

Counterexamples to Some Results on the Existence of Field Copies*

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Abstract. Several criteria are known for determining which connections A are determined uniquely by their curvature F , or by F and its covariant derivatives. On a principal bundle with semi-simple gauge group G over a 4-manifold M , a sufficient condition for F to determine A uniquely is that the linear map $B \rightarrow [F \wedge B]$ from Lie algebra-valued 1-forms to 3-forms (pulled back to M via a local gauge) be invertible on an open dense set in M . Recently F. A. Doria has claimed that this condition is also necessary. We present counterexamples to this claim, and also to his assertion that F determines A uniquely if the restriction of the bundle to every open subset of M has holonomy group equal to G and F is “not degenerate as a 2-form over spacetime.”

1. Background and Results

In recent years there has been an interest in finding criteria for the existence of field copies, that is, of gauge fields (= field strengths = curvatures of a principal bundle) which arise from two or more gauge potentials (connections) which are not gauge equivalent (see, for example, [C, DX, DD, DW, GY, H1, H2, KC, M, R, So, W, and WY]). This problem is closely related to questions of when a connection A is determined uniquely by its curvature F , or by F together with other data (in the absolute sense, that is, ignoring gauge equivalence). Among the numerous results on these questions are two types of general criteria, which we shall call algebraic and geometric.

The algebraic criterion ([C, DT, DW, H1, MS, R]), applicable when the base space M is a 4-dimensional manifold, is based on the Bianchi identity. Specialized to the trivial principal G -bundle $\mathbb{R}^4 \times G$ over $M = \mathbb{R}^4$, it says that a sufficient condition for F to determine A uniquely is that the linear transformation $\text{ad}_F: B \rightarrow [F \wedge B]$ from \mathfrak{g} -valued 1-forms to \mathfrak{g} -valued 3-forms on M ($\mathfrak{g} = \text{Lie algebra of } G$) be invertible at all x in an open dense subset of M . (See [MS] for the generalization of this, in

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bundle language, to arbitrary bundles over arbitrary 4-dimensional M .) The condition “ ad_F is invertible on an open dense set” is a generic condition on A when G is semi-simple [MS], but ad_F is never invertible if the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is not equal to \mathfrak{g} [*op. cit.*].

Our first result states that the algebraic criterion, though sufficient, is not necessary. The second part of the theorem contradicts an assertion of F. A. Doria [D2, p. 2943 and p. 2949, Theorem 3(v)].

Theorem. *The invertibility of ad_F on an open dense subset of M is not a necessary condition for the curvature F of a connection A to be the curvature of no other connection A' , at least when the gauge group G is $SU(2)$ or $SO(3)$. In fact, F may determine A uniquely even if ad_F is nowhere invertible.*

Our next result deals with geometric criteria, by which we mean the use of F and of its covariant partial derivatives up to some order to determine the connection A . Such criteria [DD, GY, M] should involve a condition on the *infinitesimal holonomy Lie algebras* $\mathfrak{g}'(p)$ [KN, p. 96], where $\mathfrak{g}'(p)$ is the linear span in \mathfrak{g} at a point p (in the total space P of the bundle) of the components of the curvature form F and of its covariant partial derivatives $D_\mu F, D_\mu D_\nu F, \dots$ of all orders, evaluated at p . For example ([M, p. 141], Remark after Corollary 1), if the center of \mathfrak{g} is 0 (for instance, \mathfrak{g} semi-simple) and if A and A' are two connections for which the respective components of curvature and of covariant partial derivatives of curvature of all orders coincide and span \mathfrak{g} at all $p \in P$, then $A = A'$. This result is trivial to prove, as is the following generalization.

Proposition. *Let $P \rightarrow M$ be any principal bundle ($\dim M$ arbitrary) with structure group G whose Lie algebra \mathfrak{g} has trivial center. Let A and A' be two connection forms on P . Suppose that at all points $p \in P$, the respective components of curvature and of covariant partial derivatives of curvature of all orders less than or equal to some integer $k > 1$ coincide with each other, and that for all points p in an open dense subset of P the components of curvature and of its covariant partial derivatives of order less than k span \mathfrak{g} . Then $A = A'$.*

Proof. Adapt the proof of the quoted results [M, p. 141], and then use continuity. Q.E.D.

A result stronger than this has been asserted by F. A. Doria [D1]. He defines a holonomy condition which he calls “full irreducibility” and claims in the main result of [D1] (Theorem 3.5 and Abstract) that if it is satisfied by a connection A with curvature F , then no other connection A' can have the same curvature, provided F is not “degenerate as a 2-form over spacetime.” In particular, if a connection A with curvature F satisfies $\mathfrak{g}'(p) = \mathfrak{g}$ at all $p \in P$, then by standard results on holonomy ([KN], or see summary in [M, p. 145]) the restriction of the bundle $P \rightarrow M$ to any open set $U \subset M$ has (ordinary) holonomy equal to the full group G (provided that G is connected). This means that F is “fully irreducible.” If, in addition, F is non-degenerate as a 2-form on M , Doria’s theorem would imply that F determines A uniquely. The following counterexample shows that the theorem cannot be true.

Counterexample. Let $P \rightarrow M$ be the trivial bundle $\mathbb{R}^4 \times G \rightarrow \mathbb{R}^4$, where $G = \text{SU}(2)$ or $\text{SO}(3)$. Let $\{t_1, t_2, t_3\}$ be a basis of \mathfrak{g} satisfying $[t_i, t_j] = \varepsilon_{ijk}t_k$. Let $x = (x_0, x_1, x_2, x_3)$ be the coordinates of \mathbb{R}^4 . Consider the two families of connections, parametrized by real constants α, β , whose pullbacks to \mathbb{R}^4 via the trivial gauge (section) $\mathbb{R}^4 \rightarrow \mathbb{R}^4 \times \{1\}$ are

$$\begin{aligned} A_\alpha &= t_1x_0dx_1 + t_2x_2dx_3 + \alpha t_1dx_3, \\ A'_\beta &= t_1x_0dx_1 + t_2x_2dx_3 + \beta t_2dx_1. \end{aligned}$$

Then all the A_α and A'_β have the curvature form

$$F = t_1dx_0 \wedge dx_1 + t_2dx_2 \wedge dx_3 + t_3x_0x_2dx_1 \wedge dx_3$$

(which is non-degenerate as a 2-form on \mathbb{R}^4) and satisfy $g'(p) = \mathfrak{g}$ at all points p . Moreover, no two of these connections are gauge equivalent to each other.

Remark 1. This Counterexample does not contradict our Proposition (taking $k = 1$), because the first covariant partial derivatives of F distinguish every pair of connections. For instance, the A_α are distinguished from each other and from A'_β by $D_3F_{23} = \partial_3F_{23} + [(A)_3, F_{23}]$, which equals αt_3 if $A = A_\alpha$ and equals 0 if $A = A'_\beta$.

Remark 2. For more general classes of counterexamples, see [DW], p. 391, examples 1, 2, and 3.

2. Proofs

Proof of Theorem. Let $P = M \times G \rightarrow M = \mathbb{R}^4$ be the trivial principal G bundle over \mathbb{R}^4 , $G = \text{SU}(2)$ or $\text{SO}(3)$, and let $x = (x_0, x_1, x_2, x_3)$ and $\{t_1, t_2, t_3\}$ be as in the Counterexample above. Consider the connection (pulled back to \mathbb{R}^4)

$$A = t_1dx_0 + t_2dx_1 + t_3x_3dx_2.$$

Its curvature is

$$F = t_3dx_0 \wedge dx_1 - t_2x_3dx_0 \wedge dx_2 + t_1x_3dx_1 \wedge dx_2 - t_3dx_2 \wedge dx_3.$$

The linear mapping ad_F maps any \mathfrak{g} -valued 1-form $B = t_iB_\mu^i dx_\mu$ (summation convention) to the \mathfrak{g} -valued 3-form

$$\begin{aligned} [F \wedge B] &= L_0dx_1 \wedge dx_2 \wedge dx_3 + L_1dx_0 \wedge dx_2 \wedge dx_3 \\ &\quad + L_2dx_0 \wedge dx_1 \wedge dx_3 + L_3dx_0 \wedge dx_1 \wedge dx_2, \end{aligned}$$

where the $L_\mu = L_\mu^i t_i$ are certain \mathfrak{g} -valued functions on \mathbb{R}^4 . In particular, $L_2 = [t_3, t_iB_3^i] = t_2B_3^1 - t_1B_3^2$, showing that $[F \wedge B]$ has no $t_3dx_0 \wedge dx_1 \wedge dx_3$ component and hence that ad_F cannot be invertible. This calculation is valid at all $x \in \mathbb{R}^4$.

Now suppose there were connection $A' = A + B$ with the same curvature F . The two Bianchi identities $dF + [A \wedge F] = 0 = dF + [A' \wedge F]$ would imply that $0 = [F \wedge (A' - A)] = [F \wedge B]$.

Lemma 1. *The continuous solutions $B = B(x)$ of $[F \wedge B] = 0$ are*

$$B = f_0t_3dx_0 + f_1t_3dx_1 + (f_0x_3t_1 + f_1x_3t_2 + f_2t_3)dx_2,$$

where f_0, f_1, f_2 are arbitrary continuous functions of $x \in \mathbb{R}^4$.

Proof of Lemma. The condition $[F \wedge B] = 0$ is equivalent to the four equations

$$0 = L_0 = x_3[t_1, B_3] - [t_3, B_1], \quad (0)$$

$$0 = L_1 = -x_3[t_2, B_3] - [t_3, B_0], \quad (1)$$

$$0 = L_2 = [t_3, B_3], \quad (2)$$

$$0 = L_3 = [t_3, B_2] + x_3[t_2, B_1] + x_3[t_1, B_0], \quad (3)$$

where $B_\mu = B_\mu^i t_i$. Equation (2) implies that $B_3^3 = 0 = B_3^2$, so that

$$B_3 = B_3^3 t_3. \quad (4)$$

Equations (0) and (4) imply that

$$0 = L_0 = -x_3 B_3^3 t_2 - B_1^1 t_2 + B_1^2 t_1, \quad (5)$$

and hence that

$$0 = x_3 B_3^3 + B_1^1 = B_1^2. \quad (6)$$

Thus

$$B_1 = B_1^i t_i = -x_3 B_3^3 t_1 + B_1^3 t_3. \quad (7)$$

Similarly, Eqs. (1) and (4) yield

$$0 = L_1 = -x_3 B_3^3 t_1 - B_0^1 t_2 + B_0^2 t_1, \quad (8)$$

implying that

$$B_0 = x_3 B_3^3 t_2 + B_0^3 t_3. \quad (9)$$

Equations (3), (7), and (9) now give

$$0 = L_3 = B_2^1 t_2 - B_2^2 t_1 + x_3(x_3 B_3^3 t_3 + B_1^3 t_1) + x_3(x_3 B_3^3 t_3 - B_0^3 t_2). \quad (10)$$

This implies

$$0 = x_3 B_1^3 - B_2^2, \quad (11)$$

$$0 = B_2^1 - x_3 B_0^3, \quad (12)$$

$$0 = 2(x_3)^2 B_3^3. \quad (13)$$

Now (13) implies that $B_3^3 = 0$ whenever $x_3 \neq 0$, and hence at all x , by continuity. This simplifies our results to

$$B = B_0^3 t_3 dx_0 + B_1^3 t_3 dx_1 + (x_3 B_0^3 t_1 + x_3 B_1^3 t_2 + B_2^3 t_3) dx_2,$$

which is equivalent to the asserted form with $f_\mu = B_\mu^3$ (note that $f_3 \equiv 0$). Conversely, our work, read backwards, shows that any B of this form satisfies $[F \wedge B] = 0$.

Q.E.D. Lemma 1

Now for A' to have curvature F , B must satisfy the additional condition

$$F = d(A + B) + \frac{1}{2}[(A + B) \wedge (A + B)],$$

or equivalently, since $F = dA + \frac{1}{2}[A \wedge A]$,

$$0 = dB + [A \wedge B] + \frac{1}{2}[B \wedge B]. \tag{14}$$

As we have seen, a necessary condition for Eq. (14) to hold is $0 = [F \wedge B]$. (Equivalently, one obtains $0 = [F \wedge B]$ by taking d of Eq. (14), just as one obtains the Bianchi identity $dF = [F \wedge A]$ by taking d of the curvature equation $F = dA + (1/2)[A \wedge A]$.)

Lemma 2. *For the specific connection A defined above, Eq. (14) admits only one differentiable (C^1) solution B , namely $B = 0$.*

Proof. Since every solution B of (14) satisfies $[F \wedge B] = 0$, we may assume that B is of the form specified in Lemma 1, with f_0, f_1, f_2 now C^1 functions of $x \in \mathbb{R}^4$. Expanding Eq. (14) in components yields 18 separate equations. In particular, the coefficients of $t_1 dx_0 \wedge dx_1$, $t_2 dx_0 \wedge dx_1$, and $t_1 dx_1 \wedge dx_2$ are, respectively,

$$-f_0 = -f_1 = x_3(\partial f_0 / \partial x_1 - (f_1)^2) + f_2 = 0,$$

implying that $f_0 = f_1 = f_2 = 0$.

Q.E.D. Lemma 2 and Theorem

Proof of Counterexample. The calculation that the curvature $dA + \frac{1}{2}[A \wedge A]$ is as claimed for all A_α and A'_β is routine.

Choose one of the connections. At any point $p = (x, g) \in \mathbb{R}^4 \times G = P$, $g'(p) = g^{-1} \cdot g'(x, 1) \cdot g[\text{KN}]$, $g'(x, 1)$ being the Lie subalgebra of \mathfrak{g} spanned by the components of F and its covariant partial derivatives of all orders at x . But the components of F alone span \mathfrak{g} except on $\{x_0 x_2 = 0\}$, and even there the components of F include t_1 and t_2 , which generate \mathfrak{g} as a Lie algebra. Hence $g'(p) = g^{-1} g g' = \mathfrak{g}$ at all $p \in P$, no matter which of the connections is used to compute the covariant partial derivatives of F .

The non-degeneracy of $F = F_{\mu\nu}$ as a 2-form over space-time follows from direct verification that $F_{\mu\nu} c_\mu = 0$ only if $(c_\mu) = 0 (c \in \mathbb{R}^4)$.

Now if the function $\varphi: \mathbb{R}^4 \rightarrow G$ defined a gauge transformation Φ on P (by $(x, g) \rightarrow (x, \varphi(x)g)$), taking one of the connections to another one, then Φ would have to take F to itself, that is, $\varphi^{-1} F \varphi = F$. On $\{x_0 x_2 \neq 0\}$ we would have $\varphi^{-1} t_i \varphi = t_i, i = 1, 2, 3$, implying that $\varphi(x)$ is in the center of G (1 if $G = \text{SO}(3)$, ± 1 if $G = \text{SU}(2)$). But then Φ would map every connection to itself. Thus no two of the connections A_α, A'_β can be gauge equivalent.

Q.E.D.

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