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A Classical Solution of the Non-Linear Complex Grassmann σ-Model with Higher Derivatives

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Abstract. We construct a soliton solution of the non-linear complex Grassmann σ-model with higher derivatives, and show that this solution, as a continuous map, represents a generator of the K-group of a sphere.

Introduction

Non-linear σ-models such as the CP^N σ-model or complex Grassmann σ-model in two dimensions are interesting objects to study not only for physicists but also mathematicians. They have non-instanton solutions with finite action other than instanton solutions. Moreover, a discrete symmetry transformation has been constructed in their solution spaces. See, in detail, [5] and its references.

In three or more dimensions, the situation is different. With usual action form, it is well known that a classical solution with finite action, which we call a soliton, does not exist, by the scaling argument of Derrick's type. Therefore we must alter the action to obtain a soliton.

In this note we construct a new Lagrangian on R^{2m} and show that it has at least one non-trivial soliton solution. Moreover we show that this one represents a generator of the K-group $\tilde{K}(S^{2m})(=Z)$ of the sphere S^{2m}.

I. The Model

We define a configuration space H which we consider hereafter. For natural numbers m, N we set

$$G_{2N,N} \equiv \{A \in M(2N;C) \mid A^2 = A, A^+ = A, \operatorname{Tr} A = N\}, \tag{1}$$

$$H_{2m} \equiv \{P: R^{2m} \to G_{2N,N}, C^\infty\text{-class}\}. \tag{2}$$

It is known that $G_{2N,N}$ is a Grassmann manifold and $G_{2N,N} \cong \mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$. We call an element P in (2) a projector.

For the space H_{2m} we define a new Lagrangian as follows

$$L(P) \equiv \tfrac{1}{2}\int d^{2m}X \operatorname{Tr}(\partial_{\mu_1} \ldots \partial_{\mu_m} P)^2, \tag{3}$$
$$\partial_{\mu_j} \equiv \partial/\partial x_{\mu_j} \quad (j = 1, \ldots, 2m).$$

Here and hereafter we adopt the Einstein rule on summation. The new Lagrangian coincides with original one for $m = 1$ [5], and was introduced by Kafief [4] for $m = 2$. Hereafter we consider only a classical configuration.

Lemma 1. *The equation of motion of (3) is given by*

$$[P, \Delta^m P] = 0, \tag{4}$$

where Δ^m stands for the m-times iteration of the Laplacian Δ on R^{2m} and $[\ ,\ ]$ stands for the Lie brackets.

The proof is easy. $P \in H_{2m}$ satisfying (4) and $L(P) < \infty$ we call a soliton. On the other hand our Lagrangian has a topological number. We explain this. For P in (2), a global form of the curvature F is defined by

$$F \equiv P dP \Lambda dP, \tag{5}$$

see [2, 6]. Then a topological index is given by

$$C_m(P) \equiv \frac{1}{2^m m!} \int \mathrm{Tr}\left(\frac{F}{2\pi\sqrt{-1}}\right)^{\Lambda m} \tag{6}$$

where Λm denotes the m-times exterior product. For example when $m = 2$ we have

$$C_2(P) = \frac{-1}{32\pi^2} \int \mathrm{Tr}\, F \Lambda F.$$

This is the first Pontrjagin number. We shall construct a soliton solution with topological index $= 1$ for any $2m (m > 0)$.

II. A Solution

For any natural number m, let $e_j (j = 1, \ldots, 2m - 1)$ be generators of the Clifford algebra, $e_i e_j + e_j e_i = 2\delta_{ij}$. Now we realize $\{e_j\}$ in $M(2^{m-1}; C)$ by the usual embedding. Then we may assume $e_j^+ = e_j (j = 1, \ldots, 2m - 1)$. We set $N = 2^{m-1}$ and

$$Z = x_{2m} 1_N + \sqrt{-1}\, x_j e_j. \tag{7}$$

Now we state our main result.

Theorem 2.

$$P \equiv \frac{1}{1 + X^2} \begin{bmatrix} 1_N & Z^+ \\ Z & X^2 1_N \end{bmatrix}; \quad X^2 = \sum_{j=1}^{2m} x_j^2 \tag{8}$$

is a non-trivial soliton solution.

Note that P in (8) is a CP^1-instanton projector for $m = 1$ and a Yang–Mills instanton projector for $m = 2$.

Before giving the proof of Theorem 2 we make some preparations. We resolve a Laplacian Δ as

$$\Delta \equiv \frac{\partial^2}{\partial X^2} + \frac{2m - 1}{X} \frac{\partial}{\partial X} + \text{(angles-parts)}$$

using the polar coordinate. Remarking that

$$\Delta^k \frac{x_j}{1+X^2} = \Delta^k \tfrac{1}{2}\partial_j \log(1+X^2) = \tfrac{1}{2}\partial_j \Delta^k \log(1+X^2), \tag{9}$$

we compute as follows:

$$\Delta^k \frac{1}{1+X^2} = \left(\frac{\partial^2}{\partial X^2} + \frac{2m-1}{X}\frac{\partial}{\partial X}\right)^k \frac{1}{1+X^2},$$

$$\Delta^k \log(1+X^2) = \left(\frac{\partial^2}{\partial X^2} + \frac{2m-1}{X}\frac{\partial}{\partial X}\right)^k \log(1+X^2).$$

Proposition 3.

$$\Delta^k \frac{1}{1+X^2} = (-4)^k \left[\sum_{j=0}^{k-1} {}_kC_j \{m-(j+2)\}\{m-(j+3)\}\cdots\{m-(k+1)\}(k+j)! \times \frac{1}{(1+X^2)^{k+j+1}} + (2k)! \frac{1}{(1+X^2)^{2k+1}} \right], \tag{10-1}$$

$$\Delta^k \log(1+X^2) = -(-4)^k \left[\sum_{j=0}^{k-1} {}_kC_j \{m-(j+1)\}\{m-(j+2)\}\cdots\{m-k\}(k+j-1)! \times \frac{1}{(1+X^2)^{k+j}} + (2k-1)! \frac{1}{(1+X^2)^{2k}} \right]. \tag{10-2}$$

Proof. The proof is by the mathematical induction on k.
When $k=m$ (10-1) and (10-2) become very simple equalities.

Corollary 4.

$$\Delta^m \frac{1}{1+X^2} = (-4)^m (2m)! \frac{1}{(1+X^2)^{2m}} \frac{1-X^2}{2(1+X^2)}, \tag{11-1}$$

$$\Delta^m \log(1+X^2) = -(-4)^m (2m-1)! \frac{1}{(1+X^2)^{2m}}. \tag{11-2}$$

From Corollary 4 and (9) we have

Corollary 5.

$$\Delta^m \frac{X^2}{1+X^2} = -\Delta^m \frac{1}{1+X^2}, \tag{12-1}$$

$$\Delta^m \frac{x_j}{1+X^2} = (-4)^m (2m)! \frac{1}{(1+X^2)^{2m}} \frac{x_j}{1+X^2}. \tag{12-2}$$

Using the above corollaries we prove

Proof of Theorem 2. From Corollaries 4 and 5 we obtain

$$\Delta^m P = (-4)^m (2m)! \frac{1}{(1+X^2)^{2m+1}} \frac{1}{2} \begin{bmatrix} (1-X^2)1_N & 2Z^+ \\ 2Z & -(1-X^2)1_N \end{bmatrix}. \tag{13}$$

Therefore

$$P\Delta^m P = \frac{1}{2}(-4)^m(2m)!\frac{1}{(1+X^2)^{2m}}P = \Delta^m P\cdot P,$$

that is, $[P, \Delta^m P] = 0$. Next we show $L(P) < \infty$. Substituting (8) into (3), we obtain a rational function on X inside integration (in polar coordinates). The highest exponent of X in the numerator of the rational function is $2(m+1)$. Therefore if

$$\int_0^\infty X^{2m-1}dX\frac{X^{2(m+1)}}{(1+X^2)^{2(m+1)}} < \infty, \tag{14}$$

then $L(P) < \infty$. We show (14). Putting $X = \tan\theta$,

$$\begin{aligned}\text{left hand side of (14)} &= \int_0^\infty \frac{dX}{1+X^2}\frac{X^{4m+1}}{(1+X^2)^{2m+1}}\\ &= \int_0^{\pi/2} d\theta\cos\theta\sin^{4m+1}\theta < \infty.\end{aligned}$$

Finally we show our solution is non-trivial. Substituting (8) into (6) we have

$$C_m(P) = 1 \tag{15}$$

in a similar way as in [3;§I]. Since (6) is a topological invariant, our P is non-trivial. QED

III. A Relation with *K* Theory

We sketch in this section the relation of our solution with K-theory. For $j = 1, \ldots, 2m$, we set

$$\phi_j = \frac{2x_j}{1+X^2};\quad \phi_{2m+1} = \frac{1-X^2}{1+X^2}. \tag{16}$$

Clearly $\Sigma\phi_j^2 = 1$. Using (16), we rewrite (8) as follows:

$$\tilde{P} \equiv \frac{1}{2}\begin{bmatrix}(1+\phi_{2m+1})1_N & \phi_{2m}1_N - \sqrt{-1}\phi_j e_j\\ \phi_{2m}1_N + \sqrt{-1}\phi_j e_j & (1-\phi_{2m+1})1_N\end{bmatrix}. \tag{17}$$

This $\tilde{P}$ represents a generator of the K-group $\tilde{K}(S^{2m})$ of S^{2m}, namely, in the diagram

$$\begin{tikzcd} R^{2m} \arrow[d, "\phi"'] \arrow[dr, "P"] & & & \\ S^{2m} \arrow[r, "\tilde{P}"'] & G_{2N,N} \arrow[r, hook] & G_{2N+2,N+1} \arrow[r, hook] & \cdots \arrow[r, hook] & BU, \end{tikzcd}$$

the homotopy class of the composite of the bottom horizontal arrows is a generator of

$$\varinjlim [S^{2m}, G_{2N,N}] \cong [S^{2m}, BU] \cong \tilde{K}(S^{2m}), \tag{18}$$

where we remark that $\tilde{K}(S^{2m}) = Z, \tilde{K}(S^{2m+1}) = 0$.

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