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The KAM Theory of Systems with Short Range Interactions, II*

C. Eugene Wayne**
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

Abstract. The proof of the results on the KAM theory of systems with short range interactions, stated in [6] is completed. Estimates on the decay of the interactions generated by the iterative procedure in the KAM theorem are proved, as well as the modification of the theorems of [2–3] needed for results.

1. Introduction

In a previous paper [6], hereafter referred to as I, we presented results on a KAM theory of systems with short range interactions. The proofs of those theorems are completed here. In referring to results from I, we shall precede the equation or theorem number by I, e.g., (I.1.1) refers to Eq. (1.1) of I, and Theorem I.1.1 to Theorem 1.1 of the same work. For a general introduction to the problem, and references to previous work in the literature, the reader should see I.

2. Decay Estimates

A sequence of lemmas is proved which in turn imply Proposition I.4.1. The first is an easy application of the chain rule.

Lemma 2.1. *Suppose g is analytic on some domain $\mathscr{D} \subset \mathbb{C}^{2N}$ and satisfies*

$$\sup_{\mathscr{D}} \left| \frac{\partial g}{\partial x_i}(\underline{x}) \right| \leqq C_i, \quad \sup_{\mathscr{D}} \left| \frac{\partial^2 g}{\partial x_i \partial x_j}(\underline{x}) \right| \leqq C^g_{ij} e^{-\kappa\delta(i,j)}, \tag{2.1}$$

for some positive constants C_i, C^g_{ij}, and $\kappa (i, j = 1, \dots, 2N)$. Here, $\delta(i,j) = |i(\text{mod}\, N) - j(\text{mod}\, N)|$. Suppose $\tilde{\underline{x}}$ is a holomorphic map from $\mathscr{D}' \to \mathscr{D}$ satisfying

$$\sup_{\mathscr{D}'} \left| \frac{\partial \tilde{x}_i}{\partial x'_j}(\underline{x}') \right| \leqq C^1_{ij} e^{-\kappa\delta(i,j)}, \tag{2.2}$$

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* This work was completed while the author was at the Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, MN, 55455, USA

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and

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2 \tilde{x}_m}{\partial x_i' \partial x_j'}(\underline{x}')\right| \leqq C_{mij}^2 e^{-\kappa\delta(i,j)},$$

for some constants C_{ij}^1 and C_{mij}^2.

Then

$$\sup_{\mathscr{D}'}\left|\frac{\partial}{\partial x_i'}(g\circ\tilde{\underline{x}})(\underline{x}')\right| \leqq D_i, \tag{2.3}$$

and

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2}{\partial x_i' \partial x_j'}(g\circ\tilde{\underline{x}})(x')\right| \leqq Ce^{-\kappa\delta(i,j)}, \tag{2.4}$$

for some constants C and D_i, $i=1,\dots,N$.

Proof. By the chain rule,

$$\left|\frac{\partial}{\partial x_i'}(g\circ\tilde{\underline{x}})(\underline{x}')\right| = \left|\sum_{m=1}^{N}\frac{\partial g}{\partial x_m}(\tilde{\underline{x}}(\underline{x}'))\cdot\frac{\partial \tilde{x}_m}{\partial x_j'}(\underline{x}')\right|.$$

Then (2.4) follows if we take $D_i = \sum_{m=1}^{N} C_m C_{mi}^1$. (We will usually estimate D_i by $N\sup_m |C_m C_{mi}^1|$.) Inequality (2.4) follows in the same fashion, and we find

$$\begin{aligned} C &= \sup_{i,j}\left(\sum_{m=1}^{N}|C_m C_{mij}^2| + \sum_{m,n=1}^{N}|C_{mn}^g C_{mi}^1 C_{nj}^1|\right) \\ &\leqq N\sup_{m,i,j}|C_m C_{mij}^2| + N^2\sup_{m,n,i,j}|C_{mn}^g C_{mi}^1 C_{nj}^1|. \end{aligned}$$

Proposition I.4.1 follows by combining Lemma 5.1 with the three following lemmas:

Lemma 2.2. *On* $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$,

$$\begin{aligned} \left|\frac{\partial z_i}{\partial z_j'}(\underline{I}',\underline{z}')\right| &\leqq 2^4 2^{|i-j|}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\ &\leqq 2^4(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \end{aligned} \tag{2.5}$$

$$\left|\frac{\partial z_i}{\partial I_j'}(\underline{I}',\underline{z}')\right| \leqq \rho_0^{-1}c_1(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.6}$$

$$\left|\frac{\partial^2 z_m}{\partial z_i' \partial z_j'}(\underline{I}',\underline{z}')\right| \leqq 2^{19}N^B(\tilde{\rho}_k^{-1}\rho_0)(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.7}$$

$$\left|\frac{\partial^2 z_m}{\partial z_i' \partial I_j'}(\underline{I}',\underline{z}')\right| \leqq \tilde{\rho}_k^{-1}c_2(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.8}$$

$$\left|\frac{\partial^2 z_m}{\partial I_i' \partial I_j'}(\underline{I}',\underline{z}')\right| \leqq (\tilde{\rho}_k\rho_0)^{-1}c_3(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.9}$$

for B some fixed constant. Also, $c_1 - c_3$ *may be chosen* $\mathcal{O}((\varepsilon_0\rho_0^{-1})^\alpha)$, *for some constant* α.

Lemma 2.3. *On* $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$,

$$\left|\frac{\partial I_i}{\partial I'_j}(\underline{I}', \underline{z}') - \delta_{ij}\right| \leqq c_1(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.10}$$

$$\left|\frac{\partial I_i}{\partial z'_j}(\underline{I}', \underline{z}')\right| \leqq \rho_0 c_2(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.11}$$

$$\left|\frac{\partial^2 I_m}{\partial I'_i \partial I'_j}(\underline{I}', \underline{z}')\right| \leqq \tilde{\rho}_k^{-1} c_3(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.12}$$

$$\left|\frac{\partial^2 I_m}{\partial I'_i \partial z'_j}(\underline{I}', \underline{z}')\right| \leqq (\tilde{\rho}_k^{-1}\rho_0) c_4(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.13}$$

$$\left|\frac{\partial^2 I_m}{\partial z'_i \partial z'_j}(\underline{I}', \underline{z}')\right| \leqq \tilde{\rho}_k^{-1}\rho_0^2 c_5(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}, \tag{2.14}$$

where the constants c_j *may once again be assumed to be* $\mathcal{O}((\varepsilon_0\rho_0^{-1})^\alpha)$.

Let $f^{k[\leqq]}(\underline{I}, \underline{z})$ and $f^{k[\geqq]}(\underline{I}, \underline{z})$ be the functions defined in (I.3.16). To simplify our notation slightly let $x_i = \{I_i$ if $1 \leqq i \leqq N$ and z_{i-N} if $N < i \leqq 2N\}$.

Lemma 2.4. *On* $W(\tilde{\rho}_k, \xi_k - 2\delta; V_k)$,

$$\left|\frac{\partial f^{k[\leqq]}}{\partial x_i}(\underline{I}, \underline{z})\right| \leqq C_i\varepsilon_k, \tag{2.15}$$

$$\left|\frac{\partial f^{k[\geqq]}}{\partial x_i}(\underline{I}, \underline{z})\right| \leqq C_i\varepsilon_k, \tag{2.16}$$

$$\left|\frac{\partial^2 f^{k[\leqq]}}{\partial x_i \partial x_j}(\underline{I}, \underline{z})\right| \leqq C_{ij}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.17}$$

$$\left|\frac{\partial^2 f^{k[\geqq]}}{\partial x_i \partial x_j}(\underline{I}, \underline{z})\right| \leqq C_{ij}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}. \tag{2.18}$$

We can take $C_i = 2^3 2^{L_k} N$ *if* $1 \leqq i \leqq N$, *and* $C_i = 2^3 2^{L_k}\rho_k N^B$ *for some constant B if* $i > N$. *In (2.17) and (2.18) we can take* $C_{ij} = 2\varepsilon_0\rho_0^{-1}N^B$ *if i and j are both less than or equal to N. If* $1 \leqq i \leqq N, N < j \leqq 2N$, *or vice versa let* $C_{ij} = 2N^B\varepsilon_0$, *and finally if both i and j are greater than N take* $C_{ij} = N^B\varepsilon_0\rho_0$.

We now prove Proposition I.4.1. By (I.3.5),

$$f^{k+1}(\underline{I}', \underline{z}') = H^{k+1}(\underline{I}', \underline{z}') - h^{k+1}(\underline{I}') = f^I(\underline{I}', \underline{z}') + f^{II}(\underline{I}', \underline{z}') + f^{III}(\underline{I}', \underline{z}'), \tag{2.19}$$

where $f^I - f^{III}$ are defined in (I.3.19)–(I.3.21). Let $x'_i = \{I'_i$ if $1 \leqq i \leqq N$ and z'_{i-N} if $N < i \leqq 2N\}$,

$$\tilde{x}_i(\underline{x}') = \{I_i(\underline{I}', \underline{z}') \text{ if } 1 \leqq i \leqq N \quad \text{and} \quad z_{i-N}(\underline{I}', \underline{z}') \text{ if } N < i \leqq 2N\}. \tag{2.20}$$

If we set $\mathscr{D} = W(\tilde{\rho}_k/2, \xi_k - 3\delta; V_k)$ and $\mathscr{D}' = W(\rho_{k+1}, \xi_{k+1}; V_k)$, $\tilde{x}$ is a holomorphic map from $\mathscr{D}'$ into $\mathscr{D}$. Then,

$$\frac{\partial^2 f^{III}}{\partial x_i' \partial x_j'}(\underline{x}') = \frac{\partial^2}{\partial x_i' \partial x_j'}(f^{k[\geqq]} \circ \tilde{\underline{x}}(\underline{x}')). \tag{2.21}$$

Combining the bounds of Lemmas 2.2–2.4 with Lemma 2.1 (where we take $g = f^{[\geqq]}$) we obtain

$$\sup_{\mathscr{D}'} \left| \frac{\partial^2 f^{III}}{\partial x_i' \partial x_j'}(x') \right| \leqq c_{III} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})\delta(i,j)}, \tag{2.22}$$

where c_{III} may be taken to be $\varepsilon_0 \rho_0^{-1} (\varepsilon_0 \rho_0^{-1})^\alpha$, for some constant α if i and j are both less than or equal to N, $\varepsilon_0 \rho_0 (\varepsilon_0 \rho_0^{-1})^\alpha$ if i and j are both greater than or equal to N, and $\varepsilon_0 (\varepsilon_0 \rho_0^{-1})^\alpha$ if $i < N$ and $j \geqq N$ or vice versa.

Next note that

$$\begin{aligned} \frac{\partial^2 f^{II}}{\partial x_i \partial x_j}(\underline{x}') = \int_0^1 dt \sum_{m=1}^N \Bigg\{ & \left[\frac{\partial}{\partial x_i'} \left(\frac{\partial f^{k[\leqq]}}{\partial x_m} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \frac{\partial \Xi_m}{\partial x_j'}(\underline{x}') \\ & + \left[\frac{\partial}{\partial x_j'} \left(\frac{\partial f^{k[\leqq]}}{\partial x_m} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \frac{\partial \Xi_m}{\partial x_i'}(\underline{x}') + \frac{\partial f^{k[\leqq]}}{\partial x_m} \circ \tilde{x}^t(\underline{x}') \cdot \frac{\partial^2 \Xi_m}{\partial x_i' \partial x_j'}(\underline{x}') \\ & + \left[\frac{\partial^2}{\partial x_i' \partial x_j'} \left(\frac{\partial f^{k[\leqq]}}{\partial x_m} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \Xi_m(\underline{x}') \Bigg\}, \end{aligned} \tag{2.23}$$

where $x_j^t(\underline{x}') = \{x_j' + t\Xi_j(x')$ if $1 \leqq j \leqq N$ and $z_j(\underline{x}')$ if $j > N.\}$. Lemmas 2.2 and 2.3 bound derivatives of $\tilde{x}^t$ if we note that $\partial \tilde{x}_j^t / \partial x_i' - \delta_{ij} = t(\partial \tilde{x}_j / \partial x_i' - \delta_{ij})$ if $j \leqq N$ and $\partial \tilde{x}_j^t / \partial x_i = \partial \tilde{x}_j / \partial x_i$ if $j > N$. Lemma 2.4 (combined with a dimensional estimate) bounds derivatives of $f^{k[\leqq]}(\underline{x})$, so derivatives of $\partial f^{k[\leqq]} / \partial x_m \circ \tilde{\underline{x}}^t(\underline{x}')$ (with respect to x') may be bounded on $W(\rho_{k+1}, \xi_{k+1}; V^k)$ by Lemma 2.1. (In this case $g = \partial f^{k[\leqq]} / \partial I_m$). Finally, derivatives of $\Xi_m(x')$ are bounded by noting that $\partial \Xi_m / \partial x_i'(\underline{x}') = \partial I_m / \partial x_i'(\underline{x}') - \delta_{mi}$, and then applying Lemma 2.3, while $\Xi_m(\underline{x}')$ is bounded by noting that $\Xi_m(\underline{x}') = \partial \Phi^k / \partial \phi_m \, (\underline{I}', \underline{\phi}(\underline{I}', \underline{\phi}'))$ by (I.3.11) and then applying (I.3.6). Collecting all the constants that arise in this process, bounding the sum over m by a factor of N and noting most importantly that this process allows us to extract a factor of $(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)\delta(i,j)}$ from each term yields

$$\sup_{\mathscr{D}'} \left| \frac{\partial^2 f^{II}}{\partial x_i' \partial x_j'}(\underline{x}') \right| \leqq c_{II} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})\delta(i,j)}, \tag{2.24}$$

where c_{II} may be chosen to equal c_{III} in (2.22).

Finally,

$$\begin{aligned} \frac{\partial^2 f^I}{\partial x_i' \partial x_j'}(\underline{x}') = \sum_{m,n=1}^N \int_0^1 ds \int_0^s dt \Bigg\{ & \left[\frac{\partial^2}{\partial x_i' \partial x_j'} \left(\frac{\partial^2 h^k}{\partial x_m \partial x_n} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \Xi_m(\underline{x}') \Xi_n(\underline{x}') \\ & + \left[\frac{\partial}{\partial x_i'} \left(\frac{\partial^2 h^t}{\partial x_m \partial x_n} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \left(\frac{\partial \Xi_m}{\partial x_j'}(\underline{x}') \Xi_n(\underline{x}') + \frac{\partial \Xi_n}{\partial x_j'}(\underline{x}') \Xi_m(\underline{x}') \right) \\ & + \left[\frac{\partial}{\partial x_j'} \left(\frac{\partial^2 h^t}{\partial x_m \partial x_n} \circ \tilde{\underline{x}}^t(\underline{x}') \right) \right] \left(\frac{\partial \Xi_m}{\partial x_i'}(\underline{x}') \Xi_n(\underline{x}') + \frac{\partial \Xi_n}{\partial x_i'}(\underline{x}') \Xi_m(\underline{x}') \right) \end{aligned}$$

$$+\frac{\partial^2 h^k}{\partial x_m \partial x_n}\circ \underset{\sim}{x}^t(\underset{\sim}{x}')\left(\frac{\partial^2 \Xi_m}{\partial x_i' \partial x_j'}(\underset{\sim}{x}')\Xi_n(\underset{\sim}{x}') + \frac{\partial \Xi_m}{\partial x_i'}(\underset{\sim}{x}')\frac{\partial \Xi_n}{\partial x_j'}(\underset{\sim}{x}')\right.$$
$$\left.\left. + \frac{\partial \Xi_n}{\partial x_i'}(\underset{\sim}{x}')\frac{\partial \Xi_m}{\partial x_j'}(\underset{\sim}{x}') + \Xi_m(\underset{\sim}{x}')\frac{\partial^2 \Xi_m}{\partial x_i' \partial x_j'}\right)\right\}. \tag{2.25}$$

We bound Ξ_m and its derivatives, and the derivatives of $\underset{\sim}{x}^t$ as we did just above. Derivatives of $h^k(\underset{\sim}{x})$ are bounded by (I.2.5), (I.2.6) and dimensional estimates. Then Lemma 2.1 bounds derivatives (with respect to $\underset{\sim}{x}'$) of $\partial^2 h^k/\partial x_m \partial x_n \circ \underset{\sim}{x}^t$. Finally, bounding the sum over m and n by $2N$, collecting all the constants that arise we find

$$\sup_{\mathscr{D}'}\left|\frac{\partial^2 f^I}{\partial x_i' \partial x_j'}(\underset{\sim}{x}')\right| \leqq c_I(\varepsilon_0 \rho_0^{-1})^{(1-\eta_{k+1})\delta(i,j)}, \tag{2.26}$$

with c_I chosen equal to c_{III} in (2.22). Finally noting that

$$\sup\left|\frac{\partial^2 f^{k+1}}{\partial \phi_i' \partial \phi_j'}(\underset{\sim}{I}', \underset{\sim}{z}')\right| \leqq N^B \sup\left|\frac{\partial^2 f^{k+1}}{\partial z_i \partial z_j}(\underset{\sim}{I}', \underset{\sim}{z}')\right|$$

and

$$\sup\left|\frac{\partial^2 f^{k+1}}{\partial \phi_i' \partial I_j'}(\underset{\sim}{I}', \underset{\sim}{z}')\right| \leqq N^{B'} \sup\left|\frac{\partial^2 f^{k+1}}{\partial z_i' \partial I_j'}(\underset{\sim}{I}', \underset{\sim}{z}')\right|,$$

for some constants B and B', we obtain Proposition I.4.1 by adding estimates (2.22), (2.24) and (2.26). We now turn to the proof of Lemmas 2.2–2.4. Their proofs depend on the following two results.

Lemma 2.5. *On* $W(\tilde{\rho}_k/2, \xi_k - 2\delta; V_k)$

$$\sup\left|\frac{\partial^2 \Phi^k}{\partial z_i \partial z_j}(\underset{\sim}{I}', \underset{\sim}{z})\right| \leqq \rho_0 c_1(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.27}$$

$$\sup\left|\frac{\partial^2 \Phi^k}{\partial I_i' \partial z_j}(\underset{\sim}{I}', \underset{\sim}{z})\right| \leqq c_2(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}, \tag{2.28}$$

$$\sup\left|\frac{\partial^2 \Phi^k}{\partial I_i' \partial I_j'}(\underset{\sim}{I}', \underset{\sim}{z})\right| \leqq \rho_0^{-1} c_3(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}. \tag{2.29}$$

Again, $c_1 - c_3$ *are* $\mathcal{O}((\varepsilon_0 \rho_0^{-1})^\alpha)$.

Lemma 2.6. *If* M *is an* $n \times n$ *matrix whose elements satisfy* $|M_{ij}| \leqq ce^{-\kappa|i-j|}$, *and* c *and* κ *are such that* $\sum_{j=0}^{\infty} e^{-\kappa j} \leqq 1$ *and* $c < 1/4$, *then* $(\mathbb{1} - M)^{-1}$ *exists and its matrix elements satisfy*

$$|[(\mathbb{1} - M)^{-1}]_{ij}| \leqq 2^3 2^{|i-j|} e^{-\kappa|i-j|}. \tag{2.30}$$

This is a straightforward lemma that may be proved in a number of ways. We sketch a proof in the appendix based on ideas from statistical mechanics and field theory [1], [5].

To prove estimate (2.5) note that (I.3.8), (I.3.10), and the chain rule imply

$$\frac{\partial z_i}{\partial z'_j}(\underline{I}',\underline{z}')=\delta_{ij}e^{i\Delta_i(\underline{I}',\underline{z}')}-iz'_i\sum_{l=1}^{N}\frac{\partial^2\Phi^k}{\partial I'_i\partial z_l}(\underline{I}',\underline{z}')\cdot\frac{\partial z_l}{\partial z'_j}(\underline{I}',\underline{z}')e^{i\Delta_i(\underline{I}',\underline{z}')},\tag{2.31}$$

with $\partial^2\Phi^k/\partial I'_i\partial z_l$ evaluated at the point $(\underline{I}',\underline{z}(\underline{I}',\underline{z}'))$. Note that (2.31) may be written as a matrix equation

$$D=\Lambda+MD,\tag{2.32}$$

where

$$D_{ij}=\frac{\partial z_i}{\partial z'_j}(\underline{I}',\underline{z}'),\ \Lambda_{ij}=\delta_{ij}e^{i\Delta_i(\underline{I}',\underline{z}')},$$

and

$$M_{ij}=-iz'_i\frac{\partial^2\Phi^k}{\partial I'_i\partial z_j}(\underline{I}',\underline{z}(\underline{I}',\underline{z}'))e^{i\Delta_i(\underline{I}',\underline{z}')}.$$

Thus,

$$D_{ij}=[(\mathbb{1}-M)^{-1}\Lambda]_{ij}=(\mathbb{1}-M)^{-1}_{ij}\Lambda_{jj}.\tag{2.33}$$

We first note that (I.3.7) and (I.3.10) imply that on $W(\tilde{\rho}_k/2,\ \xi_k-2\delta;\ V_k)$, $\sup|\Lambda_{jj}|\leqq 2$. By (2.28) we see that on $W(\tilde{\rho}_k/4,\ \xi_k-3\delta;\ V_k)$,

$$\sup|M_{ij}|\leqq 2e^{\xi_k-2\delta}\cdot c_2(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}.\tag{2.34}$$

Since $c_2\sim\mathcal{O}((\varepsilon_0\rho_0^{-1})^{\alpha})$, and $e^{\xi_k-2\delta}=N^B$, we can choose ε_0 sufficiently small that the hypotheses of Lemma 2.6 are satisfied, and we find

$$|(\mathbb{1}-M)^{-1}_{ij}|\leqq 2^3 2^{|i-j|}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}.\tag{2.35}$$

Combining (2.33), (2.35) and our observation concerning Λ_{jj} yields (2.5).

If we again use (I.3.8), (I.3.10) and apply the chain rule to $\partial^2 z_m/\partial z'_i\partial z'_j(\underline{I}',\underline{z}')$, we obtain an expression which may be written in matrix form as

$$D^{(2)}=\Lambda^{(2)}+MD^{(2)},\tag{2.36}$$

where $D^{(2)}_{ij}=\partial^2 z_m/\partial z'_i\partial z'_j(\underline{I}',\underline{z}')$, M is the same matrix that appears in (2.32) and

$$\begin{aligned}\Lambda^{(2)}_{ij}=&-i\delta_{mj}e^{i\Delta_m}\cdot\sum_{l=1}^{N}\frac{\partial^2\Phi^k}{\partial I'_m\partial z_l}\cdot\frac{\partial z_l}{\partial z'_i}-i\delta_{mi}e^{i\Delta_m}\sum_{l=1}^{N}\frac{\partial^2\Phi^k}{\partial I'_m\partial z_l}\cdot\frac{\partial z_l}{\partial z'_j}\\&-z'_m e^{i\Delta_m}\cdot\sum_{l,n=1}^{N}\frac{\partial^2\Phi^k}{\partial I'_m\partial z_l}\cdot\frac{\partial^2\Phi^k}{\partial I'_m\partial z_n}\cdot\frac{\partial z_n}{\partial z'_i}\cdot\frac{\partial z_l}{\partial z'_j}\\&-iz_m e^{i\Delta_m}\cdot\sum_{l,n=1}^{N}\frac{\partial^3\Phi^k}{\partial I'_m\partial z_l\partial z_n}\cdot\frac{\partial z_l}{\partial z'_j}\cdot\frac{\partial z_m}{\partial z'_i}.\end{aligned}\tag{2.37}$$

(We have omitted the arguments of the functions on the right-hand side of (2.37) to save space.) Lemma 2.5, (2.5) and a dimensional estimate imply that we may extract a factor of $(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$ from each term in the definition of $\Lambda^{(2)}_{ij}$, giving a bound

$$|\Lambda_{ij}^{(2)}| \leqq 2^{13}\tilde{\rho}_k^{-1}\rho_0 N^2 e^{\xi_k} 2^{|i-j|}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}. \tag{2.38}$$

(We have taken a supremum of the right-hand side of (2.37) over $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$.) By (2.36),

$$\frac{\partial^2 z_m}{\partial z_i' \partial z_j'}(\underline{I}', \underline{z}') = D_{ij}^{(2)} = [(\mathbb{1} - M)^{-1}\Lambda^{(2)}]_{ij}. \tag{2.39}$$

Combining (2.38) and (2.35) then gives

$$\begin{aligned}\sup\left|\frac{\partial^2 z_m}{\partial z_i' \partial z_j'}(\underline{I}', \underline{z}')\right| &\leqq (\tilde{\rho}_k^{-1}\rho_0)2^{19}N^2 e^{\xi_k}(|i-j|+1)2^{|i-j|}(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \\ &\leqq 2^{19}N^B(\tilde{\rho}_k^{-1}\rho_0)(\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|},\end{aligned} \tag{2.40}$$

for some constant B, and the supremum runs over $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$.

To prove (2.6) note that (I.3.8), (I.3.10) and the chain rule imply

$$\frac{\partial z_i}{\partial I_j'}(\underline{I}', \underline{z}') = -z_i' e^{i\Delta_i(\underline{I}', \underline{z}')}\left\{+\frac{\partial^2 \Phi^k}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}') + \sum_{l=1}^{N}\frac{\partial^2 \Phi^k}{\partial I_i' \partial z_l}(\underline{I}', \underline{z}')\cdot\frac{\partial z_l}{\partial I_j'}(\underline{I}', \underline{z}')\right\}. \tag{2.41}$$

The proof follows the now standard procedure. Let $D^{(3)}$ be the matrix with elements $D_{ij}^{(3)} = \partial z_i/\partial I_j'$, and $\Lambda^{(3)}$ the matrix with elements

$$\Lambda_{ij}^{(3)} = -z_i' e^{i\Delta_i(\underline{I}', \underline{z}')}\cdot\frac{\partial^2 \Phi^k}{\partial I_i' \partial I_j'}(\underline{I}', \underline{z}(\underline{I}', \underline{z}')).$$

Then (2.41) implies

$$\frac{\partial z_i}{\partial I_j'}(\underline{I}', \underline{z}') = D_{ij}^{(3)} = [(\mathbb{1} - M)^{-1}\Lambda^{(3)}]_{ij}. \tag{2.42}$$

Bound $\Lambda_{ij}^{(3)}$ by Lemma 2.5 and we use (2.35) to obtain (2.6), with $c_1 \sim \mathcal{O}((\varepsilon_0\rho_0^{-1})^\alpha)$. Inequalities (2.8) and (2.9) are now proved in exactly the same fashion. Let $\tilde{D}$ be the matrix with elements $\tilde{D}_{mj} = \partial^2 z_m/\partial z_i' \partial I_j'(\underline{I}', \underline{z}')$ or $\partial^2 z_m/\partial I_i' \partial I_j'(\underline{I}', \underline{z}')$, depending on the circumstances. Using (I.3.8) and (I.3.10) show that

$$\tilde{D}_{mj} = [(\mathbb{1} - M)^{-1}\tilde{\Lambda}]_{mj}, \tag{2.43}$$

for some matrix $\tilde{\Lambda}$. Then use Lemma 2.5, and (2.5) and (2.6) to show that $|\tilde{\Lambda}_{mj}| \sim (\varepsilon_0\rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}$, for all m. Combining this observation with (2.35) yields (2.8) and (2.9). We won't write all the details.

With Lemma 2.2 in hand, Lemma 2.3 is just an application of the chain rule. For instance combining (I.3.12) with (I.3.11) we see that

$$\begin{aligned}\frac{\partial I_i}{\partial I_j'}(\underline{I}', \underline{z}') - \delta_{ij} &= i\frac{\partial z_i}{\partial I_j'}(\underline{I}', \underline{z}')\frac{\partial \Phi^k}{\partial z_i}(\underline{I}', \underline{z}(\underline{I}', \underline{z}')) \\ &\quad + iz_i(\underline{I}', \underline{z}')\frac{\partial^2 \Phi^k}{\partial I_j' \partial z_i}(\underline{I}', \underline{z}(\underline{I}', \underline{z})) \\ &\quad + i\sum_{l=1}^{N} z_i(\underline{I}', \underline{z}')\frac{\partial^2 \Phi^k}{\partial z_l \partial z_i}(\underline{I}', \underline{z}(\underline{I}', \underline{z}'))\frac{\partial z_l}{\partial I_j'}(\underline{I}', \underline{z}'),\end{aligned} \tag{2.44}$$

on $W(\tilde{\rho}_k/2, \xi_k - 2\delta; V_k)$. Taking the supremum of the magnitude of both sides of (2.44) over $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$, and using (I.3.6), Lemma 2.2, and Lemma 2.5, we obtain (2.10). (Note that if $(\underline{I}', \underline{z}') \in W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$, $(\underline{I}(\underline{I}', \underline{z}'), \underline{z}(\underline{I}', \underline{z}')) \in W(\tilde{\rho}_k/2, \xi_k - 2\delta; V_k)$, so $|z_i(\underline{I}', \underline{z}')| < e^{\xi_k}$.) Similarly,

$$\frac{\partial I_i}{\partial z_j'}(\underline{I}', \underline{z}') = i \frac{\partial z_i}{\partial z_j'}(\underline{I}', \underline{z}') \frac{\partial \Phi^k}{\partial z_i}(\underline{I}', \underline{z}(\underline{I}', \underline{z}'))$$
$$+ i z_i(\underline{I}', \underline{z}') \cdot \sum_{l=1}^{N} \frac{\partial^2 \Phi^k}{\partial z_i \partial z_l}(\underline{I}', \underline{z}(\underline{I}', \underline{z}')) \frac{\partial z_l}{\partial z_j'}(\underline{I}', \underline{z}'), \tag{2.45}$$

on $W(\tilde{\rho}_k/4, \xi_k - 3\delta; V_k)$. Applying Lemma 2.2 and Lemma 2.5 plus (I.3.6) to estimate $\partial \Phi^k / \partial z_i$, we immediately obtain (2.11).

The last three estimates follow in exactly the same fashion—one applies the chain rule to (I.3.12) and (I.3.11) and notes that Lemmas 2.2 and (2.5) allow one to extract a factor of $(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-2\beta_k)|i-j|}$ from each of the resulting terms.

It now remains to prove Lemmas 2.4 and 2.5. First note that the method used to prove (I.3.25) easily bounds $|f^{k[\geqq]}(\underline{I}, \underline{z})|$ by $[(\varepsilon_0 \rho_0 N)(2\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)L_k+1} + 2^{L_k+2} N \varepsilon_k \rho_k e^{(2\ln 2 - 2\delta)(M_k+1)}]$ on $W(\rho_k, \xi_k; V_k)$. Estimate (2.16) then follows via a dimensional estimate.

$$\sup |f^{k[\leqq]}(\underline{I}, \underline{z})| \leqq \sum_{\substack{\underline{v} \in \mathbb{X}_k \\ \underline{v} \neq \underline{0}}} \varepsilon_k \rho_k e^{-\delta |\underline{v}|} \leqq 2^2 N \varepsilon_k \rho_k 2^{L_k}, \tag{2.46}$$

where the supremum runs over $W(\rho_k, \xi_k - \delta; V^k)$. The first inequality used the fact that (I.2.4), plus Cauchy's theorem implies $\sup |f_{\underline{v}}^k(\underline{I})| \leqq \varepsilon_k \rho_k e^{-(\xi_k - \delta)|\underline{v}|}$, and then the sum over $\underline{v}$ is bounded just as in (I.3.6). Combining (2.46) with a pair of dimensional estimates yields (2.15). The two remaining estimates of Lemma 2.4 as well as those of Lemma 2.5 are proved with the help of

Lemma 2.7. *Suppose* $|g_{\underline{v}}| \leqq \min(c_1 e^{-\kappa_1 |i-j|} |\underline{v}|^2 e^{-\delta' |\underline{v}|}, c_2 e^{-\kappa_2 d(\operatorname{supp} \underline{v})} |\underline{v}|^2 e^{-\delta' |\underline{v}|})$. *If* $\delta' \geqq 6$ *and* $\kappa_2 \geqq 2 \ln 2$, *then*

$$\sum_{\underline{v}} |g_{\underline{v}}| \geqq 4N(c_1 2^{|i-j|} e^{-\kappa_1 |i-j|} + c_2 2^{|i-j|} e^{-\kappa_2 |i-j|}). \tag{2.47}$$

Proof.

$$\sum_{\underline{v}} |g_{\underline{v}}| \leqq \sum_{v: d(\operatorname{supp} v) \leqq |i-j|} c_1 |\underline{v}|^2 e^{-\kappa_1 |i-j|} e^{-\delta |\underline{v}|}$$
$$+ \sum_{v: d(\operatorname{supp} \underline{v}) > |i-j|} c_2 |\underline{v}|^2 e^{-\kappa_2 d(\operatorname{supp} v)} e^{-\delta' |\underline{v}|}. \tag{2.48}$$

Estimate the number of terms in each of the sums with $|\underline{v}| = M$ and $d(\operatorname{supp} \underline{v}) = L$ by $N 2^L 2^{2M}$. Then sum from $L = 0$ to $|i-j|$ and $M = 0$ to ∞ in the first sum in (2.48), and $L = |i-j| + 1$ to ∞, $M = 1$ to ∞ in the second sum in (2.48). Summing the resulting geometric series yields (2.47).

Next note that

$$\frac{\partial^2 f^{k[\leqq]}}{\partial z_i \partial z_j}(\underline{I}, \underline{z}) = \sum_{\substack{\underline{v} \in \mathbb{X}_k \\ \underline{v} \neq \underline{0}}} f_{\underline{v}}^k(\underline{I})(v_i v_j) \underline{z}^{\underline{v}} / (z_i z_j). \tag{2.49}$$

(We are assuming that $|i-j| \geqq 1/2(3/2)^k$ since (2.17) and (2.18) can be obtained by combining (2.46), and the remark that precedes it with a dimensional estimate otherwise.) Note that the expression for $\partial^2 f^{k[\geqq]}/\partial z_i \partial z_j$ is exactly the same except that the sum runs over $\underline{\nu} \notin \mathbb{X}_k$. Also, all terms on the right-hand side of (2.49) vanish unless both i and j are in supp $\underline{\nu}$. Thus, define $g_\nu = \sup |f_{\underline{\nu}}^k(\underline{I})(\nu_i \nu_j)\underline{z}^{\nu}/(z_i z_j)|$ if i and j are in supp $\underline{\nu}$, $g_\nu = 0$ otherwise. This supremum is over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$. By (I.2.7), $|g_\nu| \leqq \varepsilon_0 \rho_0 (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp}\nu)} |\underline{\nu}|^2 e^{2\xi_k} e^{-\delta|\underline{\nu}|}$. Since $d(\operatorname{supp}\underline{\nu}) \geqq |i-j|$ for all non-zero g_ν, the hypotheses of Lemma 2.7 are satisfied with $c_1 = c_2 = \varepsilon_0 \rho_0 e^{2\xi_k}$, $\delta' = \delta$ and $\kappa_1 = \kappa_2 = (1-\eta_k)\ln(\varepsilon_0 \rho_0^{-1})$. Since $|\partial^2 f^{k[\leqq]}/\partial z_i \partial z_j|$ and $|\partial^2 f^{k[\geqq]}/\partial z_i \partial z_j|$ are both bounded by $\sum_{\underline{\nu}} g_{\underline{\nu}}$, (2.47) implies both (2.17) and (2.18) in the case $z_i = \chi_{i+N}$ and $z_j = \chi_{j+N}$, where we used the fact that $2^4 N e^{2\xi_k 2|i-j|} (\varepsilon_0 \rho_0^{-1})^{\beta_k |i-j|} \leqq 1$ if $|i-j| \geqq 1/2(3/2)^k$.

Now consider

$$\frac{\partial^2 f^{k[\leqq]}}{\partial z_i \partial I_j}(\underline{I}, \underline{z}) = \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{\partial f_{\nu}^k}{\partial I_j}(\underline{I})(\nu_i)\underline{z}^{\nu}/z_i. \tag{2.50}$$

Once again $\partial^2 f^{k[\geqq]}/\partial z_i \partial I_j$ is given by the same expression but the sum runs over $\underline{\nu} \notin \mathbb{X}_k$. Define $g_\nu = \sup |(\partial f_{\underline{\nu}}^k/\partial I_j)(\underline{I})(\nu_i)\underline{z}^\nu/z_i|$ if $i \in \operatorname{supp}\underline{\nu}$ (the sup runs over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$), zero otherwise. By (I.2.7), $|g_\nu| \leqq \varepsilon_0 e^{\xi_k} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} |\underline{\nu}| e^{-\delta|\underline{\nu}|}$. On the other hand (I.2.7) and a dimensional estimate also imply $|g_\nu| \leqq 2\varepsilon_0 \rho_k^{-1} \rho_0 e^{\xi_i} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp}\underline{\nu})} |\underline{\nu}| e^{-\delta|\underline{\nu}|}$. Combining these two bounds we find $|g_{\underline{\nu}}| \leqq \varepsilon_0 (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \times (\varepsilon_0 \rho_0^{-1})^{\beta_k/2(1-\eta_k)d(\operatorname{supp}\nu)} |\underline{\nu}| e^{-\delta|\underline{\nu}|}$. (We have again assumed that $|i-j| \geqq \frac{1}{2}(\frac{3}{2})^k$ for the reason stated above.) Thus, we can apply Lemma 2.7 with $c_1 = \varepsilon_0 e^{\xi_k}$, $\kappa_1 = (1-\eta_k)$, $c_2 = \varepsilon_0 (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$, and $\kappa_2 = (\beta_k/2)|\ln \varepsilon_0 \rho_0^{-1}|$, and (2.17) and (2.18) follow in the cases $I_j = x_j$ and $z_i = x_{i+N}$, or vice versa.

Finally

$$\frac{\partial^2 f^{k[\leqq]}}{\partial I_i \partial I_j}(\underline{I}, \underline{z}) = \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{\partial^2 f_{\underline{\nu}}^k}{\partial I_j \partial I_j}(\underline{I})\underline{z}^{\nu}, \tag{2.51}$$

and a corresponding expression holds for $\partial^2 f^{k[\geqq]}/\partial I_i \partial I_j$, with the sum running over $\underline{\nu} \notin \mathbb{X}_k$. Let g_ν be the supremum over $W(\tilde{\rho}_k, \xi_k - \delta; V^k)$ of the summand. By (I.2.7) we see that $|g_\nu| \leqq \varepsilon_0 \rho_0^{-1} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} e^{-\xi_k|\underline{\nu}|}$, and $|g_\nu| \leqq 4\varepsilon_0 \rho_0 \rho_k^{-2} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp}\underline{\nu})} e^{-\xi_k|\underline{\nu}|}$. Combining these observations with the fact that we may assume $|i-j| \geqq \frac{1}{2}(\frac{3}{2})^k$, we find

$$|g_{\underline{\nu}}| \leqq (\varepsilon_0 \rho_0^{-1})(\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} \times (\varepsilon_0 \rho_0^{-1})^{\beta_k/2(1-\eta_k)d(\operatorname{supp}\underline{\nu})} e^{-\delta|\nu|}. \tag{2.52}$$

This allows us to apply Lemma 2.7 and the remaining cases of (2.17) and (2.18) follow.

The last task is to prove Lemma 2.5. It is very similar to the proof we just finished,

$$\frac{\partial^2 \Phi^k}{\partial z_i \partial z_j}(\underline{I}', \underline{z}) = \sum_{\substack{\underline{\nu} \in \mathbb{X}_k \\ \underline{\nu} \neq \underline{0}}} \frac{f_{\underline{\nu}}^k(\underline{I}')(\nu_i \nu_j)\underline{z}^{\nu}}{i\langle \underline{\omega}^k(\underline{I}'), \underline{\nu}\rangle (z_i z_j)}. \tag{2.53}$$

Just as before we will assume that $|i-j| \geqq (\frac{1}{2})(\frac{3}{2})^k$. Let g_ν equal the supremum of the absolute value of this summand over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$ if $\nu \in \mathbb{X}_k$ and both i and j are in supp $\underline{\nu}$. Let $g_\nu = 0$ otherwise. Then by the estimates above, plus (I.3.2), $|g_{\underline{\nu}}| \leqq 2C\varepsilon_0 \rho_0 e^{2\xi_k} e^{2L_k} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)d(\operatorname{supp}\nu)} |\underline{\nu}|^2 e^{(\delta - 3/2)|\nu|}$

Since $d(\operatorname{supp}\underline{v}) \geqq |i-j|$, and $\delta > (\frac{3}{2}) + 6$ we may apply Lemma 2.7 and (2.27) follows.

Next note that

$$\frac{\partial^2 \Phi^k}{\partial z_i \partial I'_j}(\underline{I}', \underline{z}) = \sum_{\substack{\underline{v} \in \mathbb{X}_k \\ \underline{v} \neq \underline{0}}} \left\{ \frac{\frac{\partial f^k_{\underline{v}}}{\partial I'_j}(\underline{I}')(v_i)\underline{z}^{\underline{v}}}{i\langle \underline{\omega}^k(\underline{I}'), \underline{v}\rangle z_i} - i \frac{\left\langle \frac{\partial \underline{\omega}^k}{\partial I'_j}(\underline{I}'), \underline{v} \right\rangle f_{\underline{v}}(\underline{I}')(v_i)\underline{z}^{\underline{v}}}{(i\langle \underline{\omega}^k(\underline{I}'), \underline{v}\rangle)^2 z_i} \right\}. \tag{2.54}$$

Let $g_{\underline{v}}$ equal the supremum of the integrand over $W(\tilde{\rho}_k, \xi_k - \delta; V_k)$ if $\underline{v} \in \mathbb{X}_k$, $\underline{v} \neq \underline{0}$, and $i \in \operatorname{supp} \underline{v}$ and zero otherwise. Remark that

$$\left| \frac{\partial f^k_{\underline{v}}}{\partial I'_j}(\underline{I}')(v_i)\underline{z}^{\underline{v}}/(i\langle \underline{\omega}^k(\underline{I}'), \underline{v} >) \right| \leqq \min(2^2 \varepsilon_0 C e^{\xi_k} e^{L_k} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)|i-j|} |\underline{v}| e^{-(\delta - 3/2)|\underline{v}|}$$
$$2\varepsilon_0 C e^{\xi_k} e^{L_k} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|} (\varepsilon_0 \rho_0^{-1})^{(\beta_k/2)(1-\eta_k) d(\operatorname{supp}\underline{v})} |\underline{v}| e^{-(\delta - 3/2)|\underline{v}|}),$$

by our observations above, and (I.3.2). If we note that $\langle (\partial \underline{\omega}^k / \partial I'_j)(\underline{I}'), \underline{v} \rangle = \sum_{l=1}^{N} (\partial^2 h^k / \partial I'_j \partial I'_l)(\underline{I}') v_l$, then combining (I.2.5), (I.2.6), (I.2.7) and (I.3.2), we obtain

$$\left| \left\langle \frac{\partial \underline{\omega}^k}{\partial I'_j}(\underline{I}'), \underline{v} \right\rangle f_{\underline{v}}(\underline{I}')(v_i)\underline{z}^{\underline{v}}/(i\langle \underline{\omega}^k(\underline{I}'), \underline{v}\rangle)^2 z_i \right|$$
$$\leqq 2^3 C^2 \varepsilon_0 \rho_0 e^{\xi_k} e^{2L_k} (\varepsilon_0 \rho_0^{-1})^{(1-\eta_k)(1-\beta_k)|i-j|}$$
$$\times (\varepsilon_0 \rho_0^{-1})^{(\beta_k/2)(1-\eta_k) d(\operatorname{supp}\underline{v})} |\underline{v}| e^{-(\delta-3)|\underline{v}|}.$$

These two bounds result in a bound on $g_{\underline{v}}$, and allow us to apply Lemma 2.7 to obtain (2.28).

Our last bound is on $(\partial^2 \Phi^k / \partial I'_i \partial I'_j)(\underline{I}', \underline{z})$. Once again one just writes out the expression for this quantity using (I.3.3). Each term is bounded using (I.2.4)–(I.2.7), and (I.3.2), and one then applies Lemma 2.7. This is a straightforward exercise and we leave the details to the reader.

3. The Proof of Theorem 1.2

Large parts of the proof of Theorem 1.2 are identical to the proofs of the main theorems in [2] and [3]. We need, however, a better estimate on the amount of phase space lost at each step in the inductive procedure used to prove the theorem. This improvement results from the short range property of the initial interactions.

We will construct an infinite sequence of canonical transformations, which at each stage reduce the size of the perturbation of the Hamiltonian, resulting finally in an integrable system. Consider the Hamiltonian $H \circ C(\underline{I}, \underline{z}) \equiv \hat{H}^0(\underline{I}, \underline{z}) = \hat{h}^0(\underline{I}) + \hat{f}^0(\underline{I}, \underline{z})$ constructed in Theorem I.1.1. If

$$\sup \left\{ \left| \frac{\partial \hat{f}^0}{\partial \underline{I}} \right| + \hat{\rho}_0^{-1} \left| \frac{\partial \hat{f}^0}{\partial \underline{\phi}} \right| \right\} \leqq \hat{\varepsilon}_0, \tag{3.1}$$

$$\sup \left| \frac{\partial \hat{h}^0}{\partial \underline{I}} \right| \leqq \hat{E}_0, \tag{3.2}$$

$$\sup \left| \left(\frac{\partial \hat{h}^0}{\partial \underline{I}\, \partial \underline{I}} \right)^{-1} \right| \leqq \hat{\eta}_0, \tag{3.3}$$

and

$$\frac{\partial^2 \hat{h}^0}{\partial I_i \partial I_j}(\underline{I}) \equiv \frac{\partial \hat{\omega}_i^0}{\partial I_j}(\underline{I}) = \delta_{ij}(1+\hat{\chi}_i^0(\underline{I})) + (1-\delta_{ij})\frac{\partial \hat{\omega}_i^0}{\partial I_j}(\underline{I}), \tag{3.4}$$

then (I.1.16)–(I.1.20) insure that we can take $\hat{\varepsilon}_0 = \rho_0(\varepsilon_0\rho_0^{-1})^N$, $\hat{E}_0 = 2E_0$, and $\hat{\rho}_0 = \rho_{k_0}$, where $k_0 = [(\ln N)(\ln 3/2)^{-1}] + 1$, and ρ_{k_0} is defined inductively by (I.2.8). Here, the quantities with no hats (e.g., ε_0, ρ_0) refer to the Hamiltonian $H(\underline{I}, \underline{z})$ of I. Also, $\sup|\hat{\chi}_i^0(\underline{I})| \leqq (2^2N)^{-1}$ and $\sup|(1-\delta_{ij})(\partial\hat{\omega}_i^0/\partial I_j)| \leqq (\varepsilon_0\rho_0^{-1})\cdot(\varepsilon_0\rho_0^{-1})^{(1/8)|i-j|}$, and all suprema are evaluated over the region $W(\hat{\rho}_0, \hat{\xi}_0; \hat{V})$, where $\hat{V} = V^{k_0-1}$. From the proof of Theorem I.1.1 we see we may pick $\hat{\xi}_0 = 1$. Define inductively the sequence of parameters

$$\begin{aligned}
\hat{\delta}_j &= \hat{\xi}_0/16(1+j^2),\\
\hat{C}_k &= (1+k^2)\hat{C}_0,\\
\hat{\xi}_{k+1} &= \hat{\xi}_k - 4\hat{\delta}_k,\\
\hat{E}_{k+1} &= \hat{E}_k + \hat{\varepsilon}_k,\\
\hat{\eta}_{k+1} &= \hat{\eta}_k(1+4N^2\hat{\eta}_k\hat{\varepsilon}_k\hat{\rho}_k^{-1}),\\
\hat{M}_k &= 2\hat{\delta}_k^{-1}\ln(\hat{\varepsilon}_k\hat{C}_k\hat{\delta}_k^N)^{-1},\\
\hat{\rho}_{k+1} &= \hat{\rho}_k/2^7N\hat{E}_k\hat{C}_k\hat{M}_k,\\
\hat{\varepsilon}_{k+1} &= (\hat{\varepsilon}_k\hat{C}_k)^{(3/2)^k}\hat{E}_k(\hat{E}_k\hat{C}_k)\hat{M}_k^{N+1}\hat{\delta}_k^{-(N+1)},
\end{aligned} \tag{3.5}$$

with $\hat{C}_0$ a constant to be determined later. Defining $\tilde{\rho}_k = 2^3\rho_{k+1}$, we construct a sequence of regions $\hat{V} \supset \hat{V}_0 \supset \hat{V}_1 \ldots$. Set

$$\hat{R}(k, \hat{h}^k; \hat{V}_{k-1}) = \{\underline{I} \,|\, \underline{I} \in \hat{V}^{k-1}; |\langle \underline{\hat{\omega}}^k(\underline{I}), \underline{v}\rangle| < \hat{C}_k|\underline{v}|^N \text{ for some } \underline{v} \text{ with } 0 < |\underline{v}| < M_k\}. \tag{3.6}$$

(By convention take $\hat{V}_{-1} = \hat{V}$.) Then

$$\bar{\hat{V}}_k \equiv \{\underline{I} \,|\, \underline{I} \in \hat{V}_{k-1}; \operatorname{dist}(\underline{I}, \partial\hat{V}_{k-1}) \geqq \tilde{\rho}_k\} \backslash \hat{R}(k, \hat{h}^k, \hat{V}_{k-1}),$$

and $\hat{V}_k \equiv \bigcup_{I \in \bar{\hat{V}}_k} S(\underline{I}, \tilde{\rho}_k/2)$, with $\operatorname{dist}(\underline{I}, \underline{I}') = |\underline{I} - \underline{I}'|$.

Following [2] or [3] one constructs canonical transformations

$$\hat{C}^k\colon (\underline{I}', \underline{z}') \to (\underline{I}, \underline{z}), \qquad \tilde{\hat{C}}^k\colon (\underline{I}, \underline{z}) \to (\underline{I}', \underline{z}'), \tag{3.7}$$

where both transformations are defined on $W(\tilde{\rho}_k/2,\ \hat{\xi}_k - 2\hat{\delta}_k;\ \hat{V}^k)$ and map $W(\tilde{\rho}_k/4,\ \hat{\xi}_k - 3\hat{\delta}_k;\ \hat{V}^k)$ into $W(\tilde{\rho}_k/2,\ \hat{\xi}_k - 2\hat{\delta}_k;\ \hat{V}^k)$. Defining

$$H^k(\underline{I}', \underline{z}') = \hat{H}^{k-1} \circ \hat{C}^{k-1}(\underline{I}', \underline{z}') \equiv \hat{h}^k(\underline{I}') + \hat{f}^k(\underline{I}', \underline{z}'), \tag{3.8}$$

one has

$$\sup\left|\frac{\partial \hat{h}^k}{\partial \underline{I}}\right| \leqq \hat{E}_k, \tag{3.9}$$

$$\sup\left|\left(\frac{\partial^2 \hat{h}^k}{\partial \underline{I}\, \partial \underline{I}}\right)^{-1}\right| \leqq \eta_k, \tag{3.10}$$

and

$$\sup\left\{\left|\frac{\partial \hat{f}^k}{\partial \underline{I}}\right| + \hat{\rho}_k^{-1}\left|\frac{\partial \hat{f}^k}{\partial \underline{\phi}}\right|\right\} \leqq \hat{\varepsilon}_k. \tag{3.11}$$

The conditions which guarantee that the iterative procedure of (3.7)–(3.11) may be arbitrarily repeated can be combined in the single inequality

$$\hat{\varepsilon}_0 < \hat{C}_0^{-1}\hat{B}_2\hat{\eta}_0^{-2}(\hat{\rho}_0\hat{E}_0^{-1})^2(\hat{E}_0\hat{C}_0)^{-16}\hat{g}'(N). \tag{3.12}$$

where $\hat{g}'(N) = (N!)^{-4}N^{-10N}e^{-152N}$ and $\hat{B}_2$ is some small ($\ll 1$) constant. (This is essentially inequality (3.58) of [3] and we do not repeat its derivation.) Given this sequence of canonical transformations we must estimate the value of $\hat{C}_0$ which insures that not too much phase space gets thrown away. We will need the fact that the integrable parts of the Hamiltonians constructed above are given by

$$\hat{h}^{k+1}(\underline{I}) = \hat{h}^0(\underline{I}) + \sum_{j=0}^{k} \hat{f}_0^j(\underline{I}). \tag{3.13}$$

As in I note that

$$\mathrm{vol}(\hat{V}^{k-1}\backslash\hat{V}^k) \leqq \mathrm{vol}\,\hat{B} + \mathrm{vol}\,\hat{R}(k,\hat{h}^k;\hat{V}_{k-1}), \tag{3.14}$$

with

$$\hat{B} = \{\underline{I}\,|\,\underline{I}\in\hat{V}^{k-1} \text{ and } \mathrm{dist}(\underline{I},\partial\hat{V}^{k-1}) \leqq \hat{\rho}_k\}. \tag{3.15}$$

As in Sect. 5 of I,

$$\mathrm{vol}\,\hat{B} \leqq |1-(1+\tilde{\rho}_k/\rho_k)^N|. \tag{3.16}$$

It is easy to show that $\tilde{\hat{\rho}}_k/\hat{\rho}_k \leqq 2^{-4}(1+k^2)^{-1}N^{-1}2^{-N}(\hat{E}_0\hat{C}_0)^{-1}$. Thus,

$$\mathrm{vol}\,\hat{B} \leqq 2^{-2}N2^{-N}(1+k^2)^{-1}(\hat{E}_0\hat{C}_0)^{-1}\,\mathrm{vol}\,\hat{V}. \tag{3.17}$$

Since $\hat{\underline{\omega}}^k(\underline{I})$ is single-valued (a fact we verify below),

$$\mathrm{vol}(\hat{R}(k,\hat{h}^k;\hat{V}_{k-1})) = \int_{\hat{\underline{\omega}}^k(\hat{R}(k,\hat{h}^k;\hat{V}_{k-1}))} \left|\det\left(\frac{\partial\underline{\omega}^k}{\partial\underline{I}}\right)^{-1}\right| d\underline{\omega}. \tag{3.18}$$

Bound $|\det(\partial\hat{\underline{\omega}}^k/\partial\underline{I})^{-1}|$ by 2^3 using (I.5.5)–(I.5.10), with $c_1 = (\frac{1}{2})$. Since $\hat{\underline{\omega}}^k(\underline{I}) = \hat{\underline{\omega}}^0(\underline{I}) + \sum_{j=0}^{k-1}(\partial\hat{f}_0^j/\partial\underline{I})(\underline{I})$, verifying that the matrix $\mathbb{D}$ defined there has diagonal entries bounded by $c_1N^{-1} + 2\sum_{j=0}^{k-1}\hat{\varepsilon}_j\hat{\rho}_j^{-1}$, and off diagonal entries bounded by $(\varepsilon_0\rho_0^{-1})N^{-1}$ is an easy exercise. Definitions (3.5) and inequality (3.12) imply $\hat{\varepsilon}_j\hat{\rho}_j^{-1} \leqq (\hat{\varepsilon}_0\hat{C}_0)^{(1/16)(3/2)^j}$, and we then bound the sum over j by a geometric series in $(\hat{\varepsilon}_0\hat{C}_0)$. The single-valuedness of $\hat{\underline{\omega}}^k$ follows by noting that (I.5.7) implies $|\hat{\underline{\omega}}^0(\underline{I}') - \hat{\underline{\omega}}^0(\underline{I})| \geqq (\frac{1}{2})|\underline{I}-\underline{I}'|$, while the argument of Appendix H of [3] guarantees that

$$\left|\sum_{j=0}^{k-1}\frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}') - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I})\right| \leqq 3|\underline{I}'-\underline{I}'|\sum_{j=0}^{k-1}\hat{\varepsilon}_j\hat{\rho}_j^{-1} \leqq (\tfrac{1}{4})|\underline{I}-\underline{I}'|.$$

Thus,

$$-\hat{\underline{\omega}}^k(\underline{I}') - \hat{\underline{\omega}}^k(\underline{I})| \geqq |\hat{\underline{\omega}}^0(\underline{I}) - \hat{\underline{\omega}}^0(\underline{I}')| - \left|\sum_{j=0}^{k-1}\frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}') - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I})\right| \geqq (\tfrac{1}{4})|\underline{I}'-\underline{I}|, \tag{3.19}$$

and

$$\operatorname{vol}(\hat{R}(k,\hat{h}^k;\hat{V}_{k-1})) \leqq 2^3 \int_{\hat{\omega}^k(\hat{R}(k,\hat{h}^k;\hat{V}_{k-1}))} d\underline{\omega}$$
$$\leqq 2^3 \sum_{\underline{v}\neq\underline{0}} \int_{\substack{|\langle\omega,\nu\rangle|<\hat{C}_k^{-1}|\nu|^{-(N+1)}\\ \|\omega-\omega^k(0)\|\leqq r(1+\tau)}} d\underline{\omega}, \tag{3.20}$$

where $\|\cdot\|$ is the ordinary Euclidean norm and τ is defined below. This estimate follows by noting that (I.5.12) implies $\|\hat{\underline{\omega}}^0(\underline{I}') - \hat{\underline{\omega}}^0(\underline{0})\| \leqq (1 + 2^2/N)r$ for all $\underline{I}\in\hat{V}$. By the estimate of [3] used above,

$$\left|\sum_{j=0}^{k-1}\frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}) - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{0})\right| \leqq 3|\underline{I}|\sum_{j=0}^{k-1}\hat{\varepsilon}_j\hat{\rho}_j^{-1}.$$

For any vector $\underline{x}$, $\|\underline{x}\| \leqq |\underline{x}| \leqq N\|\underline{x}\|$, so for any $\underline{I}\in R(k,h^k;\hat{V}_{k-1})$,

$$\|\hat{\underline{\omega}}^k(\underline{I}) - \hat{\underline{\omega}}^k(\underline{0})\| \leqq \|\hat{\underline{\omega}}^0(\underline{I}) - \hat{\underline{\omega}}^0(\underline{0})\| + \left\|\sum_{j=0}^{k-1}\frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{I}) - \frac{\partial\hat{f}_0^j}{\partial\underline{I}}(\underline{0})\right\| \leqq (1+\tau)r, \tag{3.21}$$

with $\tau = 2^2/N + 3N\sum_{j=0}^{k-1}\hat{\varepsilon}_j\hat{\rho}_j^{-1}$.

A simple geometrical argument gives

$$\int_{\substack{|\langle\bar{\omega},\nu\rangle|\leqq\hat{C}_k|\nu|^{-(N+1)}\\ \|\omega-\omega^k(0)\|\leqq r(1+\tau)}} d\underline{\omega} \leqq \hat{C}_k^{-1}\pi^{(N-1)/2}[r(1+\tau)]^{N-1}[\Gamma(1+(N-1)/2)]^{-1}|\underline{v}|^{-(N+1)}. \tag{3.22}$$

Also, $\operatorname{vol}\hat{V} \geqq (1-\lambda)\operatorname{vol}V = (1-\lambda)\pi^{N/2}r^N/\Gamma(1+N/2)$, and $\Gamma(1+N/2)/\Gamma(1+(N-1)/2) \leqq 2^2[(N/4)+1]^{1/2}$, so

$$\sum_{\underline{v}\neq\underline{0}} \int_{\substack{|\langle\omega,\nu\rangle|\leqq\hat{C}_k^{-1}|\nu|^{-(N+1)}\\ \|\omega-\omega^k(\underline{0})\|\leqq r(1+\tau)}} d\underline{\omega} \leqq 2^N(1-\lambda)^{-1}(1+k^2)^{-1}N^{1/2}(\rho_0\hat{C}_0)^{-1}\operatorname{vol}\hat{V}, \tag{3.23}$$

where the last inequality used the fact that $\rho_0 < r$. Combining (3.16), (3.20) and (3.23) we find

$$\sum_{k=0}^{\infty}\operatorname{vol}(\hat{V}_{k-1}\backslash\hat{V}_k) \leqq \sum_{k=0}^{\infty}\{N2^{-(N+1)}(1+k^2)^{-1}(\hat{E}_0\hat{C}_0)^{-1}$$
$$+ 2^{3+N}(1-\lambda)^{-1}(1+k^2)^{-1}N^{1/2}(\rho_0\hat{C}_0)^{-1}\}\operatorname{vol}\hat{V}$$
$$\leqq \hat{\lambda}\operatorname{vol}\hat{V}, \tag{3.24}$$

provided $\hat{C}_0^{-1} = 2^{5+N}(1-\lambda)^{-1}N(\rho_0\hat{\lambda})^{-1}$.

Finally we estimate the parameter $\hat{\eta}_0$. Writing $(\partial^2\hat{h}^0/\partial\underline{I}\partial\underline{I}) = (\hat{\mathbb{D}} - \hat{M})^{-1}$, where $\hat{\mathbb{D}}$ is a diagonal matrix and $\mathbb{M}$ a purely off diagonal one, (I.1.18)–(I.1.20) allow one to estimate the elements of $\hat{\mathbb{D}}$ and $\hat{M}$ by $|\hat{D}_{ii}| > (1/2)$, and $|\hat{M}_{ij}| < \varepsilon_0\rho_0^{-1}$ respectively. Using this information it is easy to bound $\sup|(\partial^2\hat{h}^0/\partial\underline{I}\partial\underline{I})_{ij}^{-1}|$ by 2^2 by estimating the Neumann series for $(\hat{\mathbb{D}} - \hat{M})_{ij}^{-1}$. Thus,

$$\sup\left|\left(\frac{\partial^2\hat{h}^0}{\partial\underline{I}\partial\underline{I}}\right)^{-1}\right| \leqq 2^2N^2 \equiv \hat{\eta}_0. \tag{3.25}$$

Inserting the expressions for $\hat{\eta}_0$ and $\hat{C}_0$ into (4.12), the inequality which allows arbitrarily many canonical transformations to be performed becomes

$$\hat{\varepsilon}_0 < \rho_0 B_3(1-\lambda)^{17}(\hat{\rho}_0\hat{E}_0^{-1})^2(\rho_0\hat{E}_0^{-1})^{-16}\hat{\lambda}^{17}\hat{g}(N), \tag{3.26}$$

with $\hat{g}(N)=(N!)^{-4}N^{-10N}e^{-190N}$. This yields inequality (I.1.21). Given that one may iterate the canonical transformation arbitrarily often while losing arbitrarily little phase space, the remainder of the statements in Theorem 1.2 follow word-for-word from [2] or [3] and we don't repeat the proofs here.

Appendix

By Lemma 1.1 of [1],

$$[(\mathbb{1}-M)^{-1}]_{ij}=\sum_{\Omega:i\to j}\Lambda_{jj}^{-n}\left(\prod_{l\in L}\Lambda_{ll}^{-n(l,\Omega)}\right)\left(\prod_{s\in\Omega}\tilde{M}_s\right). \tag{A.1}$$

On the right-hand side of (A.1), $\Lambda_{jj}=1-M_{jj}$, and Ω is a random walk on the lattice $L=\{1,\dots,N\}$, i.e. a set of pairs $\{(i_1,i_2),\dots,(i_k,i_{k+1})\}$, $i_j\in\{1,\dots,N\}$. Each of the pairs is referred to as a step, s, with $|\Omega|$ the number of steps in the walk, and Ω: $i\to j$ means $i_1=i$, $i_{k+1}=j$. Finally $\tilde{M}_s=\tilde{M}_{(i_j,i_j+1)}=(1-\delta_{i_j,i_{j+1}})\cdot M_{(i_j,i_{j+1})}$, $|s|=|i_{j+1}-i_j|$, and $n(j,\Omega)$ is the number of times j appears as the first element of some step in Ω. Note that the matrix M is not symmetric here (not does it necessarily have positive entries) as in [1], but this just requires us to keep track of the direction of each step in the walk Ω. The hypotheses of Lemma 2.6 imply that each term on the right-hand side of (A.1) is bounded in magnitude by $(4/3)^{|\Omega|+1}c^{|\Omega|}e^{-\kappa L(\Omega)}$, where $L(\Omega)=\sum_{s\in\Omega}|s|$. If we note that every walk from i to j has $L(\Omega)\geqq|i-j|$, and that the number of walks starting at i, with $L(\Omega)=L$, $|\Omega|=M$ is bounded by 2^L2^M, we can bound the magnitude of (A.1) by

$$(4/3)\sum_{L=|i-j|}^{\infty}\sum_{M=0}^{\infty}(8c/3)^M2^Le^{-\kappa L}. \tag{A.2}$$

Summing the geometric series yields (2.30).

References

1. Brydges, D., Fröhlich, J., and Spencer, T.: The random walk representation of classical spin systems and correlation inequalities. Commun. Math. Phys. **83**, 123 (1982)
2. Chierchia, L., Gallavotti, G.: Smooth prime integrals for quasi-integrable Hamiltonian systems. Il Nuovo Cimento **67B**, 277 (1982)
3. Gallavotti, G.: 'Perturbation theory for classical Hamiltonian systems.' In: Scaling and Self-Similarity in Physics, Fröhlich, J. (ed.). Boston, MA: Birkhäuser Boston, Inc., 1983
4. Pöschel, J.: Commun. Pure Appl. Math. **35**, 653 (1982)
5. Symmanzik, K.: Euclidean quantum field theory. In: Local Quantum Theory Jost, (ed.), New York, London: Academic Press, 1969
6. Wayne, C. E.: The KAM Theory of Systems with Short Range Interactions, I. Institute for Mathematics and its Applications, Preprint Series No. 32 (1983). Commun. Math. Phys. 1984

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