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On the Structure of Tensor Operators in SU3*

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Abstract. A global algebraic formulation of SU3 tensor operator structure is achieved. A single irreducible unitary representation (irrep), V, of $\mathfrak{so}(6,2)$ is constructed which contains every SU3 irrep precisely once. An algebra of polynomial differential operators $\mathscr{A}$ acting on V is given. The algebra $\mathscr{A}$ is shown to consist of linear combinations of all SU3 tensor operators with polynomial invariant operators as coefficients. By carrying out an analysis of $\mathscr{A}$, the multiplicity problem for SU3 tensor operators is resolved.

1. Introduction

The theory of tensor operators has been fully developed only for the symmetry group SU2—the quantal angular momentum group—and the resulting theory is of fundamental importance in almost all applications of angular momentum in physics [1]. It is to be expected that the development of an analogous theory of tensor operators for the symmetry group SU3 might possibly be of comparable importance since this symmetry—$\mathrm{SU}^{\mathrm{color}}$—is held to be exact and of fundamental importance in hadronic physics; a distinct SU3 symmetry is known to be of practical importance as an approximate symmetry of the nuclear shell model.

Quite early in the development of the theory of tensor operators, Wigner [2] achieved a classification of those symmetry groups for which a direct analog to the SU2 tensor operator construction was possible. Such groups were termed *simply reducible*, and there are two conditions: the group must be (a) ambivalent (g and g^{-1} belong to the same class) and (b) multiplicity free (in the reduction of the Kronecker product of two irreps, no irrep occurs more than once). The group SU3 fails both criteria; it is neither ambivalent nor multiplicity-free. Mackey [3], however, showed that ambivalence could be weakened to *quasi-ambivalence* (the group possesses an involutive anti-automorphism). SU3 is indeed quasi-ambivalent [4] (using conjugation as the involution), but the problem of multiplicity is far more difficult to resolve.

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A constructive resolution of the multiplicity problem for SU3 was achieved in [5], but the construction was computationally-based and complicated. What was needed for the full acceptance of this result was a global, coordinate-free resolution of the multiplicity problem in purely conceptual terms. Toward this end a basic theorem extending the concept of the universal enveloping algebra for SU3 has recently been obtained [6].

The purpose of the present paper is to develop this result, in a more physically oriented manner, and to use it to resolve the multiplicity problem for SU3 in a remarkable, and satisfying, way which permits a global view of the set of all SU3 tensor operators to be achieved. In the course of this we actually find a new proof of the main result of [6].

The essential element in this development is the demonstration that the set of all tensor operators for SU3 can be identified as a *simple algebra*, with no (non-trivial) two-sided ideal, which is a quotient algebra $\mathscr{A}$ of the enveloping algebra of $\mathfrak{so}_8$. The Hilbert space on which these tensor operators act is shown to be a *single* irreducible unitary representation of the Lie algebra $\mathfrak{so}(6,2)$ (a non-compact real form of $\mathfrak{so}_8$ having Cartan index -4).

The plan of the paper is as follows: in Sect. 2, we review previous results on tensor operators for SU3 and in Sect. 3 use the basic structural result (the product law, Eq. (2.11)) to demonstrate the embedding in $\mathfrak{so}_8$. With benefit of the insights achieved in Sect. 2 and 3, we then develop, *ab initio*, a more elementary construction of both the underlying Hilbert space (Sect. 4) and the algebra $\mathscr{A}$ (Sect. 5), as well as the explicit embedding in $\mathfrak{so}_8$ (Sect. 6).

The decomposition of the algebra $\mathscr{A}$ under the Lie algebra $\mathfrak{so}_8$ is carried out in Sect. 7. The basic result of the present paper—the decomposition under SU3 of the algebra $\mathscr{A}$ (the algebra of SU3 tensor operators) is then achieved (Theorem (8.12) in Sect. 8).

The concluding section, Sect. 9, discusses the results obtained and the significance of these results in effecting a complete resolution of the multiplicity problem for SU3 tensor operators.

2. Resumé of Previous Work

We wish to review briefly the basic results on tensor operators belonging to SU3 symmetry [7–9]. (A reader familiar, or partly familiar, with this material is encouraged to go to Sect. 3).

Let $\mathfrak{g}$ be the Lie algebra of the Lie group SU3. The unitary irreps (*irr*educible *rep*resentations) of SU3 are uniquely labelled by two-rowed Young frames $[m_{13}m_{23}0]$, with the integers m_{i3} obeying $m_{13} \geqq m_{23} \geqq 0$. The orthonormal vectors $|(m)\rangle$ belonging to each irrep are uniquely labelled by the Gelfand–Weyl pattern:

$$(m) = \begin{pmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{pmatrix} \tag{2.1}$$

where the integers m_{ij} obey the "betweenness" conditions:

$$m_{13} \geqq m_{12} \geqq m_{23} \geqq m_{22} \geqq m_{33} = 0, \tag{2.2}$$

$$m_{12} \geqq m_{11} \geqq m_{22}. \tag{2.3}$$

The Hilbert space, $\mathscr{H}$, on which the tensor operators act is defined to be the direct sum of the vector spaces belonging to the unitary irreps of SU3, *each irrep* (and hence each Gelfand–Weyl vector) *occurring once and only once.*

In order to preserve the underlying symmetry between the labels 1, 2, 3—and moreover to make this symmetry evident—it is convenient, in defining the space on which the tensor operators act, to admit formally U3 irreps having three-rowed Young frames, $[m_{13}m_{23}m_{33}]$, but then to declare an equivalence relation on U3 irreps:

$$[m_{13}+k \quad m_{23}+k \quad m_{33}+k] \sim [m_{13}m_{23}m_{33}]. \tag{2.4}$$

The Hilbert space $\mathscr{H}$ then consists of a direct sum of U3 irreps, exactly one representative taken from each equivalence class.

Let us now recall the definition [10] of a tensor operator on $\mathscr{H}$ belonging to SU3 symmetry. A tensor operator is a *set* of linear operators $\mathcal{O}(M)$ indexed by SU3 Gelfand–Weyl patterns (M) and obeying the equivariance condition:

$$U_g \mathcal{O}(M) U_g^{-1} = \mathcal{O}(g(M)) \tag{2.5}$$

for every $g \in \mathrm{SU3}$, where U_g is the unitary transformation of $\mathscr{H}$ associated with g.

An *irreducible* tensor operator is indexed by patterns belonging to a single SU3 irrep $[M_{13}M_{23}0]$. The unit tensor operators are not uniquely specified by the SU3 labels (M) alone. A linear basis for the tensor operators in SU3 is provided by the *unit* tensor operators (operators with unit norm phased conventionally), having scalar operators as multipliers [7]. The unit tensor operators are continuous, and may be uniquely determined by giving all matrix elements on the basis $\{|(m)\rangle\}$ of $\mathscr{H}$.

It has been shown [7] that irreducible unit tensor operators having the same (M) index may be distinguished by a second triangular pattern of integers (obeying the constraints of a Gelfand–Weyl pattern) called the *operator pattern* (Γ):

$$(\Gamma) \equiv \begin{pmatrix} M_{13} & & M_{23} & & 0 \\ & \gamma_{12} & & \gamma_{22} & \\ & & \gamma_{11} & & \end{pmatrix}. \tag{2.6}$$

The signifance of this operator pattern may be seen in this way: the action of a tensor operator is a transformation of $\mathscr{H}$ into $\mathscr{H}$; for unit tensor operators, the operator label (Γ) specifies the *shift* $(\Delta_1\Delta_2\Delta_3)$ induced by the operator when acting on vectors belonging to the irrep $[m]$. That is:

$$\mathcal{O}_\Gamma : [m] = [m_{13}m_{23}m_{33}] \to [m+\Delta] = [m_{13}+\Delta_1, m_{23}+\Delta_2, m_{33}+\Delta_3], \tag{2.7}$$

where:

$$\Delta_1 = \gamma_{11}, \tag{2.8a}$$

$$\Delta_2 = \gamma_{12} + \gamma_{22} - \gamma_{11}, \tag{2.8b}$$

$$\Delta_3 = M_{13} + M_{23} + M_{33} - \gamma_{12} - \gamma_{22}. \tag{2.8c}$$

(Recall that we deal with equivalence classes in $\mathscr{H}$, which permits this more symmetrical notation with $\Delta_3 \neq 0$. It is useful to extend this notation to the operators as well.)

The shift labels (Δ) do not in general distinguish among all unit tensor operators with the same Gelfand–Weyl label but the complete operator pattern (Γ) does provide such a distinction. Since the Gelfand–Weyl pattern and the operator pattern share a common row (the irrep label $[M]$), it is convenient to designate a unique irreducible unit tensor operator by writing the two patterns together:

$$\left\langle \begin{matrix} & & \gamma_{11} & & \\ & \gamma_{12} & & \gamma_{22} & \\ M_{13} & & M_{23} & & M_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{matrix} \right\rangle, \tag{2.9}$$

or more briefly by $\left\langle \begin{matrix} (\gamma) \\ [M] \\ (m) \end{matrix} \right\rangle$ or $\langle M \rangle$.

The fundamental irreducible unit tensor operators, $\langle 100 \rangle$ and $\langle 110 \rangle$, are distinguished in several ways:

(a) the shift label (Δ) suffices to determine the operator pattern itself,

(b) the matrix elements for these operators are completely known and can be given quite simply in terms of the pattern calculus rules [8], and

(c) these operators provide an algebraic "basis" for constructing all unit tensor operators.

Remark 2.10. It is convenient to make use of (a), above, to simplify the notation for the fundamental operators. Since the Gelfand–Weyl pattern is also uniquely determined by the SU3 weights (w) of this pattern (just as the shifts (Δ) determined the operator pattern), we may conveniently label the fundamental operators by: $\begin{pmatrix} \Delta \\ w \end{pmatrix}$. This abbreviated notation uniquely designates the 18 operators $\langle 100 \rangle$ and $\langle 110 \rangle$, and proves very convenient in the sections to follow.

The details of the construction noted in (c) above are, as yet, not completely explicit, which is one of the points addressed in this paper. What *is* known is the formal algebraic structure of the products of unit tensor operators. The basic relation is the *product law* [7].

$$\left\langle \begin{matrix} (\gamma') \\ [M'] \\ (m') \end{matrix} \right\rangle \left\langle \begin{matrix} (\gamma) \\ [M] \\ (m) \end{matrix} \right\rangle = \sum_{[M''],(m''),(\gamma''),(\Gamma)} \left\langle \begin{matrix} [M''] \\ (m'') \end{matrix} \right| \left\langle \begin{matrix} (\Gamma) \\ [M'] \\ (m') \end{matrix} \right\rangle \left| \begin{matrix} [M] \\ (m) \end{matrix} \right\rangle$$

$$\cdot \left\langle \begin{matrix} (\gamma'') \\ [M''] \\ (m'') \end{matrix} \right\rangle \left\langle \begin{matrix} [M''] \\ (\gamma'') \end{matrix} \right| \left\{ \begin{matrix} (\Gamma) \\ [M'] \\ (\gamma') \end{matrix} \right\} \left| \begin{matrix} [M] \\ (\gamma) \end{matrix} \right\rangle \tag{2.11}$$

Expressed in words, this law states that the product of two unit tensor operators (the left-hand side) is a sum over unit tensor operators—the terms $\left\langle \begin{matrix} (\gamma'') \\ [M''] \\ (m'') \end{matrix} \right\rangle$ on the right-hand side above—multiplied by both *numerical* (Wigner or "(3j)" coupling) coefficients—the terms $\left\langle \begin{matrix} [M''] \\ (m'') \end{matrix} \middle| \left\langle \begin{matrix} (\Gamma) \\ [M'] \\ (m') \end{matrix} \right\rangle \middle| \begin{matrix} [M] \\ (m) \end{matrix} \right\rangle$ above—and *invariant operator* (Racah or "(6j)") coefficients—the terms $\left\langle \begin{matrix} [M''] \\ (\gamma'') \end{matrix} \middle| \left\{ \begin{matrix} (\Gamma) \\ [M'] \\ (\gamma') \end{matrix} \right\} \middle| \begin{matrix} [M] \\ (\gamma) \end{matrix} \right\rangle$ above. Note that the shifts in a product of operators—like the weights-are additive.

• For the fundamental unit tensor operators, all terms occurring in the product law are explicitly known [9].

3. Derivation of the $\mathfrak{so}_8$ Commutation Rules using Structural Considerations

The fact that the fundamental unit tensor operators $\langle 100 \rangle$ and $\langle 110 \rangle$ generate, algebraically, the space of all unit tensor operators in SU3 is very suggestive. In particular, an investigation of the Lie algebra generated by these fundamental operators seems clearly indicated. As we shall see, there are difficulties in any such straightforward attempt.

Let us begin by considering the product of two $\langle 100 \rangle$ operators having the *same* upper pattern. The abstract product $[100] \times [100]$ can only lead to $[200]$ and $[110]$, which are symmetric and antisymmetric, respectively. Since, however, the upper patterns are the same and the shifts extremal, the shift of the product operator is also extremal, and hence belongs uniquely to the irrep $[200]$. It follows that the commutator, being antisymmetric, must *vanish*. An analogous result follows for the $[110]$ operators.

Thus we have shown [8] **Proposition 3.1**: *For the* 18 *fundamental operators* $\begin{pmatrix} \Delta \\ w \end{pmatrix}$, *any two operators having the same operator shift* (upper label, Δ) *commute.*

This is, however, as far as one can go in a direct approach and obtain simple results; further commutators fail to vanish and generate more and more new operators.

The way out of this impasse is to recognize (with some surprise) that the fundamental unit tensor operators are *not the proper elementary objects to consider.* The correct objects are the fundamental unit tensor operators *re-normalized to remove the* (SU3 invariant) *denominators* (defined in [8]).

In order to understand why this renormalization is so essential, let us consider again the product of two $\langle 100 \rangle$ operators, but this time having the same lower (Gelfand–Weyl) pattern. Once again, the product pattern is extremal and hence only $\langle 200 \rangle$ can occur. *But we can no longer appeal to symmetry to eliminate the* *commutator* since an invariant (Racah) operator appears (unsymmetrically) in the product law (2.11). The key observation is that *removing the* SU3 *denominator has the effect of removing this invariant operator so that the result is symmetric and the commutator vanishes.*

Thus we have obtained **Proposition 3.2**: *For the* 18 *re-normalized fundamental operators* $\begin{pmatrix} \Delta \\ w \end{pmatrix}$, *any two operators having the same* SU3 *weight* (w) *commute.*

Remark 3.3. Note that re-normalization does not affect the previous result (3.1). Hereafter we shall consider the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ always to be renormalized, and denote the renormalized fundamental operators by $\langle 100 \rangle_R$ and $\langle 110 \rangle_R$.

Now let us consider the product of $\langle 100 \rangle_R$ with $\langle 110 \rangle_R$, where the shifts are such that the sum is a permutation of (210). Since the resultant (product) shift is extremal, only [210] operators can occur on the right-hand side of the product law, Eq. (2.11). *It is remarkable that*, once again, *the renormalization removes the invariant* (Racah) *operator that occurs for this term with extremal shift.* Since the numerical ("(3j)") coefficient in the product is *symmetric*, we obtain, once again, a vanishing commutator.

Thus we have shown **Proposition 3.4**:

- *The three* $\begin{pmatrix} 001 \\ \cdots \end{pmatrix}$ *commute with the three* $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ *and the three* $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$.
- *The three* $\begin{pmatrix} 010 \\ \cdots \end{pmatrix}$ *commute with the three* $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ *and the three* $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$.
- *The three* $\begin{pmatrix} 100 \\ \cdots \end{pmatrix}$ *commute with the three* $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$ *and the three* $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$.

The results (3.1, 2, 4) establish *all* the vanishing commutators. Let us consider next the non-zero commutators.

Consider the commutator of two $\langle 100 \rangle_R$ operators, where, to avoid vanishing, the two shifts and the two weights involved both differ. Because of the re-normalization the invariant operator in (2.11) multiplying the $\langle 200 \rangle_R$ term has been replaced by unity. Since the numerical (Wigner) coefficient in (2.11) multiplying the $\langle 200 \rangle_R$ term is symmetric, we see that the commutator can only result in an operator $\langle 110 \rangle$. This argument is not enough to show that the commutator is a *numerical* multiple of $\langle 110 \rangle_R$ with the appropriate Δ and w; the possibility of an invariant operator multiple cannot *a priori* be ruled out. But using the pattern calculus [8] it is easy to evaluate the explicit form of the operators involved verifying that $\langle 110 \rangle_R$ does indeed occur with a numerical coefficient.

Using similar considerations, it can be verified that the $\langle 110 \rangle_R$ commutators, when non-zero, yield numerical multiples of the $\langle 100 \rangle_R$ operators.

Finally let us consider the commutators of $\langle 100 \rangle_R$ with $\langle 110 \rangle_R$. To be non-vanishing, the shifts of these two operators must be complementary, so that the resultant (product) shift is (111). The right hand side of (2.11) now contains, in general, three possible terms: operators transforming as the adjoint irrep [210], for which there are two possible types, the so-called F and D operators (the generators and "non-generators" of SU3, respectively) and operators transforming invariantly, i.e. as [000]. Once again the remarkable effect of re-normalizing intervenes to remove the Racah operator multiplying the non-generator $\langle 210 \rangle$ terms, so that symmetry eliminates all such terms.

Using the pattern calculus rules verifies that the commutators of $\langle 100 \rangle_R$ with $\langle 110 \rangle_R$, when non-vanishing, close on the Lie algebra $\mathscr{g}$ of SU3 (the generators $\langle 210 \rangle$) and three SU3 invariant linear operators, X_i, whose matrix elements on $\mathscr{H}$ are:

$$\langle (m)|X_i|(m) \rangle = (p_{13} + p_{23} + p_{33} - 3p_{i3}), \tag{3.5}$$

where we have used the symmetrical notation:

$$p_{i3} \equiv m_{i3} + 3 - i. \tag{3.6}$$

Only two of the three X_i are independent, since it is clear that $\sum_i X_i = 0$.

The commutation relations for the SU3 generators, $\mathscr{g}$, with the 18 operators $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ are, of course, known to yield numerical multiples of the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$, since this is a consequence of the tensor operator condition (2.5). Clearly the commutators $[\mathscr{g}, X_i] = 0$.

The commutators of the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ with the invariant operators X_i do not vanish (since the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ effect changes (shifts) in the irrep labels of the vectors in $\mathscr{H}$), but the results are *numerical* multiples of the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ since the matrix elements of the X_i, Eq. (3.6), are *linear* in the p_{i3}.

It follows that the commutators among the 18 $\begin{pmatrix} \Delta \\ w \end{pmatrix}$, the 8 $\mathscr{g}$ and the 2 independent invariants, say, X_1 and X_2, *close*. The resulting Lie algebra of these 28 operators is isomorphic, as we shall show below, to the complex simple Lie algebra $\mathscr{so}_8$.

We shall deal much more explicitly with this Lie algebra in the following sections. Our purpose in the discussion above is intended to show how information on the SU3 tensor operator structures allows one to "understand" the commutation relations *a priori*—from the structural properties of the product law (2.11)—once the all-important condition of re-normalization has been imposed.

Remark 3.7. The fact that the unit tensor operators $\langle 100 \rangle$ and $\langle 110 \rangle$ obey the simplest commutation relations only after renormalization has important further implications.

First of all, let us recall that the re-normalization consisted of removing the

SU3 norms of the $\langle 100 \rangle$ and $\langle 110 \rangle$ tensor operators. The use of normalized operators originated in the requirement that the operators be continuous, and hence defined everywhere by their matrix elements on a given basis.

The removal of the normalizing factors thus means that we achieve the simplest commutational rules for unbounded operators. This suggests that these more elementary operators, the $\begin{pmatrix} \Delta \\ w \end{pmatrix}$, can be realized by polynomials (over $\mathbb{C}$) in the boson operators, in particular *without* adjoining square roots of rational functions of invariant operators as given by the normal form [9].

Once we have recognized the role of polynomial operators it is clear that one should go further and introduce *unnormalized* state vectors as basis vectors of $\mathcal{H}$. Such a step is familiar from angular momentum theory, where the matrix elements of the raising-lowering operators, which have the (un-normalized) form:

$$J_+ \to [(m_{12} - m_{11})(m_{11} - m_{22} + 1)]^{1/2}$$

$$J_- \to [(m_{11} - m_{22})(m_{12} - m_{22} + 1)]^{1/2} \text{ acting on states } \left| \begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix} \right\rangle,$$

take on the much simpler polynomial form:

$$J_+ \to (m_{12} - m_{11}), J_- \to (m_{11} - m_{22}),$$

when one uses *un-normalized* state vectors:

$$\left| \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \right\rangle_R = (a_{12})^{m_{22}} (a_1)^{m_{11} - m_{22}} (a_2)^{m_{12} - m_{11}} |0\rangle.$$

(Here a_{12} is the determinant of two pairs of independent bosons.)

Thus we see that considerably simpler structures enter if we use orthogonal but unnormalized state vectors, which are polynomials in the bosons (multiplied on the right by the ground state ket), and also use polynomials in the bosons as operators.

It is also useful to recognize that the commutation relations (for the renormalized operators) introduce invariant operators of a very special type (see Eq. (3.6)). There are in SU3 two invariant operators, the quadratic invariant I_2,

$$18 I_2 \to [(p_{13} - p_{23})^2 + (p_{23} - p_{33})^2 + (p_{33} - p_{13})^2 - 6], \tag{3.8}$$

and the cubic invariant I_3,

$$162 I_3 \to (p_{13} + p_{23} - 2p_{33})(2p_{13} - p_{23} - p_{33})(p_{13} - 2p_{23} + p_{33}), \tag{3.9}$$

where we have used the partial hook variables:

$$p_{i3} = m_{i3} + 3 - i, \tag{3.6}$$

instead of the Young frame labels $[m_{13} m_{23} m_{33}]$ of the U3 state $|(m)\rangle$ on which I_2 and I_3 act.

Note that the operators I_2 and I_3 are invariant to a shift in the m_{i3} by a constant, and are *symmetric under an* S3 *group acting on the indices* $i = 1, 2, 3$ of the p_{i3}. (It was to achieve this latter symmetry [8] that required the use of the partial hook variables.)

The invariant operators, X_ι, introduced by the commutation relations are quite different, with eigenvalues *linear in the partial hooks*. In order to express the symmetric variables p_{i3} in terms of the SU3 invariant operators I_2 and I_3, it is necessary to explicitly *break* the S3 symmetry (by taking roots). Thus an *ordering* has been imposed on the structure, and this ordering is the same as the ordering that defined the highest weight vectors in the Gelfand–Weyl basis $|(m)\rangle$.

The operators X_i are, however, still invariant to constant shifts in the m_{i3}. (A more symmetrical way to proceed would be to introduce all three X_i operators together with the constraint $\sum_i X_i = 0$, but for brevity we shall not do so.)

• We propose in the next section to utilize these insights and to begin *ab initio*, using boson polynomials over $\mathbb{C}$ both as operators and as states to realize this simplest of algebraic structures for SU3.

4. The Representation V

We shall use two sets of three bosons for SU3. The first set: $a_i, \bar{a}_i$ ($i = 1, 2, 3$), with

$$[\bar{a}_i, a_j] = \delta_{ij}, \tag{4.1}$$

and

$$[a_i, a_j] = [\bar{a}_i, \bar{a}_j] = 0 \tag{4.2}$$

realizes the representation [100] under the commutation action of $\mathscr{g}$, the complexified Lie algebra of SU3:

$$E_{ij} = a_i\bar{a}_j, (i \neq j; i,j = 1,2,3), \tag{4.3}$$

$$\mathscr{h}_1 \equiv E_{11} - E_{22} = a_1\bar{a}_1 - a_2\bar{a}_2, \tag{4.4}$$

$$\mathscr{h}_2 \equiv E_{22} - E_{33} = a_2\bar{a}_2 - a_3\bar{a}_3. \tag{4.5}$$

The second set of three bosons will not be the (usual) determinantal forms in pairs of independent bosons but will instead be a second triplet of bosons, indexed, however, by pairs of integers:

$$a_{12}, a_{23}, a_{31} \text{ and } \bar{a}_{12}, \bar{a}_{23}, \bar{a}_{31}, \tag{4.6}$$

Obeying the boson commutation rules:

$$[\bar{a}_\alpha, a_\beta] = \delta_{\alpha\beta}, \tag{4.7}$$

$$[\bar{a}_\alpha, \bar{a}_\beta] = [a_\alpha, a_\beta] = 0, (\alpha, \beta = 12, 23, 31). \tag{4.8}$$

Moreover, all $a_i, \bar{a}_i$ are defined to commute with all $a_\alpha, \bar{a}_\beta$.

In order that this second triplet realize the representation [110] (conjugate to [100]) we extend the definition of the SU3 generators, $\mathscr{g}$, now to be:

$$E_{12} = a_1\bar{a}_2 - a_{31}\bar{a}_{23}, E_{23} = a_2\bar{a}_3 - a_{12}\bar{a}_{31}, \qquad \text{(4.9a), (4.9b)}$$

$$E_{13} = a_1\bar{a}_3 - a_{12}\bar{a}_{23}, E_{21} = a_2\bar{a}_1 - a_{23}\bar{a}_{31}, \qquad \text{(4.9c), (4.9d)}$$

$$E_{32} = a_3\bar{a}_2 - a_{31}\bar{a}_{12}, E_{31} = a_3\bar{a}_1 - a_{23}\bar{a}_{12}, \qquad \text{(4.9e), (4.9f)}$$

$$\mathscr{h}_1 = E_{11} - E_{22} = a_1\bar{a}_1 - a_2\bar{a}_2 + a_{31}\bar{a}_{31} - a_{23}\bar{a}_{23}, \tag{4.9g}$$

$$\mathscr{h}_2 = E_{22} - E_{33} = a_2\bar{a}_2 - a_3\bar{a}_3 + a_{12}\bar{a}_{12} - a_{31}\bar{a}_{31}. \tag{4.9h}$$

This structure is defined so that the involution:

$$\mathscr{J}: a_i \to \bar{a}_{jk}, \bar{a}_i \to a_{jk}, (ijk) = (123)\,\text{cyclic} \tag{4.10}$$

acts as a conjugation on $\mathscr{g}$, that is,

$$\mathscr{J}(\mathscr{g}) = -\mathscr{g}. \tag{4.11}$$

Now let us consider the set of all polynomials over $\mathbb{C}$ formed from the six boson (creation) operators; that is, the polynomial ring

$$W = \mathbb{C}[a_1, a_2, a_3, a_{12}, a_{23}, a_{31}]. \tag{4.12}$$

The action of the Lie algebra $\mathscr{g}$ on the polynomial ring W is by commutation:

$$g: w \to w' \equiv [g, w], w \in W, g \in \mathscr{g}. \tag{4.13}$$

The polynomial ring W can be made into a space of ket vectors, as is customary in physics by first defining the vacuum ket $|0\rangle$ by:

$$\bar{a}_i|0\rangle = \bar{a}_{jk}|0\rangle = 0, \quad \text{all } i, j, k, \tag{4.14}$$

and defining for $w \in W$ the associated ket vector $|w\rangle$ by:

$$|w\rangle \equiv w|0\rangle. \tag{4.15}$$

Noting now that (from Eqs. (4.9)):

$$\mathscr{g}|0\rangle = 0, \tag{4.16}$$

we see that the action of the Lie algebra on this ket-vector space agrees with the action on the polynomial ring, that is: (for $g \in \mathscr{g}$)

$$g: |w\rangle \to |w\rangle' \equiv g|w\rangle = gw|0\rangle = [g, w]\,|0\rangle = |[g, w]\rangle, \tag{4.17}$$

which accords with Eq. (4.13).

The action of $\mathscr{g}$ on the space of ket vectors $\{|w\rangle\}$, $w \in W$ defines W as a representation of $\mathscr{g}$. It is clear, however, from the fact that the polynomial:

$$M_+ \equiv a_1 a_{23} + a_2 a_{31} + a_3 a_{12}, \tag{4.18}$$

commutes with $\mathscr{g}$, i.e.,

$$[\mathscr{g}, M_+] = 0, \tag{4.19}$$

that the vector $|M_+\rangle$ is annihilated by all $\mathscr{g}$. Thus the identity subrepresentation is not unique in W.

To achieve uniqueness let us introduce the operator M_-:

$$M_- \equiv \bar{a}_1 \bar{a}_{23} + \bar{a}_2 \bar{a}_{31} + \bar{a}_3 \bar{a}_{12}. \tag{4.20}$$

This operator, M_-, and the polynomial M_+ (viewed as an operator) together with the operator M_0:

$$M_0 \equiv a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 + a_{12}\bar{a}_{12} + a_{23}\bar{a}_{23} + a_{31}\bar{a}_{31} + 3, \tag{4.21}$$

generate an $\mathscr{sl}_2$ Lie algebra.

• Let us now impose the constraint $M_- \to 0$ on the vector space W, thereby defining a new space V:

Definition 4.22 V *is the set of all* $w \in W$ *such that* $M_-|w\rangle = 0$.

We observe that the new space V is a representation of $\mathscr{g}$, since it is easily seen that:

$$[\mathscr{g}, M_-] = 0. \tag{4.23}$$

(In fact, one sees that $[\mathscr{g}, M_+] = [\mathscr{g}, M_0] = 0$ also.)

The importance of the space V lies in the fact—which we shall now demonstrate—that V *contains every irreducible representation of* SU3 *each irrep occurring once and only once.*

Consider the homogeneous polynomials of degree k (k an integer $\geqq 0$) in W; call this the space P^k. Let H^k be the subspace of polynomials h in P^k which obey the constraint: $[M_-, h]|0\rangle = 0$.

The space P^k may be characterized group-theoretically as the carrier space of an irreducible representation ($k00$) of the symplectic group Sp6 whose generators are defined in this way:

Using the fact that $\mathscr{sp}_6 \supset \mathscr{g}$, we take for 8 of the 21 generators of $\mathscr{sp}_6$ the Lie algebra $\mathscr{g}$ of SU3. Introduce the generator:

$$\mathscr{h}_3 = a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3 - a_{12}\bar{a}_{12} - a_{23}\bar{a}_{23} - a_{31}\bar{a}_{31}, \tag{4.24}$$

which commutes with the Lie algebra $\mathscr{g}$ (and forms the third commuting element of an $\mathscr{sp}_6$ Cartan sub-algebra, the other two being $\mathscr{h}_1$ and $\mathscr{h}_2$). The remaining 12 generators transform as [200] and [220] under $\mathscr{g}$ and have the form:

[200] $\mathscr{sp}_6$ *generators* $\equiv S_{ij}$:

$$S_{11} = 2a_1\bar{a}_{23}, S_{22} = 2a_2\bar{a}_{31}, S_{33} = 2a_3\bar{a}_{12}, \tag{4.25}$$

$$S_{12} = a_1\bar{a}_{31} + a_2\bar{a}_{23}, S_{23} = a_2\bar{a}_{12} + a_3\bar{a}_{31},$$

$$S_{31} = a_3\bar{a}_{23} + a_1\bar{a}_{12}. \tag{4.26}$$

[220] $\mathscr{sp}_6$ *generators* $\equiv T_{ij}$:

$$T_{11} = 2a_{23}\bar{a}_1, T_{22} = 2a_{31}\bar{a}_2, T_{33} = 2a_{12}\bar{a}_3, \tag{4.27}$$

$$T_{12} = a_{31}\bar{a}_1 + a_{23}\bar{a}_2, T_{23} = a_{31}\bar{a}_3 + a_{12}\bar{a}_2,$$

$$T_{31} = a_{12}\bar{a}_1 + a_{23}\bar{a}_3. \tag{4.28}$$

It is now easily seen that the polynomials in W generated from the monomial $(a_1)^k$ carry an irreducible representation ($k00$) of Sp6. To verify that this is precisely the space P^k requires only checking the dimensions: both spaces have the dimension $\binom{k+5}{5}$.

Now consider the space: M_+P^k. Clearly this space is contained in P^{k+2}, and hence in the irrep $(k+200)$. Because M_+ commutes with $\mathscr{g}$, every SU3 irrep belonging to P^k also belongs to $(k+200)$.

Next consider the polynomial associated with the Sp6 highest weight vector in $(k+200)$. This vector is of the form: $|(a_1)^{k+2}\rangle$.

Applying the $\mathscr{sp}_6$ lowering operator T_{31} to this vector, we obtain a non-zero multiple of the SU3 highest weight vector $|(a_1)^{k+1}a_{12}\rangle$ belonging to the SU3 irrep $[k+210]$ in H^{k+2}.

Successively applying the same lowering operator, we obtain in H^{k+2} non-vanishing vectors belonging to the sequence of SU3 irreps:

$$[k+200], [k+210], [k+220], \ldots [k+2 \quad k+1 \quad 0], [k+2 \quad k+2 \quad 0]. \quad (4.29)$$

This sequence of SU3 irreps, combined with the SU3 irreps from M_+P^k, completely exhausts the space of the Sp6 irrep $(k+200)$ as demonstrated by a dimensionality argument:

- The dimensionality of the space (4.29) is given by:

$$\sum_{i=0}^{k+2} \dim [k+2i0] = \frac{(k+3)(k+4)^2(k+5)}{12}.$$

- This is the same as $\dim(k+200) - \dim(k00)$.
- There can be no overlap between M_+P^k and (4.29).

Accordingly we have demonstrated:

$$P^k = H^k \oplus M_+ P^{k-2}, \quad (4.30)$$

and hence by applying this result successively,

$$P^k = \sum_{0 \leqq p \leqq k/2} \oplus \, (M_+)^p H^{k-2p}, \quad (4.31)$$

since $P^0 = H^0$ and $P^1 = H^1$.

Moreover this result shows that the space H^k is itself the sum of SU3 irreps given by:

$$H^k = [k00] \oplus [k\,10] \oplus \ldots \oplus [kk0]. \quad (4.32)$$

We have thus proven: **Proposition 4.33.** *The* SU3 *representation V is a multiplicity free sum of all finite dimensional irreducible representations of* SU3.

Moreover we see that:

Corollary 4.34. *The algebra of operators on V generated by $\mathcal{g}$ is isomorphic to the universal enveloping algebra of $\mathcal{g}$.*

5. The Algebra $\mathscr{A}$

We have now obtained V, a space of (unnormalized) boson polynomial ket-vectors which is, by Proposition 4.33, a multiplicity-free sum of all irreps of SU3. This is the precise equivalent—except for normalization—of the Hilbert space $\mathscr{H}$ which was introduced, *by fiat*, as the appropriate space for discussion of SU3 tensor operators in Sect. 2. It will be recognized that the space V is obtained much more naturally and in a completely elementary way.

The next step is the construction of the minimal set of tensor operators—containing at least $\langle 100 \rangle$ and $\langle 110 \rangle$—which closes under commutation. This set closes on $\mathfrak{so}_8$—as we know from Sect. 3—and the objective now is to determine this set of operators as boson polynomials. We simply state the result.

Let us define the six boson polynomial operators—denoted again by $\begin{pmatrix} \Delta \\ w \end{pmatrix}$—to be:

$$\begin{pmatrix}100\\100\end{pmatrix}=2a_1+a_1^2\bar{a}_1+a_1a_2\bar{a}_2+a_1a_3\bar{a}_3+a_1a_{12}\bar{a}_{12}+a_1a_{31}\bar{a}_{31}$$

$$-a_2a_{31}\bar{a}_{23}-a_3a_{12}\bar{a}_{23}, \tag{5.1}$$

$$\begin{pmatrix}010\\100\end{pmatrix}=a_{12}\bar{a}_2-a_{31}\bar{a}_3, \tag{5.2}$$

$$\begin{pmatrix}001\\100\end{pmatrix}=\bar{a}_{23}, \tag{5.3}$$

$$\begin{pmatrix}110\\110\end{pmatrix}=2a_{12}+a_{12}^2\bar{a}_{12}+a_{12}a_{23}\bar{a}_{23}+a_{12}a_{31}\bar{a}_{31}+a_1a_{12}\bar{a}_1+a_2a_{12}\bar{a}_2$$

$$-a_1a_{23}\bar{a}_3-a_2a_{31}\bar{a}_3, \tag{5.4}$$

$$\begin{pmatrix}101\\110\end{pmatrix}=-a_1\bar{a}_{31}+a_2\bar{a}_{23}, \tag{5.5}$$

$$\begin{pmatrix}011\\110\end{pmatrix}=\bar{a}_3, \tag{5.6}$$

• Calculation shows that *each of these operators carries the space V into itself.* Define twelve more operators on V.
For $\Delta = 100, 010, 001$:

$$\begin{pmatrix}\Delta\\010\end{pmatrix}=\left[E_{21},\begin{pmatrix}\Delta\\100\end{pmatrix}\right],\begin{pmatrix}\Delta\\001\end{pmatrix}=\left[E_{32},\begin{pmatrix}\Delta\\010\end{pmatrix}\right]. \tag{5.7}$$

For $\Delta' = 110, 101, 011$:

$$\begin{pmatrix}\Delta'\\101\end{pmatrix}=-\left[E_{32},\begin{pmatrix}\Delta'\\110\end{pmatrix}\right],\begin{pmatrix}\Delta'\\011\end{pmatrix}=-\left[E_{21},\begin{pmatrix}\Delta'\\101\end{pmatrix}\right]. \tag{5.8}$$

• *The algebra of operators on V generated by the eighteen $\begin{pmatrix}\Delta\\w\end{pmatrix}$ will be denoted $\mathscr{A}$.* (To avoid confusion, let us note explicitly that products of operators (and not just commutator products) are admitted in $\mathscr{A}$, as well as linear combinations.)

Observe that $\mathscr{A}$ contains the SU3 Lie algebra $\mathscr{g}$ and hence also the enveloping algebra of $\mathscr{g}$.

$$E_{12}=\left[\begin{pmatrix}101\\101\end{pmatrix},\begin{pmatrix}010\\100\end{pmatrix}\right],E_{21}=\left[\begin{pmatrix}101\\011\end{pmatrix},\begin{pmatrix}010\\010\end{pmatrix}\right], \tag{5.9}$$

$$E_{23}=\left[\begin{pmatrix}101\\110\end{pmatrix},\begin{pmatrix}010\\010\end{pmatrix}\right],E_{32}=\left[\begin{pmatrix}101\\101\end{pmatrix},\begin{pmatrix}010\\001\end{pmatrix}\right]. \tag{5.10}$$

We can therefore view $\mathscr{A}$ as the space of an $\mathfrak{su}_3$ representation ρ through the action: $\rho(x)a = [x, a]$ *for* $x \in \mathscr{g}$, $a \in \mathscr{A}$.

Each of the eighteen generators of $\mathscr{A}$ has been written in the form $\begin{pmatrix}\Delta\\w\end{pmatrix}$, where (as before) we refer to Δ and w as the *upper* and *lower labels.* These labels are interpreted as weights: the upper label is the "weight" of the SU3 *operator pattern* uniquely associated to the operator and the lower label is the weight of the Gelfand–Weyl pattern associated with the operator. As discussed in Sect. 2, the upper label is the *shift* induced by the action of the operator: if $\lambda = [pq0]$ is an irrep of V then $\begin{pmatrix} \Delta \\ w \end{pmatrix} \Big| [pq0] \Big\rangle$ is (possibly the null space) contained in the irrep $\lambda + \Delta$ of V. (Recall that $\lambda + \Delta \sim [p + \Delta_1 - \Delta_3, q + \Delta_2 - \Delta_3, 0]$.)

The lower label w of the operator $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ is a weight of the SU3 representation in $\mathscr{A}$ carried by the tensor operator. In the action of $\begin{pmatrix} \Delta \\ w \end{pmatrix}$ on the SU3 vectors belonging to V, the weight w of the operator adds to the weight of the vector to give the weight of the new vector.

• It is useful to note at this point that the algebra $\mathscr{A}$ is exactly the right "size" for the study of all the SU3 tensor operators. It is shown in [6] that: *if U is any finite dimensional vector subspace of V, and T is any linear transformation U into U, then there exists an element of $\mathscr{A}$ whose restriction to U equals T.*

It follows from this result that:

i) *If a transformation T of V commutes with $\mathscr{A}$ then T is a scalar multiplication.*

ii) *The center of $\mathscr{A}$ is $\mathbb{C}$, the scalar multiplications.*

iii) *V is a simple $\mathscr{A}$-module.*

6. The $\mathscr{so}_8$ Lie Algebra

Direct calculation with the eighteen generators of $\mathscr{A}$ shows that the same three useful rules for the commutators hold as for the (similarly denoted) operators in Sect. 2. That is:

- The three operators with a given upper label commute.
- The three operators with a given lower label commute.
- The three $\begin{pmatrix} 001 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$.
- The three $\begin{pmatrix} 010 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 011 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$.
- The three $\begin{pmatrix} 100 \\ \cdots \end{pmatrix}$ commute with the three $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$ and the three $\begin{pmatrix} 110 \\ \cdots \end{pmatrix}$.

We define six more elements of $\mathscr{A}$:

$$\begin{aligned}
H_1 &= -1 - a_2\bar{a}_2 - a_3\bar{a}_3 - a_{23}\bar{a}_{23}, \\
H_2 &= -1 - a_1\bar{a}_1 - a_3\bar{a}_3 - a_{31}\bar{a}_{31}, \\
H_3 &= -1 - a_1\bar{a}_1 - a_2\bar{a}_2 - a_{12}\bar{a}_{12}, \\
H_4 &= -1 - a_{12}\bar{a}_{12} - a_{23}\bar{a}_{23} - a_{31}\bar{a}_{31}, \\
X &= 1 + a_1\bar{a}_1 + a_2\bar{a}_2 + a_3\bar{a}_3, \\
Y &= -H_4.
\end{aligned}$$

Notice that X and Y commute with $\mathscr{g}$. On the $[pq0]$ sub-representation of V, X acts as scalar multiplication by $p - q + 1$ and Y as scalar multiplication by $q + 1$.

The outcome of a great many laborious commutation calculations may now be summarized by the following result (which is exactly the result established earlier and more abstractly for the operators $\langle 100 \rangle_R$, $\langle 110 \rangle_R$ in Sect. 3).

• *The eighteen generators of $\mathscr{A}$, $\mathscr{g}$, X, and Y span a twenty-eight dimensional complex Lie algebra isomorphic to $\mathscr{so}_8$.* (6.1)

Moreover it is clear from the construction:

• *$\mathscr{A}$ is isomorphic to a quotient of the universal enveloping algebra of $\mathscr{so}_8$.* (6.2)

• *V may be viewed as an irreducible representation of $\mathscr{so}_8$.*

It is useful to give explicitly the isomorphism with the Lie algebra $\mathscr{so}_8$.

Let $J = (\delta_{i,9-i})$ be the 8×8 matrix all of whose entries are zero except those on the second diagonal which are equal to one. We will take for $\mathscr{so}_8$ the Lie algebra of 8×8 complex matrices A such that ${}^tAJ + JA = 0$. These are precisely the 8×8 matrices which are antisymmetric with respect to the second diagonal.

The identification of matrices in $\mathscr{so}_8$ with elements of $\mathscr{A}$ is given in Table 1, where F_{ij} is the 8×8 matrix all of whose entries are zero except the ij^{th} which is one.

We wish to find a real form of the $\mathscr{so}_8$ given in (6.1) for which the representation V is unitary. It will have a maximal compact subalgebra $\mathscr{k}$ whose action on V splits up into finite-dimensional sub-representations. Clearly the sub-algebra $\mathscr{su}_3$ of $\mathscr{g}$, as well as X and Y, belongs to $\mathscr{k}$. Of the 18 remaining basis elements of $\mathscr{so}_8$ we can eliminate immediately all those having $\Delta = (100)$ on the following grounds: If any operator with $\Delta = (100)$ lies in $\mathscr{k}$, then so must $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ since $\mathscr{su}_3 \subset \mathscr{k}$. But the action of $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ on V is unbounded, since $\begin{pmatrix} 100 \\ 100 \end{pmatrix}$ on any highest weight vector of any SU3 irrep λ of V yields a non-vanishing vector of $\lambda + [100]$. A similar argument eliminates operators with $\Delta = (110)$. Since we seek a compact structure we must admit the conjugate operator to every admitted operator as well: this eliminates the operators $\Delta = (011)$ and $\Delta = (001)$, conjugate to $\Delta = (100)$ and (110) respectively.

We are thus left with the six operators: $\begin{pmatrix} 010 \\ \cdots \end{pmatrix}$ and $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$.

The following results are then easily verified:

• *$\mathscr{g}$, $X - Y$, and six Hermitian combinations of the $\begin{pmatrix} 010 \\ \cdots \end{pmatrix}$ and the $\begin{pmatrix} 101 \\ \cdots \end{pmatrix}$ span a fifteen dimensional real Lie algebra $\mathscr{su}_4$.*

• *Each of the subspaces H^k of V is irreducible as a representation of this $\mathscr{su}_4$.*

• *Adjoining $X + Y$ we see that the subalgebra $\mathscr{k}$ is isomorphic to $\mathscr{su}_4 \oplus \mathscr{u}_1$.*

• *In fact, $\mathscr{k}$ is a maximal compact subalgebra of a real form of $\mathscr{so}_8$ corresponding to the* (noncompact) *Lie algebra* $\mathscr{so}(6,2) \simeq D_4^{\mathrm{H}} \simeq D_4^{\mathbb{R},2}$ [11].

Let L be the 8×8 diagonal matrix, $L = \operatorname{diag}(1,1,1,1,-1,-1,-1,-1)$. We can take for $\mathscr{so}(6,2)$ the real Lie algebra of all 8×8 complex matrices A in $\mathscr{so}_8$ such that $-L{}^t\bar{A}L = A$.

Table 1. $\mathscr{A}$ and so_8

$E_{12}=F_{12}-F_{78}$	$E_{21}=F_{21}-F_{87}$
$E_{13}=F_{13}-F_{68}$	$E_{31}=F_{31}-F_{86}$
$E_{23}=F_{23}-F_{67}$	$E_{32}=F_{32}-F_{76}$
$\begin{pmatrix}010\\100\end{pmatrix}=F_{14}-F_{58}$	$\begin{pmatrix}101\\011\end{pmatrix}=-F_{41}+F_{85}$
$\begin{pmatrix}010\\010\end{pmatrix}=F_{24}-F_{57}$	$\begin{pmatrix}101\\101\end{pmatrix}=-F_{42}+F_{75}$
$\begin{pmatrix}010\\001\end{pmatrix}=F_{34}-F_{56}$	$\begin{pmatrix}101\\110\end{pmatrix}=-F_{43}+F_{65}$
$\begin{pmatrix}001\\100\end{pmatrix}=F_{15}-F_{48}$	$\begin{pmatrix}110\\011\end{pmatrix}=-F_{51}+F_{84}$
$\begin{pmatrix}001\\010\end{pmatrix}=F_{25}-F_{47}$	$\begin{pmatrix}110\\101\end{pmatrix}=-F_{52}+F_{74}$
$\begin{pmatrix}001\\001\end{pmatrix}=F_{35}-F_{46}$	$\begin{pmatrix}110\\110\end{pmatrix}=-F_{53}+F_{64}$
$\begin{pmatrix}100\\100\end{pmatrix}=F_{62}-F_{73}$	$\begin{pmatrix}011\\011\end{pmatrix}=-F_{26}+F_{37}$
$\begin{pmatrix}100\\010\end{pmatrix}=F_{83}-F_{61}$	$\begin{pmatrix}011\\101\end{pmatrix}=-F_{38}+F_{16}$
$\begin{pmatrix}100\\001\end{pmatrix}=F_{71}-F_{82}$	$\begin{pmatrix}011\\110\end{pmatrix}=-F_{17}+F_{28}$
$H_i=F_{ii}-F_{9-i,9-i}$	$i=1,2,3,4.$

It follows from the main theorem of [15] that:

- *The irreducible representation V of $so(6,2)$ is unitary.*

7. The Decomposition of the Algebra $\mathscr{A}$ under so_8

We have already noted that the complexified SU3 Lie algebra, denoted by g, acts on $\mathscr{A}$ to define a representation ρ by the action:

$$\rho(x)a \equiv [x,a] \quad \text{for } a\in\mathscr{A}, x\in g. \tag{7.1}$$

Since the so_8 Lie algebra contains g we can extend the action of g on $\mathscr{A}$ to the so_8 Lie algebra, thereby obtaining a representation, also denoted ρ, of so_8 on $\mathscr{A}$:

$$\rho(x)a \equiv [x,a] \quad \text{for } a\in\mathscr{A}, x\in so_8. \tag{7.2}$$

Our objective in this section is to obtain an explicit decomposition of ρ into irreps of so_8. The result is remarkably simple:

Proposition. 7.3. *The representation* ρ *is a direct sum of* irreps *of* so_8 *of the form* (0p00) *each* irrep *occurring once and only once.* That is:

$$\rho = \bigoplus_{p=0}^{\infty} (0p00).$$

(The notation for the orthogonal group irreps is that of [12], with (0000) being the identity irrep and (0100) the 28 dimensional adjoint irrep of so_8.)

To prove this result we shall make use of the known branching rules [13] for the decomposition of the orthogonal groups, using in particular the sub-group chain: $D_4 \supset B_3 \supset D_3 \simeq A_3$. Using the branching rule $A_3 \supset A_2$ (encoded by the Gelfand–Weyl pattern) we can then determine the SU3 irreps contained in each (0p00) so_8 irrep.

Let us begin by noting two facts that are evident upon examining the $B_3 \supset A_3 \supset A_2$ branching rules.

- Every finite dimensional B_3 irrep contains the identity irrep [000] of SU3 *at least once*, (7.4)
- Every B_3 irrep of the form (ab0) contains *precisely one* [000] irrep of SU3. (7.5)

To proceed further we use the $D_4 \supset B_3$ branching rule for the irrep (0p00):

$$(0p00)_{D_4} = \bigoplus_{k=0}^{p} (k \quad p-k \quad 0)_{B_3}. \tag{7.6}$$

Combining this result with the fact (7.5) noted above we see that:

- *The irrep* (0p00) *of* D_4 *reduced with respect to* A_2 *in the chain*

  $D_4 \supset B_3 \supset A_3 \supset A_2$ *contains precisely* ($p+1$) SU3 irreps [000].

In terms of the algebra $\mathscr{A}$, this means that if the sub-representation (0p00) occurs in $\mathscr{A}$, then this sub-representation contains precisely ($p+1$) SU3 invariant operators.

Consider now the algebra $\mathscr{A}$. It is generated as an algebra by the 28 generators of so_8 which belong to the irrep (0100). By considering the highest weight generator—that is, the operator $\begin{pmatrix} 011 \\ 110 \end{pmatrix}$ which commutes with the four raising operators: E_{12}, E_{23}, $\begin{pmatrix} 010 \\ 001 \end{pmatrix}$ and $\begin{pmatrix} 001 \\ 001 \end{pmatrix}$—we see that we certainly must get in the algebra $\mathscr{A}$ all products $\left[\begin{pmatrix} 011 \\ 110 \end{pmatrix}\right]^p$, and hence $\mathscr{A}$ must contain every irrep (0p00) *at least once*.

But we argue that the irrep (0p00) cannot appear *more than once* since each such irrep contains ($p+1$) SU3 invariants and there are *exactly* ($p+1$) linearly independent homogeneous SU3 invariants of degree p in the two independent invariant operators X and Y of the so_8 generating algebra.

Thus: *each irrep* (0p00) *occurs exactly once.*

The same argument establishes that there are no other so_8 irreps in the representation ρ, aside from $\bigoplus_{p=0}^{\infty}(0p00)$, since any additional irrep would necessarily contain one or more SU3 invariants, and the direct sum $\oplus\,(0p00)$ has exhausted all such invariants.

Closer examination of this proof shows that it contains an assumption, which—though valid—nonetheless itself requires proof. This is the assumption that: *the sub-algebra of* $\mathscr{A}$, *denoted* $\mathscr{B}_0^0$, *of all* SU3 *invariant operators is the algebra of polynomials in the two invariants X and Y.* That is:

$$\mathscr{B}_0^0 = \mathbb{C}[X, Y]. \tag{7.7}$$

One can establish this result as follows. Let $h \in \mathscr{B}_0^0$. Since h is SU3-invariant, it will act as scalar multiplication by $f(k,j)$ on the subspace $[k+jj0]$ of V, where the number $f(k,j)$ is determined by the condition that $h|a_1^k a_{12}^j\rangle = f(k,j)|a_1^k a_{12}^j\rangle$. Because h can be expressed as a polynomial differential operator in the a_i and the a_α, the function f must be a polynomial in k and j. After a linear change of variables from (k,j) to (X, Y), f becomes a polynomial in X and Y.

There is an equivalent form of the result (7.3), using now an algebraic formulation, which is of interest:

• *The polynomial algebra* $\mathbb{C}\left[\begin{pmatrix} 011 \\ 110 \end{pmatrix}\right]$ *is the commutant of*

$$\left\{E_{12}, E_{23}, \begin{pmatrix} 010 \\ 001 \end{pmatrix}, \begin{pmatrix} 001 \\ 001 \end{pmatrix}\right\} \text{ in the algebra } \mathscr{A}. \tag{7.8}$$

Since the four operators $\{\ldots\}$ in (7.8) are the raising operators in $\mathfrak{so}_8$, and $\begin{pmatrix} 011 \\ 110 \end{pmatrix}$ is the operator of highest weight, the equivalence of (7.8) to (7.3) is clear.

8. The SU3 Decomposition of the Algebra $\mathscr{A}$

From Sect. 7 we know that the algebra $\mathscr{A}$ splits into a direct sum of the $\mathfrak{so}_8$ representations $(0p00)$.

Let us denote the operators belonging to the $\mathfrak{so}_8$ irrep $(0p00)$ as *the operators belonging to level p*. The space of operators at level p which are *scalar multiples* (by which we mean multiples by elements of $\mathbb{C}[X, Y]$) of operators at lower levels will be called *the old operators at level p*. By the *new operators at level p* we shall mean an $\mathfrak{su}_3$ subspace of $(0p00)$ complementary to the space of old operators.

The identity operator spans the space of new operators at level 0. There are no nonzero level 0 old operators.

The operators belonging to level 1 are just the Lie algebra of $\mathfrak{so}_8$, and among them the operators of highest $\mathfrak{su}_3$ weight are just the ones that commute with E_{12} and E_{23}, namely the nine operators:

$$X, Y, E_{13}, \begin{pmatrix} 100 \\ 100 \end{pmatrix}, \begin{pmatrix} 010 \\ 100 \end{pmatrix}, \begin{pmatrix} 001 \\ 100 \end{pmatrix}, \begin{pmatrix} 110 \\ 110 \end{pmatrix}, \begin{pmatrix} 101 \\ 110 \end{pmatrix}, \begin{pmatrix} 011 \\ 110 \end{pmatrix}. \tag{8.1}$$

The level 1 old operators are spanned by X and Y. The last seven operators of (8.1) are new at level 1.

The operators (8.1) generate a nine dimensional Lie algebra. They also generate a sub-algebra of $\mathscr{A}$, which we shall denote by $\mathscr{B}$. We shall see that there are relations among the elements of $\mathscr{B}$ so that the generators, (8.1), are not independent. One of our major objectives is to prove that the commutant of $\{E_{12}, E_{23}\}$ in $\mathscr{A}$ is precisely the subalgebra $\mathscr{B}$.

Let us now examine the operators at level 2. Decomposing (0200) in terms of su_3 irreps using the branching rules, we obtain the following:

New operators at level 2:

$$1[420], 3[320], 3[310], 6[220], 7[210], 6[200]. \tag{8.2}$$

Old operators at level 2:

$$2 \times (\text{new operators at level } 1) + 3 \times (\text{new operators at level } 0). \tag{8.3}$$

The identification of the set of new operators through level 2 by su_3 irrep and shift labels is unambiguous with the exception of the operators transforming as the adjoint irrep [210]. The [210] tensor operator at level 1 and *one* of the new [210] tensor operators at level 2 both have shift label (111).

This problem was not unforeseen. The shift labels $\left(\text{the upper labels in } \begin{pmatrix} \Delta \\ w \end{pmatrix}\right)$ are inadequate since there are in general several distinct operators with the same su_3 irrep and shift labels. This is because the operators in $\mathscr{A}$, being generic operators, actually achieve maximal multiplicity, which is known to be [5, 14] equal to the number of distinct Gelfand–Weyl patterns having the same su_3 irrep and shift labels.

Notation 8.4. We shall denote by $\Gamma_t(pq0)$ $(1 \leqq t \leqq \min(p-q+1, q+1))$ the set of all weights (shifts), each taken once, that occur in $[pq0]$ with multiplicity t or greater.

Simply as a way of enumerating the independent new operators at a fixed level, we will write:

$$\begin{pmatrix} \Gamma_t \\ pq0 \end{pmatrix} \tag{8.5}$$

for a set of highest weight tensor operators $\langle pq0 \rangle$ with shifts belonging to $\Gamma_t(pq0)$.

Remark 8.6. It is one of the consequences of Theorem 8.12 that this labelling is unambiguous.

For example, the shift (111) has multiplicity 2 for the irrep [210], so that there are 7 operators in the set $\begin{pmatrix} \Gamma_1 \\ 210 \end{pmatrix}$, and one operator in the set $\begin{pmatrix} \Gamma_2 \\ 210 \end{pmatrix}$. (Of course, there are for each element in these sets $\dim[210] = 8$ independent components of the associated tensor operator.)

The new operator at level 0 is labelled $\begin{pmatrix} \Gamma_1 \\ 000 \end{pmatrix}$. At level 1, (see 8.1), we can label new operators by $\begin{pmatrix} \Gamma_2 \\ 210 \end{pmatrix}$, $\begin{pmatrix} \Gamma_1 \\ 100 \end{pmatrix}$, and $\begin{pmatrix} \Gamma_1 \\ 110 \end{pmatrix}$.

In this notation, the 26 new operators at level 2, (8.2), are given by the following six labels:

$$\begin{pmatrix}\Gamma_3\\420\end{pmatrix},\begin{pmatrix}\Gamma_2\\320\end{pmatrix},\begin{pmatrix}\Gamma_2\\310\end{pmatrix},\begin{pmatrix}\Gamma_1\\220\end{pmatrix},\begin{pmatrix}\Gamma_1\\210\end{pmatrix},\begin{pmatrix}\Gamma_1\\200\end{pmatrix}. \tag{8.7}$$

We are obliged to check that the new operators, (8.2), actually effect the shifts implied by the notation (8.7).

In order to do this, we consider the highest weight operators at level 2 as symmetrized products of the nine operators in (8.1). (We must use symmetrized products in order to guarantee that the resulting operators belong entirely to (0200). There are 45 such symmetrized products but, from (8.2) and (8.3), just 26 new tensor operators and $14+3=17$ old tensor operators at level 2. There must be precisely two relations among the symmetrized products:

$$\begin{pmatrix}100\\100\end{pmatrix}\begin{pmatrix}011\\110\end{pmatrix}+\begin{pmatrix}011\\110\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix}-\begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}101\\110\end{pmatrix}-\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}=2XE_{13}, \tag{8.8}$$

$$\begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}101\\110\end{pmatrix}+\begin{pmatrix}101\\110\end{pmatrix}\begin{pmatrix}010\\100\end{pmatrix}-\begin{pmatrix}001\\100\end{pmatrix}\begin{pmatrix}110\\110\end{pmatrix}-\begin{pmatrix}110\\110\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix}=2YE_{13}. \tag{8.9}$$

These relations allow us to eliminate two of the symmetrized products, and we choose to eliminate:

$$\begin{pmatrix}100\\100\end{pmatrix}\begin{pmatrix}011\\110\end{pmatrix}+\begin{pmatrix}011\\110\end{pmatrix}\begin{pmatrix}100\\100\end{pmatrix}\text{ and }\begin{pmatrix}001\\100\end{pmatrix}\begin{pmatrix}110\\110\end{pmatrix}+\begin{pmatrix}110\\110\end{pmatrix}\begin{pmatrix}001\\100\end{pmatrix}. \tag{8.10}$$

The 26 remaining symmetrized products *not involving X or Y* thus enumerate exactly the new operators at level 2.

There is only one possibility for a [420] operator: the product $(E_{13})^2$ which has shift (222). Since for [420] the shift (222) has multiplicity 3, and moreover the set of shifts with multiplicity $\geqq 3$ is exactly (222) itself, we see that the resultant [420] operator is to be labelled by Γ_3 as shown in (8.7).

Similarly the (symmetrized) products of E_{13} with the three $\begin{pmatrix}\cdots\\100\end{pmatrix}$ operators yield three [310] operators with shifts: (211), (121) and (112). Since these shifts have multiplicity 2, and since the set of shifts with multiplicity $\geqq 2$ is just these three shifts; we conclude that the resulting operators are to be enumerated by $\begin{pmatrix}\Gamma_2\\310\end{pmatrix}$. The $\begin{pmatrix}\Gamma_2\\320\end{pmatrix}$ operators follow in the same way.

The symmetric products of $\begin{pmatrix}\cdots\\100\end{pmatrix}$ with themselves yield the 6 operators [200] with the shifts: (200), (020), (002), (110), (101), (011) all of multiplicity 1 (and the only shifts possible). These operators are accordingly denoted by $\begin{pmatrix}\Gamma_1\\200\end{pmatrix}$, and similarly for $\begin{pmatrix}\Gamma_1\\220\end{pmatrix}$.

The enumeration of the [210] operators at level 2 is quite significant. We have seen that the symmetric products of $\begin{pmatrix}\cdots\\100\end{pmatrix}$ and $\begin{pmatrix}\cdots\\110\end{pmatrix}$ lead to 9 [210] operators of which—using Eq. (8.8) and (8.9)—only 7 are independent. Of these 7 operators, 6 have shifts which are permutations of (210); the 7th has the shift (111). *These are precisely the set of shifts that occur with multiplicity* 1 *or greater, and accordingly we enumerate these operators as the set* $\begin{pmatrix} \Gamma_1 \\ 210 \end{pmatrix}$.

Thus we have verified the consistency of the notation in (8.7), since this enumeration exactly accounts for all the new operators at level 2 as given by (8.2).

We are now in a position to analyze the SU3 structure of all the tensor operators that belong to level p.

Theorem 8.11 *A basis for the* SU3 *operators of highest weight belonging to the irrep* (0p00) *in* $\mathscr{A}$ *is given by the totally symmetrized products:*

$$\mathscr{S}\left[X^a Y^b (E_{13})^c \begin{pmatrix} 010 \\ 100 \end{pmatrix}^d \begin{pmatrix} 101 \\ 110 \end{pmatrix}^e \begin{pmatrix} 100 \\ 100 \end{pmatrix}^f \begin{pmatrix} 011 \\ 110 \end{pmatrix}^g \begin{pmatrix} 001 \\ 100 \end{pmatrix}^h \begin{pmatrix} 110 \\ 110 \end{pmatrix}^i \right],$$

where $\mathscr{S}[\ldots]$ *denotes symmetrization and with:* $a+b+c+d+e+f+g+h+i=p$, *and* $fg=hi=0$.

This result may be stated in a more structural way in the equivalent form:

Theorem 8.12. *The* SU3 *decomposition of the new tensor operators in* $\mathscr{A}$ *belonging to the* $\mathfrak{so}_8$ irrep (0p00) (*or, what is the same, the operators in 8.11 with* $a=b=0$) *is given by:*

$$\begin{matrix} & & & \begin{pmatrix} \Gamma_{p+1} \\ 2p\,p\,0 \end{pmatrix} & & & \\ & & \begin{pmatrix} \Gamma_p \\ 2p-1\,p\,0 \end{pmatrix} & & \begin{pmatrix} \Gamma_p \\ 2p-1\,p-1\,0 \end{pmatrix} & & \\ & \begin{pmatrix} \Gamma_2 \\ p+1\,p\,0 \end{pmatrix} & & \begin{pmatrix} \Gamma_2 \\ p+1\,p-1\,0 \end{pmatrix} & \cdots & \begin{pmatrix} \Gamma_2 \\ \overline{p+1\,1\,0} \end{pmatrix} & \\ \begin{pmatrix} \Gamma_1 \\ p\,p\,0 \end{pmatrix} & & \begin{pmatrix} \Gamma_1 \\ p\,p-1\,0 \end{pmatrix} \cdots & & & \cdots & \begin{pmatrix} \Gamma_1 \\ p\,0\,0 \end{pmatrix}. \end{matrix}$$

Notice that we have already verified this for levels $p=0, 1$, and 2.

We wish to prove the two theorems together so that we can understand the relationship between the structure in (8.12) and the exponents in (8.11). The requirement $fg=hi=0$ has been imposed to take account of the relations (8.8) and (8.9).

We begin by discussing the elements of the bottom row of (8.12), the Γ_1. This is the main part of the analysis from which all else will easily follow.

Lemma 8.13. *The highest weight operators in the set* $\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix}$ *correspond one-to-one with the set of elements*

$$\mathscr{S}\left[\begin{pmatrix} 010 \\ 100 \end{pmatrix}^d \begin{pmatrix} 101 \\ 110 \end{pmatrix}^e \begin{pmatrix} 100 \\ 100 \end{pmatrix}^f \begin{pmatrix} 011 \\ 110 \end{pmatrix}^g \begin{pmatrix} 001 \\ 100 \end{pmatrix}^h \begin{pmatrix} 110 \\ 110 \end{pmatrix}^i \right]$$

with: $d+f+h=p, e+g+i=q, fg=hi=0$ *and with*
shift $\Delta=(\Delta_1\Delta_2\Delta_3)$ *with* $\Delta_1=e+f+i$, $\Delta_2=d+g+i$, $\Delta_3=e+g+h$.

We give the proof for the case $p \geqq q$, the opposite case being similar. Let us first consider arbitrary symmetrized products of the six operators $\begin{pmatrix} \cdots \\ 100 \end{pmatrix}$ and $\begin{pmatrix} \cdots \\ 110 \end{pmatrix}$.

Consider a product of degree p in the $\begin{pmatrix} \cdots \\ 100 \end{pmatrix}$, which need not be symmetrized since these operators all commute. The resultant operator must belong to the irrep $[p00]$, for which all of the weights—and hence all possible shifts—occur with multiplicity 1. Hence the product yields exactly those operators of highest SU3 weight denoted by $\begin{pmatrix} \Gamma_1 \\ p00 \end{pmatrix}$. Similarly the product of q operators $\begin{pmatrix} \cdots \\ 100 \end{pmatrix}$ yields precisely those operators denoted by $\begin{pmatrix} \Gamma_1 \\ qq0 \end{pmatrix}$.

Now consider arbitrary products of the six operators $\begin{pmatrix} \cdots \\ 100 \end{pmatrix}$ and $\begin{pmatrix} \cdots \\ 110 \end{pmatrix}$, restricted, however, to p operators $\begin{pmatrix} \cdots \\ 100 \end{pmatrix}$ and q operators $\begin{pmatrix} \cdots \\ 110 \end{pmatrix}$. Any such product will have a definite shift, Δ, and will belong to $[p+q\,q\,0]$ since the factors have highest weight. Symmetrizing over all possible arrangements of the factors will preserve Δ, but the problem is to determine the number of *distinct* symmetrized products with the same shift Δ. Since the ordering is not relevant to this distinctness, one sees that the number of distinct such products is exactly the multiplicity of the shift Δ in the set of products

$$\begin{pmatrix} \Delta' \\ p00 \end{pmatrix}\begin{pmatrix} \Delta'' \\ qq0 \end{pmatrix}, \Delta=\Delta'+\Delta''.$$

This multiplicity exactly equals the multiplicity of the *weight* Δ in the (SU3) tensor product $[p00]\otimes[qq0]$. We make use of the known decomposition (for $p \geqq q$):

$$[p00]\otimes[qq0]=[p+q\,q\,0]+[p+q-1\,q\,1]+\cdots+[pqq]. \tag{8.14}$$

To enumerate the weights on the right-hand side we use another lemma.

Lemma 8.15. *A weight w has multiplicity $r \geqq 1$ in $[a\,b\,c]$ if and only if it has multiplicity $r+1$ in $[a+1\,b\,c-1]$.*

Proof. Examine Gelfand–Weyl patterns. The multiplicities of $w=(w_1\,w_2\,w_3)$ (with $w_1+w_2+w_3=a+b+c$) in the two irreps equal the numbers of patterns possible below with $\alpha+\beta=w_1+w_2$.

$$\begin{pmatrix} a & & b & & c \\ & \alpha & & \beta & \\ & & w_1 & & \end{pmatrix} \qquad \begin{pmatrix} a+1 & & b & & c-1 \\ & \alpha & & \beta & \\ & & w_1 & & \end{pmatrix}$$

There is clearly one more possibility for $[a+1\,b\,c-1]$ than for $[a\,b\,c]$.

Now let us re-consider the weights belonging to the right-hand side of (8.14), this time as shifts. The shifts of multiplicity t in $[p+q\,q\,0]$ (for $t = 1, 2, \ldots, (q+1)$) appear—according to Lemma 8.15—as weights of each of the irreps $[p+q-r\,q\,r]$ for $0 \leqq r \leqq t-1$. It follows, therefore, from the multiplicities with which the weights appear—as given by the lemma—that the set of shifts denoted by the sum:

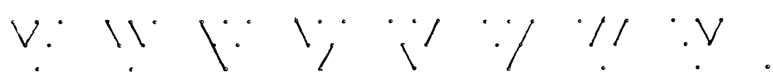
$$\sum_{t=1}^{q+1} t\Gamma_t(p+q\,q\,0), \tag{8.16}$$

(where t is a multiplicity) actually exhausts the set with multiplicities of shifts Δ appearing in the symmetrized products $\mathscr{S}\left[\begin{pmatrix}\cdots\\100\end{pmatrix}^p\begin{pmatrix}\cdots\\110\end{pmatrix}^q\right]$.

We must next come to grips with the restrictions imposed on the exponents in (8.13). We start with the shifts of multiplicity one.

Lemma 8.17. *To every shift Δ of multiplicity one in $[p+q\,q\,0]$ there corresponds an operator of* (8.13) *with the additional restriction that $de = 0$.*

The weights which are of multiplicity 1 belong to the set of patterns of the types:



(A line here signifies that two integers in the pattern are equal.) Corresponding operators can be written out explicitly in terms of the entries in the patterns, and we shall do so at the conclusion of the proof of (8.13). The eight pattern types shown correspond to the eight ways of choosing one exponent to be zero from each of the pairs $\{d, e\}$, $\{f, g\}$, $\{h, i\}$. We shall in fact see that the correspondence of (8.17) is bijective.

We now count the operators that can be built with Lemmas 8.15 and 8.17.

Each shift Δ of multiplicity t in $[p+q\,q\,0]$ is by (8.15) of multiplicity 1 in $[p+q+2-2t\;\, q+1-t\;\, 0]$. An operator with shift Δ and of this latter $\mathfrak{su}_3$ type can be constructed (by 8.17) from the product of $p+1-t$ of the $\begin{pmatrix}\cdots\\100\end{pmatrix}$ and $q+1-t$ of the $\begin{pmatrix}\cdots\\110\end{pmatrix}$. We next multiply this product operator by $\left[\begin{pmatrix}010\\100\end{pmatrix}\begin{pmatrix}101\\110\end{pmatrix}\right]^k$ $\left[\begin{pmatrix}100\\100\end{pmatrix}\begin{pmatrix}011\\110\end{pmatrix}\right]^l\left[\begin{pmatrix}001\\100\end{pmatrix}\begin{pmatrix}110\\110\end{pmatrix}\right]^m$ with $k+l+m = t-1$. There are $\dfrac{t(t+1)}{2}$ possible choices of the k, l, m.

From (8.16) we learn that the multiplicity of Δ in $[p00] \otimes [qq0]$ is also $t(t+1)/2$. Thus we have constructed *all* the operators of shift Δ arising as a product of p of the $\begin{pmatrix}\cdots\\100\end{pmatrix}$ and q of the $\begin{pmatrix}\cdots\\110\end{pmatrix}$. Lemma 8.13 now follows from the observation that exactly one of these operators satisfies the criteria of (8.13), the one with $k = t-1$ and $l = m = 0$.

It is essential that we give the association established by Lemma (8.13) explicitly, and we pause to do so before continuing with the proof of (8.11) and (8.12). The labels Γ_1—which denote the highest weight operators, $\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix}$, that have shifts of multiplicity 1 or greater—can now be written as fully explicit operator patterns:

$$\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix} = \begin{pmatrix} & & \gamma_{11} & & \\ & \gamma_{12} & & \gamma_{22} & \\ p+q & & q & & 0 \end{pmatrix}. \tag{8.18}$$

The shift associated to this pattern is: $(\Delta_1 \Delta_2 \Delta_3)$ with $\Delta_1 = \gamma_{11}$,

$$\Delta_2 = \gamma_{12} + \gamma_{22} - \gamma_{11} \text{ and } \Delta_3 = p + 2q - \gamma_{12} - \gamma_{22}.$$

Remark. For each shift Δ of $[p+q\,q\,0]$ there is a *unique* pattern which is special in that either γ_{12} or γ_{22} is "tied" to its largest or smallest value respectively. This is often expressed by saying that the pattern is "stretched". We always select for Γ_1 the stretched patterns.

To correlate the associated operator with the basis set (8.13) we distinguish four cases:

(a) $\gamma_{12} = p+q$ and $\gamma_{11} \geqq q$:

$$\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix} \leftrightarrow \mathscr{S}\left[\begin{pmatrix} 100 \\ 100 \end{pmatrix}^{\gamma_{11}-q} \begin{pmatrix} 010 \\ 100 \end{pmatrix}^{p+q-\gamma_{11}} \begin{pmatrix} 110 \\ 110 \end{pmatrix}^{\gamma_{22}} \begin{pmatrix} 101 \\ 110 \end{pmatrix}^{q-\gamma_{22}}\right], \tag{8.19a}$$

(b) $\gamma_{12} = p+q$ and $\gamma_{11} \leqq q$:

$$\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix} \leftrightarrow \mathscr{S}\left[\begin{pmatrix} 010 \\ 100 \end{pmatrix}^{p} \begin{pmatrix} 110 \\ 110 \end{pmatrix}^{\gamma_{22}} \begin{pmatrix} 011 \\ 110 \end{pmatrix}^{q-\gamma_{11}} \begin{pmatrix} 101 \\ 110 \end{pmatrix}^{\gamma_{11}-\gamma_{22}}\right], \tag{8.19b}$$

(c) $\gamma_{22} = 0$ and $\gamma_{11} \geqq q$:

$$\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix} \leftrightarrow \mathscr{S}\left[\begin{pmatrix} 100 \\ 100 \end{pmatrix}^{\gamma_{11}-q} \begin{pmatrix} 010 \\ 100 \end{pmatrix}^{\gamma_{12}-\gamma_{11}} \begin{pmatrix} 001 \\ 100 \end{pmatrix}^{p+q-\gamma_{12}} \begin{pmatrix} 101 \\ 110 \end{pmatrix}^{q}\right] \tag{8.19c}$$

(d) $\gamma_{22} = 0$ and $\gamma_{11} \leqq q$:

$$\begin{pmatrix} \Gamma_1 \\ p+q\,q\,0 \end{pmatrix} \leftrightarrow \mathscr{S}\left[\begin{pmatrix} 010 \\ 100 \end{pmatrix}^{\gamma_{12}-q} \begin{pmatrix} 001 \\ 100 \end{pmatrix}^{p+q-\gamma_{12}} \begin{pmatrix} 011 \\ 110 \end{pmatrix}^{q-\gamma_{11}} \begin{pmatrix} 101 \\ 110 \end{pmatrix}^{\gamma_{11}}\right]. \tag{8.19d}$$

These operators are independent since they belong to *distinct* shifts (as implied by the definition of Γ_1).

Let us now complete the proof of Theorems 8.11 and 8.12.

We first note that the operators of (8.11) are linearly independent. This can be shown by direct calculation [8].

We proceed by induction. Let us assume that both (8.11) and (8.12) have been established for all levels less than p; we investigate level p.

The old operators of (8.11) are those with $a > 0$ or $b > 0$ or both. The rest are new.

Factoring out all powers of X and Y, one is led to enumerate the old operators at level p by the symbolic formula:

$$\sum_{k=0}^{p-1} (p-k+1) \times (\text{new operators at level } k). \tag{8.20}$$

Factoring out all powers of E_{13}, one is (via 8.13) led to enumerate the new operators at level p by the symbolic formula:

$$\sum_{k=0}^{p} E_{13}^{p-k} \times \left(\begin{pmatrix} \Gamma_1 \\ kk0 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ k\,k-1\,0 \end{pmatrix} \cdots \begin{pmatrix} \Gamma_1 \\ k00 \end{pmatrix} \right). \tag{8.21}$$

Since E_{13} effects no shift and has group label $[210] \sim [1\,0\,-1]$, we can use Lemma 8.15 to write

$$E_{13}^{p-k} \begin{pmatrix} \Gamma_1 \\ k\,k-r\,0 \end{pmatrix} = \begin{pmatrix} \Gamma_{p+1-k} \\ 2p-k\,p-r\,0 \end{pmatrix}. \tag{8.22}$$

The combination of (8.20), (8.21), and (8.22) establishes Theorem 8.12.

It remains only to show that the operators belonging to level p as given by (8.11) actually exhaust the complete set of independent SU3 operators in the $\mathfrak{so}_8$ irrep $(0p00)$. That this is so follows directly from the branching law for the chain $D_4 \supset B_3 \supset D_3 \simeq A_3 \supset A_2$, completing the proof of (8.11).

Theorem 8.23. *The algebra $\mathscr{B}$ is the full commutant of $\{E_{12}, E_{23}\}$ in $\mathscr{A}$.*

Proof. The commutant is the space of all SU3 highest weight operators in $\mathscr{A}$. All such are explicitly written down in (8.11) and are seen to lie in $\mathscr{B}$.

9. The Structure of Tensor Operators in SU3

In this concluding section we shall summarize the results obtained in earlier sections and discuss the implications of these results for the structure of the tensor operators in SU3. We will show that the algebra $\mathscr{A}$ contains a qualitative, complete, and globally defined description of all tensor operators in SU3, which can be exploited to resolve all multiplicities.

Let us recall the basic results established. First of all the Hilbert space, V, on which the tensor operators act, was shown to be a single irreducible unitary representation of $\mathfrak{so}(6,2)$ which contains every irrep of SU3 exactly once. The tensor operators—the algebra $\mathscr{A}$—were shown to be a direct sum of $\mathfrak{so}_8$ irreps, each $\mathfrak{so}_8$ irrep of the form $(0p00)$, $p = 0, 1, 2, \ldots$ occurring once and only once.

The most important structural result is Theorem 8.12, which gives a detailed description of the SU3 decomposition of the operators in each $\mathfrak{so}_8$ irrep $(0p00)$.

How do these results establish a splitting of the SU3 multiplicity?

To answer this, let us recall that only those SU3 tensor operators whose shift labels are *exactly* of multiplicity 1 are free of multiplicity problems. To enumerate the set of operators having fixed SU3 irrep and shift labels (called a *multiplicity set*) we introduced (in Sect. 8) the operator pattern labels Γ_t, with $t = 1, 2, \ldots$.

Consider the SU3 tensor operators transforming as the irrep $[p\,q\,0]$. To enumerate the multiple occurrences, we use the sequence:

$$\left\langle \begin{matrix} \Gamma_1 \\ p\,q\,0 \end{matrix} \right\rangle, \left\langle \begin{matrix} \Gamma_2 \\ p\,q\,0 \end{matrix} \right\rangle, \ldots, \left\langle \begin{matrix} \Gamma_N \\ p\,q\,0 \end{matrix} \right\rangle, N = \min(p - q + 1, q + 1). \tag{9.1}$$

By $\left\langle \begin{matrix} \Gamma_t \\ p\,q\,0 \end{matrix} \right\rangle$ we mean the set of tensor operators whose highest weight components are the $\begin{pmatrix} \Gamma_t \\ p\,q\,0 \end{pmatrix}$ of (8.12) and (8.11). This set contains at most one operator in any multiplicity set.

This listing is not entirely satisfactory. Upon tracing it to its source one finds that it depends, through the definition of the $\begin{pmatrix} \Gamma_t \\ p\,q\,0 \end{pmatrix}$, upon the arbitrary choice (8.10).

Now observe the remarkable fact implicit in Theorem 8.12.

- The definition of that $\langle p\,q\,0 \rangle$ tensor operator of a multiplicity t set which occurs last in the list (9.1) (or equivalently, which belongs to lowest level) is independent of the choice (8.10). The operator is determined as a subset of the $\mathscr{so}_8$ irrep $(0\,p + 1 - t\,0\,0)$.

For example, the 27-plet operator belonging to the multiplicity 3 set with shift (333)—denoted by $\left\langle \begin{matrix} \Gamma_3 \\ 420 \end{matrix} \right\rangle$—is uniquely determined by being the only SU3 operator $\langle 420 \rangle$ in the $\mathscr{so}_8$ irrep (0200).

Suppose the *first operator* $\langle p\,q\,0 \rangle$ in a multiplicity set (the last in the list (9.1), or equivalently, the one belonging to lowest level) belongs to level s. Theorem 8.12 shows that the level $s + 1$ contains *exactly one operator* of the same multiplicity set linearly independent of $X\langle p\,q\,0 \rangle$ and $Y\langle p\,q\,0 \rangle$.

At each higher level, one more operator in the multiplicity set is defined. Aside from the first operator in the multiplicity set the operators are satisfactorily defined only up to linear combinations of previously defined operators. (This situation is exactly the same as exists for the Poincaré–Birkhoff–Witt theorem which establishes a *basis* for the enveloping algebra of a Lie algebra, but *not* a unique basis.)

To fully resolve the multiplicity problem, however, one seeks, if possible, to go further. To achieve this goal, we must return to the *orthonormal* vectors and *orthonormal* tensor operators of Sect. 2. Take the ordered (from right to left) basis from (9.1) of tensor operators in a fixed multiplicity set. The first operator is defined without having made any arbitrary choices. By carrying out a Gram–Schmidt process on the second operator in the basis, we produce a tensor operator orthogonal to the first which is also defined choice-free. We can continue the process, in effect simply orthogonalizing a lower triangular ordered set.

- *The ordered basis of (9.1) becomes a tensor operator basis free of arbitrary choices upon orthogonalizing. This resolves all multiplicities in the set of* SU3 *tensor operators.*

Note, however, that in orthogonalizing we have left the polynomial algebra $\mathscr{A}$, since square roots of rational functions have been admitted.

The SU3 decomposition of the algebra $\mathscr{A}$ established in Theorem 8.12 has another aspect:

• *Every operator in* $\mathscr{A}$ *is a linear combination of the* SU3 *tensor operators* $\begin{pmatrix} \Gamma_t \\ p\,q\,0 \end{pmatrix}$ *with invariant operators* (the polynomials $\mathbb{C}[X, Y]$) *as coefficients.*

The algebra $\mathscr{A}$ is accordingly a global formulation of the set of all SU3 tensor operators which resolves structurally the problem of multiplicity.

References

1. Biedenharn, L. C., Louck, J. D.: Angular Momentum in Quantum Physics. In: Encyclopedia of Mathematics and Its Applications. Rota, G. C. (Ed.). Reading, MA: Addison-Wesley Publishing Co. 1981
2. Wigner, E. P.: Am. J. Math. **63**, 57–63 (1941)
3. Mackey, G. W.: Pacific J. Math. **8**, 503–510 (1958)
4. Sharp, W. T., Biedenharn, L. C., de Vries, E., van Zanten, J.: Can. J. Math. **27**, 246–255 (1975)
5. Biedenharn, L. C., Giovannini, A., Louck, J. D.: Canonical definition of Wigner coefficients in U_n, J. Math. Phys. **8**, 691–700 (1967)
6. Flath, D. E., Biedenharn, L. C.: Beyond the enveloping algebra of $\mathscr{sl}_3$. 1982
7. Louck, J. D., Biedenharn, L. C.: Canonical unit adjoint tensor operators in $U(n)$, J. Math. Phys. **11**, 2368–2414 (1970)
8. Biedenharn, L. C., Louck, J. D.: A pattern calculus for tensor operators in the unitary groups, Commun. Math. Phys. **8**, 89–131 (1968)
9. Lohe, M. A., Biedenharn, L. C., Louck, J. D.: Structural properties of the self-conjugate SU(3) tensor operators, J. Math. Phys. **18**, 1883–1891 (1977), and references cited there
10. Michel, L.: Application of Group Theory to Quantum Physics; Algebraic Aspects. In: Lecture Notes in Physics. Battelle Rencontres. Bargmann, V. (Ed.), pp. 36–143. Berlin, Heidelberg, New York: Springer-Verlag, 1970
11. Tits, J.: Tabellen zu den einfachen Lieschen Gruppen und Darstellungen. In: Lecture Notes in Mathematics. Vol. 40. Berlin, Heidelberg, New York: Springer-Verlag 1967
12. McKay, W. G., Patera, J.: Tables of dimensions, indices and branching rules for representations of simple Lie algebras. New York: M. Dekker 1981
13. Gel'fand, I. M., Tseitlin, M. L.: Dokl. Akad. Nauk, SSSR, **71**, 825–828 (1950)
14. Klimyk, A. U.: A. Math. Soc. Trans., Series 2, Vol. **76**, Am. Math. Soc. (Providence) 1968, pp. 63–73
15. Enright, T. J., Howe, R., Wallach, N. R.: "A classification of unitary highest weight modules," Preprint (1981)

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