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The Algebraic Complete Integrability of Geodesic Flow on $SO(N)$

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Abstract. We study for which left invariant diagonal metrics λ on $SO(N)$, the Euler-Arnold equations

$$\dot{X}=[X,\lambda(X)],\, X=(x_{ij})\in so(N),\, \lambda(X)_{ij}=\lambda_{ij}x_{ij},\, \lambda_{ij}=\lambda_{ji}$$

can be linearized on an abelian variety, i.e. are solvable by quadratures. We show that, merely by requiring that the solutions of the differential equations be single-valued functions of complex time $t\in\mathbb{C}$, suffices to prove that (under a non-degeneracy assumption on the metric λ) the only such metrics are those which satisfy Manakov's conditions $\lambda_{ij}=(b_i-b_j)\,(a_i-a_j)^{-1}$. The case of degenerate metrics is also analyzed. For $N=4$, this provides a new and simpler proof of a result of Adler and van Moerbeke [3].

Introduction

Recently the question of understanding the complete integrability (or the non-integrability) of a Hamiltonian system has regained considerable interest. For example, Adler and van Moerbeke [2, 3] have discussed and used a criterion to decide what they propose to call the algebraic complete integrability of a Hamiltonian system and, in a completely different vein, Ziglin [16, 17] has proved the (global) non-analytic integrability of the motion of a rigid body around a fixed point in the presence of gravity except in the three famous well known cases of Euler, Lagrange and Kowalewski (see also Holmes and Marsden [11]). One of the most fascinating common features of all these investigations is the connection between the question of the complete integrability of a Hamiltonian system and the behaviour of its solutions as functions of *complex* time. Up to now, this connection is not understood in general. In this note, we provide an interesting new example of this connection by showing that, merely by requiring that the solutions of the differential equations be single-valued functions of $t\in\mathbb{C}$, suffices to

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single out the left invariant diagonal metrics on $SO(N)$ for which the Euler-Arnold equations, associated to geodesic motion on $SO(N)$, form an algebraically completely integrable system (in the sense of [3]).

The concept of algebraic complete integrability is a geometric requirement which explains why, in many examples, most solutions of a Hamiltonian completely integrable system have to be meromorphic in $t \in \mathbb{C}$ and in fact, must blow up after a finite complex time. Namely, given a completely integrable system with polynomial invariants, it happens quite often that the real invariant tori obtained from Liouville's theorem are part of abelian varieties (=complex algebraic tori) and the flows (run with complex time) generated by the constants of the motion are straight lines on these abelian varieties. When this does happen, the phase variables, restricted to a generic complex level manifold defined by the intersection of the constants of the motion, are meromorphic (abelian) functions on a compact complex torus, which forces them to blow up along a codimension one divisor. Hence, the solutions of the differential equations admit Laurent series expansions in $t \in \mathbb{C}$ near this divisor and the system possesses "large" families of complex pole solutions (depending on "dim phase space -1" free parameters). As was shown by Adler and van Moerbeke [2, 3], the existence of such pole solutions can be exploited as a criterion to decide the algebraic complete integrability of a Hamiltonian system. In [3], the criterion is used to detect the algebraically completely integrable geodesic flows on $SO(4)$ for a left invariant diagonal metric.

Now, trying to apply this method to the geodesic flow on $SO(N)$ leads to formidable computations which are in fact impossible to carry out even with the help of a computer! To understand this, it is important to realize that the algebraic complete integrability hypothesis is an assumption on the "general" solution of the differential equations; namely, any solution with initial condition on a generic complex level manifold defined by the intersection of the constants of the motion, must blow up in a meromorphic fashion. In Sect. 1, we prove that such a generic assumption certainly implies that all solutions of the differential equations have to be single-valued functions of $t \in \mathbb{C}$. Now, it turns out that the Euler-Arnold equations associated to the geodesic flow on $SO(N)$ possess a lot of linear invariant manifolds on which they reduce to geodesic flow on $SO(3)$. So, the solutions of the differential equations with initial conditions on these invariant manifolds are elliptic functions and this, without any condition on the metric. Surprisingly, looking at solutions near these special a priori known solutions and imposing these solutions to be single-valued functions of $t \in \mathbb{C}$, will suffice to single out the algebraically completely integrable geodesic flows on $SO(N)$ for a left invariant diagonal metric. For the sake of clarity we shall argue in two steps: Sect. 3 deals with the case $N=4$ while the general case will be treated in Sect. 4 by restricting the flow to various linear invariant manifolds on which it reduces to geodesic flow on $SO(4)$. As a by-product, this will provide a very simple new proof of Adler-van Moerbeke's result [3]. In Sect. 2, we recall some more or less well known results concerning a class of algebraically completely integrable geodesic flows on $SO(N)$ first discovered by Manakov [12] and in Sect. 4, it is proved that for $N \geqq 5$, Manakov's metrics are the only left invariant diagonal metrics on $SO(N)$ for which the Euler-Arnold equations are algebraically completely integrable.

1. Algebraic Complete Integrability and Single-Valuedness of Solutions

In this section we prove that all solutions of an algebraically completely integrable system (in the sense of Adler and van Moerbeke [3]) have to be single-valued functions of $t \in \mathbb{C}$. By this we mean that, if it is possible to perform the analytic continuation of a solution along some closed path in the complex t-plane, the result must be a single-valued function of $t \in \mathbb{C}$. As emphasized in the introduction, merely this property will suffice to single out the algebraically completely integrable geodesic flows on $SO(N)$ for a left invariant diagonal metric. But, first of all, we need to explain the concept of algebraic complete integrability.

Consider

$$\dot{x} = J\frac{\partial H}{\partial x};\, x \in \mathbb{R}^m, \qquad m = 2n+k, \qquad J(x) \text{ polynomial in } x, \tag{1.1}$$

a Hamiltonian completely integrable system with $n+k$ functionally independent invariants $H_1, \ldots, H_{n+k}$ of which k lead to zero vector fields $J\left(\frac{\partial H_{n+j}}{\partial x}\right) = 0$ $(j = 1, \ldots, k)$ [1], the n remaining ones being in involution $(\{H_i, H_j\} = 0)$. By the Arnold-Liouville theorem, if the invariant manifolds $\bigcap_{i=1}^{n+k} \{H_i = c_i\} \subset \mathbb{R}^m$ are compact then, for most values of $c_i \in \mathbb{R}$, their connected components are diffeomorphic to real tori $\mathbb{R}^n/\text{Lattice}$ and the flows $\varphi_i^t(x)$ defined by the vector fields X_{H_i} $(i = 1, \ldots, n)$ are straight lines on these tori.

Let now $x \in \mathbb{C}^m$, $t \in \mathbb{C}$. By the functional independence of the integrals, the map

$$I : (H_1, \ldots, H_{n+k}) : \mathbb{C}^m \to \mathbb{C}^{n+k} \tag{1.2}$$

is submersive [i.e. $dH_1(x), \ldots, dH_{n+k}(x)$ are linearly independent] on a non-empty Zariski open set $S \subset \mathbb{C}^m$. Let $\Delta = I(\mathbb{C}^m \backslash S)$ be the set of critical values of I, i.e.

$$\Delta = \{c = (c_i) \in \mathbb{C}^{n+k} | \exists x \in I^{-1}(c) \quad \text{with} \quad dH_1(x), \ldots, dH_{n+k}(x)\} \tag{1.3}$$
$$\text{linearly dependent}$$

and denote by $\bar{\Delta}$ the Zariski closure of Δ in $\mathbb{C}^{n+k}$. Following Adler and van Moerbeke [3], we define

Definition 1.1. The system (1.1) will be called *algebraically completely integrable with abelian functions* x_i *(in short* a.c.i.) when, for every $c \in \mathbb{C}^{n+k} \backslash \bar{\Delta}$, the fibre $A_c = I^{-1}(c)$ is the affine part of an abelian variety $\tilde{A}_c \approx \mathbb{C}^n/L_c$ and moreover, the flows $\varphi_i^t(x)$, $x \in A_c$, $t \in \mathbb{C}$, defined by the vector fields X_{H_i} $(i = 1, \ldots, n)$ are straight lines on $\mathbb{C}^n/L_c$, i.e.

$$[\varphi_i^t(x)]_j = f_j(p + t(k_1^i, \ldots, k_n^i)) \tag{1.4}$$

with $f_j(t_1, \ldots, t_n)$ abelian (meromorphic) functions on $\mathbb{C}^n/L_c$, $f_j(p) = x_j$ $(j = 1, \ldots, m)$.

From (1.4), it is clear that the algebraic complete integrability assumption implies that any solution of (1.1) (or of any other commuting vector field X_{H_i}) with initial condition on a non-critical level manifold $A_c = I^{-1}(c)$ is a meromorphic function of $t \in \mathbb{C}$ and thus, *a fortiori*, it is a single-valued function of $t \in \mathbb{C}$. That this

1 Usually, Hamiltonian completely integrable systems possess trivial invariants

forces in fact all solutions of (1.1) to be single-valued functions of $t \in \mathbb{C}$ will now be established but, before this, we need some Lemmas.

Lemma 1.2. *The set*

$$A = \{x \in \mathbb{C}^m | I(x) \in \mathbb{C}^{n+k} \setminus \bar{\Delta}\} \tag{1.5}$$

is a non-empty Zariski open set in $\mathbb{C}^m$. Hence A is everywhere dense in $\mathbb{C}^m$ for the usual topology.

Proof. Since a polynomial map is continuous for the Zariski topology, $A = I^{-1}(\mathbb{C}^{n+k} \setminus \bar{\Delta})$ is certainly a Zariski open set in $\mathbb{C}^m$. Suppose it is empty, i.e. $I(\mathbb{C}^m) \subset \bar{\Delta}$. By the functional independence of the integrals, the map I is submersive on a non-empty Zariski open set $S \subset \mathbb{C}^m$ and thus $I(S)$ is open in $\mathbb{C}^{n+k}$. Now, by Sard's lemma for varieties [13, p. 42], $\mathbb{C}^{n+k} \setminus \bar{\Delta}$ is a non-empty Zariski open set (hence everywhere dense for the usual topology) in $\mathbb{C}^{n+k}$. So, $I(S) \cap (\mathbb{C}^{n+k} \setminus \bar{\Delta}) \neq \phi$, a contradiction. This proves Lemma 1.2.

Lemma 1.3. *Suppose you have a system*

$$\dot{x} = f(x);\ x \in \mathbb{C}^m,\ f \text{ holomorphic on } \mathbb{C}^m, \tag{1.6}$$

such that all solutions of (1.6) *with initial conditions in a dense set $A \subset \mathbb{C}^m$ are (analytic) single-valued functions of $t \in \mathbb{C}$. Then*

(i) *all solutions of* (1.6) *are single-valued functions of $t \in \mathbb{C}$,*

(ii) *if $\varphi(t)$ is a particular solution of* (1.6) *holomorphic along some closed path l in the complex t-plane, the analytic continuation along l of any solution of the variational (linearized) equations:*

$$\dot{\delta} = \frac{\partial f}{\partial x}(\varphi(t))\delta \tag{1.7}$$

has to be single-valued.

Proof. (i) Let $x^0 \in \mathbb{C}^m$, $x^0 \notin A$ and $\varphi(t, x^0)$ be the solution of (1.6) with $\varphi(0, x^0) = x^0$ holomorphic in t for $|t|$ small enough. Let l_1, l_2 be two paths from 0 to $w \in \mathbb{C}$, $w \neq 0$, along which this solution can be analytically continued and denote by $\varphi^i(t, x^0)$ $(i = 1, 2)$ these analytic continuations. Let also V_i $(i = 1, 2)$ be simply connected neighborhoods of l_i $(i = 1, 2)$ in which $\varphi^i(t, x^0)$ are holomorphic as in Fig. 1. By the theorem of analytic dependence of initial conditions [5, Theorem 8.2, p. 35], there exists then a ball $B(x^0, r) \subset \mathbb{C}^m$ of center x^0 and radius r such that for every $x \in B(x^0, r)$ the system (1.6) has solutions $\varphi^i(t, x)$ with $\varphi^i(0, x) = x$ holomorphic in $t \in V_i$ $(i = 1, 2)$ and $x \in B(x^0, r)$. Since A is dense, we can find a sequence of points

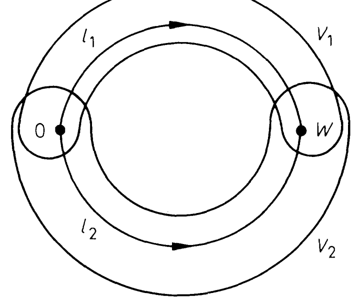

Fig. 1

$x^k = x^0 + \varepsilon_k u^k$, $\|u^k\| = 1$, $x^k \in A$ which converges to x^0 and so, for k big enough, the system (1.6) has solutions $\varphi^i(t, x^0 + \varepsilon_k u^k)$ with $\varphi^i(0, x^0 + \varepsilon_k u^k) = x^0 + \varepsilon_k u^k = x^k$, holomorphic in V_i ($i = 1, 2$), which can be expanded in convergent series:

$$\varphi^i(t, x^0 + \varepsilon_k u^k) = \varphi^i(t, x^0) + \varepsilon_k \sum_{s=1}^{m} \frac{\partial \varphi^i}{\partial x_s}(t, x^0) u_s^k + O(\varepsilon_k^2).$$

By hypothesis, since $x^k = x^0 + \varepsilon_k u^k \in A$, one has

$$0 \equiv \varphi^1(t, x^k) - \varphi^2(t, x^k) = \varphi^1(t, x^0) - \varphi^2(t, x^0) + O(\varepsilon_k)$$

in some neighborhood of w.

Letting $k \to \infty$, $\varepsilon_k \to 0$ and since $|u_s^k| \leqq 1$, this obviously implies:

$$\varphi^1(t, x^0) - \varphi^2(t, x^0) \equiv 0 \text{ in some neighborhood of } w$$

which proves (i).

(ii) Consider now a [necessarily single-valued by (i)] solution $\varphi(t, x^0)$ of (1.6) with $\varphi(0, x^0) = x^0$, which can be analytically continued along two paths l_1 and l_2 as in Fig. 1. Given $\delta^0 \in \mathbb{C}^m$, δ^0 arbitrary, the system (1.6) has, for ε small enough, solutions $\varphi^i(t, x^0 + \varepsilon\delta^0)$ with $\varphi^i(0, x^0 + \varepsilon\delta^0) = x^0 + \varepsilon\delta^0$ holomorphic in V_i ($i = 1, 2$) which can be expanded in convergent series:

$$\varphi^i(t, x^0 + \varepsilon\delta^0) = \varphi^i(t, x^0) + \varepsilon\delta^i(t, \delta^0) + O(\varepsilon^2),$$

where $\delta^i(t, \delta^0)$ are the solutions of the variational equations (1.7) in V_i such that $\delta^i(0, \delta^0) = \delta^0$. By (i), for ε small enough, one has

$$0 \equiv \varphi^1(t, x^0 + \varepsilon\delta^0) - \varphi^2(t, x^0 + \varepsilon\delta^0) = \varepsilon(\delta^1(t, \delta^0) - \delta^2(t, \delta^0)) + O(\varepsilon^2)$$

in some neighborhood of w which implies $\delta^1(t, \delta^0) - \delta^2(t, \delta^0) \equiv 0$ in some neighborhood of w.

This finishes the proof of Lemma 1.3.

Corollary 1.4. *If* (1.1) *is algebraically completely integrable with abelian functions* x_i, *then the conclusions* (i) *and* (ii) *of Lemma 1.3 hold.*

Proof. By the algebraic complete integrability assumption, every solution of (1.1) with initial condition in the set A (1.5) is a meromorphic function of $t \in \mathbb{C}$ and thus, a fortiori, a single-valued function of $t \in \mathbb{C}$. By Lemma 1.2, A is everywhere dense in $\mathbb{C}^m$. This proves Corollary 1.4.

Conjecture. If (1.1) is a.c.i. with abelian functions x_i, then all its solutions are meromorphic functions of $t \in \mathbb{C}$, i.e. the analytic continuation of any solution of (1.1) can lead to *at worst* pole singularities.

2. Manakov's Geodesic Flows on $SO(N)$

From now on until the end of this note, we shall be concerned with the following system of differential equations:

$$\dot{X} = [X, \lambda(X)],$$

$$X = (x_{ij}) \equiv \sum_{i<j} x_{ij} e_{ij} \in so(N), \tag{2.1}$$

$$\lambda(X)_{ij} = \lambda_{ij} x_{ij}, \ \lambda_{ij} = \lambda_{ji}.$$

Defining

$$A_{ij}^{kl} = \lambda_{ij} - \lambda_{kl}\,, \tag{2.2}$$

(2.1) takes the explicit form

$$\dot{x}_{ij} = \sum_{k=1}^{N} A_{kj}^{ki} x_{ik} x_{kj}\,. \tag{2.3}$$

Equations (2.1) are the well known Euler-Arnold equations associated to geodesic motion on $SO(N)$ for a left invariant metric λ diagonal in the usual basis $(e_{ij})_{1 \leqq i < j \leqq N}$ of the Lie algebra $so(N)$. They form a Hamiltonian system with regard to the usual Kostant-Kirillov symplectic structure on each orbit in $so(N)^* \approx so(N)$ defined by setting the $\left[\frac{N}{2}\right]$ polynomial invariants $\operatorname{Tr} X^{2i}$ $\left(1 \leqq i \leqq \left[\frac{N}{2}\right]\right)$ equal to generic constants. Functions H defined on the orbit lead to Hamiltonian vector fields

$$\dot{X} = [X, \nabla H(X)]\,. \tag{2.4}$$

In particular $H = \frac{1}{2} \sum_{i<j} \lambda_{ij} x_{ij}^2$ leads to (2.1) while the orbit invariants $\operatorname{Tr} X^{2i}$ lead to identically zero vector fields (see [4, Appendix 2]). Since the dimension of the phase space is $\frac{N(N-1)}{2} - \left[\frac{N}{2}\right]$, one needs $\frac{1}{2}\left(\frac{N(N-1)}{2} - \left[\frac{N}{2}\right]\right)$ non-trivial invariants to make the system completely integrable. One such instance is known and was first discovered by Manakov [12], namely if the metric λ satisfies:

$$\lambda_{ij} = \frac{b_i - b_j}{a_i - a_j} \Leftrightarrow [X, b] + [a, \lambda(X)] = 0, \ \forall X$$

with (2.5)

$$a = \operatorname{diag}(a_1, \ldots, a_N),\ b = \operatorname{diag}(b_1, \ldots, b_N),\ \prod_{i<j} (a_i - a_j) \neq 0\,.$$

Then Eqs. (2.1) become equivalent to

$$(X + ah)^{\boldsymbol{\cdot}} = [X + ah, \lambda(X) + bh] \tag{2.6}$$

with a formal indeterminate h. Equation (2.6) is an example of what is now usually called a "Lax equation with a parameter." It means that, for each $h \in \mathbb{C}$, the spectrum of the matrix $X + ah$ is time independent or, what amounts to the same information, if we form the characteristic polynomial

$$\det(X + ah - zI) = Q_{00}^2(X) + \sum_{r=1}^{N-2} \sum_{i+j=r} Q_{ij}(X) z^i h^j + \prod_{k=1}^{N} (a_k h - z)\,, \tag{2.7}$$

the $Q_{ij}(X)$ are constants of the motion where, because of the antisymmetry of X,

$$\begin{aligned} &Q_{00}(X) \equiv \sqrt{\det X} = \text{pfaffian of } X\,, \\ &Q_{ij}(X) \equiv 0 \quad \text{for} \quad \begin{cases} i+j & \text{even when } N \text{ is odd} \\ i+j & \text{odd when } N \text{ is even}\,. \end{cases} \end{aligned} \tag{2.8}$$

An elementary count shows one produces in this way the right number ($=\frac{1}{2}$dim phase space) of non-trivial invariants. A nice proof of their independence and their involutivity can be found in [15].

Also, the flows $X_{Q_{ij}}$ ($j\neq 0$) can be linearized on an abelian variety as a consequence of a recent general theory [1, 9]. Namely, consider the spectral curve:

$$R:\{(z,h)\in\mathbb{C}^2|\det(X+ah-zI)=0 \quad \text{with} \quad Q_{ij}(X)=c_{ij}\}. \tag{2.9}$$

For generic $c=(c_{ij})$, R is a non-singular algebraic plane curve of degree N, so its genus is $g=\dfrac{(N-1)(N-2)}{2}$. Because of the antisymmetry of X, the map $\iota:R\to R:(z,h)\curvearrowright(-z,-h)$ is an involution on R with fixed points the N distinct points over $h=\infty$, together with the origin $(0,0)$ in the case N is odd. By the Riemann-Hurwitz formula, the quotient curve $R_0=R/\iota$ has genus:

$$g_0=\text{genus}(R_0=R/\iota)=\begin{cases}\dfrac{(N-2)^2}{4} & N \text{ even}\\[2ex] \dfrac{(N-1)(N-3)}{4} & N \text{ odd},\end{cases}$$

and so, the Prym variety $\text{Prym}_\iota R$ in the Jacobi variety of R has dimension $g-g_0=\frac{1}{2}\left(\frac{N(N-1)}{2}-\left[\frac{N}{2}\right]\right)=\frac{1}{2}\dim$ (phase space). Let now A_c be the complex affine variety defined by the intersection of the constants of the motion:

$$A_c=\bigcap_{i,j}\{Q_{ij}(X)=c_{ij}\}\subset\mathbb{C}^{\frac{N(N-1)}{2}}. \tag{2.10}$$

Clearly, by (2.7), the finite group action

$$\begin{gathered} G:X\curvearrowright DXD^{-1}\\ \text{with } D=\text{diag}(1,d_1,\dots,d_{N-1}), d_i=\pm 1, \prod_{i=1}^{N-1} d_i=1 \text{ when } N \text{ is even}^2 \end{gathered} \tag{2.11}$$

leaves A_c invariant. Let also $\varphi_j(1\leqq j\leqq g_0)$, $\psi_k(1\leqq k\leqq g-g_0)$ be a basis of holomorphic differentials on R (2.9) satisfying $\iota^*(\varphi_j)=\varphi_j$ and $\iota^*(\psi_k)=-\psi_k$. One has then the following

Theorem 1 [1, 6, 10]. *The system* (2.1) *with* λ *satisfying* (2.5) *is algebraically completely integrable with abelian functions* x_{ij}.

More precisely, there is a one-to-one algebraic map from the affine variety A_c/G *onto a Zariski open set* $\mathscr{U}\subset\text{Prym}_\iota(R)$:

$$A_c/G\xrightarrow{\sim}\mathscr{U}\subset\text{Prym}_\iota(R):X\in A_c/G\curvearrowright D=\sum_{i=1}^{g}\mu_i\in\mathscr{U},$$

such that any smooth Hamiltonian $H(Q_{ij}(X))$ *leads to a linear flow on* $\text{Prym}_\iota(R)$ *i.e.*

$$\frac{d}{dt}\sum_{i=1}^{g}\int_{0_i}^{\mu_i(t)}(\varphi_1,\dots,\varphi_{g_0},\psi_1,\dots,\psi_{g-g_0})=(0,\dots,0,k_1,\dots,k_{g-g_0}).$$

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2 When N is even, the restriction $\Pi d_i=1$ must be made so as to leave the pfaffian $Q_{00}(X)=\sqrt{\det X}$ invariant

Moreover,

$$\mathrm{Prym}_\iota(R)\backslash\mathscr{U}=\theta\cap\mathrm{Prym}_\iota(R)$$

with θ some translate of the θ-divisor in Jac(R).

Finally, the variety A_c (2.10) itself is the affine part of an abelian variety $\tilde{A}_c$ which is an unramified cover of degree 2^{N-1} (when N is odd) or 2^{N-2} (when N is even) of Prym$_\iota(R)$, *obtained by multiplying some periods of* Prym$_\iota(R)$ *by two.*

Remarks. 1. Theorem 1 is part of the following wider statement, a proof of which can be found in [1, 6]. Namely, consider

$$M_c=\{X\in gl(n,\mathbb{C})\left|\begin{array}{l}\mathrm{diag}(X)=0 \text{ and spectrum of } X+ah \text{ is fixed}\\ \text{for each } h\in\mathbb{C} \text{ with } a=\mathrm{diag}(a_1,\dots,a_N)\}\,.\end{array}\right.$$

On M_c, the group Π of invertible diagonal matrices acts by conjugation and it can be shown that $M_c/\Pi=\mathrm{Jac}R\backslash\{$a translate of the θ-divisor$\}$ with R the algebraic curve defined by $R:\det(X+ah-zI)=0$. Functions defined on M_c invariant under the action of Π (like $x_{ij}x_{ji}$) are thus abelian functions on Jac(R). Explicit formulae can be found in [7, p. 82]. In particular, the divisor structure of $x_{ij}x_{ji}$ is given by:

$$(x_{ij}x_{ji})=\theta_{ij}+\theta_{ji}-\theta_0^2 \tag{2.12}$$

with θ_0, θ_{ij} suitable translates of the θ-divisor of Jac(R).

2. In the skew-symmetric case, as explained above, the curve R has an involution $\iota:(z,h)\curvearrowright(-z,\ -h)$,the action of Π on the variety A_c (2.10) reduces to the action G defined in (2.11) and $A_c/G=\mathrm{Prym}_\iota(R)\backslash(\theta_0\cap\mathrm{Prym}_\iota(R))$. The fact that A_c itself is the affine part of an abelian variety $\tilde{A}_c$ covering Prym$_\iota(R)$ was first stated and proved by the author in [10] for $N=4$. In that case, the curve R (2.9) has genus 3, the variety A_c (2.10) is an intersection of 4 quadrics in $\mathbb{C}^6$ and the abelian variety $\tilde{A}_c$ can be identified as a Prym variety Prym$_\iota(C)$ of *another* genus 3 curve C nicely connected with the geometry of the situation and which is only isogenous to Prym$_\iota(R)$ (see [10, Theorem 4, p. 457]). As such this result does not extend to $N\geqq 5$. However it is easy to prove that A_c (2.10) is still the affine part of an abelian variety $\tilde{A}_c$ by using (2.12) and observing that

$$\theta_{ij}\cap\mathrm{Prym}_\iota(R)=\theta_{ji}\cap\mathrm{Prym}_\iota(R)\,,$$

which shows that the functions x_{ij}^2 restricted to A_c/G are perfect squares of abelian functions defined on an abelian variety covering Prym$_\iota(R)$.[3]

The two next sections of this note will be devoted to the proof that (2.5) are the only left invariant diagonal non-degenerate (i.e. all λ_{ij} distinct) metrics on $SO(N)$ for which (2.1) is algebraically completely integrable with abelian functions x_{ij}.

3. Geodesic Flow on $SO(4)$

In this section we provide a very simple new proof that Manakov's metrics (2.5) are the only left invariant non-degenerate diagonal metrics on $SO(4)$ for which (2.1) is algebraically completely integrable with abelian functions x_{ij} which was first

3 A similar situation arises in the Neumann problem, where $\tilde{A}_c$ is a covering of the Jacobi variety of a hyperelliptic curve and is discussed in full detail in [14, pp. 87–88]

proved in [3]. As explained in Sect. 1 (Corollary 1.4) it will be sufficient to prove that they are the only metrics for which the solutions of (2.1) are single-valued functions of $t \in \mathbb{C}$.

Theorem 2. *For $N=4$, if the solutions of* (2.1) *are single-valued (analytic) functions of $t \in \mathbb{C}$, then either*

$$\begin{aligned} K \equiv\ & \lambda_{12}\lambda_{34}(\lambda_{23}+\lambda_{14}-\lambda_{13}-\lambda_{24})+\lambda_{23}\lambda_{14}(\lambda_{13}+\lambda_{24}-\lambda_{12}-\lambda_{34}) \\ & +\lambda_{13}\lambda_{24}(\lambda_{12}+\lambda_{34}-\lambda_{23}-\lambda_{14})=0\,, \end{aligned} \tag{3.1}$$

or

$$\lambda_{12}=\lambda_{34} \quad \textit{and} \quad \lambda_{23}=\lambda_{14} \quad \textit{and} \quad \lambda_{13}=\lambda_{24}\,, \tag{3.2}$$

and this without any (non-degeneracy) assumption on the metric λ.

Remarks. 1. Conditions (3.1) and (3.2) were discovered by Adler and van Moerbeke in [3] using their a.c.i. criterion. In that paper, it is assumed that all λ_{ij} are distinct. Looking at the proof, it is sufficient to assume $\Pi(\lambda_{ij}-\lambda_{ik}) \neq 0$, which explains the occurrence of (3.2).

2. Condition (3.1) defines an algebraic variety in $\mathbb{C}^6$, a Zariski open set of which can be rationally parametrized by (2.5) and corresponds thus to Manakov's flows (see [3, p. 310, Lemma 3] or Lemma 4.1 in the next section). Under condition (3.2), Eqs. (2.1) ($N=4$) decouple into two copies of geodesic flow on $SO(3)$ in the variables $x_{32} \pm x_{41}$, $x_{13} \pm x_{42}$, $x_{21} \pm x_{43}$ corresponding to the well known decomposition $so(4)=so(3)\oplus so(3)$. So, in this case, the system linearizes on a product of two elliptic curves.

Proof of Theorem 2. From the explicit form of the differential Eqs. (2.3), it is easy to see that the system (2.1) ($N=4$) possesses the following five linear invariant manifolds:

$$\begin{gathered} \Gamma_i \subset \mathbb{C}^6 (1 \leqq i \leqq 5): \\ \Gamma_i \equiv \{x_{ij}=0,\ 1 \leqq j \leqq 4\} \quad (1 \leqq i \leqq 4)\,, \end{gathered} \tag{3.3}$$

and

$$\Gamma_5 \equiv \{x_{32}=x_{41}, x_{13}=x_{42}, x_{21}=x_{43}\}\,, \tag{3.4}$$

corresponding to five copies of $so(3)$ sitting in $so(4)$ on which (2.1) ($N=4$) reduces to geodesic flow on $SO(3)$. Hence, the solutions of (2.1) ($N=4$) restricted to Γ_i ($1 \leqq i \leqq 5$) are elliptic functions. By Lemma 1.3 (ii), if the solutions of (2.1) ($N=4$) have to be single-valued, the same must then hold for the solutions of the variational Eqs. (1.7) along the special a priori known $so(3)$ solutions. This will now be analyzed. For the sake of clarity, we shall first assume all $A_{ij}^{ik} \neq 0$.

Case 1. $\Pi A_{ij}^{ik} \neq 0$. Eqs. (2.1) ($N=4$) restricted to Γ_1 reduce to:

$$\begin{aligned} &\dot{x}_{23}=A_{42}^{43}x_{24}x_{34}; \quad &&\dot{x}_{24}=A_{34}^{32}x_{23}x_{34}; \quad &&\dot{x}_{34}=A_{23}^{24}x_{23}x_{24}; \\ &\dot{x}_{12}=0; &&\dot{x}_{13}=0; &&\dot{x}_{14}=0\,. \end{aligned} \tag{3.5}$$

As is well known, (3.5) is solved in terms of elliptic functions (depending on two free parameters) with four distinct simple poles in their smallest common period parallelogram. Around each of these poles, one has the following Laurent series expansions:

$$\begin{aligned} x_{23}(t,\alpha,\beta) &= t^{-1}(x_{23}^0 + (\alpha+\beta)x_{23}^0 t^2 + \ldots),\\ x_{24}(t,\alpha,\beta) &= t^{-1}(x_{24}^0 - \alpha x_{24}^0 t^2 + \ldots),\\ x_{34}(t,\alpha,\beta) &= t^{-1}(x_{34}^0 - \beta x_{34}^0 t^2 + \ldots), \end{aligned} \tag{3.6}$$

with

$$\begin{gathered} (x_{23}^0)^2 = \frac{1}{A_{23}^{24}A_{34}^{32}};\quad (x_{24}^0)^2 = \frac{1}{A_{23}^{24}A_{42}^{43}};\quad (x_{34}^0)^2 = \frac{1}{A_{34}^{32}A_{42}^{43}};\\ A_{23}^{24}A_{34}^{32}A_{42}^{43}x_{23}^0 x_{24}^0 x_{34}^0 = -1\,. \end{gathered} \tag{3.7}$$

The variational Eqs. (1.7) along the solutions (3.6) decouple into:

$$\begin{aligned} \dot{\delta}_{12} &= A_{31}^{32}x_{23}(t,\alpha,\beta)\delta_{13} + A_{41}^{42}x_{24}(t,\alpha,\beta)\delta_{14},\\ \dot{\delta}_{13} &= A_{23}^{21}x_{23}(t,\alpha,\beta)\delta_{12} + A_{41}^{43}x_{34}(t,\alpha,\beta)\delta_{14},\\ \dot{\delta}_{14} &= A_{24}^{21}x_{24}(t,\alpha,\beta)\delta_{12} + A_{34}^{31}x_{34}(t,\alpha,\beta)\delta_{13}, \end{aligned} \tag{3.8}$$

and three other equations.

Now (for each α, β) Eqs. (3.8) form a linear system of differential equations with regular singular points. According to the classical Fuchsian theory of such systems [5, Chap. 4], to have single-valuedness of the solutions, all the roots of the indicial equation have to be integers[4]. Using (3.7), a simple computation shows that the indicial equation is given by:

$$(1-r)\left(r^2 + r - \frac{K}{A_{24}^{23}A_{32}^{34}A_{43}^{42}}\right) = 0,$$

with K defined by (3.1). So we must have

$$K = A_{24}^{23}A_{32}^{34}A_{43}^{42}r_1(r_1+1);\quad r_1 \in \mathbb{Z}. \tag{3.9.1}$$

Repeating the same argument with Γ_2, Γ_3, and Γ_4 gives:

$$K = A_{13}^{14}A_{34}^{31}A_{41}^{43}r_2(r_2+1);\quad r_2 \in \mathbb{Z}, \tag{3.9.2}$$

$$K = A_{14}^{12}A_{21}^{24}A_{42}^{41}r_3(r_3+1);\quad r_3 \in \mathbb{Z}, \tag{3.9.3}$$

$$K = A_{12}^{13}A_{23}^{21}A_{31}^{32}r_4(r_4+1);\quad r_4 \in \mathbb{Z}. \tag{3.9.4}$$

To handle Γ_5, we first change coordinates to

$$\begin{aligned} y_1 &= x_{32} + x_{41},\ y_2 = x_{13} + x_{42},\ y_3 = x_{21} + x_{43},\\ y_4 &= x_{32} - x_{41},\ y_5 = x_{13} - x_{42},\ y_6 = x_{21} - x_{43}, \end{aligned} \tag{3.10}$$

4 Of course this does not guarantee the absence of logarithmic terms in the solutions of (3.8), but we shall not need to worry about this point

corresponding to the decomposition $so(4)=so(3)\oplus so(3)$. In these new coordinates, Eqs. (2.1) ($N=4$) restricted to $\Gamma_5\equiv\{y_4=y_5=y_6=0\}$ become:

$$\begin{aligned}&\dot{y}_1=B_{32}y_2y_3; \quad \dot{y}_2=B_{13}y_1y_3; \quad \dot{y}_3=B_{21}y_1y_2;\\ &\dot{y}_4=0; \quad \dot{y}_5=0; \quad \dot{y}_6=0,\end{aligned} \tag{3.11}$$

and the variational Eqs. (1.7) along the solutions of (3.11) decouple into:

$$\begin{aligned}\dot{\delta}_4&=\mu_{36}y_3(t)\delta_5-\mu_{25}y_2(t)\delta_6,\\ \dot{\delta}_5&=-\mu_{36}y_3(t)\delta_4+\mu_{14}y_1(t)\delta_6,\\ \dot{\delta}_6&=\mu_{25}y_2(t)\delta_4-\mu_{14}y_1(t)\delta_5,\end{aligned} \tag{3.12}$$

plus three other equations, with $B_{ij}=\mu_i-\mu_j$ and

$$\begin{aligned}&2\mu_1=\lambda_{23}+\lambda_{14}, 2\mu_{14}=\lambda_{23}-\lambda_{14},\\ &2\mu_2=\lambda_{13}+\lambda_{24}, 2\mu_{25}=\lambda_{13}-\lambda_{24},\\ &2\mu_3=\lambda_{12}+\lambda_{34}, 2\mu_{36}=\lambda_{12}-\lambda_{34}.\end{aligned} \tag{3.13}$$

Assume first $B_{21}B_{13}B_{32}\neq 0$. Then, as above, Eqs. (3.11) are solved in terms of elliptic functions and Eqs. (3.12) form a linear system of differential equations with regular singular points. The indicial equation reads:

$$r\left(r^2+\frac{\mu_{14}^2}{B_{21}B_{13}}+\frac{\mu_{25}^2}{B_{32}B_{21}}+\frac{\mu_{36}^2}{B_{13}B_{32}}\right)=0. \tag{3.14}$$

Imposing the roots of (3.14) to be integers, by using (3.13), we find

$$K=2(1-r_5^2)B_{21}B_{13}B_{32}; \quad r_5\in\mathbb{Z}. \tag{3.15}$$

If $K\neq 0$, from the identity

$$8B_{21}B_{13}B_{32}=2K+A_{24}^{23}A_{32}^{34}A_{43}^{42}+A_{13}^{14}A_{34}^{31}A_{41}^{43}+A_{14}^{12}A_{21}^{24}A_{42}^{41}+A_{12}^{13}A_{23}^{21}A_{31}^{32}, \tag{3.16}$$

and from the conditions (3.9.i) ($1\leqq i\leqq 4$) and (3.15) one deduces immediately:

$$4=(1-r_5^2)\left(2+\frac{1}{r_1(r_1+1)}+\frac{1}{r_2(r_2+1)}+\frac{1}{r_3(r_3+1)}+\frac{1}{r_4(r_4+1)}\right).$$

Since $K\neq 0$, $r_i(r_i+1)>0$ ($1\leqq i\leqq 4$) and the above equality implies $r_5=0$. But then the indicial Eq. (3.14) has zero as a triple root. This always leads to logarithmic terms in the solutions of (3.12) except if $\mu_{14}=\mu_{25}=\mu_{36}=0$, i.e. $\lambda_{12}=\lambda_{34}$, $\lambda_{13}=\lambda_{24}$, and $\lambda_{23}=\lambda_{14}$.

In the case where $B_{21}B_{13}B_{32}=0$, (3.9.i) ($1\leqq i\leqq 4$) and (3.16) immediately imply $K=0$.

Case 2. $\Pi A_{ij}^{ik}=0$. If some $A_{ij}^{ik}=0$, we just replace condition (3.9.l) ($\{i,j,k,l\}=\{1,2,3,4\}$) by $A_{ij}^{ik}A_{jk}^{ji}A_{ki}^{kj}=0$, and the previous argument goes through leading always to $K=0$. This finishes the proof of Theorem 2.

Remark. In [16], Ziglin has applied Melnikov's method to prove that the motion of a non-symmetric (i.e. all moments of inertia distinct) rigid body in the presence of gravity does not have any additional real analytic first integral besides the known ones, by rescaling the variables so that the system can be viewed as a small perturbation of Euler's rigid body motion. In principle, such a rescaling can also be made in the case of geodesic flow on $SO(4)$. For example, corresponding to Γ_1 (3.3) the variables can be rescaled as follows:

$$x_{23}=z_{23}, x_{24}=z_{24}, x_{34}=z_{34}, x_{12}=\varepsilon z_{12}, x_{13}=\varepsilon z_{13}, x_{14}=\varepsilon z_{14},$$

so that (2.1) ($N=4$) becomes equivalent to $\dot{z}=f(z,\varepsilon)$. However, the reason why the method does not apply here is that, to prove a global result by Melnikov's method, one must express (by some rescaling) the originally given Hamiltonian system as a perturbation of a *completely integrable* situation, i.e. the simplified system $\varepsilon=0$ must possess besides the two orbit invariants and the Hamiltonian a 4^{th} independent invariant, which does not seem possible in this case.

4. Geodesic Flow on $SO(N)$

In this section we prove

Theorem 3. *Under the non-degeneracy assumption on the diagonal metric λ that all λ_{ij} be distinct, the system* (2.1) *is algebraically completely integrable with abelian functions x_{ij} if and only if the metric λ satisfies* (2.5).

As announced in the introduction, this theorem will be a straightforward consequence of the fact that all solutions of an algebraically completely integrable system have to be single-valued functions of $t\in\mathbb{C}$ (Corollary 1.4) combined with the characterization of the left invariant diagonal metrics on $SO(4)$ for which the solutions of (2.1) are single-valued (Theorem 2). But first, we need to introduce some notations.

Let $X=(x_{ij})\in so(N)$. To each choice of four distinct indices $1\leqq m,n,r,s\leqq N$, there corresponds a copy Γ_{mnrs} of $so(4)$ in $so(N)$:

$$\Gamma_{mnrs}\equiv\{x_{ij}=0 \quad \text{if} \quad i \quad \text{or} \quad j\notin\{m,n,r,s\}\}. \tag{4.1}$$

Relative to each such copy we define

$$\begin{aligned} K_{mnrs}=&\lambda_{mn}\lambda_{rs}(\lambda_{nr}+\lambda_{ms}-\lambda_{mr}-\lambda_{ns})\\ &+\lambda_{nr}\lambda_{ms}(\lambda_{mr}+\lambda_{ns}-\lambda_{mn}-\lambda_{rs})\\ &+\lambda_{mr}\lambda_{ns}(\lambda_{mn}+\lambda_{rs}-\lambda_{nr}-\lambda_{ms}). \end{aligned} \tag{4.2}$$

The following expression for K_{mnrs} will be useful

$$K_{mnrs}=A_{nr}^{nm}A_{rs}^{rm}A_{sn}^{sm}+A_{nm}^{ns}A_{rm}^{rn}A_{sm}^{sr}. \tag{4.3}$$

Note $K_{mnrs}=-K_{nrsm}=-K_{nmrs}$. Let K be the algebraic variety in $\mathbb{C}^{\frac{N(N-1)}{2}}$ defined by

$$K=\{\lambda=(\lambda_{ij})\in\mathbb{C}^{\frac{N(N-1)}{2}}\,|\,K_{mnrs}=0\;\forall 1\leqq m,n,r,s\leqq N\}. \tag{4.4}$$

Lemma 4.1. *The Zariski open set $K \cap \{\Pi A_{ij}^{ik} \neq 0,\ 1 \leqq i, j, k \leqq N\}$ of the algebraic variety K (4.4) can be rationally parametrized by*

$$\lambda_{ij} = \frac{b_i - b_j}{a_i - a_j}$$

$a_i,\ b_i \in \mathbb{C},\ \prod_{i<j} (a_i - a_j) \neq 0$. *In particular* $\dim K = 2N - 3$.

The proof of Lemma 4.1 is delayed to the end of this section.

Proof of Theorem 3. Suppose (2.1) is algebraically completely integrable with abelian functions x_{ij}, all λ_{ij} distinct. Then, by Corollary 1.4, the solutions of (2.1) must be single-valued functions of $t \in \mathbb{C}$. From the explicit form of the differential Eqs. (2.3), one sees easily that each Γ_{mnrs} $(1 \leqq m, n, r, s \leqq N)$ (4.1) is a linear invariant manifold of (2.1) on which (2.1) reduces to geodesic flow on $SO(4)$, i.e.

$$\begin{aligned} \dot{x}_{ij} &= \sum_{k \in \{m,n,r,s\}} A_{kj}^{ki} x_{ik} x_{kj} \quad &&\text{if} \quad i \text{ and } j \in \{m, n, r, s\}\,, \\ \dot{x}_{ij} &= 0 &&\text{if} \quad i \text{ or } j \notin \{m, n, r, s\}\,. \end{aligned} \tag{4.5}$$

So, in particular, the solutions of (4.5) have to be single-valued. By Theorem 2, this implies $K_{mnrs} = 0$ and by Lemma 4.1, it follows then that λ must satisfy (2.5). This establishes Theorem 3.

Now, from Theorem 2 and the proof of Theorem 3, the question arises naturally whether allowing some λ_{ij} to become equal could lead to new algebraically completely integrable flows. That this is not the case is insured by the following

Proposition 4.2. *Let $N \geqq 5$ and λ be an arbitrary diagonal metric on $SO(N)$. Then, if the solutions of* (2.1) *are (analytic) single-valued functions of $t \in \mathbb{C}$, $\lambda \in K$* (4.4).

In proving this proposition, the two following lemmas will be useful:

Lemma 4.3. *Let $1 \leqq m, n, r, s \leqq N$ and assume some $A_{ij}^{ik} = 0$ with $i, j, k \in \{m, n, r, s\}$. Then $K_{mnrs} = 0$ if and only if $(\lambda_{ij} = \lambda_{ik},\ \lambda_{lj} = \lambda_{lk})$ or $(\lambda_{ij} = \lambda_{ik} = \lambda_{il})$ or $(\lambda_{ij} = \lambda_{ik} = \lambda_{jk})$ with l defined by $\{i, j, k, l\} = \{m, n, r, s\}$.*

Proof. Obvious from the formula (4.3) for K_{klij}.

Lemma 4.4. *Let $N = 5$. If the metric λ satisfies*

(i) $\lambda_{12} = \lambda_{34} = \lambda_{35}, \lambda_{13} = \lambda_{24} = \lambda_{25}, \lambda_{23} = \lambda_{14} = \lambda_{15}$ *and* $A_{12}^{13} A_{23}^{21} A_{31}^{32} \neq 0$ *or*

(ii) $\lambda_{12} = \lambda_{34},\ \lambda_{13} = \lambda_{24},\ \lambda_{23} = \lambda_{14},\ \lambda_{51} = \lambda_{52} = \lambda_{53} = \lambda_{54}$ *and* $A_{12}^{13} A_{23}^{21} A_{31}^{32} \neq 0$, *all solutions of* (2.1) $(N = 5)$ *cannot be single-valued functions of $t \in \mathbb{C}$.*

The proof of Lemma 4.4 is delayed to the end of this section.

Proof of Proposition 4.2. Assume first $N = 5$ and suppose the solutions of (2.1) $(N = 5)$ are single-valued. Then, as in (4.5), for each choice of indices $1 \leqq m, n, r, s \leqq 5$, the solutions of (2.1) $(N = 5)$ restricted to Γ_{mnrs} must be single-valued and so, by Theorem 2, one of the two following conditions has to be satisfied:

$$(I_{mnrs}: \lambda_{mn}=\lambda_{rs}, \lambda_{mr}=\lambda_{ns}, \lambda_{ms}=\lambda_{nr}) \quad \text{or} \quad (II_{mnrs}: K_{mnrs}=0).$$

We now distinguish several cases.

Case 1. At least three "I conditions" are satisfied. For example I_{1234}, I_{1235}, and I_{1245} are satisfied, i.e. $\lambda_{12}=\lambda_{34}=\lambda_{35}=\lambda_{45}$ and $\lambda_{13}=\lambda_{23}=\lambda_{14}=\lambda_{24}=\lambda_{15}=\lambda_{25}$. By Lemma 4.3, this immediately implies $K_{1234}=K_{1235}=K_{1245}=K_{1345}=K_{2345}=0$.

Case 2. Two "I conditions" are satisfied. For example I_{1234} and I_{1235} are satisfied, i.e. $\lambda_{12}=\lambda_{34}=\lambda_{35}$, $\lambda_{13}=\lambda_{24}=\lambda_{25}$ and $\lambda_{23}=\lambda_{14}=\lambda_{15}$. By Lemma 4.3, these conditions imply $K_{1245}=K_{1345}=K_{2345}=0$. If $A_{12}^{13}A_{23}^{21}A_{31}^{32}\neq 0$, by Lemma 4.4 (i), all solutions of (2.1) ($N=5$) cannot be single-valued. Thus $A_{12}^{13}A_{23}^{21}A_{31}^{32}=0$ implying $K_{1234}=K_{1235}=0$.

Case 3. One "I condition" is satisfied. For example I_{1234}, II_{1235}, II_{1245}, II_{1345}, and II_{2345} are satisfied. If $\Pi A_{ij}^{ik}\neq 0$, II_{1245}, II_{1345}, and II_{2345} imply $K_{1234}=0$ (see the proof of Lemma 4.1). Suppose now some $A_{ij}^{ik}=0$. If $1\leqq i,j,k\leqq 4$, from I_{1234} and Lemma 4.3 it follows that $K_{1234}=0$. So the only cases still to be considered are:

(a) $A_{i5}^{ij}=0$ or (b) $A_{5i}^{5j}=0$ $\quad (1\leqq i,j\leqq 4)$.

We do this schematically. Define k, l by $\{i,j,k,l\}=\{1,2,3,4\}$

(a) $\lambda_{i5}=\lambda_{ij}, \lambda_{ij}=\lambda_{kl}, \lambda_{ik}=\lambda_{jl}, \lambda_{il}=\lambda_{jk}$

$\quad\downarrow II_{ijk5}$

- $\lambda_{ij}=\lambda_{i5}=\lambda_{ik}$ $\;\rightarrow\;$ $K_{1234}=0$
- $\lambda_{ij}=\lambda_{i5}$, $\lambda_{kj}=\lambda_{k5}$ $\;\xrightarrow{II_{ijl5}}$
  - $\lambda_{ij}=\lambda_{i5}=\lambda_{il}$ $\;\rightarrow\;$ $K_{1234}=0$
  - $\lambda_{ij}=\lambda_{i5}$, $\lambda_{lj}=\lambda_{l5}$ $\;\xrightarrow{II_{ikl5}}\;$ $K_{1234}=0$
  - $\rightarrow\;\lambda_{ij}=\lambda_{i5}=\lambda_{j5}$
- $\lambda_{ij}=\lambda_{i5}=\lambda_{j5}$ $\;\xrightarrow{II_{kil5},\ II_{ljk5}}\;$ $A_{il}^{ik}A_{5l}^{ij}A_{ij}^{5k}=0$, $A_{ik}^{ij}A_{ij}^{il}A_{5k}^{5l}=0$ $\;\rightarrow\;$ $K_{1234}=0$ or $\lambda_{51}=\lambda_{52}=\lambda_{53}=\lambda_{54}$

(b) $\lambda_{5i}=\lambda_{5j}, \lambda_{ij}=\lambda_{kl}, \lambda_{ik}=\lambda_{jl}, \lambda_{il}=\lambda_{jk}$

$\downarrow II_{ijk5}$

- $\lambda_{5i}=\lambda_{5j}$, $\lambda_{ki}=\lambda_{kj}$ $\rightarrow$ $K_{1234}=0$
- $\lambda_{5i}=\lambda_{5j}=\lambda_{5k}$ $\xrightarrow{II_{ijl5}}$
  - $\lambda_{5i}=\lambda_{5j}$, $\lambda_{li}=\lambda_{lj}$ $\rightarrow$ $K_{1234}=0$
  - $\lambda_{5i}=\lambda_{5j}=\lambda'_{5l}$ $\rightarrow$ $\lambda_{51}=\lambda_{52}=\lambda_{53}=\lambda_{54}$
  - $\lambda_{5i}=\lambda_{5j}=\lambda_{ij}$ $\rightarrow$ go to (a)
- $\lambda_{5i}=\lambda_{5j}=\lambda_{ij}$ $\rightarrow$ go to (a)

The upshot is that Case 3 always leads to $K_{1234}=0$ or ($K_{1234}\neq 0$ and $\lambda_{51}=\lambda_{52}=\lambda_{53}=\lambda_{54}$). By Lemma 4.4 (ii) the last case does not correspond to a situation where all solutions of (2.1) ($N=5$) are single-valued and must be rejected. This proves Proposition 4.2 for $N=5$.

If $N>5$, one considers the invariant manifolds $\Gamma_{mnrsl}\equiv\{x_{ij}=0$ if i or $j \notin \{m,n,r,s,l\}\}$ on which (2.1) reduces to geodesic flow on $SO(5)$. From the previous discussion, it follows then immediately that, if the solutions of (2.1) are single-valued, all $K_{mnrs}=0$, i.e. $\lambda\in K$. This establishes Proposition 4.2.

We end this section with the proofs of Lemmas 4.1 and 4.4.

Proof of Lemma 4.1. Assume first $N=4$. From

$$\lambda_{ij}=\frac{b_i-b_j}{a_i-a_j},\ a_1=b_1=0\,, \tag{4.6}$$

it follows that

$$A_{31}^{34}a_3+A_{43}^{41}a_4=0;\qquad A_{21}^{24}a_2+A_{42}^{41}a_4=0;\qquad A_{21}^{23}a_2+A_{32}^{31}a_3=0\,. \tag{4.7}$$

Since $a_2a_3a_4\neq 0$, by (4.3), (4.7) implies $K_{1234}=0$. Conversely, if $\lambda\in K\cap\{\Pi A_{ij}^{ik}\neq 0,\ 1\leq i,j,k\leq 4\}$, (4.6) can be solved for a_i and b_i by

$$\begin{aligned} &a_2=cA_{32}^{31},\ a_3=cA_{23}^{21},\ a_4=c\frac{A_{24}^{21}A_{32}^{31}}{A_{42}^{41}}\,,\\ &b_2=\lambda_{12}a_2,\ b_3=\lambda_{13}a_3,\ b_4=\lambda_{14}a_4\,, \end{aligned} \tag{4.8}$$

with c an arbitrary non-zero constant, $\prod_{i<j}(a_i-a_j)\neq 0$.

If $N \geqq 5$ and $\lambda \in K \cap \{\Pi A_{ij}^{ik} \neq 0\}$, from $K_{123j}=0$ and (4.8) it follows that

$$\begin{aligned} &\lambda_{12}=\frac{b_2}{a_2}, \lambda_{13}=\frac{b_3}{a_3}, \lambda_{23}=\frac{b_2-b_3}{a_2-a_3} \\ &\lambda_{1j}=\frac{b_j}{a_j}, \lambda_{2j}=\frac{b_2-b_j}{a_2-a_j}, \lambda_{3j}=\frac{b_3-b_j}{a_3-a_j}, \end{aligned} \tag{4.9}$$

with

$$a_2=cA_{32}^{31}, a_3=cA_{23}^{21}, a_j=c\frac{A_{2j}^{21}A_{32}^{31}}{A_{j2}^{j1}},$$
$$b_2=\lambda_{12}a_2, b_3=\lambda_{13}a_3, b_4=\lambda_{14}a_4 .$$

Equations $K_{12ij}=0$ imply

$$\begin{aligned} &\lambda_{12}=\frac{b_2'}{a_2'}, \lambda_{1i}=\frac{b_i'}{a_i'}, \lambda_{2i}=\frac{b_2'-b_i'}{a_2'-a_i'}, \\ &\lambda_{1j}=\frac{b_j'}{a_j'}, \lambda_{2j}=\frac{b_2'-b_j'}{a_2'-a_j'}, \lambda_{ij}=\frac{b_i'-b_j'}{a_i'-a_j'}, \end{aligned} \tag{4.10}$$

with

$$a_2'=c'A_{i2}^{i1}, a_i'=c'A_{2i}^{21}, a_j'=c'\frac{A_{2j}^{21}A_{i2}^{i1}}{A_{j2}^{j1}},$$
$$b_2'=\lambda_{12}a_2', b_3'=\lambda_{13}a_3', b_4'=\lambda_{14}a_4' .$$

Choosing $c'=cA_{32}^{31}(A_{i2}^{i1})^{-1}$ shows one may take $a_i'=a_i$, $b_i'=b_i$ in (4.9) and (4.10). This establishes Lemma 4.1.

Proof of Lemma 4.4. Consider the following linear invariant manifold Γ of (2.1) $(N=5)$:

$$\Gamma \equiv \{x_{32}=x_{41}, x_{13}=x_{42}, x_{21}=x_{43} \quad \text{and} \quad x_{51}=x_{52}=x_{53}=x_{54}=0\} .$$

Changing coordinates to y_i $(1 \leqq i \leqq 6)$, x_{5i} $(1 \leqq i \leqq 5)$ with y_i defined by (3.10) and since, by hypothesis, in both cases (i) and (ii) $\lambda_{12}=\lambda_{34}$, $\lambda_{13}=\lambda_{24}$, and $\lambda_{23}=\lambda_{14}$, Eqs. (2.1) $(N=5)$ restricted to $\Gamma \equiv \{y_4=y_5=y_6=x_{51}=x_{52}=x_{53}=x_{54}=0\}$ reduce to:

$$\begin{aligned} &\dot{y}_1=A_{12}^{13}y_2y_3, \dot{y}_2=A_{23}^{21}y_1y_3, \dot{y}_3=A_{31}^{32}y_1y_2 , \\ &\dot{y}_4=0, \dot{y}_5=0, \dot{y}_6=0 , \\ &\dot{x}_{51}=0, \dot{x}_{52}=0, \dot{x}_{53}=0, \dot{x}_{54}=0 . \end{aligned} \tag{4.11}$$

As in the proof of Theorem 2, (4.11) is solved in terms of elliptic functions and the variational Eqs. (1.7) along these solutions decouple into

$$\begin{aligned} &2\dot{\delta}_{51}=A_{21}^{25}y_3(t)\delta_{52}+A_{35}^{31}y_2(t)\delta_{53}+A_{23}^{45}y_1(t)\delta_{54} , \\ &2\dot{\delta}_{52}=A_{15}^{12}y_3(t)\delta_{51}+A_{32}^{35}y_1(t)\delta_{53}+A_{13}^{45}y_2(t)\delta_{54} , \\ &2\dot{\delta}_{53}=A_{13}^{15}y_2(t)\delta_{51}+A_{25}^{23}y_1(t)\delta_{52}+A_{12}^{45}y_3(t)\delta_{54} , \\ &2\dot{\delta}_{54}=A_{15}^{23}y_1(t)\delta_{51}+A_{25}^{13}y_2(t)\delta_{52}+A_{35}^{12}y_3(t)\delta_{53} , \end{aligned} \tag{4.12}$$

and six other equations. Equations (4.12) form a linear system with regular singular points. In case (i) the roots of the indicial equation are $-1, 0, \frac{1}{2}, \frac{1}{2}$ while, in case (ii), putting $\lambda_{5i}=0$ (which can always be done), one sees easily that $-\frac{1}{2}$ is always a root of the indicial equation. So, in both cases, all solutions of (2.1) ($N=5$) cannot be single-valued. This achieves the proof of Lemma 4.4.

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