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Difficulties with Massless Particles?

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Abstract. Some difficulties with sharp momentum (one-particle) states for massless particles are indicated, in the framework of unitary irreducible representations of the Poincaré group. It is shown that a Poincaré covariant set of such states requires the introduction, in the spatial direction opposite to the point stabilized, of momentum generalized eigenstates which (when the helicity is nonzero) have a nontrivial orbital transformation. The relevance of these generalized momentum eigenstates for massless theories is then shown

1. Introduction

The basis of all the existing modern fundamental particle theories is a family of interacting massless fields. This is so in gravity, in the electroweak model before spontaneous symmetry breaking, and in quantum chromodynamics (at least in the original version). For this reason massless particles became even more important in the last decade or so

In spite of a few recurring claims that from some point of view the massless limit can be considered smooth [1], singularities do appear in the massless limit. Even the kinematics of one-particle massless particles presents a picture that is entirely different from massive particle kinematics.

In this paper we shall show that the concept of sharp momentum states, used elsewhere in physics with relative safety, has to be treated with great care as far as massless particles are concerned. In fact, distributions of the δ-function type are not relevant everywhere—sometimes they must be replaced by "twisted δ-functions" that, though supported on a single point, carry angular momentum.

A Paradox

Consider a massless state with momentum $\mathbf{p}$ and helicity $j \neq 0$, commonly defined by

$$\frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\mathbf{J}|p,j\rangle = j|p,j\rangle \tag{1}$$

If $\mathbf{p}$ points along the positive (negative) third coordinate axis, then the third component of angular momentum is $+j$ $(-j)$. On the other hand, consider the generator J_3 of angular momentum, in the canonical form [2] of the unitary irreducible representation (UIR) of the Poincaré group $\mathscr{P}$ with mass zero and helicity j (denoting $\partial/\partial p_k$ by ∂_k):

$$J_3 = -i(p_1\partial_2 - p_2\partial_1) + j \tag{2}$$

Then on δ-functions localized at $p_1 = p_2 = 0$, $p_3 \neq 0$ fixed arbitrarily, irrespective of the sign of p_3, we have $J_3 = j$. On the other hand, on δ-functions localized at $p_2 = p_3 = 0$, $p_1 \neq 0$ fixed, (with $p_0 = |\mathbf{p}|$)

$$J_1 = -i(p_2\partial_3 - p_3\partial_2) + jp_1(p_0 + p_3)^{-1} \tag{3}$$

reduces to $J_1 = j(\operatorname{sgn} p_1)$. A similar result is obtained for all other rotation group generators except that of the little group, in accordance with (1). We thus get the paradoxical result that the little group rotation generator (the only one with a regular analytic expression) has a strange action on δ-functions localized on its rotation axis, while this does not happen for the other rotation generators.

The Resolution

Normalized states are of the form

$$\int \psi(\mathbf{p})|\mathbf{p}, j\rangle d^3p.$$

In order to justify the above formal considerations, it would be necessary to assume that one can find two sequences of wave functions that converge respectively to $\delta(p_1)\,\delta(p_2)\,\delta(p_3 - 1)$ and $\delta(p_1)\,\delta(p_2)\,\delta(p_3 + 1)$. In the massive case this would pose no problem. In the massless case, as we shall see in the next section by studying the space of differentiable vectors for the mass zero helicity j representations of the Poincaré group (in a canonical formalism), this hypothesis is incorrect. In fact, the correct space of (generalized) wave functions has the very interesting structure of the dual space to a space [3] of differentiable sections of a nontrivial complex line bundle over the vertexless forward light cone. In the above formulation, it then turns out that $\delta(p_1)\,\delta(p_2)\,\delta(p_3 + 1)$ does not belong to this space and has to be replaced by a "twisted δ-function" carrying angular momentum $-2j$, the "twist" being caused by the transition functions.

Remarks

(a) The above paradox is caused by difficulties arising, in the massless case for $j \neq 0$, in what can be called p-space localizability. Similar difficulties occur in x-space localization of massless particles: only generalized localizability exists [4, 5]. The usual Wigner–Newton localizability (derived from the massive case) works in the massless case only for spinless particles or for two-helicity $(\pm\frac{1}{2})$ massless "neutrinos." In the latter case, since the direct sum of helicity $+\frac{1}{2}$ and $-\frac{1}{2}$ UIR of $\mathscr{P}$ is equivalent to the massless limit of a spin $\frac{1}{2}$ UIR of $\mathscr{P}$, which does not exhibit a singularity for $p_3 = -|\mathbf{p}| \neq 0$, the problem of p-space localizability can also be avoided (it is pushed back to the intertwining operator realizing the equivalence).

For $|j| > 1/2$, constraints are needed to eliminate redundant components arising in the massless limit, and the localizability problem cannot be avoided.

(b) The absence of the vertex of the light cone, on which the massless representations are realized, is the source of another particularity of massless representations (which occurs even in the zero helicity case): the space of all differentiable vectors [6], i.e. the largest subspace of the Hilbert space of the representation on which the enveloping algebra of $\mathscr{P}$ is defined, is not nuclear. In the massive case it is nuclear [7]. [The topology on the space of differentiable vectors is defined in a natural way by the action of the enveloping algebra. Nuclearity would ensure that the dual contain all the generalized eigenstates appearing in the spectral resolutions of the essentially self-adjoint elements in this enveloping algebra, and is thus a useful technical property.] The difference is due to the fact that in the massless case there is an integrability condition around $\mathbf{p} = 0$.

(c) The massless discrete helicity UIR of $\mathscr{P}$ are the only ones which are extendable, and uniquely so [8], to a (most degenerate, so-called ladder) UIR of the conformal group SU(2, 2). The space of all differentiable vectors for this conformal group extension is nuclear [7], and invariant under the Poincaré group action. It thus provides a natural nuclear space of differentiable vectors for $\mathscr{P}$.

2. Singularities in Zero-Mass UIR'S of the Poincaré Group

The massless, discrete helicity, positive energy representations U of the Poincaré group $\mathscr{P}$ are traditionally realized on the space $H = L^2(\Omega_0, d^3p/p_0)$ of square integrable functions on the forward light cone

$$\Omega_0 = \{p \in R^4,\ p^2 \equiv p_0^2 - \mathbf{p}^2 = 0,\ p_0 > \mathbf{0}\}.$$

The action $U(a, \Lambda)$ representing an element (a, Λ) of $\mathscr{P}$, with $a \in R^4$ and $\Lambda \in \mathrm{SL}(2, C)$ is given by

$$[U(a, \Lambda)f](p) = e^{ia\,p} Q_j(p, \Lambda) f(\Lambda^{-1}p). \tag{4}$$

Here j is the helicity and Q_j is a function that satisfies the multiplier condition

$$Q_j(p, \Lambda_1\Lambda_2) = Q_j(p, \Lambda_1) Q_j(\Lambda_1^{-1}p, \Lambda_2). \tag{5}$$

If the representation is to be unitary, then Q_j is a complex number of modulus one, and can be taken to be the exponential of i times a real valued 1-cocycle of the Poincaré group—in fact, of the Lorentz group, since Eq. (5) shows that it is trivial on the translations.

When the representation is built as an induced representation in the sense of Mackey, then the inducing subgroup is $R^4 \otimes \tilde{E}(2)$, where the "little group" $\tilde{E}(2)$ is the two-fold covering of the Euclidean group of the plane. In this case Q_j is a representation of $\tilde{E}(2)$, trivial on the translations, and takes the form

$$Q_j(p, \Lambda) = Q_j(A(p)^{-1} \Lambda A(\Lambda^{-1}p)),$$

where $A(p)$ is a Lorentz transformation that maps the point stabilized by the little group to p. Various choices of $A(p)$ all lead to the same multiplier, namely [5, 9]

$$Q_j(p, \Lambda) = \left[\frac{\beta\zeta + \delta}{|\beta\zeta + \delta|}\right]^{-2j}, \tag{6}$$

where $\Lambda = \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix}$ and $\zeta = -(p_1 + ip_2)(p_0 + p_3)^{-1}$. The usual infinitesimal generators can then be computed, with following result:

$$P_\mu = p_\mu, \quad M_{12} = -i(p_1\partial_2 - p_2\partial_1) + j,$$

$$M_{23} = -i(p_2\partial_3 - p_3\partial_2) + j\frac{p_1}{p_0 + p_3}, \quad M_{31} = -i(p_3\partial_1 - p_1\partial_3) + j\frac{p_2}{p_0 + p_3},$$

$$M_{01} = -ip_0\partial_1 - j\frac{p_2}{p_0 + p_3}, \quad M_{02} = -ip_0\partial_2 + j\frac{p_1}{p_0 + p_3}, \quad M_{03} = -ip_0\partial_3. \tag{7}$$

Here $\partial_k \equiv \partial/\partial p_k$. The little group $\tilde{E}(2)$ is generated by M_{12} and $M_{32} + M_{02}, M_{31} + M_{01}$; together with M_{03} they generate a four-parameter, solvable subgroup of SL(2, C), the largest subgroup that possesses non-singular generators.

One notices that, when $j \neq 0$, a singularity appears, both in the global form (6) of the multiplier Q_j and in the infinitesimal form (7), at $p_0 + p_3 = 0$. This is unimportant as far as the Hilbert space representation is concerned, but it becomes relevant if we need continuity or differentiability properties. It turns out that this singularity has an interesting and profound effect on the construction of generalized eigenstates of momentum.

Generalized momentum eigenstates do not belong to the representation space, but an action of the representation on such states can be defined by duality on the Gel'fand triplet $D \subset H \subset D'$. If D is a complete, invariant, topological vector space of differentiable vectors for the representation, dense in H, then one restricts the representation U to D and extends to D' by duality. Since the Lie algebra generators leave D invariant, they too can be extended to D'. Thus, for a massive representation of the Poincaré group, one can take for D the space of C^∞ functions with compact support on the mass hyperboloid; the dual D' contains the generalized functions $\delta(p - \tilde{p})$ that are nothing else than the generalized momentum eigenstates $|\tilde{p}\rangle$. We have $P_\mu|\tilde{p}\rangle = p_\mu|\tilde{p}\rangle$ and all the other generators are also defined on these states.

In the massless case one may attempt to exclude the singular set $p_0 + p_3 = 0$ from the support of the functions of D; but then this space is not invariant under the group action and the construction fails. For every $|\tilde{p}\rangle$ there will be Lorentz transformations that are singular on it (those that take $\tilde{p}$ to the singular set) and for every Lorentz transformation not belonging to the regular subgroup (that generated by the little group and by M_{03}) there will be a $|\tilde{p}\rangle$ for which it is singular. (That is, there will be a $|\tilde{p}\rangle$ that is transformed outside D'.)

Alternatively, one can take for D an invariant space of differentiable vectors for the representation of $\mathscr{P}$—or of its extension to the conformal group, endowed with its natural Fréchet topology—which is nuclear in the case of the conformal extension. In this case not all momentum eigenstates $\delta(p - \tilde{p})$ will belong to D'. A simple demonstration of this is the following. Consider first a rotation $R_\theta^{\hat{n}}$ of angle θ around the axis $(1, 0, 0) \equiv \hat{n}$,

$$[U(R_\theta^{\hat{n}})f](p) = \exp\left[2ij \arctan \frac{p_1 \sin\frac{\theta}{2}}{(p_0 + p_3)\cos\frac{\theta}{2} + p_2 \sin\frac{\theta}{2}}\right] f(R_\theta^{\hat{n}} p). \tag{8}$$

In the limits $f(p) \to \delta_{\pm \hat{n}} \equiv \delta(p_0 \mp p_1)\, \delta(p_2)\, \delta(p_3)$, we get

$$U(R_\theta^{\hat{n}})\delta_{\pm \hat{n}} = e^{\pm ij\theta}\delta_{\pm \hat{n}}, \quad \hat{n} \neq (0,0,1). \tag{9}$$

As indicated, the same result is obtained for any axis of rotation, except for the special case when $\hat{n}$ coincides with the singular third axis $(0,0,1)$. In that exceptional case the limits $f(p) \to \delta_{\pm \hat{n}}(p)$ still exist, but now

$$U(R_\theta^{\hat{n}})\delta_{\pm \hat{n}} = e^{+ij\theta}\delta_{\pm \hat{n}}, \quad \hat{n} = (0,0,1). \tag{10}$$

In fact, from (1) and (3), one sees that the space $\mathscr{D}$ of C^∞ functions with compact support on the forward light cone is not invariant under U: it is transformed into a larger subspace of H. Taking sequences $R_\theta^{\hat{n}}$ of rotations and $\delta_{-\hat{n}}$ of eigenstates for $\hat{n} \to (0,0,1)$, formulas (9) and (10) show that there is no way to remedy this situation and extend U to a continuous representation $\mathscr{P} \times E \to E$ on a topological vector space E of distributions containing H and *all* the generalized momentum eigenstates $\delta_{\hat{n}}$ (when $j \neq 0$).

A suitable set of momentum eigenstates will be constructed in the next section. To be complete one may mention that using a space of functions as given by Mackey theory [10, 11], over the Lorentz (or Poincaré) group, will eliminate the need for a multiplier in the realization of the representation (it will be hidden in the space) but this will also change the expression of the representation.

3. Poincaré Covariant Momentum Eigenstates in Zero-Mass UIR of $\mathscr{P}$

(a) *A more Appropriate Expression of these* UIR

We have seen that, when $j \neq 0$, the generalized eigenstates $\delta(p - \tilde{p})$ cannot belong to the dual of a dense space D of differentiable vectors for these UIR's of $\mathscr{P}$ when $\tilde{p}$ is on the singular semi-axis, and that under a Lorentz transformation which takes $\tilde{p}$ to the singular semi-axis, the transforms of these states $\delta(p - \tilde{p})$ are not even defined in the Schwartz distribution space $\mathscr{D}'$. A more attentive look at the definition of the representation will explain this anomaly. For this purpose the p-space parametrization (4) and (7) of the representation is not the most appropriate, due to the fact that the forward vertexless light cone is diffeomorphic to $R \times S^2$ which needs two charts to be parametrized. We shall use a parametrization similar to that given by Rideau [3], by two charts and stereographic projections of the sphere S^2 on the complex plane $C, R \times (S^2 - \{p_3 = -1, p_1 = p_2 = 0\})$ being parametrized by

$$t = \log p_0, \quad \zeta = -\frac{p_1 + ip_2}{p_0 + p_3}, \tag{11}$$

and the opposite chart by $(t, -\zeta^{-1})$.

Then the representation (4) takes the form

$$(U(a,\Lambda)f)(t,\zeta) = e^{ia.p}Q_j(\zeta,\Lambda)f(t_\Lambda,\zeta_\Lambda), \tag{4'}$$

where $Q_j(\zeta,\Lambda) = (\beta\zeta + \delta/|\beta\zeta + \delta|)^{-2j}$ for $\Lambda = \begin{pmatrix}\alpha\beta\\ \gamma\delta\end{pmatrix} \in \mathrm{SL}(2,C)$ and

$$t_\Lambda = t - \log\frac{|\alpha\zeta + \gamma|^2 + |\beta\zeta + \delta|^2}{1 + |\zeta|^2}, \quad \zeta_\Lambda = \frac{\alpha\zeta + \gamma}{\beta\zeta + \delta},$$

In order for the group action written as (4′) to be always well-defined and differentiable, the functions f must satisfy a condition at the limit $\zeta \to \infty$, adapted to the form of the multiplier Q_j, and be differentiable. Therefore we shall choose the f belonging to the space D_j of infinitely differentiable functions in t, $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$, with compact support in t, such that the function

$$\check{f}:(t,\zeta) \to (\zeta^{-1}|\zeta|)^{+2j} f(t, -\zeta^{-1}) \tag{12}$$

has the same properties; D_j is endowed with a $\mathscr{D}$-type topology (uniform convergence, together with all derivatives, on bounded subsets of $R \times S^2$, the support in t being inside a fixed compact set). This is a nuclear dense invariant subspace of the (Fréchet nuclear) space of differentiable vectors for the (uniquely defined [8]) extension of U to SU(2,2), which is defined in a similar manner but with a $\mathscr{S}$-type of behavior in e^t (instead of the $\mathscr{D}$-type preferred here).

The differentiability condition of f and $\check{f}$ is nothing but a more explicit way to say that D_j is a space of differentiable sections of a nontrivial complex line bundle on $R \times S^2$, the transition functions being $(\zeta^{-1}|\zeta|)^{+2j}$. It may be of interest, though *a posteriori* not surprising (as we shall see in Subsection (c)) to note that these factors have been also introduced around 1960 by E. Wigner [12] in his extension of U to the orthochronous Poincaré group $\mathscr{P}_+$.

(*b*) *Poincaré Covariant Momentum Eigenstates*

In the above parametrization, $\zeta = 0 (t \in R)$ describes the semi-axis passing through the stabilized point $(1,0,0,1)$ while $\zeta = \infty (t \in R)$ describes the "singular" semi-axis. For any finite value $\tilde{\zeta}$ of ζ, we can define a Dirac measure on the S^2 sphere $\tilde{p}_0 = 1$ by

$$|\tilde{p}\rangle : f(t,\zeta) \to f(0,\tilde{\zeta}), \tag{13}$$

where $(\tilde{p}_0 = 1, \tilde{\mathbf{p}})$ is related to $(\tilde{t} = 0, \tilde{\zeta})$ by the parametrization (11). For $\tilde{\zeta} = \infty$ we have to pass to the other chart, or equivalently to take the limit, in this chart of S^2, of suitable functions with compact support "around $\zeta = \infty$" which after inversion, $\zeta \to -\zeta^{-1}$, and multiplication by the transition function $(\zeta^{-1}|\zeta|)^{2j}$ are C^∞ with support around $\zeta = 0$. Thus for $\tilde{p} = (1,0,0,-1)$ we have to define the state $|\tilde{p}\rangle$ by

$$|\tilde{p}\rangle = |(1,0,0,-1)\rangle : f(t,\zeta) \to \lim_{\zeta \to 0} (\zeta^{-1}|\zeta|)^{2j} f(0, -\zeta^{-1}). \tag{14}$$

In a similar manner (taking any fixed value $\tilde{t}$) we define the states $|\tilde{p}\rangle$ for multiples of the preceding four-vector $\tilde{p}$. Formula (13) defines a usual Dirac δ, but (14) defines a "twisted delta," which we shall denote by $'\delta$ and may call "twelta", belonging to the dual D'_j of D_j. The duality is defined by the Gel'fand triplet $D_j \subset H \subset D'_j$.

We can now check that, since in the covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$,

$$\pm \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \to R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

a rotation in the (1,2) plane, we have

$$(e^{i\theta M_{12}} f)(t,\zeta) = (U(R_3(\theta))f)(t,\zeta) = e^{ij\theta} f(t, e^{i\theta}\zeta), \tag{15}$$

and therefore

$$`\delta(U(R_3(\theta))f) = e^{ij\theta} \lim_{\zeta\to 0} (e^{-i\theta}\zeta^{-1}|\zeta|)^{2j} f(0, e^{i\theta}(-\zeta^{-1}))$$

$$= e^{-ij\theta} \lim_{\zeta\to 0} (\zeta^{-1}|\zeta|)^{2j} f(0, -\zeta^{-1}) = e^{-ij\theta} `\delta(f). \tag{16}$$

In other words, by using an appropriate set of generalized eigenstates $|p\rangle$, we have restored the identity

$$\mathbf{p}\cdot\mathbf{J}|p\rangle = j|\mathbf{p}||p\rangle, \tag{1$'$}$$

and the Poincaré covariance of the momentum eigenstates. The use of transition functions amounts to the fact that the eigenstates of the "twelta" type have a built-in angular momentum $-2j$. Note that, while in (1) the p outside and inside the ket are identical, we still have the identity $\mathbf{P}\cdot\mathbf{J} = jP_0$, where $(P_0, \mathbf{P})$ are the translation generators of (7), on any differentiable vector. This is a special case of the defining relation $\varepsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} = 2jP_\mu$ which (together with $P^2 = 0$) characterizes the massless, helicity j representation of $\mathscr{P}$.

(c) *Parity and Massless* UIR. (*The Orthochronous Poincaré Group*)

As is well known, in contrast to the massive (or massless spinless) case, the massless (nonzero helicity) UIR of $\mathscr{P}$ cannot be extended to the orthochronous Poincaré group ($\mathscr{P}$ together with space reflection Σ): Σ has to be represented by a (unitary) operator exchanging the $+j$ helicity space D_j with the $-j$ helicity space D_{-j} (the Hilbert spaces H_j and H_{-j} are also exchanged, but their analytic definitions are identical since the singularity which requires the line bundle formulation for the differentiable vectors is unimportant at the Hilbert space level). More precisely, if $\mathscr{P}$ is represented by (4) or (4′), then we have [12]:

$$U(\Sigma)f_j(p_0, \mathbf{p}) = \frac{(-p_1 - ip_2)^{2j}}{(p_1^2 + p_2^2)^j} f_{-j}(p_0, -\mathbf{p}) = \left(\frac{\zeta}{|\zeta|}\right)^{2j} f_{-j}(t, \Sigma\zeta), \tag{17a}$$

$$U(\Sigma)f_{-j}(p_0, \mathbf{p}) = \frac{(p_1 - ip_2)^{2j}}{(p_1^2 + p_2^2)^j} f_j(p_0, -\mathbf{p}) = \left(-\frac{\bar{\zeta}}{|\zeta|}\right)^{2j} f_j(t, \Sigma\zeta), \tag{17b}$$

where $\Sigma\zeta = (p_1 + ip_2)/(p_0 - p_3)$. In particular, the momentum eigenstates $|p_0, \mathbf{p}, j\rangle$ are transformed to $(\zeta^{-1}|\zeta|)^{-2j}|p_0, -\mathbf{p}, -j\rangle$. It is not surprising to see the transition functions appear here, since the geometrical parity operator Σ exchanges the stabilized semi-axis $p_0 = -p_3$ with the singular semi-axis $p_0 = p_3$, and therefore exchanges the two charts which were chosen to cover the sphere S^2. It is interesting to note that they have been used by Wigner about twenty years before the line bundle formulation was made explicit.

The representations of the orthochronous Poincaré group must therefore be realized in a space $H_j \oplus H_{-j}$. One can then be tempted to utilize the limit $m\to 0$ of the representation $D(m,|j|)$ of mass m and spin $|j|$ of $\mathscr{P}$, the usual expression of which has a singularity only for $p_0 = 0$ (there is no singular semi-axis). However this limit is not written as a direct sum (to which it is equivalent) and when projecting out on an irreducible subspace of helicity $\pm |j|$ a singular semi-axis will appear, so that the line-bundle formulation cannot be avoided (for $|j| = \frac{1}{2}$, this formulation will be needed at least for the parity operator).

To end this discussion one should mention that for physical fields (satisfying some field equation), the representations of the Poincaré group which appear are not the irreducible ones but extensions of some of these, extensions which are in general (i.e. when gauge fields appear) non-trivial. For instance, for the photon, we have an extension of the representations with helicities $1, 0, -1$, and 0: the restriction to $E(2)$ of the covariance representation $D(\frac{1}{2}, \frac{1}{2})$ of the Lorentz group (acting on vector fields) is an extension of the inducing representations of $E(2)$ parametrized by $1, 0, -1$ and 0. The representations parametrized by $\varepsilon = \pm 1$ contain the states $(j_3 = \varepsilon, \mathbf{p})$ and $(j_3 = -\varepsilon, -\mathbf{p})$, and the parity invariance of the field equation is clear in this case. On the other hand, for the neutrino, the covariance representation of the massless Dirac equation is the representation usually denoted by $D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$ of the Lorentz group, hence, the splitting of this 4-component equation (if complex 4×4 matrices are allowed) into two 2-component factors, the Weyl and the anti-Weyl equations. Each contains both energy signs (in first quantized theories) and both helicity signs, as a direct sum of 2 representations. The parity operator exchanges both factors (and changes the sign of helicity), while the particle-antiparticle operator (PC) exchanges the components within the same factor. Furthermore, in this case, there exists [3] a non-trivial extension between the $+\frac{1}{2}$ and $-\frac{1}{2}$ helicity representations (with the same energy sign): it would be of interest to look (in second quantized theories) for a physical interpretation of these extensions, which looks like a "neutrino gauge theory."

4. Consequences for Massless Particles

As we saw, for $j \neq 0$ massless UIR's of the Poincaré group, the generalized eigenvectors of the energy-momentum four-vector are radically different from those of the massive case. It is also well understood from physical reasons that in any direct particle interpretation (as in S-matrix theories), sharp momentum states have a fundamental importance (this is true for the massless and massive cases).

It is therefore quite clear that difficulties with sharp momentum eigenstates in the massless case have additional consequences for the already ill-defined notion of S-matrix in massless theories. Also the fact that the most interesting massless theories we have are gauge theories, and that therefore the space of the field theory (the indefinite metric one) is different from the physical states space (positive-definite Hilbert space), is related to the smoothness problem discussed before in connection with massless particles.

With respect to the infrared singularities appearing in interacting theories containing massless particles, there occurs an interesting possibility that "tweltas" carrying nontrivial angular momentum might, at least partially, correct the bad behavior of these singularities. Moreover, the fact that the dual of the (nuclear) space of differentiable vectors for the conformal group (expressed as equivariant functions on $\mathbb{R}^4$ in massless discrete helicity representations) contains [13] elements having support in $p=0$ can also be used to attempt to cancel these singularities.

In the remainder of this paper we shall analyze in the light of what precedes a recent no-go theorem by Weinberg and Witten [14], where extensive use is made of the sharp momentum states for massless UIR's of the Poincaré group.

As a step towards showing that all massless field theories for helicity $|j|>1/2$ must be gauge theories, Weinberg and Witten have recently attempted to show that in any theory allowing the construction of a Poincaré covariant conserved four-vector current J^μ (respectively energy-momentum tensor $\theta^{\mu\nu}$), massless particles with helicity $|j|>1/2$ (respectively $|j|>1$) cannot be contained if the conserved charge $\int J^0 d^3x$ (respectively the energy-momentum four-vector $\int \theta^{\mu 0} d^3x$) is nonvanishing. Evidently, as the authors remark, this no-go result does not apply to gravity, pure Yang–Mills, supergravity, etc...., since the associated tensors will not be Lorentz covariant due to the presence of gauges.

If $|p\rangle$ denotes a sharp momentum generalized eigenstate (for a fixed helicity j, index which we shall drop from the notation), they look at the Lorentz covariance of matrix elements $\langle p'|J^\mu(x)|p\rangle$ and $\langle p'|\theta^{\mu\nu}(x)|p\rangle$.

Rigorously speaking, these matrix elements are "twisted kernels" (i.e. belonging, in p and p', to the dual of a space of C^∞ sections of a bundle)-valued distributions in x-space. However, it is not difficult to show that the x-dependence of these matrix elements is entirely of the form $\exp(i(p-p')x)$, which multiplies a constant (in x) matrix element. Therefore, though J^μ and $\theta^{\mu\nu}$ are *a priori* operator-valued distributions (in x), it makes sense to write symbolically this constant matrix element as $\langle p'|J^\mu|p\rangle$ and $\langle p'|\theta^{\mu\nu}|p\rangle$, and to forget about the x-dependence when expressing the Poincaré covariance. The limits of these matrix elements (if defined) when $p'\to p$ are then guessed to be respectively $ep^\mu/E(2\pi)^3$ and $p^\mu p^\nu/E(2\pi)^3$, where e is the one-particle charge. They then say that they get a contradiction by showing that if $(p'-p)^2\neq 0$, then these matrix elements vanish when $|j|>1/2$ and 1 (respectively), the basis of the proof being the transformation properties of these matrix elements, for $p=\tilde{p}=(\tilde{p}_0,\tilde{\mathbf{p}})\in\Omega_0$ and $p'=(\tilde{p}_0,-\tilde{\mathbf{p}})\in\Omega_0$, under rotations $R(\theta)$ around the $(p'-p)$ axis.

A first remark is that the continuity assumption of a kernel around the diagonal $p=p'$, which is replaced in a footnote by a plausibility argument based on measurement, is by no means an obvious property. In particular the argument becomes empty if the kernel has its support concentrated on the diagonal. As a matter of fact, it is not clear whether continuity can be proved for any realistic massless particle theory. Indeed, Sudarshan [15] has recently produced examples of massless (free) fields for which this assumption does not hold (even for the $j=1/2$ case, where no gauges are present).

Let us now come back to the proof of Weinberg and Witten. Since $(p'-p)^2\neq 0$, they choose a Lorentz frame for which $p=\tilde{p}=(\tilde{p}_0,\tilde{\mathbf{p}})$ and $p'=\tilde{p}'=(\tilde{p}_0,-\tilde{\mathbf{p}})$ (if the Poincaré representation U is already given, one can get this configuration by acting with a suitable Lorentz transformation, provided this is legitimate on the given momentum states). Then they state that under a rotation $R(\theta)$ around such a $\mathbf{p}'-\mathbf{p}$ axis, one has

$$U(R(\theta))|\tilde{p}\rangle = e^{ij\theta}|\tilde{p}\rangle \quad \text{while} \quad U(R(\theta))|\tilde{p}'\rangle = e^{-ij\theta}|\tilde{p}'\rangle, \tag{18}$$

and utilize the Lorentz covariance of the J^μ and $\theta^{\mu\nu}$ to get

$$e^{2ij\theta}\langle\tilde{p}'|J^\mu|\tilde{p}\rangle = R(\theta)^\mu_\nu\langle\tilde{p}'|J^\mu|\tilde{p}\rangle. \tag{19}$$

The contradiction arises because the eigenvalues of R are $e^{\mathrm{i}m\theta}$ with $m = \pm 1$ or 0.

How can this be justified in the known Poincaré representations? Let us take for $R(\theta)$ a rotation, $R_1(\theta)$, say around the p_1 axis, on which are located $\tilde{p}$ and $\tilde{p}'$ and consider C^∞ functions $f_{\tilde{p}}$ and $f_{\tilde{p}'}$, of the variable $p \in \Omega_0$, with compact support concentrated around $\tilde{p}$ and $\tilde{p}'$ (respectively), and invariant under $R_1(\theta)$. Consider the limits $f_{\tilde{p}}(p) \to \delta(p - \tilde{p})$ and $f_{\tilde{p}'}(p) \to \delta(p - \tilde{p}')$ (in the $\mathscr{D}'$ topology, or in D_j'). Then from (8) we have

$$U(R_1(\theta))f_{\tilde{p}}(p) = \exp\left(2ij\arctan\frac{p_1\sin\frac{\theta}{2}}{(p_0+p_3)\cos\frac{\theta}{2}+p_2\sin\frac{\theta}{2}}\right)f_{\tilde{p}}(p),$$

and similarly for $f_{p'}(p)$, whence (18) in the limit. The same thing is true for any other rotation axis except the one passing through the stabilized point. For the latter we have shown earlier (cf(15) and (16)) that (18) still holds, provided that the states $|p'\rangle$ are of the "twelta" type (if $\tilde{p}$ is the stabilized point). Therefore this part of the Weinberg–Witten argument is rigorously established. However, the only conclusion one should draw from Weinberg–Witten's argument is the discontinuity of current matrix elements around $p = p'$. Such a consequence is not too surprising, and indeed Sudarshan's argument [15] mentioned before shows that such discontinuities exist for some massless wave equations even in the free case and for low spins.

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