introduces new matched pairs of legs in the numerator and powers of  in the denominator producing terms of the form (5.12).

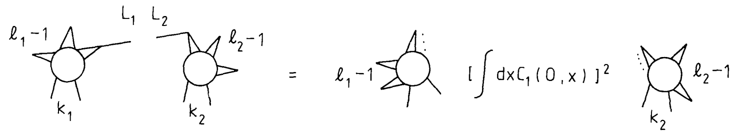
In the limit $t \to 0$ the measure in (5.5) becomes $d\mu_{C_1}$. Hence by Wick's theorem $D^k f(0)$ is a linear combination of products of connected graphs without self-lines, with vertices and lines as in the statement of the theorem as well as one-legged vertices which match up in $M-1$ pairs as depicted in (5.12), divided by ( $)^M$. Thus there is one power of  for each matched pair of legs, with one power left over. The unmatched leg in (5.12) should be thought of as being matched to the corresponding leg of (5.6), and the extra power of  in the denominator as corresponding to these legs. As we will now show, at $t=0$ each factor of  in the denominator serves to link together one matched pair of legs to create a connected graph.

We will now show that at $t=0$



, (5.13)

where each circle denotes a connected graph with no vertices other than those explicitly drawn. In fact, each of the lines L_1 and L_2 must be connected to a multi-legged vertex; choose these to be the vertices fixed at zero when evaluating the graphs. Then the numerator can be written



(5.14)

where the dashed lines indicate the absence of L_1 and L_2. One of the factors $\int dxC_1(0,x)$ on the right side of Eq. (5.14) cancels the denominator on the left side of Eq. (5.13). The remaining factor serves to link up the two graphs on the right side of Eq. (5.14). To see this, take one of the graphs under the integral $\int dxC_1(0,x)$ and use translation invariance to fix the fixed vertex of that graph at x instead of at the origin. Since the remaining graph has one vertex fixed at zero, $C_1(0,x)$ links the two graphs together. This proves Eq. (5.13).

Theorem 5.3 now follows by repeated application of Eq. (5.13) to see that at $t=0$ the $M-1$ matched pairs of legs in (5.12) can be joined by cancelling $M-1$ factors of ——— in the denominator, and that the single unmatched leg of (5.12) can be joined to the appropriate unmatched leg of (5.16) by cancelling the remaining factor of ——— in the denominator, resulting in a connected graph. □

To identify the topological structure of the graphs given by Theorem 5.3 we employ an irreducibility test introduced in [20] as used in [3]. However, this test applies to graphs having fixed vertices. Since all but one of the vertices in the graphs of Theorem 5.3 are integrated over, we will introduce space-time dependent coupling constants with respect to which partial differentiation yields fixed vertex graphs. These space-time dependent coupling constants force the external field to be also space-time dependent to preserve irreducibility, i.e., we deal with the effective action rather than the effective potential. To simplify the analysis we introduce a lattice analogue of the effective potential $E(\hbar)$ which generates graphs having the same topological structure has those in $D^N E(0)$ but assigns values to the graphs in such a way that the irreducibility test described below can be applied.

Definition 5.4. A topological graph is a collection of finitely many vertices, each having a finite number of legs (half-lines joined at one end to the vertex), such that every leg of every vertex is paired with some other leg to form a line. □

We now explain the method of [20] for testing a graph for one-particle irreducibility in the context we need. For a fixed positive integer m, we consider the lattice L_{2m} of $2m$ points $\{x_1, \ldots, x_{2m}\}$, thought of as consisting of the two sublattices $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{2m}\}$. Write $m_1^2 = U_0''(a)$ as usual and let

$$C(\lambda) = m_1^{-4} \begin{bmatrix} R_1 & \lambda R \\ \lambda R & R_2 \end{bmatrix}, \qquad \lambda \in [0,1], \tag{5.15}$$

where

$$(R_1)_{ij} = \begin{cases} m_1^2, & i=j, \\ r_{ij}, & i \neq j, \end{cases} \qquad (R_2)_{ij} = \begin{cases} m_1^2, & i=j, \\ r_{m+i,m+j}, & i \neq j, \end{cases}$$

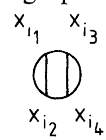
and $R_{ij} = r$ for all i,j. The matrices R, R_1, and R_2 are all $m \times m$, the r_{ij} are strictly positive with $r_{ij} \leqq r$, $r_{ij} = r_{ji}$ for all i and j, and $r > 0$ is chosen sufficiently small that

$C(\lambda)$ is positive definite for all $\lambda \in [0, 1]$ and all $r_{ij} \in (0, r)$. [It is possible to so choose r since for $r=0$, $C(\lambda)=m_1^{-2}I$.] The variable λ measures the coupling between the sets $\{x_1, x_2, \ldots, x_m\}$ and $\{x_{m+1}, x_{m+2}, \ldots, x_{2m}\}$.

Definition 5.5. Let L_{2m} (the lattice of $2m$ points) consist of the $2m$ points labeled $\{x_1, \ldots, x_{2m}\}$. A topological graph G is *imposed on* L_{2m} by assigning each vertex of G to a different point in L_{2m}. Such an assignment is called an *imposition of* G *on* L_{2m}. An *admissible imposition* (AI) is an imposition for which at least one vertex is assigned to each of the sublattices $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{2m}\}$. □

Now consider a graph with $2m$ vertices or less that has been imposed on L_{2m}.

For example, G =



, where the i_j are different elements of $\{1, \ldots, 2m\}$. The rule for evaluating such a graph is to form the product with one factor of $C(\lambda)_{i_j i_k}$ for each line joining x_{i_j} to $x_{i_{j'}}$. The graphs G depicted above has the value $G(\lambda) = C(\lambda)_{i_1 i_2}^2 C(\lambda)_{i_1 i_3} C(\lambda)_{i_2 i_4} C(\lambda)_{i_3 i_4}^2$.

The test for irreducibility is the following [20].

Lemma 5.6. *A topological graph G with V vertices is* 1-PI *if and only if* $DG(0) = G(0) = 0$ *for every AI of G on L_{2m}, for some $m \geqq V$.*

Proof. G is 1-PI if and only if two lines of G join $\{x_1, \ldots, x_m\}$ to $\{x_{m+1}, \ldots, x_{2m}\}$ for every AI of G on L_{2m}, i.e., if and only if $G(\lambda) = O(\lambda^2)$ for every AI. □

We now introduce the lattice theory. The lattice interaction in an external field $\mu \in \mathbb{R}^{2m}$ is given by

$$I_\mu(h, g, x) = \sum_{i=1}^{2m} \left[\sum_{k=2}^{n} \sum_{j=0}^{n/2} q_{kj} h^j g_{kji} x_i^k - \mu_i x_i \right], \tag{5.16}$$

where $\mu = (\mu_1, \ldots, \mu_{2m}) \in R^{2m}$, $x = (x_1, \ldots, x_{2m}) \in R^{2m}$, and the q_{kj} are defined in Eq. (5.4). The variable g_{kji} serves to label the quantity $h^j x_i^k$ in I. The vector g has components $g_{kji} \left(k=2, \ldots, n; j=0, \ldots, \frac{n}{2}; i=1, \ldots, 2m \right)$ and is restricted to lie in the subset $\bar{C}_\varepsilon \subset R^{N_m}$, $N_m = 2m\left(\frac{n}{2}+1\right)(n-1)$, defined as follows. The positive constant ε will be fixed below.

Definition 5.7. For $\varepsilon > 0$, $C_\varepsilon \subset R^{N_m}$ is the open cone with vertex at the origin, axis the line segment $\{(t, t, t, \ldots, t) \in R^{N_m}: 0 < t < 1\}$, and radius ε at its wide end. □

Let $P: \bar{C}_\varepsilon \to [0, 1]$ denote the mapping which takes a vector in $\bar{C}_\varepsilon$ to the first component of its orthogonal projection on the axis of $\bar{C}_\varepsilon$.

Lemma 5.8. *For any $g \in \bar{C}_\varepsilon$ and any component g_{kji} of g,*

$$|g_{kji} - Pg| \leqq \varepsilon Pg\,.$$

Proof. Let $P_1 g$ denote the projection of $g \in \bar{C}_\varepsilon$ on the axis of $\bar{C}_\varepsilon$. By the triangle inequality $|g_{kji} - Pg| \leqq |g - P_1 g|$. But by the cone geometry, $|g - P_1 g| \leqq \varepsilon \frac{|P_1 g|}{\sqrt{N_m}} = \varepsilon Pg$. □

The import of this lemma is that by choosing ε small, we can make the coefficients of $\sum_{k=2}^{n}\sum_{j=0}^{n/2} q_{kj}\hbar^j g_{kji} x_i^k$ near to those of $Pg\sum_{k=2}^{n} q_k(\hbar)x_i^k$.

The analogue of the pressure in the lattice theory is given by

$$T_{2m}(\hbar, g, \mu, \lambda)=\ln\int \exp\left[\frac{-1}{\hbar}:I_\mu(\hbar, g, x):\right]d\gamma_{\hbar C(\lambda)},$$
$$(\hbar, g, \mu, \lambda)\in(0,\infty)\times\bar{C}_\varepsilon\times\mathbb{R}^{2m}\times[0,1], \tag{5.17}$$

where $d\gamma_D=(2\pi)^{-m}(\det D)^{-1/2}\exp[-\frac{1}{2}xD^{-1}x]\,dx$ and the Wick dots are with respect to the covariance $\hbar C(\lambda)$. The lattice analogue of $E(\hbar)$ is the Legendre transform Γ_{2m} (evaluated with the classical field equal to zero) given by

$$\Gamma_{2m}(\hbar, g, \lambda)=\sup_{\mu\in R^{2m}}[-\hbar T_{2m}(\hbar, \mu, g, \lambda)],\quad (\hbar, g, \lambda)\in(0,\infty)\times C_\varepsilon\times[0,1]. \tag{5.18}$$

The following lemma will be used in the proof that Γ_{2m} is finite.

Lemma 5.9. *Let $dv=g(x)dx$ be a finite positive measure on R^1, with $g>0$ and $e^{\pm jx}\in L^1(dv)$. Let $dv_j=\dfrac{e^{jx}dv}{\int e^{jx}dv}$. Then $\lim\limits_{j\to\pm\infty}\int x dv_j=\pm\infty$.*

Proof. It suffices to prove that $\lim\limits_{j\to\infty}\int x dv_j=+\infty$, since $\int x dv_{-j}=-\int x dv_j^-$, where $dv_j^-=\dfrac{e^{jx}g(-x)dx}{\int e^{jx}g(-x)dx}$, and $dv^-=g(-x)dx$ satisfies the hypotheses of the lemma. To prove the $j\to+\infty$ case, we begin by showing that given any $a<1$ and $y>0$ there is a $J(y)$ such that

$$\int_y^\infty dv_j\geqq a\quad\text{for every}\quad j\geqq J(y).$$

In fact, let $\varepsilon>0$ and choose $x_0<y$ such that $\int_{x_0}^{y} dv\leqq\varepsilon$. Choose J_0 such that $e^{j(x_0-y)}\leqq\varepsilon$ for $j\geqq J_0$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{jx}dv&\leqq e^{jx_0}\int_{-\infty}^{x_0} dv+e^{jy}\varepsilon+\int_y^\infty e^{jx}dv\\&\leqq e^{jy}\left[\varepsilon\int_{-\infty}^{x_0} dv+\varepsilon+e^{-jy}\int_y^\infty e^{jx}dv\right],\end{aligned}$$

so

$$\int_y^\infty dv_j\geqq\left[e^{-jy}\int_y^\infty e^{jx}dv\right]\left[\varepsilon\int_{-\infty}^{x_0} dv+\varepsilon+e^{-jy}\int_y^\infty e^{jx}dv\right]^{-1}\geqq a$$

for ε sufficiently small. But for $y>0$ and $j\geqq J(y)$,

$$\int_{-\infty}^{\infty} x dv_j=\int_{-\infty}^{-y} x dv_j+\int_{-y}^{y} x dv_j+\int_y^\infty x dv_j\geqq\int_{-\infty}^{-y} x dv_j+(2a-1)y.$$

And if $y>0$ and $j>0$, then

$$\left|\int_{-\infty}^{-y} x d\nu_j\right| = \int_{y}^{\infty} x d\nu_{-j}^{-} \leqq \int_{0}^{\infty} x d\nu_{-j}^{-} \leqq \frac{\int_{0}^{\infty} x g(-x)dx}{\int_{-\infty}^{0} g(-x)dx} \equiv c,$$

so $\int_{-\infty}^{\infty} x d\nu_j \geqq -c+(2a-1)y$ if $j \geqq J(y)$. The lemma then follows by taking $a=\frac{3}{4}$. □

Theorem 5.10. *The lattice Legendre transform $\Gamma_{2m}(\hbar, g, \lambda)$ is finite for $(\hbar, g, \lambda) \in (0, \infty) \times C_\varepsilon \times [0,1]$, and the supremum in its definition is attained at a unique point $\mu(\hbar, g, \lambda)$.*

Proof. The variables $\hbar, g, \lambda, m$ play no role in the proof, so we drop them from the notation and simply write $\Gamma = -\inf_\mu T(\mu)$. By Hölder's inequality T is strictly convex, so if T is bounded below then its infimum is attained at a unique point. By a standard theorem [17, Theorem 27.2], T is bounded below if $\lim_{t\to\infty} \frac{\partial}{\partial t} T(t\mu) > 0$ for every $\mu \neq 0$. We use Lemma 5.9 to show that in fact $\lim_{t\to\infty} \frac{\partial}{\partial t} T(t\mu) = +\infty$. By definition of T,

$$\frac{\partial}{\partial t} T(t\mu) = \frac{\int \mu x \exp[-:I_0(x): + t\mu x] d\gamma_C}{\int \exp[-:I_0(x): + t\mu x] d\gamma_C}.$$

Expand the Wick dots, write $d\gamma_C = \text{const}\, e^{-\frac{1}{2}xC^{-1}x}dx$, and choose an i for which $\mu_i \neq 0$. Let $z=\mu \cdot x$ and $y=(x_1, \ldots, \hat{x}_i, \ldots, x_{2m}) \in R^{2m-1}$. Then for some polynomial P in $2m$ variables,

$$\frac{\partial}{\partial t} T(t\mu) = \frac{\int z e^{tz} (\int e^{-P(z,y)} dy) dz}{\int e^{tz} (\int e^{-P(z,y)} dy) dz},$$

which goes to $+\infty$ as $t \to \infty$ by Lemma 5.9. □

Let $D_\varrho = \{\mu \in \mathbb{R}^{2m} : |\mu| < \varrho\}$.

Theorem 5.11. *There exist $\gamma, \varepsilon, \varrho, r_0 > 0$ such that for all $r < r_0$ $\hbar T_{2m}(\hbar, g, \mu, \lambda)$ is C^∞ in $(\hbar, g, \mu, \lambda) \in [0, \gamma) \times \bar{C}_\varepsilon \times D_\varrho \times [0,1]$.*

Proof. Let $J_\mu(\hbar, g, \lambda, x) = :I_0(\hbar, g, x):_{\hbar C(\lambda)} - \mu x + \frac{1}{2} x C(\lambda)^{-1} x$. By Lemma 5.8 and the fact that elements of $C(\lambda)^{-1}$ differ from those of $m_1^2 I$ by at most r, for r, $\hbar$, and ε sufficiently small $J_0(\hbar, g, \lambda, x)$ has coefficients close to those of

$$\sum_{i=1}^{2m} \left[Pg \sum_{k=3}^{n} q_k(0) x^k + \frac{1}{2} m_1^2 x_i^2 \right].$$

Hence if $|\mu|$ is also small, then $J_\mu(\hbar, g, \lambda, \cdot)$ has a uniquely attained global minimum at say $\xi(\hbar, g, \mu, \lambda)$, with ξ C^∞ and

$$K(\hbar, g, \mu, \lambda, x) \equiv J_\mu(\hbar, g, \lambda, x + \xi(\hbar, g, \mu, \lambda)) - J_\mu(\hbar, g, \lambda, \xi(\hbar, g, \mu, \lambda)) \geqq c|x|^2 \tag{5.19}$$

for some $c>0$. Translating x by ξ in Eq. (5.17) and then scaling by $\hbar^{1/2}$ gives

$$\hbar T_{2m}(\hbar, g, \mu, \lambda) = -J_\mu(\hbar, g, \lambda; \xi(\hbar, g, \mu, \lambda)) + \hbar \ln \left[\frac{\int \exp\left[\frac{-1}{\hbar} K(\hbar, g, \mu, \lambda; \hbar^{1/2} x)\right] dx}{\int \exp\left[-\frac{1}{2} x C(\lambda)^{-1} x\right] dx} \right]. \tag{5.20}$$

Using the bound (5.19) and Lebesgue's Dominated Convergence Theorem, the second term on the right side can be differentiated under the integral sign with respect to

$$(t=\hbar^{1/2}, g, \mu, \lambda) \in [0, \gamma^{1/2}) \times \bar{C}_\varepsilon \times D_\varrho \times [0,1].$$

The only thing to check is that odd t derivatives vanish at $t=0$, i.e.,

$$\frac{\partial^k}{\partial t^k} D_g^\alpha D_\mu^\beta \frac{\partial^l}{\partial \lambda^l} [t^2 T_{2m}(t^2, g, \mu, \lambda)] \to 0 \quad \text{as} \quad t \downarrow 0, \quad \text{if } k \text{ is odd}. \tag{5.21}$$

To see this, note that by (5.19),

$$\frac{1}{t^2} K(t^2, g, \mu, \lambda; tx) \geqq c|x|^2$$

$$\text{for all} \quad (t, g, \mu, \lambda, x) \in (-\gamma^{1/2}, \gamma^{1/2}) \times \bar{C}_\varepsilon \times D_\varrho \times [0,1] \times R^{2m},$$

so that in fact the second term on the right side of Eq. (5.20) is C^∞ in $(t, g, \mu, \lambda) \in (-\gamma^{1/2}, \gamma^{1/2}) \times \bar{C}_\varepsilon \times D_\varrho \times [0,1]$. But by scaling,

$$\int \exp\left[\frac{-1}{t^2} K(t^2, g, \mu, \lambda; tx)\right] dx = t^{-2m} \int \exp\left[\frac{-1}{t^2} K(t^2, g, \mu, \lambda; x)\right] dx.$$

Therefore, the second term on the right side of Eq. (5.20) is invariant under $t \to -t$, and Eq. (5.21) follows. □

In the next lemma, we use the notation

$$\langle \cdot \rangle_{\hbar, g, \mu, \lambda} = \frac{\int \cdot \exp\left[\frac{-1}{\hbar} :I_\mu(\hbar, g, x):\right] d\gamma_{\hbar C(\lambda)}}{\int \exp\left[\frac{-1}{\hbar} :I_\mu(\hbar, g, x):\right] d\gamma_{\hbar C(\lambda)}}.$$

Lemma 5.12. *The following limits are uniform in* $(g, \mu, \lambda) \in \bar{C}_\varepsilon \times D_\varrho \times [0,1]$.

(i) $$\lim_{\hbar \downarrow 0} \hbar T_{2m}(\hbar, g, \mu, \lambda) = -J_\mu(0, g, \lambda; \xi(0, g, \mu, \lambda)),$$

(ii) $$\lim_{\hbar \downarrow 0} \langle x_i \rangle_{\hbar, g, \mu, \lambda} = \xi_i(0, g, \mu, \lambda),$$

(iii) $$\lim_{\hbar \downarrow 0} h^{-1} \langle x_i ; x_j \rangle_{\hbar, g, \mu, \lambda} = (M^{-1})_{ij},$$

where $M_{ab} = \left.\frac{\partial}{\partial x_a \partial x_b}\right|_{x=0} K(0, g, \mu, \lambda; x)$ *is invertible by* (5.19). *In particular,* $\lim_{\hbar \downarrow 0} \hbar^{-1} \langle x_i ; x_j \rangle_{\hbar, g, 0, \lambda} = C(\lambda)_{ij}$.

Proof. (i) is immediate by (5.20) and (5.19).

(ii) follows by differentiating (5.20) with respect to μ_i and using (5.19) to see that the derivative of the second zerm on the right side of (5.20) is $O(\hbar)$.

(iii) Translation of x by ξ followed by scaling by $\hbar^{1/2}$ gives

$$\hbar^{-1}\langle x_i; x_j\rangle_{\hbar,g,\mu,\lambda} = \frac{\int (x_i; x_j) \exp\left[\frac{-1}{\hbar} K(\hbar, g, \mu, \lambda; \hbar^{1/2}x)\right] dx}{\int \exp\left[\frac{-1}{\hbar} K(\hbar, g, \mu, \lambda; \hbar^{1/2}x)\right] dx}.$$

By (5.19) the right side approaches $\int (x_i; x_j) e^{-\frac{1}{2}xMx} dx / \int e^{-\frac{1}{2}xMx} dx = (M^{-1})_{ij}$. □

Lemma 5.13. $\lim_{\hbar \downarrow 0} \mu(\hbar, g, \lambda) = 0$ *uniformly in* $(g, \lambda) \in \bar{C}_\varepsilon \times [0, 1]$.

Proof. To simplify the notation, let $f(\hbar, \mu) = \hbar T_{2m}(\hbar, g, \mu, \lambda)$ and $f(\mu) = -J_\mu(0, g, \lambda; \xi(0, g, \mu, \lambda))$. By Lemma 5.12(i), $\lim_{\hbar \downarrow 0} f(\hbar, \mu) = f(\mu)$ uniformly in g and λ. Since $f(\hbar, \cdot)$ is convex, so is f. Also, f is smooth for small $|\mu|$ and $f(\mu) \geqq 0$ with $f(\mu) = 0$ only if $\mu = 0$. Let $\varepsilon \in (0, \varrho)$ and set $\alpha = \min_{\substack{s = \pm\varepsilon \\ |\hat{\mu}| = 1}} \left|\frac{\partial}{\partial s} f(s\hat{\mu})\right|$. Then $\alpha > 0$ and for any $|\hat{\mu}| = 1$,

$$\frac{\partial}{\partial s} f(s\hat{\mu}) \begin{cases} \leqq -\alpha, & s = -\varepsilon, \\ \geqq \alpha, & s = +\varepsilon. \end{cases}$$

But by Lemma 5.12(ii) there is a $\delta > 0$ such that

$$\left|\frac{\partial}{\partial s} f(\hbar, s\hat{\mu}) - \frac{\partial}{\partial s} f(s\hat{\mu})\right| < \frac{\alpha}{2} \quad \text{for all} \quad h < \delta,$$

and so

$$\frac{\partial}{\partial s} f(\hbar, s\hat{\mu}) \begin{cases} \leqq -\frac{\alpha}{2}, & s = -\varepsilon, \\ \geqq \frac{\alpha}{2}, & s = +\varepsilon \end{cases} \quad \text{for all} \quad \hbar < \delta, \quad |\hat{\mu}| = 1.$$

It follows that the minimum of $f(\hbar, \mu)$ is attained at some point $s(\hbar)\hat{\mu}(\hbar)$ with $|\hat{\mu}(\hbar)| = 1$ and $s(\hbar) < \varepsilon$. □

Theorem 5.14. $\Gamma_{2m}(\hbar, g, \lambda)$ *is* C^∞ *in* $(\hbar, g, \lambda) \in [0, \gamma) \times \bar{C}_\varepsilon \times [0, 1]$.

Proof. We first show smoothness of $\Gamma_{2m}(\hbar, g, \lambda) = -\hbar T_{2m}(\hbar, g, \mu(\hbar, g, \lambda), \lambda)$ in the open set $(0, \gamma) \times C_\varepsilon \times (0, 1)$. By Lemma 5.13, $\mu(\hbar, g, \lambda) \in D_\varrho$ for $\hbar < \gamma$ sufficiently small. Therefore, by Lemma 5.12(iii),

$$\det\left[\frac{\partial^2}{\partial\mu_i \partial\mu_j}\bigg|_{\mu(\hbar, g, \lambda)} \hbar T_{2m}(\hbar, g, \mu, \lambda)\right] = \det\left[\hbar^{-1}\langle x_i, x_j\rangle_{\hbar, g, \mu, \lambda}\right] \geqq C > 0 \tag{5.22}$$

uniformly in $\hbar$, g, and λ if ε and γ are sufficiently small. By Eq. (5.22) and the implicit function theorem $\mu(\hbar, g, \lambda)$ is C^∞ in $(\hbar, g, \lambda) \in (0, \gamma) \times C_\varepsilon \times (0, 1)$, and hence so is Γ_{2m}

by Theorem 5.11. The extension of smoothness to $[0,\gamma)\times\bar{C}_\varepsilon\times[0,1]$ poses no difficulty since derivatives of Γ_{2m} can be seen to be uniformly bounded in $(\hbar,g,\lambda)\in(0,\gamma)\times C_\varepsilon\times(0,1)$ using Eq. (5.22) and the fact that derivatives of T_{2m} are uniformly bounded by Theorem 5.11. □

The following theorem allows us to analyze the graphs occurring in $D_1^N\Gamma_{2N}(0,g,\lambda)$ instead of those in $D_1^N E(0)$.

Theorem 5.15. *For $N\geqq 2$, $-D_1^N\Gamma_{2N}(0,g,\lambda)$ is given by a finite linear combination of graphs which is topologically identical to the sum of graphs equal to $-D_1^N E(0)$ (as given in Theorem 5.3), with the following rules of evaluation:*

1. *Whereas a vertex in $-D_1^N E(0)$ takes a factor $-q_{kj}{:}\phi^k(\mathbb{R}^2){:}$, a vertex in $-D_1^N\Gamma(0,g,\lambda)$ takes a factor $-\sum_{i=1}^{2N} q_{kj}g_{kji}{:}x_i^k{:}$.*
2. *No vertex is fixed – all are summed over the lattice.*
3. *A line joining x_i to x_j contributes $C(\lambda)_{ij}$.*

Proof. Since

$$\Gamma_{2N}(\hbar,g,\lambda)$$
$$=-\hbar\ln\int\exp\left[-\sum_{i=1}^{2N}\left[\sum_{k=2}^{n}\sum_{j=0}^{n/2}q_{kj}\hbar^{\frac{j+k-1}{2}}g_{kji}{:}x_1^k{:}-\hbar^{-1/2}\mu_i(\hbar,g,\lambda)x_i\right]\right]d\gamma_{C(\lambda)}$$

and

$$E(\hbar)=-\hbar\lim_{\Lambda}\frac{1}{|\Lambda|}\ln\int\exp\left[-\int_\Lambda\left[\sum_{k=2}^{n}\sum_{j=0}^{n/2}q_{kj}\hbar^{\frac{j+k-1}{2}}{:}\phi^k{:}-\hbar^{-1/2}\mu(\hbar)\phi\right]\right]d\mu_{C_1},$$

differentiation of Γ with respect to $t=\hbar^{1/2}$ is formally very similar to differentiation of E with respect to $t=\hbar^{1/2}$, and with the rules 1–3 above, yields graphs of the form (5.6) with f replaced by $b(t,g,\lambda)\equiv t^{-1}\mu(t^2,g,\lambda)$. A mechanism similar to that described in the proof of Theorem 5.3 is responsible for hooking the graphs $D_1^l b(t,g,\lambda)$ onto the corresponding legs. □

Corollary 5.16.

$$D_1^N\Gamma_{2N}(0,1,\lambda)=\sum_{|\alpha|\leqq N}\frac{1}{\alpha!}D_2^\alpha D_1^N\Gamma_{2N}(0,0,\lambda),$$

where α is a multi-index with $2N\left(\frac{n}{2}+1\right)(n-1)$ components.

Proof. By Theorem 5.15, $D_1^N\Gamma_{2N}(0,g,\lambda)$ is a polynomial in g of degree N, so the Corollary follows by Taylor's theorem. □

The following lemma shows that when $g=0$ the interaction [defined in (5.16)] occurring in the lattice pressure $T(\hbar,g,\mu(\hbar,g,\lambda),\lambda)$ vanishes.

Lemma 5.17. $\mu(\hbar,0,\lambda)=0$ *for* $(\hbar,\lambda)\in[0,\gamma)\times[0,1]$.

Proof. Since $\hbar T_{2N}(\hbar,0,\mu,\lambda)=\hbar\ln\int\exp\left[\frac{1}{\hbar}\mu x\right]d\gamma_{\hbar C(\lambda)}$ is strictly convex as a function of μ and $\hbar T_{2N}(\hbar,0,-\mu,\lambda)=\hbar T_{2N}(\hbar,0,\mu,\lambda)$, it follows that $\inf_{\mu\in R^{2N}}\hbar T_{2N}(\hbar,0,\mu,\lambda)$ occurs at $\mu(\hbar,0,\lambda)=0$. □

To simplify the notation for derivatives with respect to components of g, given indices k_l, j_l, i_l, we write $g_l = g_{k_l j_l i_l}$, and denote derivatives with respect to g_l with a subscript l, e.g., $\Gamma_{12\ldots N} = \frac{\partial^N}{\partial g_1 \ldots \partial g_N}$, and we drop the subscript $2N$ from Γ_{2N} and T_{2N}.

The following lemma is the first step in identifying the graphs contributing to $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$.

Lemma 5.18. *For $\hbar < \gamma$, $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$ is a finite sum of graphs with the N vertices $-q_{k_l j_l} \hbar^{j_l + \frac{1}{2}k_l - 1} : x_{i_l}^{k_l} :$ $(l = 1, \ldots, N)$ and lines $C(\lambda)$. No self-lines can appear. Graphs enter the sum with either a plus sign or a minus sign, but all those with minus signs are* 1-PR. *Furthermore, every* 1-PI *graph with the mentioned vertices enters the sum with a plus sign. The combinatorial factor of a* 1-PI *graph is the same as for* $T_{12\ldots N}(\hbar, 0, 0, \lambda)$.

Lemma 5.18 will be improved in Theorem 5.20 where it will be shown that all the 1-PR graphs in $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$ cancel.

Proof of Lemma 5.18. The variables $\hbar$ and λ play no significant role in the proof so we drop them from the notation. Derivatives are denoted by subscripts and an implicit summation convention is used. In the following, all derivatives of T are evaluated at $(g, \mu(g))$.

Differentiating the equation $-\Gamma(g) = T(g, \mu(g))$ with respect to g_1 gives $-\Gamma_1 = T_1 + T_\mu \mu_1 = T_1$, since $T_\mu = 0$. Note that in T_1 the g dependence of μ is not differentiated. Differentiating $-\Gamma_1 = T_1$ with respect to g_2 gives

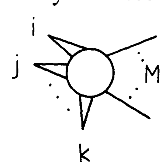
$$-\Gamma_{12} = T_{12} + T_{1\mu}\mu_2 .$$

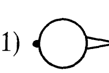
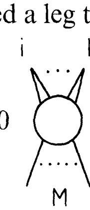
To compute μ_i, differentiate the equation $T_\mu = 0$ with respect to g_i to obtain

$$\mu_i = -T_{\mu\mu}^{-1} T_{\mu i},$$

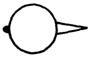
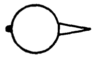
where the inverse on the right side is a matrix inverse. Therefore,

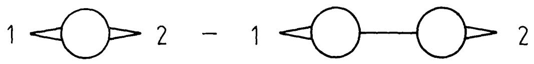
$$-\Gamma_{12} = T_{12} - T_{1\mu} T_{\mu\mu}^{-1} T_{\mu 2} . \tag{5.23}$$

Note that when $g = 0$, $(g, \mu(g)) = (0, 0)$ by Lemma 5.17 and we have a free theory. A derivative of the form $T_{ij\ldots k\underbrace{\mu\ldots\mu}_{M}}$ at $g = 0$ is the sum of all connected graphs with fixed vertices as specified by the g_l's, and M fixed one-legged vertices. A factor $T_{\mu\mu}^{-1}$ serves to link up graphs in a free theory. We use a graph notation for the derivatives as follows. Denote $T_{ij\ldots k\underbrace{\mu\ldots\mu}_{M}}$ by 

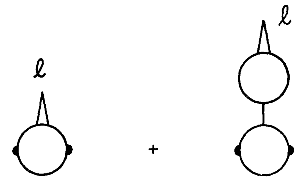
and μ_i by (-1)  i, where the dot on the $\mu_i = -T_{\mu\mu}^{-1} T_{\mu i}$ graph indicates that a $T_{\mu\mu}^{-1}$ has amputated a leg that was brought down by differentiation with respect to μ. When $g = 0$ 

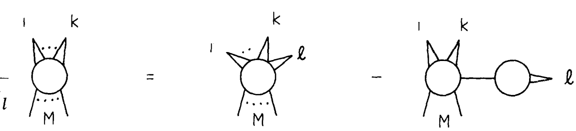
or

 i is given by a sum of connected lattice graphs without self-lines. (In particular, at $g=0$,  $=0$.) In this notation, Eq. (5.23) becomes

$-\Gamma_{12}=$ 

The theorem now follows by repeated differentiation of Eq. (5.23) using the following facts:

$$\frac{\partial}{\partial g_l} T_{\mu\mu}^{-1} = -T_{\mu\mu}^{-1}(T_{\mu\mu l} + T_{\mu\mu\mu}\mu_l)T_{\mu\mu}^{-1} = -$$ 

$$\frac{\partial}{\partial g_l} T_{\underbrace{ij\ldots k\mu\ldots\mu}_{M}} = \frac{\partial}{\partial g_l}$$ 

Clearly, all graphs occurring in $-\Gamma_{12\ldots N}(0)$ with a minus sign are 1-PR, because a minus sign is introduced with every factor of $T_{\mu\mu}^{-1}$ (and in no other way) and a factor of $T_{\mu\mu}^{-1}$ corresponds to a line whose removal disconnects the graph. Furthermore, $-\Gamma_{12\ldots N}(0)$ contains the term $+T_{12\ldots N}(0,0)$ which is the sum of all connected graphs (with combinatorial factors) having vertices as in the statement of the lemma, and hence contains as a subset all 1-PI graphs. □

The following theorem, inspired by [3], is the key to obtaining the cancellation of all 1-PR graphs in $-\Gamma_{12\ldots N}(\hbar,0,0)$.

Theorem 5.19. *Given $g_l = g_{k_l j_l i_l}$ $(l=1,\ldots,N)$, if at least one i_l is an element of $\{1,\ldots,N\}$ and at least one i_l is an element of $\{N+1,\ldots,2N\}$, then for all $\hbar\in[0,\gamma)$*

$$D_3^s\Gamma_{12\ldots N}(\hbar,0,0)=0\,,\qquad s=0,1\,.$$

Proof. Since $\hbar$ plays no role in the proof it is omitted.

Beginning with the case $s=0$, since $C(0)^{-1}=m_1^4\begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1}\end{bmatrix}$ does not mix $\{x_1,\ldots,x_N\}$ and $\{x_{N+1},\ldots,x_{2N}\}$, we can write

$$T(g,\mu,0)=S_{(1)}(g(1),\mu(1))+S_{(2)}(g(2),\mu(2))\,,$$

where $\mu(1)$ and $g(1)$ [respectively, $\mu(2)$ and $g(2)$] consist of those μ_i and g_{kji} with $i\in\{1,\ldots,N\}$ (respectively, $i\in\{N+1,\ldots,2N\}$), and

$$\begin{aligned} &S_{(1)}(g(1),\mu(1)) \\ &\quad=\ln\int\exp\left[-\sum_{i=1}^{N}\left[\sum_{k=2}^{n}\sum_{j=0}^{n/2} q_{kj}g_{kji}:x_i^k:-\mu_i x_i\right]\right]d\gamma_{m_1^{-4}R_1}(x_1,\ldots,x_N)\,,\end{aligned}$$

$$S_{(2)}(g(2), \mu(2))$$
$$= \ln \int \exp\left[- \sum_{i=N+1}^{2N} \left[\sum_{k=2}^{n} \sum_{j=0}^{n/2} q_{kj} g_{kji} :x_1^k: - \mu_i x_i \right]\right] d\gamma_{m_1^{-4} R_2}(x_{N+1}, \ldots, x_{2N}).$$

Therefore,

$$\frac{\partial}{\partial \mu_i} T(g, \mu, 0) = \begin{cases} \frac{\partial}{\partial \mu_i} S_{(1)}(g(1), \mu(1)), & i \in \{1, \ldots, N\}, \\ \frac{\partial}{\partial \mu_i} S_{(2)}(g(2), \mu(2)), & i \in \{N+1, \ldots, 2N\}. \end{cases}$$

It follows that

$$\mu_i(g, 0) = \begin{cases} \mu_i^{(1)}(g(1)), & i \in \{1, \ldots, N\}, \\ \mu_i^{(2)}(g(2)), & i \in \{N+1, \ldots, 2N\}, \end{cases}$$

and hence

$$\Gamma(g, 0) = -S_{(1)}(g(1), \mu^{(1)}(g(1)), 0) - S_{(2)}(g(2), \mu^{(2)}(g(2)), 0),$$

and the theorem follows in the case $s = 0$.

To prove the theorem in the case $s = 1$, we begin by noting that $D_2\Gamma(g, 0) = -D_3 T(g, \mu(g, 0), 0)$, since $D_2 T(g, \mu(g, \lambda), \lambda) = 0$. Denoting expectations with respect to $d\gamma_{C(\lambda)}$ by $[\,\cdot\,]_\lambda$ and expectations with respect to $\frac{\exp[-:I_\mu(g,x):]d\gamma_{C(\lambda)}}{\int \exp[-:I_\mu(g,x):]d\gamma_{C(\lambda)}}$ by $\langle \cdot \rangle_{g,\mu,\lambda}$, we have

$$\begin{aligned} D_3 T(g, \mu, 0) &= [e^{-:I_\mu(g,x):}]_0^{-1} \frac{\partial}{\partial \lambda}\bigg|_0 [e^{-:I_\mu(g,x):}]_\lambda \\ &= -\left\langle \frac{\partial}{\partial \lambda}\bigg|_0 (:I_0(g,x):_{C(\lambda)}) \right\rangle_{g,\mu,0} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^{2N} DC(0)_{ij}^{-1} (\langle x_i x_j \rangle_{g,\mu,0} - [x_i x_j]_0), \end{aligned}$$

and hence

$$\begin{aligned} D_2\Gamma(g, 0) &= \left\langle \frac{\partial}{\partial \lambda}\bigg|_0 :I_0(g,x):_{C(\lambda)} \right\rangle_{g,\mu(g,0),0} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{2N} DC(0)_{ij}^{-1} (\langle x_i x_j \rangle_{g,\mu(g,0),0} - [x_i x_j]_0). \end{aligned} \tag{5.24}$$

Now differentiate Eq. (5.24) with respect to g_a and g_b where $i_a \in \{1, 2, \ldots, N\}$ and $i_b \in \{N+1, N+2, \ldots, 2N\}$. Since $\frac{\partial}{\partial \lambda}\Big|_0 :I_0(g,x):_{C(\lambda)}$ is a sum of two polynomials: one in $x_1, \ldots, x_N$ depending only on g_l with $i_l \in \{1, \ldots, N\}$, and one in $x_{N+1}, \ldots, x_{2N}$ depending only on g_l with $i_l \in \{N+1, \ldots, 2N\}$, and since as was seen in the proof of the $s=0$ case the measure $\langle \cdot \rangle_{g,\mu(g,0),0}$ factors into a product of probability measures in $x_1, \ldots, x_N$ and $x_{N+1}, \ldots, x_{2N}$ depending only on g_l with $i_l \in \{1, \ldots, N\}$

and g_l with $i_l \in \{N+1, \ldots, 2N\}$, respectively,

$$\frac{\partial^2}{\partial g_a \partial g_b} \left\langle \frac{\partial}{\partial \lambda}\bigg|_0 :I_0(g,x):_{C(\lambda)} \right\rangle_{g,\mu(g,0),0} = 0 .$$

Next, observe that the term involving $[x_i x_j]_0$ on the right side of Eq. (5.24) does not depend on g at all and hence vanishes after taking g derivatives. It remains only to show that

$$\frac{\partial^2}{\partial g_a \partial g_b} \langle x_i x_j \rangle_{g,\mu(g,0),0} = 0 .$$

Consider the case where both i and j are in $\{1, \ldots, N\}$. Then by factorization of the measure $\langle x_i x_j \rangle_{g,\mu(g,0),0}$ depends only on the g_l with $i_l \in \{1, \ldots, N\}$ and the above derivative vanishes since $i_b \in \{N+1, \ldots, 2N\}$. The case where both i and j are in $\{N+1, \ldots, 2N\}$ is similar. Now consider the case where exactly one of i, j lies in $\{1, \ldots, N\}$. Then by factorization of the measure,

$$\langle x_i x_j \rangle_{g,\mu(g,0),0} = \langle x_i \rangle_{g,\mu(g,0),0} \cdot \langle x_j \rangle_{g,\mu(g,0),0} \,.$$

Each factor on the right side of the above equation vanishes by definition of $\mu(g,0)$. □

We now show that all 1-PR graphs occurring in $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$ cancel, and identify explicitly the remaining 1-PI graphs. As in the statement of Theorem B we write $d(a) = \frac{-1}{4\pi} \log \frac{U_0''(a)}{m^2}$.

Theorem 5.20. *The derivative* $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$ *is a polynomial in* $\hbar$ *where the coefficient of* $\hbar^m$ *is the sum of all* $d(a)$*-renormalized* m *loop* 1-PI *graphs with vertices* $-(P^{(k_l)}(a)/k_l!) x_{i_l}^{k_l}$ $(l = 1, \ldots, N)$ *and* $C(\lambda)$ *lines with self-lines allowed. Note that the vertices are fixed. Each graph takes the same combinatorial factor that it has in* $T_{12\ldots N}(\hbar, 0, 0, \lambda)$.

Proof. We first show that $-\Gamma_{12\ldots N}(\hbar, 0, \lambda)$ can be written as a sum of 1-PI graphs having $\hbar$ dependent vertices. By Theorem 5.18 we can write

$$-\Gamma_{12\ldots N} = \sum_{k=1}^{K} I_k(\hbar, \lambda) + \sum_{m=1}^{M} R_m(\hbar, \lambda) - \sum_{l=1}^{L} N_l(\hbar, \lambda) , \tag{5.25}$$

where the three sums on the right side of Eq. (5.25) are, respectively, the sum of all 1-PI graphs made of $C(\lambda)$-lines and vertices, $-q_{k_l j_l} \hbar^{j_l + \frac{1}{2}k_l - 1} :x_{i_l}^{k_l}:$ [having the same combinatorial factor as in $T_{12\ldots N}(\hbar, 0, 0, \lambda)$], the sum of all 1-PR graphs occurring in the expansion of Theorem 5.18 with a plus sign, and the sum of all 1-PR graphs occurring in the expansion with a minus sign. We now use Theorem 5.19 to show that the last two sums cancel.

In fact, treating $i_1, \ldots, i_N$ as free variables, it follows from Theorem 5.19 and Lemma 5.6 that for any admissible imposition of $x_{i_1}, \ldots, x_{i_N}$ on L_{2N}

$$\sum_{m=1}^{M} D_2^s R_m(\hbar, 0) = \sum_{l=1}^{L} D_2^s N_l(\hbar, 0), \qquad s = 0, 1 . \tag{5.26}$$

We now show that this implies that $\sum_{m=1}^{M} R_m(\hbar, \lambda)$ consists of exactly the same graphs as $\sum_{l=1}^{L} N_l(\hbar, \lambda)$.

For a graph G with vertices as in R_m or N_l, denote by $\tilde{G}$ the graph obtained from G by cancelling all factors $-q_{kj}\hbar^{j+\frac{k}{2}-1}$. Since R_1 is reducible and has N vertices, it can be imposed on L_{2N} by choosing $i_1, \ldots, i_N$ in such a way that a line of reducibility of R_1 (i.e., a line whose removal disconnects R_1) joins x_1 to x_{N+1}, and no other line joins $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$. This imposition of R_1 on L_{2N}, of course, also imposes the other R_m's and N_l's on L_{2N}. Since all these graphs are connected, at least one line crosses from $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$ for each graph. But $\frac{d}{d\lambda}\tilde{R}_m(0)$ or $\frac{d}{d\lambda}\tilde{N}_l(0)$ is zero if and only if more than one line makes the crossing from $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$. Hence for the above imposition

$$\sum_{m=1}^{M} \frac{d}{d\lambda}\tilde{R}_m(0) = \sum_{\text{one line}} \frac{d}{d\lambda}\tilde{R}_m(0) = \sum_{\text{one line}} \frac{d}{d\lambda}\tilde{N}_l(0), \tag{5.27}$$

where $\sum_{\text{one line}} \frac{d}{d\lambda}\tilde{G}_i$ denotes the sum over those i for which G_i has a single line joining $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$. But because of the form of $C(\lambda)$, for a graph $\tilde{N}_l$ on $\tilde{R}_m$ with exactly one line joining $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$, $\frac{d}{d\lambda}\tilde{N}_l(0)$ or $\frac{d}{d\lambda}\tilde{R}_m(0)$ is r multiplied by a product of r_{ij}'s ($1 \leqq i,j \leqq N$ or $N+1 \leqq i,j \leqq 2N$), because it is only when the line joining $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$ is differentiated that the result is non-zero. It follows that the second equality in Eq. (5.27) is an equality of polynomials in the r_{ij} ($1 \leqq i,j \leqq N$ or $N+1 \leqq i,j \leqq 2N$), and so the coefficients of these polynomials must agree. However, these coefficients characterize the graphs topologically. To see this, note that the r_{ij} are in a one-one correspondence with lines joining x_i to x_j. Thus a product of r_{ij}'s characterizes the parts of the graph sitting in each of the sublattices $\{x_1, \ldots, x_N\}$ and $\{x_{N+1}, \ldots, x_{2N}\}$. Because there will be only one vertex x_{i_p} in each sublattice that does not have its full quota k_p of lines provided by the sublattice graphs, there is one and only one way that the line crossing from $\{x_1, \ldots, x_N\}$ to $\{x_{N+1}, \ldots, x_{2N}\}$ can join the two sublattices, and the graph is uniquely determined. Therefore,

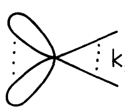
$$\sum_{\text{one line}} \tilde{R}_m = \sum_{\text{one line}} \tilde{N}_l \tag{5.28}$$

with exactly the same graphs occurring on each side of the equation. Now discard the graphs contributing to Eq. (5.28) from Eq. (5.26) and repeat the above procedure until none of the R_m remain. We now show that no graphs N_l remain, arguing by contradiction. Discarding all R_m graphs and the corresponding N_l graphs from Eq. (5.26) leaves $0 = \sum' D_2^s N_l(\hbar, 0)$, $s = 0, 1$, for every AI, where $\sum'$ denotes the sum over the remaining graphs. Therefore, $0 = \sum' \frac{d}{d\lambda}\tilde{N}_l(0)$, $s = 0, 1$, for every AI. Each term in $\sum' \frac{d}{d\lambda} \tilde{N}_l(0)$ is nonnegative, and since N_l is 1-PR, for a given l_0 the $i_1, \ldots, i_N$ can be chosen in such a way as to make $\frac{d}{d\lambda} \tilde{N}_{l_0}(0) > 0$. But this contradicts $0 = \sum' \frac{d}{d\lambda} \tilde{N}_l(0)$, and hence there can be no N_l remaining. The end result is that $\sum_{m=1}^{M} R_m(\hbar, \lambda) = \sum_{l=1}^{L} N(\hbar, \lambda)$, with exactly the same graphs on each side of the equation, and hence

$$-\Gamma_{12\ldots N}(\hbar, 0, \lambda) = \sum_{k=1}^{K} I_k(\hbar, \lambda). \tag{5.29}$$

To identify the graphs contributing to the right side of Eq. (5.29) as those stated in the theorem, we begin by obtaining an explicit formula for q_{kj}. By definition [Eq. (5.4)], $q_{kj} = \frac{1}{j!} D^j q_k(0)$, where q_k is defined in Eq. (3.9) by the requirement $\sum_{k=3}^{n} a_k : \phi^k :_{\hbar C}$ $= \sum_{k=0}^{n} q_k(\hbar) : \phi^k :_{\hbar C_1}$. Let $\tilde{a}_k = \begin{cases} a_k, & 3 \leqq k \leqq n, \\ 0, & \text{otherwise}, \end{cases}$ and extend the definition of $c_{kj} = \frac{k!}{2^j j! (k-2j)!}$ by setting $c_{kj} = 0$ if $j > \left[\frac{k}{2}\right]$. Then a simple computation gives $q_{kj} = \tilde{a}_{k+2j} c_{k+2j,j} d^j$, so a vertex

$$-q_{kj} \hbar^{j + \frac{k}{2} - 1} : x_i^k : = -\tilde{a}_{k+2j} \hbar^{j + \frac{k}{2} - 1} c_{k+2j,j} d^j : x_i^k :$$

can be interpreted as j  k, where each closed loop takes a factor of d, the combinatoric factor $c_{k+2j,j}$ counts the number of ways of choosing j pairs from $k+2j$ lines, each half-line takes a factor $\hbar^{1/2}$, and the vertex takes the factor $\frac{-1}{\hbar} \tilde{a}_{k+2j}$. This means that there is a one-one correspondence between 1-PI graphs having vertices $-q_{kj} \hbar^{j + \frac{1}{2}k - 1} : x_i^k :$ and no self-lines, and d-renormalized 1-PI graphs having vertices $\frac{-1}{\hbar} \tilde{a}_{k+2j} x_i^{k+2j}$ with self-lines allowed and each line taking a factor $\hbar$.

It remains only to identify the overall power of $\hbar$ of a graph. An unrenormalized graph has a power of $\hbar$ given by $I - V + 1$, where I is the number of lines of the graph, V is the number of vertices, and the extra $+1$ comes from the $\hbar$ in $-\Gamma = \hbar T$. But $I - V + 1$ is exactly the number of loops in the unrenormalized graph. □

In conclusion we combine the results of this section to prove Theorem B(b). By Eq. (5.1) and Corollary 5.2 we need only show that for $N \geqq 2$,

$$-v_N(a) = \frac{-1}{N!} D_1^N V(0, a) = -\frac{1}{N!} D^N q_1(0) - \frac{1}{N!} D_1^N E(0) \tag{5.30}$$

is the appropriate sum of graphs. The first term on the right side of Eq. (5.30) was identified in Eq. (5.2) to be the $d(a)$-renormalized single vertex N-loop diagram. By

Theorem 5.15 the second term is a sum of graphs topologically identical to the L_{2N}-graphs whose sum is $-\frac{1}{N!}D_1^N\Gamma(0,1,1)$, where Γ is the L_{2N} Legendre transform (evaluated at the classical field equals zero). By Corollary 5.16,

$$-\frac{1}{N!}D_1^N\Gamma(0,1,1)=\frac{-1}{N!}\frac{d^N}{dh^N}\bigg|_0\sum_{|\alpha|\leqq N}\frac{1}{\alpha!}D_2^\alpha\Gamma(h,0,1). \tag{5.31}$$

But by Theorem 5.20 the right side of Eq. (5.31) is exactly the desired sum of graphs: the different terms in the sum over α give the N-loop graphs with different kinds of vertices.

Finally, we show that the combinatorial factors are as indicated in Remark 1 under Theorem B. By Theorem 5.20 the combinatorial factor of a graph contributing to $D_2^\alpha\Gamma(\hbar,0,1)$ is the same as for $D_2^\alpha T(\hbar,0,0,1)$, namely the factor associated with the graph by Wick's theorem. The factor of $\frac{1}{\alpha!}$ occurring on the right side of Eq. (5.31) provides the factor $\frac{1}{\prod V_{jk}!}$ appearing in Remark 1. Since the $\frac{1}{N!}$ on the right side of (5.31) is cancelled by an $N!$ brought down by $\frac{d^N}{d\hbar^N}$, the combinatorial factor of a graph in $-\frac{1}{N!}D_1^N\Gamma(0,1,1)$, and hence in $-v_N(a)$, is as stated in Remark 1.

6. Proof of Theorem C

Theorem C. *Let $V(\hbar,a)$ denote the effective potential for $m=1$ and $P(a)=\left(a^2-\frac{1}{8}\right)^2-\frac{1}{2}a^2$. Then for $|a|<\frac{1}{\sqrt{8}}$, $D_1V(0,a)=-\gamma\left(\frac{1}{\sqrt{8}}\right)=0$, and for $N\geqq 2$, $\frac{-1}{N!}D_1^NV(0,a)$ is given by the sum of all N-loop <u>connected</u> graphs with no self-lines, with three- and four-legged vertices taking factors $\frac{-1}{3!}P^{(3)}\left(\frac{1}{\sqrt{8}}\right)=-2$ and $\frac{-1}{4!}P^{(4)}\left(\frac{1}{\sqrt{8}}\right)=-1$, respectively, and lines corresponding to the free covariance of mass one. Graphs take combinatorial factors as per Remark 1 under Theorem B.*

Proof. Translation of ϕ by $\pm\frac{1}{\sqrt{8}}$ in the pressure (1.2) with $m=1$ and the given interaction polynomial, followed by scaling $\phi\to\hbar^{1/2}\phi$, gives

$$p(\hbar,\mu)=\pm\frac{1}{\sqrt{8}}\mu+\hbar\lim_{\Lambda\uparrow R^2}\frac{1}{|\Lambda|}\ln\int\exp\left[-\int_\Lambda[\hbar:\phi^4:\pm\sqrt{2}\hbar^{1/2}:\phi^3:-\mu\hbar^{-1/2}\phi]\right]d\mu_C. \tag{6.1}$$

In Theorem 2.2 of [13], for sufficiently small $\hbar$ and μ the one-point function corresponding to the pressure $p(\hbar, \mu)$ is controlled using a low temperature cluster expansion. It follows from their results that

$$\left| D_2^{\pm} p(\hbar, 0) - \left(\pm \frac{1}{\sqrt{8}} \right) \right| = O(\hbar^2),$$

as perturbation theory and Eq. (6.1) would suggest. Therefore, for any $|a| < \frac{1}{8}$ there is a $\delta(a) > 0$ such that $a \in [D_2^- p(\hbar, 0), D_2^+ p(\hbar, 0)]$ for all $\hbar < \delta(a)$, and hence

$$V(\hbar, a) = \sup_{\mu} [\mu a - p(\hbar, \mu)] = -p(\hbar, 0), \quad \hbar < \delta(a).$$

In [13] an infinite volume theory corresponding to the interaction $\hbar x^4 + \sqrt{2}\hbar^{1/2} x^3$ and covariance C (with free boundary conditions) is obtained. In Sect. 6 of [13] it is shown that the perturbation series in $\hbar^{1/2}$ for a generalized Schwinger function of this theory is asymptotic. The pressure is not discussed, but it is straightforward to use the estimates of [13] to show that perturbation theory is also asymptotic for $p(\hbar, 0)$, and hence for $N \geqq 1$ and $|a| < \frac{1}{\sqrt{8}}$, $-\frac{1}{N!} D_1^N V(0, a) = \frac{1}{N!} D_1^N p(0, 0)$ is as stated in Theorem C. □

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