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Breaking and Disappearance of Tori

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Abstract. A mechanism is illustrated which can cause a torus to disappear in dissipative differential equations. Three different examples give evidence that a collision with a neighbouring unstable periodic orbit, possibly preceded by a transition into a weakly chaotic attractor, causes the sudden destruction of a torus.

1. Introduction

An open problem in dissipative dynamical systems is a satisfactory understanding of the different possible transitions away from quasiperiodic flow on a two-torus. Those transitions which lead to a turbulent flow on a strange attractor are of particular interest.

In a fundamental paper by Ruelle and Takens [1], a theoretical study of differential equations shows that a quasiperiodic motion with two independent frequencies, when followed by one with three independent frequencies, leads under generic conditions to the appearance of a strange attractor. Support of this mechanism for turbulence has been found experimentally by Gollub and Benson [2] and numerically by Yahata [3]. There are also, however, examples of transitions from a two-torus directly to a strange attractor. This has been observed in experiments (see, for example, again [2]) and, with different features, in a number of numerical studies of nonlinear differential equations. Curry [4] has found two symmetrically located tori which become unstable and give rise to a wide chaotic attractor surrounding them both. Arneodo et al. [5] and Franceschini [6] have shown that chaos can be preceded by a cascade, possibly finite, of period-doublings of a torus. Schreiber and Marek [7] have given an example in which a torus becomes a strange attractor producing some kind of foldings or wrinkles. Riela [8] has exhibited a picture showing a torus which, quite evidently on a section, disappears after development of corners.

Other approaches to the problem are possible. Shenker [9], Feigenbaum et al. [10], and Rand et al. [11, 12] have considered mappings of the circle into itself.

Their theoretical analysis is based on renormalization group techniques, already successfully used to explain Feigenbaum period-doubling onset of turbulence [13]. Chenciner and Iooss [14] have provided a set of theorems which concern bifurcations of tori, though in non-generic circumstances. Curry and Yorke [15] have studied numerically maps of the plane which exhibit bifurcations from an invariant circle. Maps of the same kind have been investigated also by Aronson et al. [16] through a combination of computer experiment and mathematical theory.

In any case, both theoretical and phenomenological analyses appear still quite incomplete. Here we describe an interesting mechanism which can lead to the destruction of a torus in a system of differential equations. As an external parameter varies, the torus grows in size and approaches a neighbouring unstable periodic orbit. The process causes the torus first to break down becoming a weakly chaotic attractor, then rapidly to disappear because of the collision with the orbit. Grebogi et al. [17, 18] have called such a phenomenon of destruction of a strange attractor a "crisis." A detailed numerical investigation of the behavior of two distinct tori in a 12-mode truncation of the planar Navier-Stokes equations sheds some light on the transition torus-strange attractor and stresses the importance of the mechanism of crisis in connection with tori. Furthermore, evidence is given for the fact that also in ref. [8] a collision with two symmetric unstable periodic orbits is responsible for the disappearance of the torus.

2. A Model of Differential Equations

We consider the following system of first-order non-linear differential equations:

$$
\begin{aligned}
\dot{x}_1 &= -x_1+5x_2(x_3+x_4)-3\sqrt{5}(x_3x_7+x_4x_{10}+x_8x_{11}+x_9x_{12}) \\
&\quad +\sqrt{10}x_6(x_8+x_9)\,, \\
\dot{x}_2 &= -x_2-5x_1(x_3+x_4)-3\sqrt{5}(x_3x_8-x_4x_9+x_7x_{11}-x_{10}x_{12}) \\
&\quad +\sqrt{10}x_5(x_{10}-x_7)\,, \\
\dot{x}_3 &= -2x_3+4\sqrt{5}(x_1x_7+x_2x_8)-5\sqrt{2}x_4(x_5+x_6)\,, \\
\dot{x}_4 &= -2x_4+4\sqrt{5}(x_1x_{10}-x_2x_9)+5\sqrt{2}x_3(x_5+x_6)\,, \\
\dot{x}_5 &= -4x_5+4\sqrt{10}x_2(x_7-x_{10})+10x_6(x_{11}-x_{12})\,, \\
\dot{x}_6 &= -4x_6-4\sqrt{10}x_1(x_8+x_9)+10x_5(x_{12}-x_{11})\,, \\
\dot{x}_7 &= -5x_7-\sqrt{5}(x_1x_3-7x_2x_{11})-3\sqrt{10}x_2x_5-2\sqrt{2}x_6x_{10}+9x_4x_8\,, \\
\dot{x}_8 &= -5x_8-\sqrt{5}(x_2x_3-7x_1x_{11})+3\sqrt{10}x_1x_6+2\sqrt{2}x_5x_9-9x_4x_7\,, \\
\dot{x}_9 &= -5x_9+\sqrt{5}(x_2x_4+7x_1x_{12})+3\sqrt{10}x_1x_6-2\sqrt{2}x_5x_8+9x_3x_{10}+R\,, \\
\dot{x}_{10} &= -5x_{10}-\sqrt{5}(x_1x_4+7x_2x_{12})+3\sqrt{10}x_2x_5+2\sqrt{2}x_6x_7-9x_3x_9\,, \\
\dot{x}_{11} &= -8x_{11}-4\sqrt{5}(x_1x_8+x_2x_7)\,, \\
\dot{x}_{12} &= -8x_{12}-4\sqrt{5}(x_1x_9-x_2x_{10})\,.
\end{aligned}
\tag{1}
$$

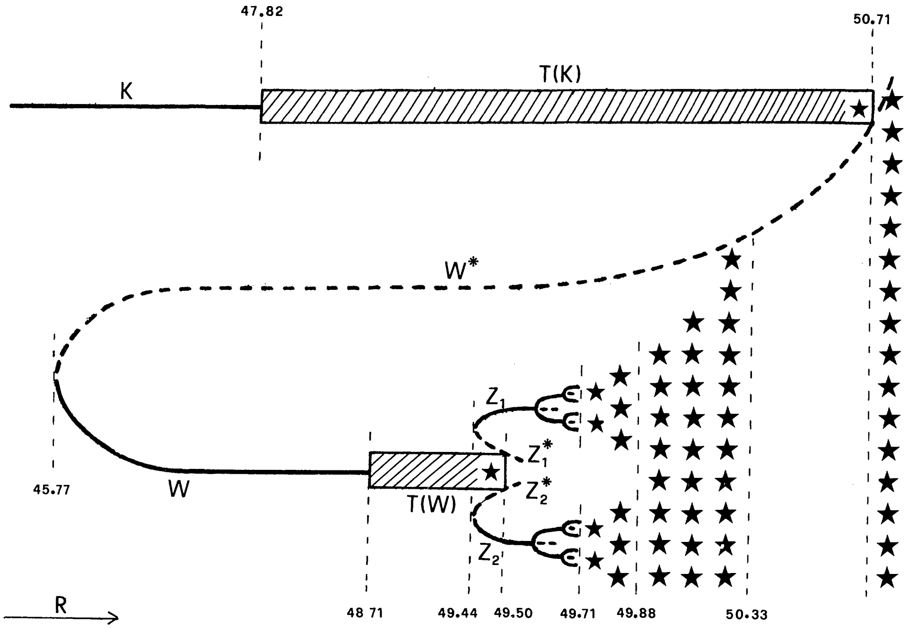


Fig. 1. Sketch of the phenomenology described here and concerning system (1). The R-scale is only roughly maintained. A continuous thick line represents a stable periodic orbit and a broken line an unstable one. A hatched tube is used to represent a torus and stars indicate chaotic behavior. The star at the end of the tubes means that the torus has evolved into a strange attractor before disappearing. The two period-doubling cascades associated to the orbits Z_1 and Z_2 are supposed to accumulate at $R=49.71$

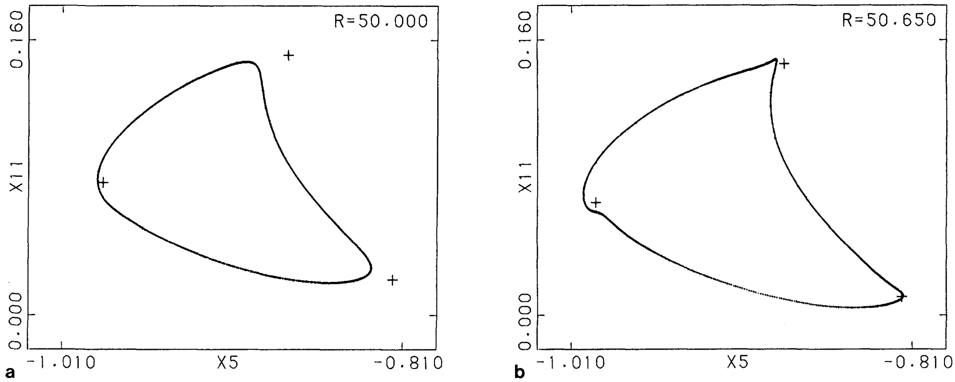
This system is obtained through a 12-mode truncation of the Fourier-expansion for the Navier-Stokes equations for an incompressible fluid on a two-dimensional torus. The modes taken into account are $\underline{k}_1=(0,1)$, $\underline{k}_2=(1,0)$, $\underline{k}_3=(1,1)$, $\underline{k}_4=(1,-1)$, $\underline{k}_5=(0,2)$, $\underline{k}_6=(2,0)$, $\underline{k}_7=(1,2)$, $\underline{k}_8=(2,1)$, $\underline{k}_9=(2,-1)$, $\underline{k}_{10}=(1,-2)$, $\underline{k}_{11}=(2,2)$, $\underline{k}_{12}=(2,-2)$. The external parameter R is the Reynolds number. We refer to [6, 19, 20] for details on the derivation of the truncated equations and for an exhaustive description of the numerical techniques useful to investigate such systems. We report here only on the phenomenology associated with tori. A complete picture of the model will be presented elsewhere.

Figure 1 represents a graphical summary of the phenomenology for the parameter range we are interested in. A stable periodic orbit K, arising via Hopf bifurcation, generates an attracting torus $T(K)$ for $R \cong 47.82$. At $R \cong 45.77$ a pair of closed orbits W and W^*, the former stable, the latter unstable, appears as a consequence of a tangent bifurcation. At $R \cong 48.71$ W also produces an attracting torus, $T(W)$. Therefore, after this value of R two distinct tori coexist, each of them with its own basin of attraction. For $R \cong 49.44$ two new pairs of stable-unstable periodic orbits arise via tangent bifurcation, (Z_i, Z_i^*), $i=1,2$[1]. At $R=49.498$ the torus $T(W)$, (or, more precisely, $T(W)$ changed into a strange attractor) can be last seen. $T(W)$ disappears because of a collision with the unstable Z_i^*'s and gives its basin of attraction to the stable Z_i's. These last orbits undergo a first period-doubling bifurcation at $R \cong 49.60$ and a second one at $R \cong 49.69$. The sequence of period-doublings, presumably infinite, leads to the formation of two symmetric strange attractors. Their size grows until they merge, giving rise to what seems to be a single strange attractor. As R increases further, it keeps growing and finally disappears at $R \cong 50.33$, colliding with the unstable periodic orbit W^*. So the torus $T(K)$ remains the only attractor present. At $R=50.709$ also $T(K)$ is destroyed; the cause is again a crisis, again due to a collision with W^*. For larger values of the parameter, a wider strange attractor appears to be responsible for the whole behavior of the system, except for possible short periodic windows.

3. Breaking of Tori and Crisis

Now, let us describe the details of the process which leads to the destruction of the torus $T(K)$. In order to follow the evolution of $T(K)$ as R increases, we construct a Poincaré map for the torus by intersecting the flow with the hyperplane $x_{12}=-1.1$. Figure 2 shows a plane projection of 2000 points, together with the three intersections of the periodic orbit W^* with the same hyperplane, for $R=50$, $R=50.65$, and $R=50.708$. The picture gives clear evidence of the fact that the torus and the periodic orbit approach each other. As this happens the torus develops three corners, each of them associated with an intersection of W^*. These corners, rather rounded at first, tend to become more pronounced as R approaches the collision point. Figure 3, representing an enlargement of the same corner for $R=50.65$ and $R=50.708$, illustrates what happens on a finer scale. While for $R=50.65$ the section curve appears completely smooth, for $R=50.708$ it clearly shows small foldings, with a point which tends to the intersection of the orbit associated with that corner. Figure 4, representing a further enlargement of the squared region of Fig. 3b, indicates an underlying complicated structure. Such a structure, observed also at the other corners (Fig. 5) and, as far as we can see, only there, gives evidence that for $R=50.708$ the attractor is no longer a torus, but it has become a strange attractor, even if weakly chaotic. This is confirmed by the computation of the Liapunov exponents, one of which is positive. On the contrary, for $R=50.65$ the two largest Liapunov exponents are both clearly zero, which indicates we are still in the presence of a torus. We are not able to define exactly when the breaking of the torus takes place. A detailed investigation of the Poincaré section indicates that the attractor is still a torus at $R=50.703$, it is a 29-point periodic cycle due to phase locking at $R=50.704$ and $R=50.705$, it is a strange attractor at $R=50.706$. It is then likely that the torus breaks down while the motion is phase-locked.

1 The periodic orbits Z_1 and Z_2, as well as Z_1^* and Z_2^*, are mutual images one of the other under the symmetry which changes the sign of the coordinates x_2, x_3, x_4, x_7, and x_{10}, and leaves invariant the remaining ones. On the contrary, K and W, and then the associated W^*, $T(K)$ and $T(W)$, are left unchanged by application of the same symmetry



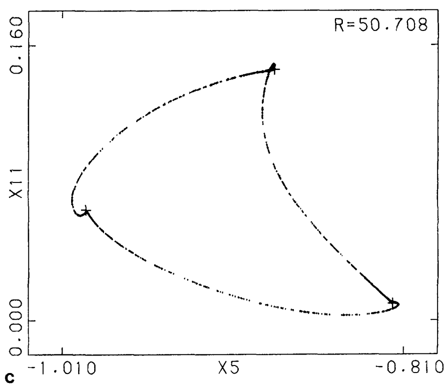


Fig. 2a–c. Plane projection of 2000 intersections of a trajectory on the torus $T(K)$ for **a** $R=50$, **b** $R=50.65$, and **c** $R=50.708$. The three crosses represent the intersections of the unstable periodic orbit W^*

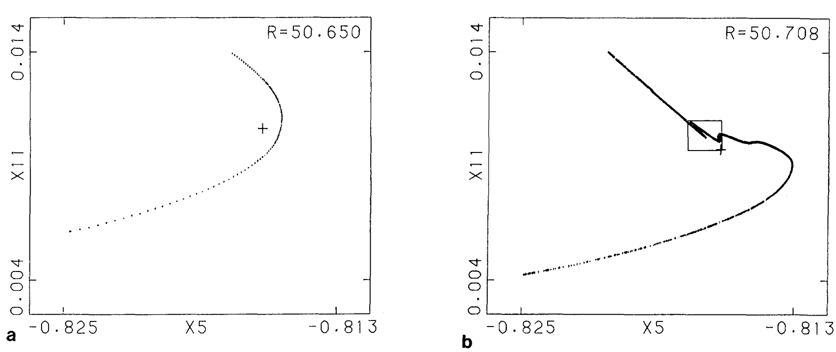


Fig. 3. a Enlargement of the rightmost corner of Fig. 2b; **b** enlargement of the same corner of Fig. 2c after addition of 8000 more intersection points

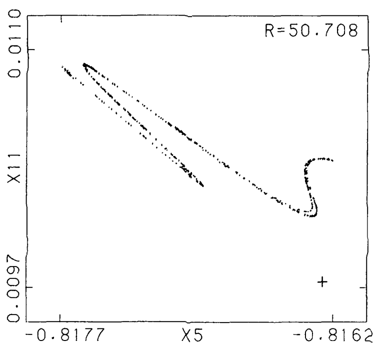


Fig. 4. Enlargement of the squared region of Fig. 3b

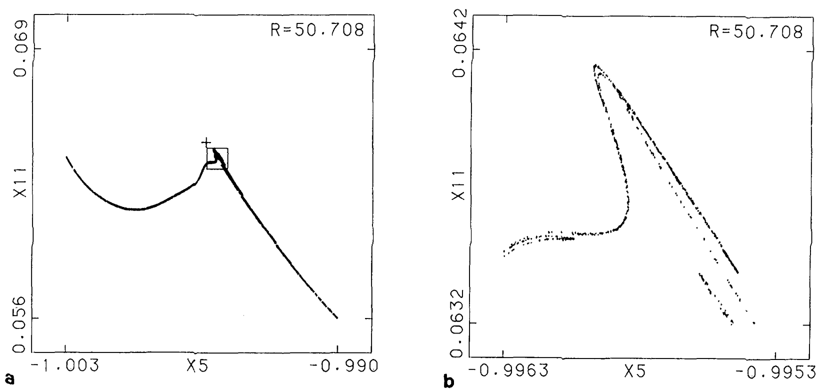
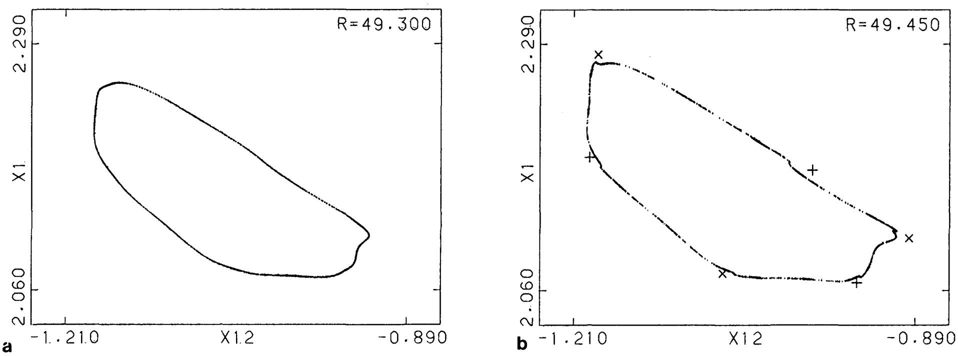


Fig. 5. Successive enlargements of the leftmost corner of Fig. 2c, also in this case after addition of 8000 extra points

As far as the torus $T(W)$ is concerned, its evolution appears completely analogous to that of $T(K)$, though complicated by phase locking. Such an evolution is displayed by Fig. 6, which shows the same Poincaré section for the torus and the periodic orbits Z_i^* for three different values of the parameter. Part a) represents $T(W)$ for $R=49.30$. Parts b) and c) depict $T(W)$ for $R=49.45$ and $R=49.49$, i.e. respectively just after the appearance of the Z_i's and Z_i^*'s and a little before it disappears. The enlargements in Fig. 7 make striking the analogy with the behavior of $T(K)$. Also in this case a positive Liapunov exponent corroborates the impression one gets by looking at the foldings of the intersection curve: we are dealing with a strange attractor. On the other hand, for $R=49.403$, $T(W)$ is still a torus because the two largest Liapunov exponents are both zero. This gives further evidence of the role played by the appearance of neighbouring unstable periodic orbits in the transition torus-strange attractor. As said before, a complication in following $T(W)$ during its evolution is the appearance on it of periodic orbits due to phase locking. We find orbits of period 175, 17, and 23, the period being the number of distinct intersections with a Poincaré section. Period 175 is found for $R=49.400$, period 17 in the parameter range (49.408, 49.447), and period 23 exists



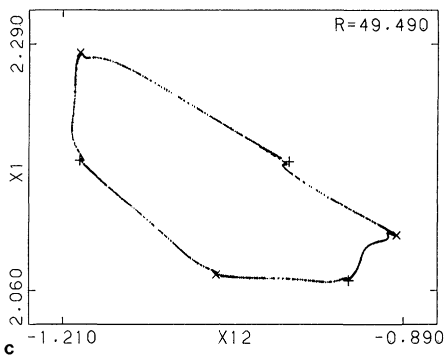


Fig. 6a–c. Plane projection of 2000 points of a Poincaré section for the torus $T(W)$ relative to the hyperplane $x_2=0$ for **a** $R=49.30$, **b** $R=49.45$, and **c** $R=49.49$. The symbols $+$ and $\times$ are used to represent the three intersections of Z_1^* and Z_2^*

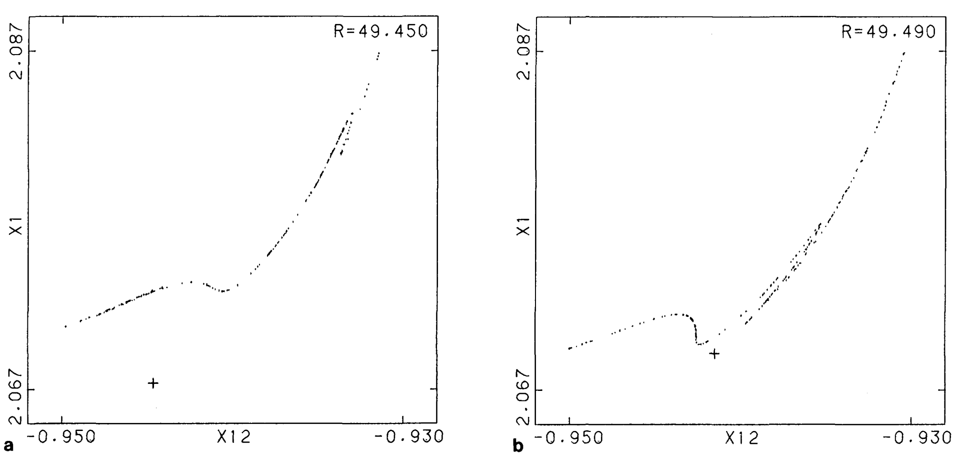


Fig. 7a and b. Enlargement of the lower right corner of **a** Fig. 6b and **b** Fig. 6c

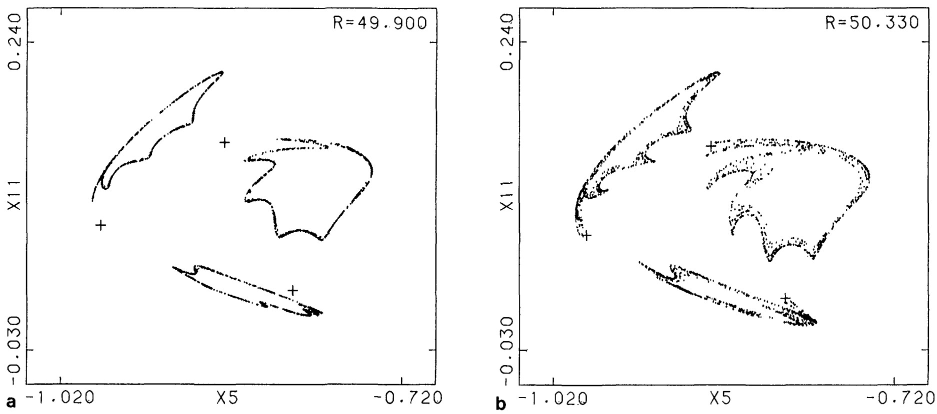


Fig. 8a and b. Plane projection of 2000 intersection points for the strange attractor present at **a** $R=49.90$ and **b** $R=50.33$. The hyperplane for the Poincaré map is $x_{12}=-1.1$. Differently from $T(K)$, the strange attractor has three distinct sections with this hyperplane. The three crosses represent the intersections of the unstable periodic orbit W^*

first for $R=49.468$ and then in the range (49.499, 49.503). For values of R larger than 49.503 no trace of $T(W)$ can be found. We remark that in this case both breaking of the torus and disappearance of the strange attractor seem to occur while the motion is phase-locked. Furthermore, we notice that for $R>49.45$ phase locking takes place on a strange attractor. This feature is not uncommon and has been previously observed in [6].

A third event of crisis is responsible for the disappearance of the strange attractor which follows the sequence of period-doublings related to the periodic orbits Z_i. Figure 8 gives evidence for the approach of the strange attractor to the unstable periodic orbit W^*. This provides a clear example of a crisis in differential equations completely in line with [17, 18]. The two cases previously described have instead the peculiarity of being essentially crises of tori. In fact they are associated with a strange attractor which has just arisen from a torus and substantially retains its geometrical shape.

Two kinds of crisis are discussed in [17, 18]: interior crisis and boundary crisis. The former takes place when the collision occurs within the basin of attraction, the latter when the unstable orbit is on the boundary of the basin. All our three crises are boundary crises. We have numerical evidence that the stable manifold of the unstable orbit W^* separates the basin of attraction of $T(K)$ from that of the coexisting attractors (see Fig. 1)[2]. Analogously, the stable manifolds of the Z_i^*'s divide the basin of attraction of $T(W)$ from the ones of the coexisting Z_i's. This analysis, besides providing a better definition of our crises, completes the description of the whole phenomenology of the system in the range of interest here.

2 A stable periodic orbit coexists with $T(K)$ when it disappears. This orbit has a long period and is present in a narrow parameter range around 50.708. The stable manifold of W^* separates the basins of the two attractors. Just after the disappearance of $T(K)$, the periodic orbit undergoes period-doubling and leads to a strange attractor

4. Another Example

We have also considered the 6-mode truncation of the Navier-Stokes equations studied by Riela in [8]. The system of differential equations considered there is the following:

$$
\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5 , \\
\dot{x}_2 &= -9x_2 + 3x_1x_3 , \\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + R , \\
\dot{x}_4 &= -5x_4 - x_1x_5 , \\
\dot{x}_5 &= -x_5 - 3x_1x_4 + \sqrt{5}x_1x_6 , \\
\dot{x}_6 &= -x_6 - \sqrt{5}x_1x_5 .
\end{aligned}
\tag{2}
$$

The motivation for further investigation of this model is given by a comparison of Fig. 2 in [8] with ours, suggesting that the process leading the torus to disappear is analogous. Evidence of this fact is given in Fig. 9, where a section of the torus is represented together with two neighbouring unstable periodic orbits. These orbits, which are symmetric with respect to the torus, arise via tangent bifurcation for $R \simeq 90.95$ at the same time as the two stable periodic orbits observed in [8] to coexist with the torus until, at $R \simeq 91.062$, it disappears. The absence of foldings in the intersection curve and, more conclusively, the Liapunov exponents, indicate that the torus disappears without evolving into a strange attractor. Although we cannot exclude a strange attractor on a practically undetectable scale or in a very narrow range of the parameter, the fact seems to represent a remarkable difference with the two cases in our system.

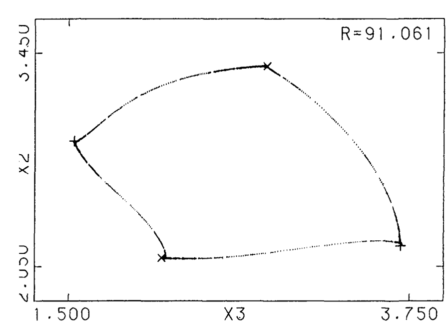


Fig. 9. Plane projection of a 1000-point section for the torus of system (2) for $R = 91.061$, i.e. just before its disappearance. As in [8] the hyperplane for the Poincaré map is $x_4 = 1.6$. The symbols $+$ and $\times$ represent the intersections of the two symmetric unstable periodic orbits responsible for the crisis

5. Return Maps

The dynamics on a torus can be studied also through a return map, that is a map of the circle into itself. To construct such a map, we parametrize the closed curve γ, intersection of the torus with the hyperplane of the Poincaré map, by a curvilinear coordinate s, $0 \leqq s \leqq 1$, and to each s we associate the abscissa $M(s)$ according to the Poincaré map. As long as the torus exists and is smooth, the return map is invertible and smooth. If, in some way, the torus breaks down, the return map must develop an inflection point or a singularity (see, for example, Gallavotti [21]).

From the computational point of view, the problem of constructing a return map lies in the parametrization of a closed curve γ known in a set of points $\{P_k\}$, $k=1, \ldots, N$, which are not ordered with respect to a fixed way of running along γ. Our algorithm to put theP_k's in order is the following. First of all, to each P_k we associate an angle θ, $0 \leqq \theta \leqq 2\pi$, by considering a plane projection of γ and a reference point interior to it. This induces immediately an order. Let us rename $\{Q_k\}$ the sequence of the points ordered with increasing angles. However, if the projection we used is not a well-shaped closed curve, i.e. without loops or foldings, such an order is incorrect. So, we always proceed to reorder the Q_k's through a second step. Let θ_1 be the angular coordinate of Q_1 and M a suitable integer with $M < N$. Among the points $Q_2, Q_3, \ldots, Q_{1+M}$, we look for that having the least euclidean distance from Q_1. If this point is different from Q_2, we exchange it with Q_2. Then we consider $Q_3, Q_4, \ldots, Q_{2+M}$ and we make Q_3 to become the point with least distance from Q_2. And so on. If Q_1 was properly chosen and N is sufficiently large, this criterium of minimum distance works and the sequence $\{Q_k\}$ at the end represents the points in the right order. By putting $s_1 = 0$ and $s_i = s_{i-1} + d(Q_{i-1}, Q_i)$, $i = 2, \ldots, N$, d being the distance, we associate to each P_k a curvilinear coordinate. This provides, after normalization, the desired parametrization of γ.

We have studied return maps for all the three tori previously discussed, in the cases of $T(K)$ and the torus of system (2) with special care. Figure 10, corresponding to $T(K)$ at $R = 50.703$, shows how one of these maps looks. A feature, common to all cases, reflects the fact that the tori draw near periodic orbits. As the collision point is approached, the maps behave as if they tended to develop periodic orbits with the same periodicity of the neighbouring unstable periodic orbits. Evidence of this is given for $T(K)$ in Fig. 11, representing the return map at $R = 50$, $R = 50.65$, and $R = 50.703$, this last being the largest value of R for which $T(K)$ is found as a torus. If we look at the return map to try to understand the cause of the breaking of $T(K)$, nothing conclusive can be stated. A loss of smoothness in a few points seems to be a plausible cause. However one fact is clear: no well defined inflection point is present.

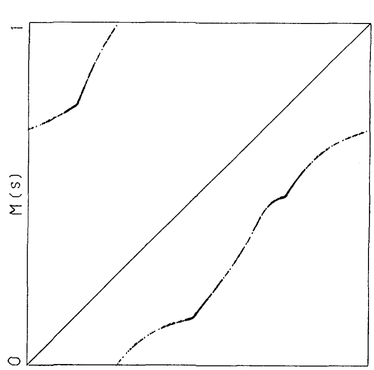
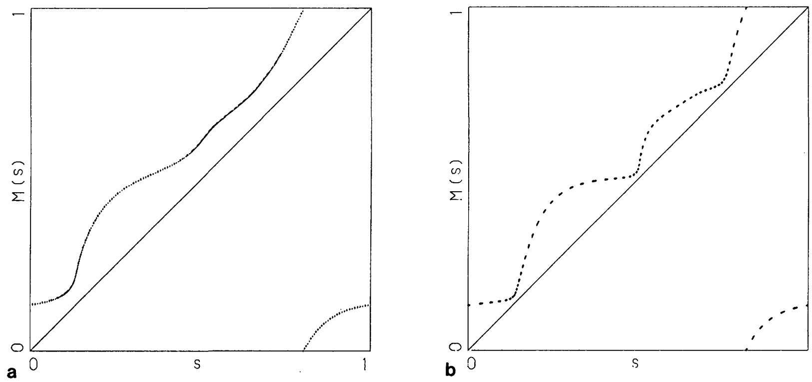


Fig. 10. Return map for $T(K)$ at $R = 50.703$



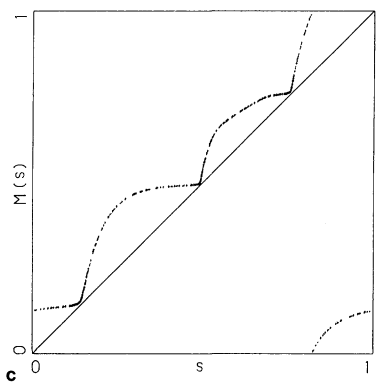


Fig. 11a–c. Return maps for $T(K)$, constructed by using the third iterate of the Poincaré map, at **a** $R=50$, **b** $R=50.65$, and **c** $R=50.703$

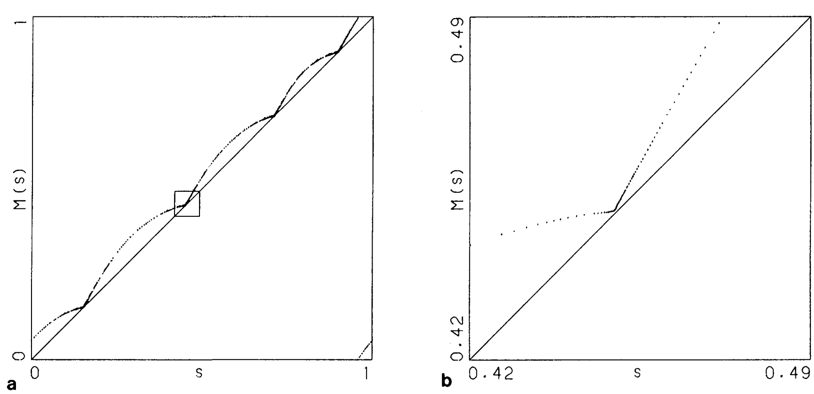


Fig. 12. a Return map, associated with the second iterate of the Poincaré map, for the torus of system (2) at $R=91.061$. **b** Enlargement of the squared part of **a**

A further remark about this subject has to be made. As said before, in order to construct a return map we must parametrize the closed curve γ. After the breaking of the torus, the section of the attractor is no longer a closed curve. So, also the parametrization is no longer possible. In practice, however, we parametrize a finite set of intersection points. As long as the points seem to belong to a closed curve, the parametrization and the associated return map we get are reliable. On the contrary, if the points do not lie on a closed curve, either for some reason we do not succeed in obtaining a parametrization, or we obtain a meaningless one with a return map which is expected to be clearly non one-to-one. Hence, the return map can be used also to establish whether an attractor is a torus or not. In agreement with this, the parametrization of the section of $T(K)$ fails at $R=50.706$ and, as far as the torus of system (2) is concerned, every return map turns out to be invertible and smooth up to $R=91.061$ (Fig. 12). In our opinion the use of return maps to distinguish between tori and weakly chaotic attractors bifurcated from tori is more efficient and requires less computer time than the use of the Liapunov exponents. In fact, as it is well known, near a bifurcation point the Liapunov exponents converge very slowly and it is often impossible to establish whether any of them is zero or very small.

6. Conclusion

To conclude, let us make some more comments about the results of our numerical investigation. We have described in details, as much as we have been able, two interesting phenomena: a transition torus-strange attractor and the disappearance of an attractor by crisis.

Concerning the latter phenomenon, we have given strong evidence for its occurrence in differential equations, to our knowledge for the first time in more than three dimensions. However, the fact which appears more relevant is a strict connection of events like crisis with tori. We have provided two different examples in which a weakly chaotic attractor, just arisen from a torus in consequence of its breaking, is destroyed by the collision with an unstable periodic orbit. A third example shows that even a torus can disappear in the same way.

We have illustrated two phenomena of breaking of a torus. Our analysis suggests three hypotheses, which seem to hold in both cases: i) the breaking is strictly connected to the presence of nearby unstable periodic orbits; ii) the breaking occurs when the flow on the torus is phase-locked; iii) the return map constructed on the section curve of the torus becomes non-smooth, rather than non-invertible. A transition torus-strange attractor with these characteristics seems to be essentially only one possible step during a process which leads the torus to destruction. In spite of this, we think that our phenomenology is significant in a wider context and provides a further small contribution to the understanding of the extremely complicated behavior of tori in dynamical systems.

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