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Monopole Charges for Arbitrary Compact Gauge Groups and Higgs Fields in any Representation

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Abstract. The topological invariants of monopoles are described for an arbitrary compact gauge group G and Higgs field Φ in any representation. The results generalize those obtained recently for compact and simply connected G and Φ in the adjoint representation. The cases when the residual symmetry group is $H = \mathrm{U}(1)$ or $H = \mathrm{U}(3)$ are worked out explicitly. This latter is needed to accommodate fractional electric charge with monopoles having one Dirac unit magnetic charge.

The general theory is illustrated on the SU(5) monopole.

1. Introduction

Let us consider a gauge theory with a compact gauge group G and the Higgs field transforming according to an arbitrary representation of G. The coupled Yang–Mills–Higgs equations admit monopole solutions (see [14] for a review). Let us consider such a monopole given by the pair (A_j, Φ), and denote by H the residual symmetry group left by the vacuum expectation value of Φ.

In the theory of monopoles a fundamental role is played by the topological invariants [2–4, 8–10, 15, 21]. The most important of these invariants is

(i) The Higgs charge $[\Phi] \in \pi_2(G/H)$ defined by the asymptotic values of the Higgs field.

(ii) If Φ belongs to the adjoint representation, we have another topological invariant-the so-called topological charge

$$I = \int_{S^2} \mathrm{Tr}\,(F \cdot \Phi), \tag{1.1}$$

where F is the gauge field strength. Equation (1.1) appears for example in the expression given by Bogomolny to the lower bound of the energy. Equation (1.1) has been generalized by Taubes [2]. In [1] we made one further step and proved that, for any $(n+1)$-linear function f on $\mathscr{G}$ the integral

$$I^{(f)} = \int_{S^2} f(F, \underbrace{\Phi, \ldots, \Phi}_{n \text{ times}}) \tag{1.2}$$

is a topological invariant whenever $D_\mu \Phi = 0$.

(iii) If the Higgs field generates a U(1) subgroup, the projection of the Yang–Mills field on the Φ-direction can be viewed as an electromagnetic field. The electric charge is then quantized [4]. The monopole's magnetic charge is expressed in an invariant integral of the type (1.2), and the electric- and magnetic charges satisfy a generalized Dirac condition.

The symmetry breaking mechanism by a Higgs field in the adjoint representation suffers however of a serious drawback: the residual symmetry group is in general not the one we would like to have in physics. For $G = \mathrm{SU}(N)$ for instance, the only possibility is [1]

$$H = S\{\mathrm{U}(i_1) \times \mathrm{U}(i_2 - i_1) \times \cdots \times \mathrm{U}(N - i_p)\}, \quad 0 < i_1 < \cdots < i_p < N. \tag{1.3}$$

Most present-day physicists believe however that the exact symmetry group should be rather that of strong- and electromagnetic interactions:

$$\mathrm{SU}(3)_c \times \mathrm{U}(1)_{\mathrm{em}}. \tag{1.4}$$

It is clearly impossible to realize (1.4) by a Higgs field in the adjoint representation (except for $G = \mathrm{SU}(4)$).

On the other hand, in grand unified theories [11] the symmetry is broken in several stages by Higgs fields which do not belong to the adjoint representation in general.

The aim of this paper is to extend the results of [1] to *any compact gauge group and Higgs field in any representation.*

First we describe $\pi_1(H)$ in some detail. We show that, for any compact and connected H,

$$\pi_1(H) = \pi_1(H_{ss}) \times Z^p, \tag{1.5}$$

where H_{ss} is the semisimple subgroup of H whose Lie algebra is $[\mathfrak{H}, \mathfrak{H}]$ and p is the dimension of the centre of H. $\pi_1(H_{ss})$ is a finite Abelian group.

As it will be seen below, it is the free part of $\pi_2(H)$ which plays a role in calculating the further topological invariants. We describe it in some more detail. To do this consider, for any loop γ in H,

$$\rho(\gamma) = \frac{1}{2\pi} z\left(\int_\gamma \theta\right) \in Z(\mathfrak{H}), \tag{1.6}$$

where z is the projection from $\mathfrak{H}$, the Lie algebra of H, to its centre $Z(\mathfrak{H})$, θ is the Maurer–Cartan form of H. Equation (1.6) depends only on the homotopy class of γ. We prove that ρ defines an isomorphism of the free part of $\pi_1(H)$ onto $z(\Gamma)$, the projection onto the centre of the unit lattice Γ of H.

Our recipe for calculating $\rho(\gamma)$ is as follow:

(i) choose a maximal torus T, find the unit lattice Γ (cf. [1]). Project Γ to the centre of $\mathfrak{H}$;

(ii) choose a Z-basis $\zeta_1, \ldots, \zeta_p$ of $z(\Gamma)$ and select $\eta_1, \ldots, \eta_p$ in Γ such that $z(\eta_k) = \zeta_k$.

(iii) define $f^j \in \mathfrak{H}^*$ by

$$f^j(\eta_k) = \delta_{jk}. \tag{1.7}$$

The f^k's are differentials of characters χ_k of H and

(iv) setting

$$m_k(\gamma)=\frac{1}{2\pi\sqrt{-1}}\int_\gamma \frac{d\chi_k}{\chi_k}, \tag{1.8}$$

we get an isomorphism between the free part and Z^p, where p is the dimension of $Z(\mathfrak{H})$. $\rho(\gamma)$ is then found as

$$\rho(\gamma)=\sum_{k=1}^{p} m_k\cdot\zeta_k. \tag{1.9}$$

Alternatively, denote by $Z_{\mathscr{G}}(\mathfrak{H})$ the centralizer of $\mathfrak{H}$ in $\mathscr{G}$ with projection $z':\mathscr{G}\to Z_{\mathscr{G}}(\mathfrak{H})$,

$$\pi^*\Omega=z'(d\theta) \tag{1.10}$$

gives a $Z_{\mathscr{G}}(\mathfrak{H})$-valued closed 2-form Ω on G/H. Let us then define

$$\rho'(\Phi)=\frac{1}{2\pi}\int_S \Phi^*\Omega\in Z_{\mathscr{G}}(\mathfrak{H}). \tag{1.11}$$

This is a homotopy invariant and it is not difficult to show that

$$\rho'(\Phi)=\rho(\delta[\Phi]), \tag{1.12}$$

where δ is the injective homomorphism from $\pi_2(G/H)$ to $\pi_1(H)$. The prime will be dropped from ρ in the sequel.

Equation (1.11) implies

$$m_k(\Phi)=m_k(\delta[\Phi])=\frac{1}{2\pi}\int_{S^2}\Phi^*\omega^k,\quad \text{where}\quad \omega^k=f^k(\Omega). \tag{1.13}$$

Let now f denote an arbitrary invariant function on $\mathscr{G}\times G/H$ which is linear in the first variable. We prove that

$$I^{(f)}=\int_S f(F,\Phi) \tag{1.14}$$

is a topological invariant (actually independent of the Yang–Mills field) and can be calculated as:

$$I^{(f)}=f(\rho(\Phi),x_0), \tag{1.15}$$

where x_0 is a reference point in the orbit G/H with stabilizer H.

There is an ambiguity in defining the electromagnetic properties: any $\zeta\in Z(\mathfrak{H})$ is admissible if it defines a U(1) subgroup (this latter condition is needed to have quantized electric charge [1,4]). If ζ is chosen to satisfy these conditions, all electric charges will be, just as in the adjoint case, multiples of a minimal charge

$$q_{\min}=e_0/|\zeta|. \tag{1.16}$$

Let us choose an invariant inner product $(\cdot,\cdot)$ on $\mathscr{G}$, and define the electromagnetic field to be the ζ-component of F. The magnetic charge of the monopole turns out to be

$$g = \frac{1}{2e_0} \cdot \frac{(\rho(\Phi), \zeta)}{|\zeta|}. \tag{1.17}$$

The electric- and magnetic charges satisfy hence a generalized Dirac condition:

$$2q_{\min} g = \frac{(\rho(\Phi), \zeta)}{(\zeta, \zeta)}. \tag{1.18}$$

The situation is particularly simple if $Z(\mathfrak{H})$ is 1-dimensional. Then $\rho(\Phi) = \zeta/M$, where ζ is the minimal generator of $Z(\mathfrak{H})$ and M is an integer, which divides the order of the finite group $Z(H)_0 \cap H_{ss}$ (the intersection of the connected component of the centre with the semisimple part).

In this case both the electric- and magnetic charges are quantized; the Dirac condition reads [4, 10, 21]

$$2q_{\min} \cdot g_{\min} = 1/M. \tag{1.19}$$

As examples, we consider the following particular cases:

(i) The Higgs field is in the adjoint representation. The previous results [1] are easily recovered.

(ii) The case $H = \mathrm{U}(1)$ is even simpler. The electric, respectively magnetic, charges satisfy the original Dirac condition.

(iii) If there exist, as conjectured, fractional electric charges and simultaneously monopoles having 1 Dirac unit magnetic charge, the residual symmetry group must be $H = \mathrm{U}(3)$, rather than $\mathrm{SU}(3) \times \mathrm{U}(1)$. (This conclusion can be obtained also from the study of to which multiplets the fermions of the theory belong [16]). The Dirac condition is modified now to

$$2q_{\min} g = m/3, \quad \text{where} \quad m \in Z. \tag{1.20}$$

The SU(5) monopole [12, 13] provides us with a nice application of our theory: at a mass scale of order 10^{14} GeV SU(5) is broken by the vacuum expectation value of a Higgs field in the adjoint representation, so (i) of Sect. 4 applies. At energies of order 10^2GeV the symmetry is broken a second time, leaving U(3) as the residual symmetry group, so we can use (iii) of Sect. 4.

2. The Higgs Charge

Let us assume that the gauge group G is a compact Lie group and let V be a finite dimensional vectorspace carrying a representation of G. The action of a $g \in G$ on a vector $v \in V$ will be denoted by g.v. If the Higgs field Φ transforms according to this representation, the usual conditions on the asymptotic behaviour of Φ imply that Φ maps S^2, the 2-sphere at infinity, into an orbit $\mathcal{O} = G \cdot x_0$ in V. This orbit can be identified with G/H, where H is the stability subgroup of the base point x.

The standard homotopy exact sequence implies that there is an injective homomorphism

$$\delta: \pi_2(G/H) \to \pi_1(H), \tag{2.1}$$

which becomes an isomorphism if G is simply connected. δ is described as follows [1]: denote $U_1 = \{x = (\theta, \rho) \in S^2 | 0 \leqq \theta < (\pi/2) + \varepsilon\}$ and $U_2 = \{x = (\theta, \rho) \in S^2 | (\pi/2) - \varepsilon < \theta \leqq \pi\} \cdot U_1 \cup U_2$ covers S^2 and $U_1 \cap U_2$ retracts to the equatorial circle S^1 $\Phi: S^2 \to G/H$ lifts on U_i according to

$$\Phi(x) = g_i(x) \cdot x_0, \quad x \in U_i. \tag{2.2}$$

Let γ denote $g_1^{-1}(x) g_2(x) | S^1$ Then γ maps S^1 into H and

$$\delta[\Phi] = [\gamma].$$

I.) We study first $[\gamma]$. To do this we need to know $\pi_1(H)$ in some more detail. Without loss of generality we assume that H is connected.

Let H denote a connected, compact Lie group, let $\mathfrak{H}$ be its Lie algebra, and denote by $Z(\mathfrak{H})$ its centre. $\mathfrak{H}$ is decomposed as

$$\mathfrak{H} = Z(\mathfrak{H}) + [\mathfrak{H}, \mathfrak{H}]. \tag{2.3}$$

Let $z: \mathfrak{H} \to Z(\mathfrak{H})$ be the projection map defined by the decomposition (2.3). Let H_{ss} denote the subgroup of H whose Lie algebra is $[\mathfrak{H}, \mathfrak{H}]$. H_{ss} is closed (hence compact) and semisimple. It is also a normal subgroup since $[\mathfrak{H}, \mathfrak{H}]$ is an ideal. Then H/H_{ss} is a compact, connected group whose Lie algebra is $Z(\mathfrak{H})$. But $Z(\mathfrak{H})$ is Abelian, hence H/H_{ss} is a torus. So $\pi_1(H/H_{ss}) \sim Z^p$, where p is the dimension of $Z(\mathfrak{H})$. On the other hand, H_{ss} is compact and semisimple, so $\pi_1(H_{ss})$ is a finite Abelian group.

The exact sequence $H_{ss} \xrightarrow{i} H \to H/H_{ss}$ gives the short exact sequence of homotopy groups

$$0 \to \pi_1(H_{ss}) \xrightarrow{i_*} \pi_1(H) \to \pi_1(H/H_{ss}) \to 0. \tag{2.4}$$

Since $\pi_1(H)$ is known to be Abelian and $\pi_1(H/H_{ss})$ is free, this sequence splits and $\pi_1(H)$ is the direct product

$$\pi_1(H) \sim \pi_1(H_{ss}) \times \pi_1(H/H_{ss}).$$

Let $\pi_1(H)_{\text{tor}}$ denote the normal subgroup of elements of finite order. Equation (2.4) shows that the inclusion map $i: H_{ss} \to H$ induces an isomorphism i_*: $\pi_1(H_{ss}) \to \pi_1(H)_{\text{tor}}$, and that $\pi_1(H)/\pi_1(H)_{\text{tor}} = Z^p$. What we have obtained is summarized in the following:

Proposition 2.1. *The first homotopy group of a compact, connected Lie group is decomposed as*

$$\pi_1(H) = \pi_1(H_{ss}) \times Z^p. \tag{2.5}$$

where p is the dimension of the centre of $\mathfrak{H}$.

The invariants we shall introduce in the sequel will be shown to depend only on the free part of $\pi_1(H)$. In what follows we focus our attention on this free part.

The relation (2.5) states an abstract isomorphism. Let us analyze it in some more detail.

Let us define first

$$\tilde{\Gamma} = \{\xi \in \mathfrak{H} \mid \exp 2\pi\xi = e\}. \tag{2.6}$$

Let us fix a maximal torus $T \subset H$ with Lie algebra $\mathfrak{T}$. Then

$$\Gamma = \tilde{\Gamma} \cap \mathfrak{T} \tag{2.7}$$

is a lattice in $\mathfrak{T}$ called the unit lattice of H. We have

$$\tilde{\Gamma} = \bigcup_{g \in H} g\Gamma g^{-1}.$$

Observe that $z(\mathrm{A}d_g\xi) = z(\xi)$, so that

$$z(\tilde{\Gamma}) = z(\Gamma). \tag{2.8}$$

Let $\theta = h^{-1}\,dh$ denote the Maurer–Cartan 1-form of H. Then $z\theta$ is a closed 1-form on H. Indeed, $d\theta = -(1/2)[\theta \wedge \theta]$ by the structure equations [7]. But the right-hand side here is in $[\mathfrak{H}, \mathfrak{H}]$ so it projects to 0 under z. Consequently $d(z \circ \theta) = z(d\theta) = 0$.

Consider now a loop γ in H and set

$$\rho(\gamma) = \frac{1}{2\pi} \int_\gamma z \circ \theta \in Z(\mathfrak{H}). \tag{2.9}$$

Since $z\theta$ is closed, it is easy to show that $\rho(\gamma)$ depends only on the homotopy class $[\gamma] \in \pi_1(H)$. Equation (2.9) provides us hence with a homomorphism $\rho: \pi_1(H) \to Z(\mathfrak{H})$. ρ plays a crucial role in the sequel.

It is a known fact from Lie group theory that, for a compact Lie group, any loop is homotopic to one of the form

$$\gamma(t) = \exp 2\pi t\xi, \quad 0 \leqq t \leqq 1 \tag{2.10}$$

for a suitable ξ from the Lie algebra. To be a loop ξ must obviously belong to $\tilde{\Gamma}$. For such a loop $\theta_{\gamma(t)}\ (\gamma'(t)) = 2\pi\xi$, and hence the integration in (2.9) is straightforward:

$$\rho([\gamma]) = z(\xi). \tag{2.11}$$

Proposition 2.2.

$$\mathrm{Ker}\,\rho = \pi_1(H)_{\mathrm{tor}}. \tag{2.12}$$

Proof. That $\mathrm{Im}\rho$ is free is obvious since it is a subgroup of a vectorspace. So $\pi_1(H)_{\mathrm{tor}} \subset \mathrm{Ker}\,\rho$. If $\rho([\gamma]) = 0$, take a representative $t \to \exp 2\pi t\xi$ of $[\gamma]$. $0 = \rho([\gamma]) = z(\xi)$ so ξ is in $[\mathfrak{H}, \mathfrak{H}]$ and hence $\exp 2\pi t\xi \in H_{ss}$ so $[\gamma] \in \pi_1(H_{ss}) = \pi_1(H)_{\mathrm{tor}}$.

Thus $\rho(\pi_1(H))$ is isomorphic to $\pi_1(H/H_{ss})$, the free part of $\pi_1(H)$.

Proposition 2.3.

$$\rho(\pi_1(H)) = z(\Gamma). \tag{2.13}$$

Proof. If $\xi \in \Gamma$, then $\gamma(t) = \exp 2\pi t\xi$ is a loop in H and $\rho(\gamma) = z(\xi)$. Conversely, if $[\gamma] \in \pi_1(H)$, take a representative loop $t \to \exp 2\pi t\xi$, $\xi \in \tilde{\Gamma}$. But then $\mathrm{Ad}_g \xi \in \Gamma$ for some $g \in H$ and $\rho([\gamma]) = z(\xi) = z(\mathrm{Ad}_g \xi) \in z(\Gamma)$.

Denote $\Gamma_z = \Gamma \cap Z(\mathfrak{H})$, and let $Z(H)_0$ be the connected component of the centre of H which has $Z(\mathfrak{H})$ as its Lie algebra.

Proposition 2.4. *The exponential map sends $2\pi\rho(\pi_1(H))$ onto $Z(H)_0 \cap H_{ss}$ with kernel $2\pi\Gamma_z$. In other words,*

$$2\pi\Gamma_z \to 2\pi\rho(\pi_1(H)) \xrightarrow{\exp} Z(H)_0 \cap H_{ss} \tag{2.14}$$

is an exact sequence of Abelian groups.

Proof. If $[\gamma] \in \pi_1(H)$, choose a representative $\gamma(t) = \exp 2\pi t\xi, \xi \in \tilde{\Gamma}$. Then $\rho(\gamma) = z(\xi)$, and so $\exp 2\pi z(\xi) \in Z(H)_0$. But $\exp 2\pi z(\xi) = \exp 2\pi(z(\xi) - \xi) \in H_{ss}$, so exp maps $2\pi\rho(\pi_1(H))$ into $Z(H)_0 \cap H_{ss}$.

To see it is onto, take $g \in Z(H)_0 \cap H_{ss}, g = \exp 2\pi\xi_0 = \exp 2\pi\xi_1$ with $\xi_0 \in Z(\mathfrak{H}), \xi_1 \in [\mathfrak{H}, \mathfrak{H}]$, so $\exp 2\pi(\xi_0 - \xi_1) = e$, and hence $\gamma(t) = \exp 2\pi t(\xi_0 - \xi_1)$ is a loop in H and $\rho(\gamma) = z(\xi_0 - \xi_1) = \xi_0$.

That $\Gamma_z = \mathrm{Ker} \exp 2\pi\rho(\pi_1(H))$ can now be easily shown.

$Z(H)_0 \cap H_{ss}$ is finite group, so $\rho(\pi_1(H))$ is lattice or rank p in $Z(\mathfrak{H})$ which spans $Z(\mathfrak{H})$. If ζ is an arbitrary element in $\rho(\pi_1(H))$, then $g = \exp 2\pi\zeta$ has finite order, $g^M = e$ for some integer M. But $g^M = \exp 2\pi M\zeta$, since ζ is in the centre. So

$$M\zeta \in \Gamma_z. \tag{2.15}$$

It is easy to prove the following

Proposition 2.5. *If $f \in \mathfrak{H}^*$, then $\sqrt{-1} f$ is the differential of a character of H if and only if*

$$\text{(i) } f([\mathfrak{H}, \mathfrak{H}]) = 0$$

and

$$\text{(ii) } f(\tilde{\Gamma}) \subset Z.$$

Observe that (i) implies that f is determined by its restriction to $Z(\mathfrak{H})$ and is invariant under Ad_H, so (ii) holds as soon as it holds on Γ. By Proposition 2.3 this is equivalent to $f(\rho(\pi_1(H))) \subset Z$.

Consequently we have a one-to-one correspondence between $\hat{H}$, the set of characters of H, and the set of those elements in $Z(\mathfrak{H})^*$ which take integral values on $\rho(\pi_1(H))$. The correspondence is given by

$$dx/\sqrt{-1} = f \circ z. \tag{2.16}$$

One can show that this correspondence is actually a group homomorphism.

Proposition 2.3 allows us to find the image of ρ without first finding $\pi_1(H)$, since the unit lattice in a torus can be found directly and then projected into $Z(\mathfrak{H})$ by z.

To calculate explicitly, choose a maximal torus T in H; let Γ be its unit lattice. Let us choose a Z-basis $\zeta_1,\ldots,\zeta_p$ for $z(\Gamma)$ and select then $\eta_1,\ldots,\eta_p$ from Γ so that $z(\eta_k)=\zeta_k$, $k=1,\ldots p$. In this way we obtain the loops $\gamma_k(t)=\exp 2\pi t\eta_k$ in H which generate the free part of $\pi_1(H)$.

If $f^1,\ldots,f^p\in\mathfrak{H}^*$ vanish on $[\mathfrak{H},\mathfrak{H}]$ and satisfy

$$f^j(\zeta_k)=f^j(\eta_k)=\delta_{jk},$$

then the conditions (i) and (ii) of Proposition 2.5 are satisfied so there are characters $\chi_1,\ldots,\chi_p$ of H such that $d\chi_k=\sqrt{-1}\,f^k$,

If χ is any character of H, then

$$d\chi/\sqrt{-1}=\sum n_k f^k, \tag{2.17}$$

where the integers $n_1,\ldots,n_p$ are computed according to

$$n_k=d\chi(\zeta_k)/\sqrt{-1}, \tag{2.18}$$

and so

$$\chi=\chi_1{}^{n_1}\cdots\chi_p{}^{n_p}. \tag{2.19}$$

If χ is any character and γ any loop, then $\chi\circ\gamma$ is a map $S^1\to \mathrm{U}(1)$ and has thus a degree $m_\chi(\gamma)$ which is a homotopy invariant, and so gives us a homomorphism $m_\chi:\pi_1(H)\to Z$. If $\gamma(t)=\exp 2\pi t\xi$, $\xi\in\Gamma$, then $\chi(\gamma(t))=\exp 2\pi t d\chi(\xi)$, whose degree is $d\chi(\xi)/\sqrt{-1}$. Hence

$$m_\chi([\gamma])=d\chi(\xi)/\sqrt{-1}=d\chi(z(\xi))/\sqrt{-1}=d\chi(\rho([\gamma]))/\sqrt{-1}. \tag{2.20}$$

Using the definition (2.9) of $\rho(\gamma)$ this can be further written as

Proposition 2.6. *Each character $\chi\in\hat{H}$ determines a homomorphism m_χ: $\pi_1(H)\to Z$,*

$$m_x([\gamma])=\frac{1}{2\pi\sqrt{-1}}d\chi\Big(\int_\gamma\theta\Big)=\frac{1}{2\pi\sqrt{-1}}\int_\gamma\frac{d\chi}{\chi}. \tag{2.21}$$

If we let $m_k=m_{\chi_k}$, where the χ_k's are the basis of H we constructed earlier, then we get the map

$$(m_1,\ldots,m_p):\pi_1(H)\to Z^p \tag{2.22}$$

which is surjective and whose kernel is the torsion part of $\pi_1(H)$. The integers $m_i([\gamma])$ depend obviously on the choice of the Z-basis. Note that $m_k\,(\exp 2\pi t\sum_j n_j\eta_j)=n_k$, consequently $\rho([\gamma])$ is simply

$$\rho([\gamma])=\sum_{k=1}^{\rho} m_k\cdot\zeta_k. \tag{2.23}$$

$\rho([\gamma])$ is already independent of the choice of the basis ζ_k.

II.) As complete as it seems, the theory given above is not very convenient in actual calculations, because the construction of the map δ is not explicit. Hence one should desire an alternative description in terms of the Higgs field Φ itself. So we examine now $\pi_2(G/H)$ to some extent.

Let us introduce the centralizer of $\mathfrak{H}$ in $\mathcal{G}$:

$$Z_{\mathcal{G}}(\mathfrak{H}) = \{\xi \in \mathcal{G} \mid [\xi, \eta] = 0, \forall \eta \in \mathfrak{H}\}. \tag{2.24}$$

It is easy to see that relative to any invariant inner product on $\mathcal{G}, Z_{\mathcal{G}}(\mathfrak{H}) = [\mathcal{G}, \mathfrak{H}]^{\perp}$, so we have a direct sum decomposition

$$\mathcal{G} = Z_{\mathcal{G}}(\mathfrak{H}) + [\mathcal{G}, \mathfrak{H}], \tag{2.25}$$

and a corresponding projection

$$z' : \mathcal{G} \to Z_{\mathcal{G}}(\mathfrak{H}). \tag{2.26}$$

Of course $Z(\mathfrak{H}) = \mathfrak{H} \cap Z_{\mathcal{G}}(\mathfrak{H})$ and (2.25) is compatible with the previous decomposition (2.3) of $\mathfrak{H}$. In the adjoint representation of H on $\mathcal{G}$, H acts trivially on $Z_{\mathcal{G}}(\mathfrak{H})$, so z' is H-invariant:

$$z'(\mathrm{Ad}_h \xi) = z'(\xi) \quad \xi \in \mathcal{G}, h \in H. \tag{2.27}$$

Thus the 2-form $z'(d\theta)$ is H-invariant for the action of H on G by the right translations as well as left translations by G. Further, if $\tilde{\xi}$ is the left-invariant vectorfield on G generated by ξ, then $\tilde{\xi} \lrcorner z' \circ d\theta = z'(\mathcal{L}_{\tilde{\xi}}\theta) = 0$ if $\xi \in \mathfrak{H}$, so $z'd\theta$ descends to G/H to give an invariant 2-form. Thus we have shown:

Proposition 2.7. *There is a G-invariant closed 2-form Ω with values in $Z_{\mathcal{G}}(\mathfrak{H})$ on G/H such that*

$$\pi^* \Omega = z'(d\theta), \tag{2.28}$$

where $\pi: G \to G/H$ denotes the natural projection.

If $\Phi : S^2 \to G/H$ is a smooth map, then by analogy with (2.9) we set

$$\rho'(\Phi) = \frac{1}{2\pi} \int_{S^2} \Phi^* \Omega, \tag{2.29}$$

which is *a priori* an element of $Z_{\mathcal{G}}(\mathfrak{H})$. We now see how it is related to $\rho(\delta[\Phi])$. Note that since Ω is closed, $\rho'(\Phi)$ depends only on the homotopy class of Φ so gives a map

$$\rho' : \pi_2(G/H) \to Z_{\mathcal{G}}(\mathfrak{H}). \tag{2.30}$$

Proposition 2.8.

$$\rho'([\Phi]) = \rho(\delta[\Phi])) \text{ for all } [\Phi] \in \pi_2(G/H). \tag{2.31}$$

In particular, ρ' actually takes its values in $Z(\mathfrak{H})$.

Proof. This is proven by the analogous argument to Theorem 3.2 of our previous paper [1]. Let us consider the lifts g_i of Φ over U_i introduced in (2.2).

$$\begin{aligned} 2\pi\rho'(\Phi) &= \lim_{\varepsilon\to 0}\int_{U^2} g_2^*\pi^*\Omega + \int_{U^1} g_1^*\pi^*\Omega = \lim_{\varepsilon\to 0}\int_{U^2} g_2^* z'(\theta) + \int_{U^1} g_1^* z'(\theta) \\ &= \int_{S^1} g_2^* z'(\theta) - g_1^* z'(\theta) = \int_{S^1} z'\{Ad_{\gamma^{-1}} g_1^*\theta + \gamma^*\theta - z'(g_1^*\theta)\} \\ &= \int_{S^1} z'(\gamma^*\theta) = \int_{\gamma} z'\theta . \end{aligned}$$

But on H θ is $\mathfrak{H}$-valued and $z = z'$ on $\mathfrak{H}$. Thus the last integral is just $2\pi\rho[\gamma]$. This proves the proposition.

Hence we get the commutative diagram

$$\begin{array}{ccc} \pi_2(G/H) & \xrightarrow{\rho'} & Z_{\mathscr{G}}(\mathfrak{H}) \\ \Big\downarrow \delta & & \Big\downarrow \\ \pi_1(H) & \xrightarrow{\rho} & Z(\mathfrak{H}) \end{array} \quad . \tag{2.32}$$

Since δ is injective, the kernel of ρ' is the torsion subgroup of $\pi_2(G/H)$ and the image of ρ' is $z(\Gamma)$.

If we take a Z-basis $\zeta_1,\ldots,\zeta_p$ for z (Γ) and extend to a basis $\zeta_1,\ldots,\zeta_q$ of $Z_{\mathscr{G}}(\mathfrak{H})$, then

$$\Omega = \sum_{k=1}^{q} \omega^k \zeta_k \tag{2.33}$$

for closed, invariant 2-forms ω^k on G/H. Then

$$\frac{1}{2\pi}\int_{S^2} \Phi^*\omega^k = \begin{cases} 0 & k > p \\ m_k(\delta[\Phi]) & 1 < k < p \end{cases} \tag{2.34}$$

Thus the forms ω^k on G/H for $1 < k < p$ determine by integration the free part of the Higgs charge $[\Phi]\in\pi_2$ (G/H) of the Higgs field Φ.

Remark. If $f^1,\ldots,f^q$ is the dual basis to $\zeta_1,\ldots,\zeta_q$ then

$$\omega^k = f^k(\Omega). \tag{2.35}$$

It is easy to see that for $k > p, f^k(z'\theta)$ descends to G/H to give a 1-form α^k with $\omega^k = d\alpha^k$ which explains why the integral (2.29) only takes values in $Z(\mathfrak{H})$. In fact $Z_{\mathscr{G}}(\mathfrak{H})^*$ represents all closed invariant 2-forms on G/H, and those in $Z_{\mathscr{G}}(\mathfrak{H})^*$ which vanish on $Z(\mathfrak{H})$ are the exact invariant 2-forms. The quotient space H^2 $(G/H;R)$ is thus $Z(\mathfrak{H})^*$.

In what follows we drop the prime on ρ and denote both maps by ρ. This is justified by Proposition 2.8.

3. Generalized Invariants

In our previous paper we considered also some other invariants which are generalizations of the "topological charge" and also can be used to describe the electromagnetic properties.

I. Invariant Integrals. By an invariant function on $\mathscr{G} \times (G/H)$ we shall mean a function

$$f:\mathscr{G} \times (G/H) \to R, \tag{3.1}$$

which is linear in the first variable and satisfies

$$f(\mathrm{Ad}_g\xi, g\cdot x) = f(\xi, x), \quad \xi \in \mathscr{G}, \quad x \in G/H, \quad g \in G.$$

Such an invariant function can be viewed alternatively as a map (denoted by the same symbol f)

$$f: G/H \to \mathscr{G}^*, \langle f(x), \xi \rangle = f(\xi, x), \tag{3.2}$$

which is equivariant for the coadjoint action on $\mathscr{G}^*$. Its image is then determined by $f_0 = f(eH) = (x_0) \in \mathscr{G}^*$ and

$$f(gH) = g\cdot f(eH) = g\cdot f_0 \tag{3.3}$$

Thus the image is Gf_0, the coadjoint orbit of f_0. Note that (3.3) gives f in terms of f_0 and tells us that in order to define f by (3.3) it is necessary and sufficient that H be contained in the stabilizer of f_0. Since H is connected this is equivalent to the infinitesimal version

$$\langle f_0, [\mathfrak{H}, \mathscr{G}] \rangle = 0. \tag{3.4}$$

Equation (2.25) tells us that f_0 is determined by its restriction to $Z_{\mathscr{G}}(\mathfrak{H})$. Thus $Z_{\mathscr{G}}(\mathfrak{H})^*$ parametrizes the set of invariant functions.

It is clear that setting

$$\omega^f = \langle f_0, \Omega \rangle \tag{3.5}$$

associates a closed invariant 2-form ω^f on G/H to each invariant function f in a one-to-one manner.

Remark. If $\tilde{\omega}$ is Kostant–Kirillov–Souriau 2-form on the coadjoint orbit of f_0 [17, 18, 19], then it is easy to show that $\omega^f = f^*\tilde{\omega}$.

Suppose that (A, Φ) is Yang–Mills–Higgs pair satisfying the finite-energy condition $D\Phi = 0$. The field strength is given by the curvature $F = dA + (1/2)[A \wedge A]$ of A. If f is an invariant function, we can form a gauge-invariant 2-form $f(F, \Phi)$ on S^2. On the other hand we can pull back ω^f by Φ to give a second 2-form $\Phi^*\omega^f$. We claim that their difference is exact. This will allow us to evaluate the integral of $f(F, \Phi)$ in terms of ρ ([Φ]).

First we translate the finite energy condition in terms of the orbit $G/H \subset V$.

Proposition 3.1 *If we define the vector fields $\hat{\xi}$ on G/H* by

$$\hat{\xi}_{gH} = \frac{d}{dt}|_0((\exp t\xi)g)H, \quad \xi \in \mathscr{G},$$

then Φ satisfies the finite energy condition (i.e. $D_\mu\Phi = 0$) if and only if for every tangent vector X on S^2

$$\Phi_* X + (A_x(X))\Phi(x) = 0. \tag{3.6}$$

Proof. This follows at once from $D_X\Phi = X(\Phi) + A(X)$. since G/H is sitting in the linear space V and in that case $X(\Phi) = \Phi_* X$.

Proposition 3.2. *If f is an invariant function then*

$$d\{f(A,\Phi)\} = f(F,\Phi) + (\tfrac{1}{2})f([A \wedge A],\Phi).$$

Proof. For $\xi \in \mathscr{G}$, X a tangent vector on S^2, Proposition 3.1 gives

$$\begin{aligned} X(f(\xi,\Phi)) &= \Phi^* X(f(\xi,\cdot)) = -A_x(X)\Phi(x)(f(\xi,\cdot)) \\ &= -\frac{d}{dt}\bigg|_0 f(\xi, \exp tA_x(X)\cdot\Phi(x)) \\ &= -\frac{d}{dt}\bigg|_0 f(Ad_{\exp - tA_x(X)}\xi, \Phi(x)) \\ &= f([A(X),\xi],\Phi(x)). \end{aligned}$$

Thus for vector fields X, Y on S^2,

$$X(f(A(Y),\Phi)) = f(X(A(Y),\Phi)) + f([A(X),A(Y)],\Phi).$$

Then

$$\begin{aligned} d\{f(A,\Phi)\}(X,Y) &= Xf(A(Y),\Phi) - Yf(A(X),\Phi) - f(A([X,Y]),\Phi) \\ &= f(dA(X,Y),\Phi) + 2f([A(X),A(Y)],\Phi) \\ &= f(F(X,Y),\Phi) + f([AX),A(Y)],\Phi). \end{aligned}$$

Since $[A \wedge A](X,Y) = 2[A(X),A(Y)]$ this proves the proposition.

It remains to examine the term $f([A \wedge A],\Phi)$. We again need the finite energy condition, but this time we use the following form. If we lift Φ by g over an open set U (see 2.2). This amounts to gauging Φ to a constant. The transformed potential

$$a = \mathrm{Ad}_{g^{-1}}A + g^*\theta$$

takes its values in $\mathfrak{H}$. Thus

$$\begin{aligned} f([A \wedge A],\Phi) &= f(\mathrm{Adg}^{-1}[A \wedge A], eH) = \langle f_0, [\mathrm{Adg}^{-1}A \wedge \mathrm{Adg}^{-1}A]\rangle \\ &= \langle f_0, [(a - g^*\theta) \wedge (a - g^*\theta)]\rangle. \end{aligned}$$

Since f_0 vanishes on $[\mathfrak{H},\mathfrak{H}]$, we have

$$f([A \wedge A],\Phi) = g^*\langle f_0, [\theta \wedge \theta]\rangle.$$

The structure equation of the Maurer–Cartan form gives then

$$\begin{aligned} f([A \wedge A],\Phi) &= -2g^*\langle f_0, d\theta\rangle = -g^*\langle f_0, z'd\theta\rangle \\ &= -2g^*\langle f_0, \pi^*\Omega\rangle = -2\langle f_0, \Phi^*\Omega\rangle \\ &= -2\Phi^*\omega^f. \end{aligned}$$

Combining this with Proposition 3.2 we obtain

Theorem 3.3. *For any invariant f and any finite energy pair (A,Φ) with $\Phi: S^2 \to G/H$*

$$f(F,\Phi) = d(f(A,\Phi)) + \Phi^*\omega^f. \tag{3.8}$$

Corollary 3.4. *For any invariant function f the integral*

$$I^{(f)} = \int_{S^2} f(F,\Phi) \tag{3.9}$$

is a topological invariant which can be calculated as

$$I^{(f)} = 2\pi\langle f_0, \rho([\Phi])\rangle = 2\pi f(\rho([\Phi]), x_0). \tag{3.10}$$

Proof. The integral of the exact term vanishes so

$$\int_{S^2} f(F,\Phi) = \int_{S^2} \Phi^*\omega^f = \langle f_0, \int_{S^2} \Phi^*\Omega\rangle = 2\pi\langle f_0, \rho([\Phi])\rangle.$$

But the integral of a closed 2-form is a homotopy invariant, so (3.9) depends only on $[\Phi]\in\pi_2(G/H)$.

This statement can be reformulated in a number of ways. For instance, by (2.31) and (2.23) we obtain

Corollary 3.5.

$$\begin{aligned} I^{(f)} &= 2\pi \sum_{k=1}^{p} \langle f_0, \zeta_k\rangle \cdot m_k([\Phi]) = 2\pi \sum_{k=1}^{p} f(\zeta_k, x_0)\cdot m_k \\ &= 2\pi \sum_{k=1}^{p} f(\eta_k, x_0)\cdot m_k, \end{aligned} \tag{3.11}$$

Since f projects to the centre.

This shows that the invariant integral formed from the Higgs and the gauge field has as its values a linear combination of the Higgs charges with the coefficients given by the invariant function f and a suitable basis $\zeta_1, \ldots, \zeta_p$ of the centre of $\mathfrak{H}$[1, 20].

II.) Electromagnetic Properties. First the electromagnetic direction must be defined. This can be done in a gauge-invariant way only by a Higgs field in the adjoint representation. Our Higgs field Φ is however in some other representation in general.

The point is that, to any vector $\zeta\in Z(\mathfrak{H})$, we can associate a new Higgs field in the adjoint representation. Indeed, let us consider a local lift $g(x)$ of Φ (2.2), and let $\zeta\in Z(\mathfrak{H})$ be an arbitrary vector playing the role of a base point. Set

$$\Psi(x) = \mathrm{Ad}_g(x)\zeta. \tag{3.12}$$

Ψ (whose ζ-dependence has been omitted for simplicity) is well-defined since ζ is in the centre. It is also covariantly constant if Φ is so. This is seen in the gauge where Φ is constant, noting that $\Psi(x) = \zeta$ now and that $D\Psi = [a, \zeta] = 0$ because a is $\mathfrak{H}$-valued and ζ is in the centre of $\mathfrak{H}$. Let $(.,.)$ denote an arbitrary invariant inner product on $\mathfrak{H}$.

Let us define the electromagnetic field,

$$\mathscr{F} = (1/e_0)(F, \Psi/|\Psi|), \tag{3.13}$$

and the electric charge operator by

$$Q_{\mathrm{em}} = e_0 \Psi/|\Psi| = (e_0/|\zeta|)\Psi, \tag{3.14}$$

respectively. As explained in [4]—see also [1], Sect. 5—in order to have quantized electric charge ζ must generate a $U(1)$ subgroup. There exists then a minimal $U(1)$ generator i.e. one whose generated loop closes first at $t = 1$ parallel to ζ. Assume for simplicity that ζ itself is minimal, i.e. a generator for Γ_z. Theorem 5.2 of [1] implies then

Proposition 3.6. *All electric charges are multiples of*

$$q_{\min} = \frac{e_0}{|\zeta|}. \tag{3.15}$$

In order to calculate the magnetic charge let us notice first that the orbit (in V) of x_0 projects to the orbit (in $\mathscr{G}$) of ζ; the projection is defined by

$$\pi_\zeta(g \cdot x_0) = \mathrm{Ad}_g \zeta.$$

Observe that $\Psi(x) = \pi_\zeta(\Phi(x))$. Let us define

$$f(\xi, y) = (\xi, \pi_\zeta(y)), \quad \zeta \in \mathscr{G}, y \in \mathcal{O}_{x_0}. \tag{3.16}$$

f is an invariant function of $\mathscr{G} \times \mathcal{O}_{x_0}$, and it is also linear in ξ. The magnetic charge is thus expressed as

$$\begin{aligned} g &= (1/4\pi) \int_{S^2} \mathscr{F} = (1/4\pi e_0 |\zeta|) \int_{S^2} (F, \Psi) \\ &= (1/4\pi e_0) \int_{S^2} f(F, \Phi). \end{aligned} \tag{3.17}$$

Here we recognize the generalized invariant $I^{(f)}$. By Corollary 3.4 we get

Theorem 3.7. *The magnetic charge is given by*

$$g = \frac{1}{2e_0} \cdot \left(\rho(\Phi), \frac{\zeta}{|\zeta|} \right). \tag{3.18}$$

The electric, respectively magnetic, charges satisfy a generalized Dirac condition. Proposition 3.6 and Theorem 3.7 imply in fact

Proposition 3.8.

$$2q_{\min} g = \frac{(\rho([\Phi]), \zeta)}{(\zeta, \zeta)}. \tag{3.19}$$

The situation is particularly simple if $Z(\mathfrak{H})$ is one dimensional. Then $\pi_1(H)_{free} \simeq Z$. Let $\gamma_0(t) = \exp 2\pi t \eta_0$ be the loop which generates the free part, and let ζ denote the (unique up to sign) generator of Γ_z. Then

$$\rho([\Phi]) = \rho(\gamma) = m \cdot \rho(\gamma_0) = m \cdot z(\eta_0)$$

for some integer m.

According to (3.18) the magnetic charge is now

$$g=(m/2e_0)(z(\eta),\zeta/|\zeta|),$$

but $z(\zeta)$ is in the centre so $z(\zeta)=M\cdot\zeta$, where M is the order of $\exp 2\pi z(\eta)\in Z(H)_0\cap H_{ss}$. So, using Proposition 3.8 we get:

Proposition 3.9.

$$\text{(i) } \rho([\Phi])=m\zeta/M, m\in Z; \tag{3.20}$$

(ii) *the electric—respectively magnetic-charges are quantized,*

$$q=n\cdot q_{\min}=n\cdot e_0/|\zeta|\cdot n\in Z; \tag{3.21}$$

$$g=m\cdot g_{\min}=m\cdot|\zeta|/2e_0M, m\in Z; \tag{3.22}$$

(iii) *The generalized Dirac condition reads*

$$2g_{\min}q_{\min}=1/M. \tag{3.23}$$

This agrees with the results known previously [4, 10, 21].

4. Particular Cases

The theory outlined in the preceding sections gives a conceptual framework valid for any compact Lie group G and a Higgs field Φ in any representation of G. Now we consider some of the physically most important particular cases.

(i) *G Compact and Simply Connected, Φ in the Adjoint Representation.* The theory of Sects. 2 and 3 is consistent with the results in [1]. Observe first that G is now semisimple so the Killing form B is non-degenerate. Let us choose our maximal torus T so that $\mathfrak{T}$ contains the base point ξ_0.

The four sets of fundamental quantities introduced in [1] are: the simple roots, $\alpha_1,\dots,\alpha_r$; the fundamental weights $\mu_1,\dots,\mu_r$, and their duals $\xi_1,\dots,\xi_r$ and $\eta_1,\dots,\eta_r$ which satisfy (see 5, 6, or 1, Sect. 2.):

$$\alpha_i(\xi_j)=\delta_{ij};\ \mu_i(\eta_j)=\delta_{ij};\ B(\xi_i,\eta_j)=\delta_{ij}[2\sqrt{-1}/\alpha_i(\eta_{\alpha i})] \tag{4.1}$$

Those ξ_{i_k}'s for which $\alpha_{i_k}(\xi_0)\neq 0$ form a basis for $Z(\mathfrak{H})$, the centre of the Lie algebra of the stability group H of ξ_0. The unit lattice in turn is generated by ($\sqrt{-1}$ times) the η_i's.

The semisimple part of H is now simply connected, $\pi_1(H_{ss})=0$ so $\pi_1(H)$ is free, $\pi_1(H)\simeq Z^p$, where p is the number of the indices i_k defined above.

ρ defined in (2.9) is hence an isomorphism between π_1 and $z(\Gamma)$, the projection of the unit lattice to the centre of $\mathfrak{H}$.

Proposition 4.1. *The* $\zeta_k=z(\sqrt{-1}\,\eta_{i_k})$'s *form a basis for* $Z(\mathfrak{H})$.

Proof. The image under z of the $\sqrt{-1}\,\eta_i$'s generate the centre. On the other hand

$z(\sqrt{-1}\,\eta_i) = 0$ if $i \neq i_k$. Indeed, the decomposition (2.4) is orthogonal with respect to the Killing form B; by (4.1) $B(\xi_{i_k}, \eta_i) = 0$ if $i \neq i_k$, so $\sqrt{-1}\,\eta_i$ belongs to $[\mathfrak{H}, \mathfrak{H}]$. Those 1-forms dual to the $\sqrt{-1}\,\eta_{i_k}$ are just the $\mu_{i_k}/\sqrt{-1}$'s. Hence

$$\chi_k(\exp 2\pi\xi) = \exp 2\pi\mu_{i_k}(\xi), \xi \in \mathfrak{H}. \tag{4.2}$$

is a character of H, and, by (2.21), we have p Higgs charges $m_1, \ldots, m_p$. According to (2.21), (2.22):

Proposition 4.2. *The Higgs charges are calculated as*

$$m_k = \mu_{i_k}(\rho[\Phi])/\sqrt{-1}. \tag{4.3}$$

Observe that a loop having $m_1, \ldots, m_p$ as Higgs charges is given by

$$\gamma(t) = \exp\{2\pi\sqrt{-1}\,t \sum_{i_k} m_k \eta_{i_k}\}.$$

Indeed, its image under ρ is

$$\rho(\gamma) = \sqrt{-1} \sum_{i_k} m_k \cdot z(\eta_{ik}) = \sum_{i_k} m_k \cdot \zeta_k,$$

But $m_k = \mu_{i_k}(\rho(\gamma))/\sqrt{-1}$

The Higgs charges are expressed also as surface integrals [1, 2]: by (2.33) we get in fact

$$m_k([\Phi]) = \frac{1}{2\pi} \cdot \int_{S^2} \Phi^* \omega^k, \tag{4.4}$$

where

$$\omega^k = \mu_{i_k}(\Omega)/\sqrt{-1}, \tag{4.5}$$

which, by (4.1), reads also

$$\sqrt{-1}\,\omega^k = \left(\frac{\alpha_{i_k}(\eta_{\alpha_{i_k}})}{2\sqrt{-1}}\right) \cdot B(\xi_{i_k}, \Omega). \tag{4.6}$$

The 2-form $B(\xi_{i_k}, \Omega)$ is seen to be just $\omega(\xi_{i_k})$ of [1].

The generalized invariant (3.9) reads in turn, according to (3.10) and (3.11), simply

$$\begin{aligned} I^{(f)} &= f(z(2\pi\sqrt{-1} \sum_{i_k} m_k \eta_{i_k}, \xi_0)) \\ &= 2\pi\sqrt{-1} \sum_{i_k} f(\eta_{i_k}, \xi_0) \cdot m_k \\ &= 2\pi \sum_{i_k} f(\zeta_{i_k}, \xi_0) \cdot m_k, \end{aligned} \tag{4.7}$$

cf. [1] Theorem 3.5.

Finally, let us study the electromagnetic properties. Let $(.,.)$ denote now the Killing form B on $\mathscr{G}$. According to Sect. 2 any $\zeta \in Z(\mathfrak{H})$ can be chosen to single out the electromagnetic direction. For example, if we choose $\zeta \in \Gamma_0$ to be $r \cdot \xi_{i_k} (r > 0)$, the electric charge is given by (3.15) while the magnetic charge becomes

$$g_k = \frac{1}{4\pi e_0 |\xi_{i_k}|} B((2\pi\sqrt{-1} \sum_{ij} m_j \eta_{i_j}), \xi_{i_k}) = \frac{1}{2e_0 |\xi_{i_k}|} \left(\frac{2}{\alpha_{i_k}(\eta_{\alpha_{i_k}})} \right) m_k . \tag{4.8}$$

This is just the "partial charge" (5.18) of [1]!

Alternatively, we can choose $\zeta \in \Gamma_z$ to be parallel to ξ_0. Expanding as $\zeta = \sum_{i_k} b_k \xi_{i_k}$, we get

$$g = \frac{1}{4\pi|\zeta|} B(\rho(\Phi), \sum b_k \xi_{i_k}) = \sum_{i_k} b_k (|\xi_{i_k}|/|\zeta|) g_k , \tag{4.9}$$

as stated in [1], Theorem 5.4.

(ii) $H = \mathrm{U}(1)$. A second, even more simple case is when the residual symmetry group is $H = \mathrm{U}(1)$. We identify it obviously with $\mathrm{U}(1)_{em}$ of electromagnetism.

Let $\zeta \in \Gamma_z$ be the minimal generator of U(1). $\delta[\Phi]$ is represented by the loop

$$\gamma(t) = \exp 2\pi \operatorname{tr} \zeta, \quad r \in Z.$$

So the Higgs charge is now $r \in Z$. The integration in (2.3) is trivial, yielding

$$\rho(\Phi) = \rho(\gamma) = r\zeta. \tag{4.10}$$

Let us suppose that the Higgs field Φ is covariantly constant. The generalized invariant (3.9) becomes simply

$$I^{(f)} = 2\pi r \cdot f(\zeta, \zeta), \tag{4.11}$$

There is now no ambiguity in choosing the electromagnetic direction. The electric charge reads, by (3.15),

$$q_{\min} = e_0/|\zeta|, \tag{4.12}$$

while the magnetic charge is expressed, by (3.18),

$$g = r \cdot |\zeta|/2e_0 . \tag{4.13}$$

Consequently

Proposition 4.3. *If the residual symmetry group is* U(1), *the original Dirac condition is satisfied*:

$$2q_{\min} g = r \in Z. \tag{4.14}$$

Equation (4.14) provides us also with the physical interpretation of the integer r. It shows also that the mere existence of fractional charges and monopoles having one unit of Dirac charge imply that the residual symmetry group can not be simply U(1). In other terms non-electromagnetic interactions must exist [4].

Interestingly, the integer r is expressed as a surface integral. Indeed, by (2.34)

$$r=\frac{1}{2\pi(\zeta,\zeta)}\int_{S^2}(\zeta,\Phi^*\Omega)$$

$$r=\frac{1}{2\pi|\zeta|^2}\int_{S^2}\Phi^*\omega,\quad \text{where } \omega=(\zeta,\Omega). \tag{4.15}$$

(iii) *Breaking to* U(3). Most present-day physicists believe that the exact symmetry group in nature should be that of strong- and electromagnetic interactions:

$$H'=\mathrm{SU}(3)_c\times \mathrm{U}(1)_{\mathrm{em}}. \tag{4.16}$$

We argue here that this can be true only locally, i.e. at the level of Lie algebras:

$$\mathfrak{H}=\mathrm{su}(3)+\mathrm{u}(1), \tag{4.17}$$

and it should be replaced rather by

$$H=\mathrm{U}(3). \tag{4.18}$$

Our argument is based on the hypothesis that particles with fractional electric charge (quarks) and monopoles having 1 Dirac unit magnetic charge exist simultaneously, so that the quantization condition becomes

$$2q_{\min}g=m/3,\quad m\in Z \tag{4.19}$$

rather then the original condition (4.14) of Dirac.

The statement follows from proposition (3.9). Indeed, for (4.16) $Z(H')\cup H'_{ss}=\{1\}$, so $M=1$ in (3.22) and the Dirac condition reads $2g_{\min}q_{\min}=1$.

For $H=\mathrm{U}(3)$ we have $M=3$ since now $A(H)$ and H_{ss} intersect in 3 points, so the Dirac condition is (4.19) as required.

Theorem 4.4. *If particles with fractional electric charge and monopoles with* 1 *Dirac unit magnetic charge are to coexist in such a way that they satisfy* (4.19), *then the only possibility to have* su(3) × u(1) *as local symmetry is by having* $H=\mathrm{U}(3)$ *as exact symmetry group.*

Note that this same conclusion can be obtained alternatively from the study of to which multiplets the fermions coupled to the theory belong [16].

In what follows we analyze the symmetry breaking to U(3) in some more detail.

Let us represent $H=\mathrm{U}(3)$ by 3×3 antihermitian matrices. $\pi_1(\mathrm{U}(3))\simeq Z$ is generated by

$$\gamma(t)=\exp 2\pi t\eta_0=\exp 2\pi t\sqrt{-1}\begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}. \tag{4.20}$$

$Z(\mathfrak{H})$ is also 1-dimensional; it is generated by

$$\zeta=\sqrt{-1}\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}. \tag{4.21}$$

The projection $z: \mathrm{u}(3) \to Z(\mathrm{u}(3))$ reads:

$$z(\xi) = \mathrm{Tr}(\xi) \cdot \zeta/3, \ \xi \in \mathrm{u}(3), \tag{4.22}$$

where Tr is the trace operator on 3×3 matrices. Consequently

$$\zeta_0 = z(\eta_0) = \zeta/3 \tag{4.23}$$

is a Z-basis for $z(\Gamma)$. Thus

$$\rho(\Phi) = m \cdot z(\eta_0) = m \cdot \zeta/3 \tag{4.24}$$

Let $f \in [\mathrm{u}(3)]^*$ be defined by

$$f = \mathrm{Tr}/\sqrt{-1}. \tag{4.25}$$

f is dual to η_0 and takes integer values on Γ so it integrates to a character χ of U(3):

$$\chi(g) = \det g. \tag{4.26}$$

Obviously $m = \mathrm{Tr}(\rho(\Phi))/\sqrt{-1}$.

Proposition 4.5. *The Higgs charge $m = [\Phi]$ is expressed also as a surface integral*

$$m = \frac{1}{2\pi} \int_{S^2} \Phi^* \omega, \tag{4.27}$$

where

$$\omega = \frac{\mathrm{Tr}(\Omega)}{\sqrt{-1}}, \tag{4.28}$$

Ω here being the $Z(\mathrm{u}(3))$-valued 2-form defined in (2.28).

The generalized invariant $I^{(f)}$ becomes, by (3.10),

$$I^{(f)} = 2\pi m f(\eta, x_0) = 2\pi m f(\zeta, x_0)/3 \tag{4.29}$$

for any invariant function f on $\mathscr{G} \times \{G/\mathrm{U}(3)\}$.
The only choice for the electromagnetic direction is that given by ζ, so

$$q_{\min} = e/3, \tag{4.30}$$

where $e = \sqrt{3} e_0/2$. The magnetic charge reads in turn

$$g = m \cdot g_{\min} = m/2e. \tag{4.31}$$

So the generalized Dirac condition is (4.19) as expected.

5. The SU(5) Monopole

Let us consider the prototype GUT of Georgi and Glashow [11] with gauge group $G = \mathrm{SU}(5)$.

At energies of order 10^{16} GEV the SU(5) symmetry is broken by a Higgs field Φ in the adjoint ($\underline{24}$) representation. (i) of Sect. 4 applies to this case.

Let us choose the base point [12, 13]

$$\xi_0 = v\sqrt{-1}\left[\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & -3/2 & \\ & & & & -3/2 \end{array}\right]. \tag{5.1}$$

The residual symmetry group is

$$H = S[\mathrm{U}(3) \times \mathrm{U}(2)] \tag{5.2}$$

with the Lie algebra

$$\mathfrak{H} = \mathrm{su}(3) \times \mathrm{su}(2) \times \mathrm{u}(1). \tag{5.3}$$

H mediates strong-weak- and electromagnetic interactions. $Z(\mathfrak{H}) = u(1)$ is generated by

$$\xi_3 = (\sqrt{-1}/5)\,\mathrm{diag}(2, 2, 2; -3, -3). \tag{5.4}$$

$H_{ss} = \mathrm{SU}(3) \times \mathrm{SU}(2)$ is simply connected, so $\pi_1(H) = Z$ is generated by

$$\gamma(t) = \exp 2\pi t\sqrt{-1}\,\eta_3, \tag{5.5}$$

where $\eta_3 = \sqrt{-1}\ \mathrm{diag}(0, 0, 1, -1, 0)$. Under $z \circ \sqrt{-1}\,\eta_3$ projects to

$$\zeta = \frac{\sqrt{-1}}{6}\left[\begin{array}{ccc|cc} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ \hline & & & -3 & \\ & & & & -3 \end{array}\right] \tag{5.6}$$

which generates $z(\Gamma)$ according to Proposition (4.1). Consequently

$$\rho(\Phi) = m \cdot \zeta. \tag{5.7}$$

The 1-form dual to (5.6) is

$$f = \mu_3/\sqrt{-1} = \mathrm{Tr}_3/\sqrt{-1}, \tag{5.8}$$

where Tr_3 is the trace on the upper U(3) part. It exponentiates to the character

$$\chi(g) = \det{}_3\,(g) \tag{5.9}$$

(determinant of the U(3) part). The Higgs charge m can be recovered as $m = \mu_3(\rho(\Phi))/\sqrt{-1}$. Plainly, $m\eta_3$ generates a loop whose Higgs charge is m.

The Higgs charge is calculated also as a surface integral:

$$m = \frac{1}{2\pi}\int_{S^2} \Phi^*\omega, \tag{5.10}$$

where

$$\omega = \mu_3(\Omega)/\sqrt{-1}. \tag{5.11}$$

If f is an invariant function, the integral invariant $I^{(f)}$ is calculated as

$$I^{(f)} = 2\pi m\cdot f(\zeta, \xi_0) = 2\pi m\cdot f(\eta_3, \xi_0). \tag{5.12}$$

In particular, the trace invariant appearing in the Bogomolny bound of the energy is

$$I = 5\pi v\cdot m. \tag{5.13}$$

The base point is chosen sometimes to be rather

$$\xi_0 = v\sqrt{-1}\left[\begin{array}{ccc|c|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & -3/2+\varepsilon & \\ \hline & & & & -3/2-\varepsilon \end{array}\right], \tag{5.14}$$

where ε is of order 10^{-14} [12]. The residual symmetry group becomes now

$$H' = S[\mathrm{U}(3)\times \mathrm{U}(1)\times \mathrm{U}(1))]. \tag{5.15}$$

It consists of those matrices of the form

$$\left[\begin{array}{c|c|c} A & & \\ \hline & a_1 & \\ \hline & & a_2 \end{array}\right], \qquad \begin{array}{c} A\in \mathrm{U}(3), \\ a_1, a_2 \in \mathrm{u}(1) \\ \det A\cdot u_1\cdot u_2 = 1. \end{array} \tag{5.16}$$

Its Lie algebra $\mathfrak{H}'$ is all

$$\xi = \sqrt{-1}\left[\begin{array}{c|c|c} A & & \\ \hline & \alpha_1 & \\ \hline & & \alpha_2 \end{array}\right], \qquad \begin{array}{c} \sqrt{-1}\,A\in \mathrm{u}(3) \\ \alpha_1, \alpha_2 \in R \\ \mathrm{Tr}\,A + \alpha_1 + \alpha_2 = 0. \end{array} \tag{5.17}$$

$Z(\mathfrak{H}')$ is 2-dimensional. It is generated by ξ_3 above and by

$$\xi_4 = (\sqrt{-1}/5)\,\mathrm{diag}(1,1,1,1,-4).$$

$\pi_1(H') \simeq Z^2$ is hence generated by those loops in (5.5) and by

$$\exp 2\pi\sqrt{-1}\,t\eta_4, \tag{5.19}$$

where $\eta_4 = \sqrt{-1}\,\mathrm{diag}(0,0,0,1,-1)$. There are now two Higgs charges, m and m'.

The projection $z\colon \mathfrak{H}' \to Z(\mathfrak{H}')$ is expressed as

$$z(\xi) = \sqrt{-1}\,\mathrm{diag}\left(\frac{\mathrm{Tr}\,A}{3}, \frac{\mathrm{Tr}\,A}{3}, \frac{\mathrm{Tr}\,A}{3}, \alpha_1, \alpha_2\right). \tag{5.20}$$

Hence for $z(\Gamma)$ we get the generators

$$\zeta_1 = z(\eta_3) = \sqrt{-1}\,\mathrm{diag}(1/3, 1/3, 1/3, -1, 0) \tag{5.21}$$

$$\zeta_2 = z(\eta_4) = \sqrt{-1}\,\mathrm{diag}(0,0,0,1,-1). \tag{5.22}$$

The dual 1-forms are

$$f^1 = \mu_3/\sqrt{-1} = \mathrm{Tr}_3\sqrt{-1}, \tag{5.23}$$

$$f^2 = \mu_4/\sqrt{-1} = \mathrm{Tr}_4/\sqrt{-1}. \tag{5.24}$$

They exponentiate to the characters

$$\chi_k = \det_k, \quad k = 3, 4. \tag{5.25}$$

If $[\gamma] \simeq (m, m')$, then

$$\rho(\gamma) = m\cdot\zeta_1 + m'\cdot\zeta_2 = \sqrt{-1}\,\mathrm{diag}(m/3, m/3, m/3, -m+m', -m'). \tag{5.26}$$

The Higgs charges are recovered as $m = \mathrm{Tr}_3(\rho(\Phi))/\sqrt{-1}$ and $m' = \mathrm{Tr}_4(\rho(\Phi))/\sqrt{-1}$. Alternatively, they can also be calculated according to

$$m = \frac{1}{2\pi}\int_{S^2} \Phi^*\omega \quad \text{and} \quad m' = \frac{1}{2\pi}\int_{S^2} \Phi^*\omega', \tag{5.27}$$

where ω is the same as in (5.10) and ω' is given by

$$\omega' = \mathrm{Tr}_4(\Omega)/\sqrt{-1}. \tag{5.28}$$

If f is an invariant function, the corresponding integral reads

$$I^{(f)} = 2\pi\{m\cdot f(\zeta_1, \xi_0) + m'\cdot f(\zeta_2, \xi_0)\} = 2\pi\{m\cdot f(\eta_3, \xi_0) + m'\cdot f(\eta_4, \xi_0)\}. \tag{5.29}$$

In particular, the trace invariant (topological charge) becomes

$$I = (5 - 2\varepsilon)\pi v\cdot m + 2\varepsilon\pi v\cdot m'. \tag{5.30}$$

In the physical applications [12] $\varepsilon \sim O(10^{-14})$ so the Bogomolny bound for the energy are essentially the same as with $\varepsilon = 0$.

At much lower energies ($\sim O(10^2)$ GeV) the symmetry is broken at a second time by the vacuum expectation values of a new Higgs field χ in the (standard) fundamental representation $\{\underline{5}\}$. In order to apply our theory we have to consider the two Higgs fields as a single one, say $\Psi = (\Phi, \chi)$. Ψ belongs to the representation $\{\underline{24+5}\}$. If we require that $D_\mu\Psi = 0$, then the energy is finite.

The base point becomes now $x_0 = (\xi_0, \chi_0)$, where

$$\chi_0 = v\cdot(0,0,0,0,1), \quad (v \sim O(10^2)\,\mathrm{GeV}). \tag{5.31}$$

χ_0 alone has SU(4) for stabilizer, so the unbroken symmetry group for (5.17) becomes $K = H \cap \mathrm{SU}(4)$ (respectively $H' \cap \mathrm{SU}(4)$). Interestingly, in both cases we get

$$K = \left[\begin{array}{c|c|c} A & & \\ \hline & (\det A)^{-1} & \\ \hline & & 1 \end{array}\right] = i[\mathrm{U}(3)], \tag{5.32}$$

$A \in \mathrm{U}(3)$. So

$$\mathscr{K} = \mathrm{i}[\mathrm{su}(3)\mathrm{xu}(1)] = \left[\begin{array}{c|c|c} \alpha + \lambda 1_3 & & \\ \hline & -3\lambda & \\ \hline & & 0 \end{array}\right], \quad \alpha \in \mathrm{su}(3), \lambda \in \sqrt{-1}\,R. \tag{5.33}$$

U(3) is, as explained in Sect. 4, the physically relevant residual symmetry group: it propagates the electrostrong interactions. (iii) of Sect. 4 applies now. $\pi_1(K) \simeq Z$ is generated by

$$\exp 2\pi i \left\{ \sqrt{-1} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \right\} t = \exp 2\pi \sqrt{-1}\, t\eta_3. \tag{5.34}$$

There is one Higgs charge, say $m \cdot Z(\mathscr{K}) = Z(i(\mathrm{u}(3))) = i(I(\mathrm{u}(3)))$, where i denotes the inclusion map $i: \mathrm{u}(3) \to \mathfrak{H}c\ \mathrm{su}(5) \cdot Z(\mathscr{K})$ is generated by

$$\zeta = i \left\{ \sqrt{-1} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \right\} = \sqrt{-1} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{bmatrix}. \tag{5.35}$$

Hence, by (4.27), $z(\Gamma)$ is generated by

$$\zeta_1 = z \left(i \left(\sqrt{-1} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \right) \right) = \zeta/3, \tag{5.36}$$

so now the 1-form dual to ζ_1 is

$$f = \mathrm{Tr}_3/\sqrt{-1}. \tag{5.37}$$

The Higgs charge is recovered as $m = \mathrm{Tr}_3(\rho(\Phi))/\sqrt{-1}$.

Alternatively, it is also expressed as a surface integral:

$$r = \frac{1}{2\pi} \int_{S^2} \Psi^* \omega, \tag{5.38}$$

where

$$\omega = \mathrm{Tr}_3(\Omega)/\sqrt{-1}. \tag{5.39}$$

Observe that the Higgs charges of Ψ and of Φ are the same.

The generalized invariants are given by (4.29). It can not however, be used to calculate the lower bound of the energy.

In order to discuss the electromagnetic properties a new Higgs field has to be constructed using ζ as base point, as indicated in Sect. 2. We do not construct it here explicitly; it is sufficient to know that it does exist.

According to (iii) of Sect. 3, electric charge is quantized (since $\zeta \in \Gamma_0$) in units of

$$q_{\min} = e/3, \tag{5.40}$$

where $e = \sqrt{3}e_0/2$. (Alternatively, one can use (3.15) directly, noting that $\zeta = \sqrt{-1}(\eta_1 + 2\eta_2 + 3\eta_3) = 4\xi_3 - 3\xi_4$.) The magnetic charge becomes in turn

$$g = m/2e, \tag{5.41}$$

[12, 13, 14], so Dirac's condition reads now

$$2q_{\min} g = m/3.$$

as required.

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