

Local Relativistic Invariant Flows for Quantum Fields

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Abstract. For quantum fields with trigonometric interaction in arbitrary space dimension we construct a representation of the Lorentz group by automorphisms on a Banach space generated by the Weyl algebra.

1. Introduction

It is well known that for systems with infinitely many degrees of freedom the problem of the formulation of the dynamics is intimately connected with the “kinematical problem” of finding a suitable representation of the basic operators of the theory. The problem as stated above has been discussed both in physical and mathematical works, see e.g. the basic early work [20, 21, 23, 24, 29, 34]. From a mathematical point of view the problem has been seen as the one connected with the phenomenon of equivalence versus singularity of measures on function space, and has been discussed extensively in this way, see e.g. [26, 45, 48]. Nevertheless, despite this basic physical and mathematical problem, since the very beginnings of quantum field theory, the canonical formalism has been for a long time the most fascinating point of reference (see e.g. Dirac [18] and especially Heisenberg and Pauli [28] but also Wentzel [46]). It is certainly the most ambitious program for field quantization in as much as it is an attempt to give a direct extension of the quantum mechanics of finitely many degrees of freedom to the case of systems of infinitely many degrees of freedom. The canonical program was pursued further in the late fifties especially by Coester and Haag [12], Araki [8], Klauder [31], Streater [42], and Segal [38]. The difficulties of the canonical formalism (Haag’s theorem [27, 43], which prevents the free and interacting fields to be in the same representation of the canonical commutation relations) are well known and have to be bypassed, when starting in Fock space, by suitable limit procedures. Only in the sixties and in the last decade models satisfying the basic postulates of locality and relativistic invariance have been constructed (see any of an excellent series of surveys on constructive quantum field e.g. [41, 25]), and indeed the basic

postulates of the canonical formalism have been verified in some of these two space-time dimensional models starting with the one with trigonometric interaction [3]. On the other hand nothing of this kind is at the present time available in higher dimensions, although Wightman models have been constructed for space-time dimension three (see e.g. [19, 32] and references therein). For other approaches see e.g. [9, 35, 38, 40]. The construction of canonical operators in the Weyl form is the problem of finding unitary operators which commute up to a factor of modulus one, hence suggests connection with the representation theory of abelian groups [47, 8, 10].

In the present paper, our approach to the construction of quantum fields is based on the following observation. Let H_I^κ defined by:

$$H_I^\kappa = \int d\mu_\kappa(g) W^\zeta(g) \quad (1.1)$$

be an interaction Hamiltonian, where $W^\zeta(g)$ is a measurable projective unitary representation of a group G , on a Hilbert space \mathcal{H} . Here $d\mu_\kappa(g)$ is a bounded measure on G which depends on a positive parameter κ (the ‘‘cut-off’’). Let H_0 be the infinitesimal generator of a strongly continuous group of unitaries, which implements on \mathcal{H} the one parameter group of automorphisms $t \rightarrow \alpha_t^0$ of G

$$t \rightarrow W^\zeta(\alpha_t^0(g)) = \exp\{itH_0\} W^\zeta(g) \exp\{-itH_0\}. \quad (1.2)$$

We assume, and this is satisfied in the applications, that in the presence of the cut-off the interaction picture is valid in the sense that one has for all Φ and Ψ in H :

$$\begin{aligned} & (\Phi, \exp\{iH^\kappa t\} W^\zeta(g) \exp\{-iH^\kappa t\} \Psi) \\ &= (\Phi, \exp\{itH^\kappa\} \exp\{itH_0\} W^\zeta(\alpha_t^0(g)) \\ & \quad \cdot \exp\{itH_0\} \exp\{-itH^\kappa\} \Psi), \end{aligned} \quad (1.3)$$

where $H^\kappa = H_0 + H_I^\kappa$ is the total Hamiltonian. The special form (1.1) of the interaction H_I^κ allows to rewrite the matrix element (1.3) as the expectation value of a functional of the time zero matrix elements with respect to a Poisson measure through a Feynman path formula. Furthermore this expression extends quite naturally to a suitable linear space of bounded functionals on G , which are no longer necessarily of the form (1.3), i.e. they are not necessarily matrix elements of a unitary projective representation of G on a Hilbert space \mathcal{H} , and defines a one parameter group of automorphisms (flow) α_t^κ associated with the interaction H^κ .

A central result of this paper is that the limit for $\kappa \rightarrow \infty$ of the flow α_t^κ exists, in the case where H_I^κ stands for an ultraviolet cut-off relativistic local interaction, on a dense subset of bounded functionals. More precisely we construct a simultaneous representation on a Banach space of the Weyl canonical commutation relations and of the Poincaré group in all space dimensions for models in which the interaction (like in the sine-Gordon model) is of the type

$$H_I = \lambda \int_{\mathbb{R}^s} dx \int_{\mathbb{R}} dv(\alpha) \cos(\alpha\phi(x) + \theta), \quad 0 \leq \theta \leq 2\pi, \quad (1.4)$$

where s is the number of space dimensions, dv is an arbitrary bounded measure on \mathbb{R} and λ a real constant. Hence we get time zero fields and canonical conjugate momenta as well as a natural dynamics acting on them in the sense of

automorphisms. This theory is a nontrivial partial solution of both the kinematical and dynamical problems alluded above. Introducing the usual renormalization procedure, it is possible to show that for $s=1$, $m=0$ and λ sufficiently small, our construction coincides with the one obtained before by other means (see e.g. [16, 22]), when interpreted in the underlying Hilbert space. We also give a solution of the Schrödinger equation, first discussed by Symanzik (see e.g. [44]) for trigonometric models. Interactions of the above form with ultra-violet cut-off have been discussed before in [1, 2, 5, 15, 30]. A Schrödinger equation for them was shortly pointed out in [6], but in a different form from the present one, using the definitions of Feynman path integral by normalized oscillatory integrals given in [5].

This paper is organized as follows: in Sect. 2, we study both left and right regular projective representations of a group G by isometric operators on the Banach space $B(G)$ of bounded functionals on G (Proposition 2.1). This space contains the subspace of matrix elements (1.3) of unitary projective representations of the group G as well as the dense subset generated by the characters of G . It is a natural space for looking at solutions of Schrödinger-like equations for the time translated matrix elements of exponentials of the field operators, which for trigonometric interactions can be expressed as Poisson expectation values (Theorem 2.5). We then define a flow on the Banach space $B(G)$ which represents the perturbed evolution in the Schrödinger picture.

In Sect. 3 we give the definition of the trigonometric interaction [30] and we derive the Schrödinger equation satisfied by the time translated functionals. We then remove the ultraviolet cut-off and the space cut-off from the solution of these equations for any space time dimension. Let us observe finally that if there existed relativistic quantum field models in higher dimensions satisfying all the Wightman axioms, and in addition some suitable technical assumptions, they would appear in our representation. For smooth initial conditions with compact support, i.e. in $\mathcal{D}(\mathbb{R}^s)$, we then get a relativistic flow on a dense set of functionals in the infinite volume limit, associated with time translations. We also discuss the case of initial conditions in $\mathcal{S}(\mathbb{R}^s)$, using methods of classical statistical mechanics (see [1, 2]), obtaining the existence of the infinite volume limit for small times. We also discuss the asymptotic behaviour in time.

2. Time Translation Automorphisms Associated with Projective Representations of a Group

Let G be a (topological) group and ζ a multiplier on G , i.e. a (continuous) function from $G \times G$ to the one dimensional torus such that

$$\begin{aligned} \zeta(g_1, g_2)\zeta(g_1, g_2, g_3) &= \zeta(g_1, g_2 \cdot g_3)\zeta(g_2, g_3), \\ \zeta(e, e) &= \zeta(g, g^{-1}) = 1, \quad \forall g \in G. \end{aligned} \tag{2.1}$$

Let $B(G)$ be the Banach space of complex-valued bounded Borel functions on G equipped with the sup-norm. We define R^ζ and L^ζ to be the isometric projective representations of G on $B(G)$ given by

$$\begin{aligned} (R^\zeta(g)F)(h) &\equiv \zeta(h, g)F(hg), \\ (L^\zeta(g)F)(h) &\equiv \zeta(g^{-1}, h)F(g^{-1}h), \quad g, h \in G. \end{aligned} \quad (2.2)$$

These representations commute and moreover they allow us to implement automorphisms of G

$$U_\sigma^0 R^\zeta(g) (U_\sigma^0)^{-1} = R^\zeta(\sigma(g)), \quad (2.3)$$

where

$$(U_\sigma^0 F)(h) \equiv F(\sigma^{-1}(h)), \quad (2.4)$$

for any automorphism σ of G which leaves ζ invariant. Similar formulas hold for L^ζ .

Let $B_1(G)$ be the subspace of $B(G)$ generated by the finite linear combinations of the functions

$$h \rightarrow F_{\Phi, \Psi}^{W^\zeta}(h) = (\Phi, W^\zeta(h)\Psi), \quad (2.5)$$

where W^ζ is a projective unitary representation of G in a Hilbert space \mathcal{H} with scalar product (\cdot, \cdot) and Φ, Ψ are vectors in \mathcal{H} . $B_1(G)$ is stable with respect to both R^ζ and L^ζ . Let $t \rightarrow \alpha_t$ be a one parameter group of automorphisms of G leaving ζ invariant (this is going to represent the free evolution, in the application below). We define $B_{am}(G)$ to be the subspace of $B(G)$ of those functions F such that

$$t \in \mathbb{R} \rightarrow F(g_1 \alpha_t(g_2) g_3), \quad g_i \in G, \quad i = 1, 2, 3.$$

is Lebesgue measurable. It is clear that B, B_1 , and B_{am} are invariant subspaces with respect to both R^ζ and L^ζ .

Given any bounded complex measure μ on G we define on $B(G)$ the following operators

$$H_t^\# \equiv \lambda \int_G d\mu(g) R^\zeta(g), \quad (2.6)$$

$$H_t^b \equiv \bar{\lambda} \int_G d\bar{\mu}(g) L^\zeta(g), \quad (2.7)$$

where λ is a complex parameter and $\bar{\mu}$ denotes the complex conjugate measure to μ , i.e. $\bar{\mu}(f) = \overline{\mu(\bar{f})}$ for any function f . To the one parameter group $t \rightarrow \alpha_t$ there corresponds an isometric operator $U_{\alpha_t}^0 \equiv U_t^0$ on $B(G)$, defined by $(U_t^0 F)(h) \equiv F(\alpha_t^{-1}(h))$, and we want to find in $B(G)$ two functions $F_t^\#(g)$ and $F_t^b(g)$ solutions of the differential equations

$$i \frac{\partial}{\partial t} F_t^\#(g) = (U_{-t}^0 H_t^\# U_t^0 F_t^\#)(g), \quad g \in G, \quad (2.8)$$

with the initial condition

$$\lim_{t \downarrow 0} F_t^\#(g) = F_0(g), \quad (2.9)$$

and similar equations for F_t^b .

In what follows we shall discuss $F_t^\#$, the considerations for F_t^b being entirely similar. This Cauchy problem uniquely defines $F_t^\#$ as a one cocycle

$$F_{t+s}^\# = U_{-t}^0 (U_t^0 F_t^\#)_s, \quad t, s \in \mathbb{R}, \quad \text{see [8]}. \quad (2.10)$$

Note that $U_t^0 F_{t+s}$ is the solution of (2.8) with t replaced by s and with initial condition $U_t^0 F_{t+s}|_{s=0} = U_t^0 F_t$. It is very useful to describe the solution $F_t^{\#}$ in a compact way with the help of a Feynman path integral using a Poisson measure P_μ , along the lines of e.g. [12, 14, 16]. We recall here only that P_μ is the Poisson measure for a Poisson process running in time t on G with intensity measure the positive bounded measure $d\mu$ on G i.e., for any f bounded Borel on $[0, t] \times G$

$$\int \exp\left(i \int_{[0, t] \times G} f(t, g) dN_\omega(t, g)\right) dP_\mu(\omega) = \exp\left(\int_{[0, t] \times G} (e^{if(t, g)} - 1) d\tau d\mu(g)\right), \tag{2.11}$$

where N is the counting measure on countable subsets of $[0, t] \times G$. A realisation of $P_\mu(\omega)$ is as follows. Let Ω be the disjoint union of Ω_n , where n runs over the integers $\{0\} \cup \mathbb{N}$, Ω_0 being an arbitrary one element set $\Omega_0 = \{\omega_0\}$, and Ω_n for $n \in \mathbb{N}$ being the set of all n -tuples ω of the form

$$\omega = \{(n, t_1, g_1), \dots, (n, t_n, g_n)\}, \quad \text{with } 0 < t_1 < \dots < t_n < t, \quad g_i \in G, \quad i \in \mathbb{N}.$$

Let a_i be Lebesgue measurable subsets of $[0, t]$ with $a_i \cap a_j = \emptyset$, B_i be Borel subsets of G , $i \in \mathbb{N}$, and define

$$V_{a_i, B_i}^{(n)} = \{\omega \in \Omega_n \mid \omega = \{(n, t_i, g_i), 0 < t_1 \dots < t_n < t, t_i \in a_i, g_i \in B_i, i = 1, \dots, n\}\}.$$

For any positive bounded measure μ on G , P_μ is then characterized by

$$P_\mu(\Omega_0) = 1, \quad P_\mu(V_{a_i, B_i}^{(n)}) = \prod_{i=1}^n |a_i| \mu(B_i),$$

where $|a_i|$ is the Lebesgue measure of a_i . Equivalently, for any $F \in L^1(P_\mu)$, we have the following interpretation of above formula (2.11):

$$\int_\Omega F(\omega) dP_\mu(\omega) = \sum_{n \geq 0} \int_0^t dt_n \dots \int_0^t dt_1 \int_G d\mu(g_n) \dots \int_G d\mu(g_1) \cdot F(\{(n, t_1, g_1), \dots, (n, t_n, g_n)\}).$$

We refer to the papers quoted above for more details. Thus we have the theorem:

Theorem 2.1. *Let μ be a complex measure on a topological group G . The solution of the evolution equation (2.8), with initial condition $F_0 \in B_{am}(G)$ and with H_t given by (2.6) is given by*

$$F_t^{\#}(g) = \int P_{|\lambda\mu|}(d\omega) \exp\{i\Phi(\omega)\} \exp\{iS_0^*(\omega)\} F_0(gG_0^*(\omega)),$$

and similarly the solution of the evolution equation (2.8) with initial condition $F_0 \in B_{am}(G)$ is given by

$$F_t^b(g) = \int P_{|\lambda\mu^*|}(d\omega) \exp\{-i\Phi^*(\omega)\} \exp\{iS_0^a(\omega)\} F_0(G_0^a(\omega)g),$$

where

$$\exp\{i\Phi(n, t_i, g_i)\} \equiv (-i\lambda/|\lambda|)^n \exp\left\{i \sum_{j=1}^n \phi(g_j)\right\},$$

ϕ being defined by

$$d\mu(g) = \exp(i\phi(g)) d|\mu|(g).$$

Moreover

$$\exp \{iS_0^r(n, t_i, g_i)\} \equiv \prod_{k=2}^n \zeta(\alpha_{-t_n}(g_n) \dots \alpha_{-t_k}(g_k), \alpha_{-t_{k-1}}(g_{k-1})) \cdot \zeta(g, \alpha_{-t_n}(g_n) \dots \alpha_{-t_1}(g_1)),$$

and

$$G_0^r(n, t_i, g_i) \equiv \alpha_{-t_n}(g_n) \dots \alpha_{-t_1}(g_1),$$

whereas

$$d\mu^*(g) = d\bar{\mu}(g^{-1}),$$

$$\exp \{iS_0^a(n, t_i, g_i)\} = \prod_{k=1}^{n-1} \zeta(\alpha_{-t_k}(g_k), \alpha_{-t_{k+1}}(g_{k+1}) \dots \alpha_{-t_n}(g_n)) \cdot \zeta(\alpha_{-t_1}(g_1) \dots \alpha_{-t_n}(g_n), g),$$

and

$$G_0^a(n, t_i, g_i) \equiv \alpha_{-t_1}(g_1) \dots \alpha_{-t_n}(g_n).$$

The solutions are entire functions of λ . [The suffixes r and a stand for retarded (respectively, advanced)].

Corollary 2.2. Let $\Omega_t^\#$ be defined by

$$(\Omega_t^\# F_0)(g) \equiv F_t^\#(g),$$

where $F_t^\#$ is the solution of (2.8) given by Theorem 2.1. Then $\Omega_t^\#$ is a linear map of $B_{am}(G)$ into itself, with norm bounded as follows

$$\|\Omega_t^\#\| \leq \exp \{t|\lambda| |\mu|(G)\}.$$

Define correspondly Ω_t^b by $\Omega_t^b F_0 \equiv F_t^b$, then $\Omega_t^b \Omega_{t'}^\# = \Omega_{t'}^\# \Omega_t^b$ for all t and t' . One has

$$\Omega_{t+s} = U_t^0 \Omega_s U_t^0 \Omega_t.$$

and $U_t^\# \equiv U_t^0 \Omega_t^\#, U_t^b \equiv U_t^0 \Omega_t^b$ are groups of bounded operators on $B_{am}(G)$, $t \in \mathbb{R}$.

Proof. For the bounds one uses the boundedness of $H_t^\#$. The group property follows easily from the cocycle relation (2.10). \square

We make now a remark concerning the ordering problem for Weyl quantization.

Lemma 2.3. Let ζ and ξ be two equivalent multipliers on G , i.e. such that there exists a function λ from G to the one dimensional torus such that:

$$\zeta(g_1, g_2) = \lambda(g_1) \lambda(g_2) \bar{\lambda}(g_1 g_2) \xi(g_1, g_2), \quad g_1, g_2 \in G.$$

Then we have

$$\begin{aligned} & \prod_{k=2}^n \zeta(\alpha_{-t_n}(g_n) \dots \alpha_{-t_k}(g_k), \alpha_{-t_{k-1}}(g_{k-1})) \zeta(\alpha_{-t}(g), \alpha_{-t_n}(g_n) \dots \alpha_{-t_1}(g_1)) \\ &= \prod_{k=2}^n \xi(\alpha_{-t_n}(g_n) \dots \alpha_{-t_k}(g_k), \alpha_{-t_{k-1}}(g_{k-1})) \xi(\alpha_{-t}(g), \alpha_{-t_n}(g_n) \dots \bar{\alpha}_{-t_1}(g_1)) \\ & \cdot \prod_{\ell=1}^n \lambda(\alpha_{-t_\ell}(g_\ell)) \lambda(\alpha_{-t}(g)) \bar{\lambda}(\bar{\alpha}_{-t}(g) \bar{\alpha}_{-t_n}(g_n) \dots \bar{\alpha}_{-t_1}(g_1)). \end{aligned}$$

Remark. It follows from this that the functions S_0^r, S_0^a defined in Theorem 2.1 depend trivially on ζ within a cohomology class. The following lemma is immediate

Lemma 2.4. *If ζ is an α_t -invariant bicharacter of G , i.e.*

$$\zeta(g_1, g_2, g_3) = \zeta(g_1, g_2)\zeta(g_2, g_3), \quad g_1, g_2, g_3 \in G,$$

and

$$\zeta(g_1 \cdot g_2, g_3) = \zeta(g_1, g_2)\zeta(g_2, g_3),$$

then

$$\begin{aligned} & \prod_{k=2}^n \zeta(\alpha_{-t_n}(g_n) \dots \alpha_{-t_k}(g_k), \alpha_{-t_{k-1}}(g_{k-1})) \\ &= \prod_{i>j} \zeta(g_i, \alpha_{t_i-t_j}(g_j)). \end{aligned}$$

Remark. Let $g \rightarrow W^{\zeta}(g)$ be a unitary projective representation of the canonical commutation relations on a Hilbert space \mathcal{H} , where $t \rightarrow \alpha_t$ is unitarily implemented by a strongly continuous unitary group $t \rightarrow \exp(iH_0 t)$. Let H_I be the bounded operator on \mathcal{H} given by

$$H_I = \lambda \int_G dv(g) W^{\zeta}(g),$$

with ν a complex bounded measure on G . Then for any $\Phi, \Psi \in \mathcal{H}$, with $F_{\Phi, \Psi}^{W^{\zeta}}$ defined by (2.5),

$$(\Omega_t^{\#} F_{\Phi, \Psi}^{W^{\zeta}})(g) = (\Phi, W^{\zeta}(g) \exp(itH_0) \exp(-i(H_0 + H_I)t) \Psi),$$

is not a group operation with respect to t , but a one cocycle. On the other hand

$$(U_t^{\#} F_{\Phi, \Psi}^{W^{\zeta}})(g) = (\Phi, \exp(itH_0) W^{\zeta}(g) \exp(-i(H_0 + H_I)t) \Psi)$$

is a group operation with respect to t . Similar formulae hold for Ω_t^b and U_t^b .

Now we are in position to state a theorem which gives the perturbed free evolution in the Schrödinger picture [see formula (1.1) in the introduction]. We call Σ the cone consisting of functions $F \in B_{\text{sm}}(G)$ of ζ -positive type, i.e. such that

$$\sum_{i,j=1}^N \lambda_i \bar{\lambda}_j \zeta(g_i g_j^{-1}, g_i) F(g_i g_j^{-1}) \geq 0$$

for any choice of $\lambda_i \in \mathbb{C}, g_i \in G, i, j = 1, \dots, N, N \in \mathbb{N}$.

Theorem 2.5. *Let α_t^H be defined by*

$$\alpha_t^H \equiv U_t^0 \Omega_t^{\#} \Omega_t^b,$$

where $U_t^0(h) \equiv F(\alpha_t^{-1}(h))$, $F \in B(G)$, α_t being a one-parameter automorphism and $\Omega_t^{\#}, \Omega_t^b$ being defined in Corollary 2.2. Then α_t^H is a one parameter continuous subgroup of bounded operators on $B_{\text{sm}}(G)$ leaving invariant the cone Σ .

Proof. The proof follows easily from Theorem 2.1 and Corollary 2.2. \square

Remark. α_t^H can be explicitly computed. We give its expression in the most useful case where λ is real and $\mu = \mu^*$, which corresponds to a real potential

$$\begin{aligned}
 (\alpha_t^H F)(g) = & \sum_{m, n \geq 0} \lambda^{m+n} (-i)^{m+n} \int_0^t ds_m \dots \int_0^{s_2} ds_1 \int_0^t dt_n \dots \int_0^{t_2} dt_1 \\
 & \cdot \int_G d\mu(g_m) \dots \int_G d\mu(g_1) \int_G d\mu(h_n) \dots \int_G d\mu(h_1) \\
 & \cdot \prod_{k=1}^{m-1} \zeta(\alpha_{-s_1}(g_1) \dots \alpha_{-s_k}(g_k), \alpha_{-s_{k-1}}(g_{k-1})) \\
 & \cdot \prod_{\ell=1}^{n-1} \zeta(\alpha_{-t_{\ell+1}}(h_{\ell+1}), \alpha_{-t_\ell}(h_\ell) \dots \alpha_{-t_1}(h_1)) \\
 & \cdot \zeta(\alpha_{-s_1}(g_1) \dots \alpha_{-s_m}(g_m), \alpha_{-t}(g)) \\
 & \cdot \zeta(\alpha_{-s_1}(g_1) \dots \alpha_{-s_m}(g_m) \alpha_{-t}(g), \alpha_{-t_n}(h_n) \dots \alpha_{-t_1}(h_1)) \\
 & \cdot F(\alpha_{-s_1}(g_1) \dots \alpha_{-s_m}(g_m) \alpha_{-t}(g) \alpha_{-t_n}(h_n) \dots \alpha_{-t_1}(h_1)).
 \end{aligned}$$

The next proposition gives a Schrödinger equation for α_t^H , in the sense that it gives an affine, one parameter group of transformations of the set of states. This equation will enter as an essential tool in Sect. 4 to prove the existence of a thermodynamical limit for relativistic quantum field models. Let H_0 be the infinitesimal generator of U_t^0 defined on a dense domain $D(H_0)$ in $B_{zm}(G)$.

Theorem 2.6. *If G is an abelian group, for any F in $D(H_0)$ the following equation holds:*

$$i \frac{\partial}{\partial t} (\alpha_t^H F)(g) = (H_0 \alpha_t^H F)(g) + \lambda \int_G d\mu(h) (\zeta(g, h) - \zeta(h, g)) (\alpha_t^H F)(gh).$$

Proof. This follows easily from Theorem 2.1. \square

3. Field Theory with Trigonometric Interaction and Cut-Offs

This section is an application of the previous results to a model for quantum field theory, the trigonometric interaction in $s+1$ dimensional space time, with ultraviolet and space cut-offs.

Let $\mathcal{S}_R(\mathbb{R}^s) \equiv \mathcal{S}_R$ (respectively, $\mathcal{D}_R(\mathbb{R}^s) \equiv \mathcal{D}_R$) be the Schwartz space of real C^∞ -functions of rapid decrease (respectively with compact support). They are abelian groups under pointwise addition. We look at:

$$G = \mathcal{S}_R \times \mathcal{S}_R, \quad (\text{respectively, } \mathcal{D}_R \times \mathcal{D}_R), \quad (3.1)$$

as the space of initial conditions of a relativistic neutral scalar Bose field in $s+1$ space-time dimensions.

Corresponding to the equal time commutation relations of such a field there is a central extension of G which is given, choosing units such that $\hbar=1$, by a multiplier ζ , where

$$\zeta((f, g), (f', g')) = \exp \left\{ \frac{i}{2} \int_{\mathbb{R}^s} dx \{ f'(x)g(x) - f(x)g'(x) \} \right\}, \quad (3.2)$$

for all $(f, g), (f', g') \in G$.

Consider a projective representation of the C.C.R. in a Hilbert space \mathcal{H} with multiplier ζ , and let $\phi(f)$ be the field operator (at time zero) and $\Pi(g)$ its canonical conjugate momentum with $f, g \in \mathcal{S}_R$. Provided that $\lambda \in \mathbb{R} \rightarrow W_{\lambda f, g}^\zeta$ and $\mu \in \mathbb{R} \rightarrow W_{f, \mu g}$ are continuous, the Weyl operators are defined as the unitary operators

$$W_{g, f}^\zeta = \exp \{ i(\Pi(g) - \phi(f)) \}, \quad (3.3)$$

on \mathcal{H} . We have:

$$W_{g, f}^\zeta W_{g', f'}^\zeta = \zeta((f, g), (f', g')) W_{g+g', f+f'}^\zeta, \quad (3.4)$$

for all $(f, g), (f', g') \in G$. For all Φ in \mathcal{H} the map

$$(f, g) \in G \rightarrow (\Phi, W_{g, f}^\zeta \Phi) \quad (3.5)$$

belongs to the space $B(G)$ of bounded complex functionals on G .

As discussed in Sect. 2, $B(G)$ carries two isometric projective representations R^ζ and L^ζ of G given by (2.2), i.e.

$$\{ R^\zeta(f, g)F \} (h_1, h_2) = \exp \left\{ \frac{i}{2} \int_{\mathbb{R}^s} dx \{ f(x)h_2(x) - g(x)h_1(x) \} \right\} F(f+h_1, g+h_2), \quad (3.6)$$

$$\{ L^\zeta(f, g)F \} (h_1, h_2) = \exp \left\{ \frac{i}{2} \int_{\mathbb{R}^s} dx \{ f(x)h_2(x) - g(x)h_1(x) \} \right\} F(h_1-f, h_2-g), \quad (3.7)$$

for all $f, g, h_1, h_2 \in \mathcal{S}_R(\mathbb{R}^s)$.

Now we shall define a group of automorphisms α_t of G , leaving ζ invariant, and consider as in Sect. 2 the fourtuple $(B, \alpha_t, R^\zeta, L^\zeta)$. α_t corresponds to the free field time evolution, usually defined for free quantum fields in Fock space. We define α_t as the map of G in itself given by

$$\alpha_t(f, g) = (f_t, g_t), \quad (3.8)$$

where f_t and g_t are defined by their Fourier transform as follows:

$$\tilde{f}_t(p) = \cos(\omega(p)t) \tilde{f}(p) + \omega(p)^{-1} \sin(\omega(p)t) \tilde{g}(p), \quad (3.9)$$

$$\tilde{g}_t(p) = \partial_t \tilde{f}_t(p), \quad (3.10)$$

with $\omega(p) \equiv \sqrt{p^2 + m^2}$. Thus for $t > 0$,

$$f_t(x) = (\partial_t \Delta_r^m * f)(t, x) + (\Delta_r^m * g)(t, x), \quad (3.11)$$

$$g_t(x) = \partial_t f_t(x), \quad (3.12)$$

where Δ_r^m is the retarded relativistic propagator:

$$\Delta_r^m(t, x) = \frac{\theta(t)}{(2\pi)^s} \int_{\mathbb{R}^s} \frac{dp}{\omega(p)} \sin(\omega(p)t) e^{ipx}, \quad (3.13)$$

with

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0. \end{cases} \quad (3.14)$$

α_t extends to a group of isometric transformations $U_{\alpha_t}^0$ of $B(G)$, defined by:

$$\{U_{\alpha_t}^0 F\}(f, g) = F(\alpha_t^{-1}(f, g)). \tag{3.15}$$

$U_{\alpha_t}^0$ is continuous and isometric on the Banach space $B_{\alpha_m}(G)$.

Also $B_{\alpha_m}(G)$ carries an isometric representation of the Poincaré group in \mathbb{R}^{s+1} , which leaves the multiplier ζ invariant. The infinitesimal generators of the Poincaré group are given in [7].

Now we want to give the corresponding representation for interacting fields with trigonometric interaction. This type of interaction has been studied, e.g. in [30]. They are defined in a bounded domain A of \mathbb{R}^s , which we take for convenience to be a symmetric cubic box,

$$A = \{x \in \mathbb{R}^s; |x_i| < a, i = 1, 2, \dots, s; a > 0\}, \tag{3.16}$$

by a Hamiltonian $H^{A\kappa} = H_0 + V$ with:

$$V = \lambda \int_A dx \int_{\mathbb{R}} dv(\alpha) \exp\{-i\alpha\phi_\kappa(x)\}, \tag{3.17}$$

defined in the Fock representation of the C.C.R. Now H_0 is the free relativistic Hamiltonian and $\phi_\kappa(x) = (\phi * \chi_\kappa)(x)$ is the time zero free field regularized by an ultraviolet cut-off function $\chi_\kappa(x) = \kappa^s \chi(\kappa x)$, $\kappa > 0$, where χ is a positive symmetric C^∞ -function with support in the unit ball of \mathbb{R}^s and such that $\int_{\mathbb{R}^s} \chi(x) dx = 1$. Here v is a complex measure on \mathbb{R} and λ is a positive constant. As in Sect. 2 we denote by ϕ the phase which appears in the polar decomposition of v :

$$dv(\alpha) = \exp\{i\phi(\alpha)\} d|v|(\alpha). \tag{3.18}$$

In order to have a self-adjoint Hamiltonian, we require that

$$\bar{v}(\alpha) = v(-\alpha). \tag{3.19}$$

We remark that (3.19) implies that (3.17) can be written as

$$V = 2\lambda \int_A dx \int_0^\infty d|v|(\alpha) \cos(\alpha\phi_\kappa(x) - \phi(\alpha)).$$

A particular case of interaction of the form (3.17) is the sine-Gordon model, see e.g. [22], which corresponds to the Bernoulli measure:

$$dv(\alpha) = \frac{1}{2} \{\delta_{\alpha_0} + \delta_{-\alpha_0}\}. \tag{3.20}$$

To define the Poisson measure entering in Theorem 2.1 in our present case, we shall rewrite (3.17) in term of the Weyl operators W_{f_1, f_2}^κ , $f_i \in \mathcal{S}_R(\mathbb{R}^s)$ associated with the Fock representation of the C.C.R. We have for the interaction V with ultraviolet and space cut-offs

$$V = \lambda \int_A dx \int_{\mathbb{R}} dv(\alpha) W_{0, \alpha\chi_\kappa}^{\text{Fock}}, \tag{3.21}$$

with $\chi^x(y) \equiv \chi(x - y)$. Here $W_{f, g}^{\text{Fock}}$ stands for the Weyl operator in the Fock representation, with the usual multiplier given by (3.2).

For any two vectors Φ and Ψ in the Fock representation we then get:

$$(\Phi, W_{f,g}^{\text{Fock}} V \Psi) = (V^* F_{\Phi, \Psi}^{\text{Fock}})(f, g), \quad (\Phi, V W_{f,g}^{\text{Fock}} \Psi) = (V^b F_{\Phi, \Psi}^{\text{Fock}})(f, g), \quad (3.22)$$

where $F_{\Phi, \Psi}^{\text{Fock}}(f, g) \equiv (\Phi, W_{f,g}^{\text{Fock}} \Psi)$, $V^* \equiv \lambda \int_A dx \int_{\mathbb{R}} dv(\alpha) R_{0, \alpha \chi_k^c}^\zeta$, and similarly for V^b , with R^ζ replaced by $L^{\bar{\zeta}}$.

Furthermore α_t [from (3.8)], is unitarily implemented in the Fock representation by the weakly continuous unitary group $U_t^0 = \exp(-iH_0 t)$. We have the perturbation series:

$$\begin{aligned} & (\Phi | W_{f,g}^{\text{Fock}} \exp\{-itH_0\} \exp\{it(H_0 + V)\} \Psi) \\ &= \sum_{n \geq 0} (i\lambda)^n \int_0^t dt_n \dots \int_0^{t_2} dt_1 (\Phi | W_{f,g}^{\text{Fock}} V_{t_n} \dots V_{t_1} \Psi) \\ &= \sum_{n \geq 0} (i\lambda)^n \int_0^t dt_n \dots \int_0^{t_2} dt_1 \int_G dQ(f_n, g_n) \dots \int_G dQ(f_1, g_1) \\ & \quad \cdot \{R_{(f_n, g_n) t_n}^\zeta \dots R_{(f_1, g_1) t_1}^\zeta F_{\Phi, \Psi}^{\text{Fock}}\}(f, g), \end{aligned} \quad (3.23)$$

where $V(t) \equiv U_t^0 V U_{-t}^0$, $(f, g)_t \equiv \alpha_t(f, g) \equiv (f_t, g_t)$ [cf. (3.8)], and

$$dQ(f, g) = \int_A dx \int_{\mathbb{R}} dv(\alpha) \delta_0^{(f)} \delta_{\alpha \chi_k^c}(g),$$

the δ being the evaluation measures on $\mathcal{S}_R(\mathbb{R}^s)$. A similar formula holds for the matrix elements $(\Phi, \exp(-i(H_0 + V)) \exp(iH_0 t) W_{f,g}^{\text{Fock}} \Psi)$. We remark that the operators V^* and V^b in (3.22) can be written as

$$V^* = \lambda \int_G dQ(g) R_g^\zeta \quad \left[\text{respectively, } V^b = \lambda \int_G dQ(g) L_g^{\bar{\zeta}} \right], \quad (3.24)$$

As in Sect. 2, Theorem 2.1, it is useful to express the solution of the Schrödinger equation associated with $H_0 + V$, by a Poisson measure associated with the group G .

In the present situation we can also characterize this measure as the measure \hat{P}_e defined by

$$\begin{aligned} \int \exp\{i\langle f, \xi \rangle\} d\hat{P}_e(\xi) &= \exp \left\{ \lambda \int_0^t d\tau \int_A dx \int_{\mathbb{R}} d|v|(\alpha) \right. \\ & \quad \cdot \left. \left(\exp \left(-i\alpha \int_0^\tau d\sigma \iint dudv f(\sigma, u) A_r^m(\tau - \sigma, u - v) \chi_k^c(v) \right) - 1 \right) \right\}, \end{aligned} \quad (3.25)$$

the integration on the left hand side being over $\mathcal{S}'_R(\mathbb{R}^s) \times \mathcal{S}'_R(\mathbb{R}^s) \times C_0^\infty([0, t]')$, where

$$f \in \mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s) \times C_0^\infty(0, t),$$

hence f is indefinitely differentiable and of compact support with respect to the variable τ . In fact \hat{P}_e has support on elements of

$$\mathcal{S}'_R(\mathbb{R}^s) \times \mathcal{S}'_R(\mathbb{R}^s) \times C_0^\infty(0, t)'$$

of the form: $(\delta_0, \delta_{\alpha\chi_\kappa}, \delta_{t'})$, where $\alpha \in \text{Support}(v)$ and $t' \in [0, t]$. Let P_μ be the measure image of \hat{P}_ϱ under the transformation of $\mathcal{S}'_R \times \mathcal{S}'_R \times C^\infty_0(0, t)'$ into $(0, t) \times \Lambda \times \mathbb{R}$ induced by the inverse of the mapping $(s, x, \alpha) \rightarrow \alpha\chi_\kappa^x \otimes \delta_s$. Then P_μ is the measure introduced in Sect. 2, Theorem 2.1, with G replaced by $\Lambda \times \mathbb{R}$ and $d\mu$ by $dx \times dv(\alpha)$. Since P_μ depends on $\Lambda, t, \lambda|v|$ we shall denote it from now on by $P_{\lambda|v|}^{A,t}$.

Let us recall the essential definitions needed to state the basic result of this section. Let $U_t^0 \equiv U_{\alpha_t}^0$ be the isometric operator given by the free evolution α_t defined by (3.8). Let $\Omega_t^\#$ be the linear bounded operator on $B_{am}(G)$ given, corresponding to Theorem 2.1, by

$$(\Omega_t^\# f)(g) = \int P_{\lambda|v|}^{A,t}(d\omega) \exp\{i\Phi(\omega) + iS_0^r(\omega)\} F(g + G_0^r(\omega)), \tag{3.26}$$

where $g \in G; F \in B_{am}(G)$,

$$\exp\{i\Phi(n, t_j; x_j; \alpha_j)\} = \exp\left(-i\frac{\pi}{2}n + i\sum_{j=1}^n \phi(\alpha_j)\right), \tag{3.27}$$

with ϕ given by (3.18) and

$$\begin{aligned} \exp\{iS_0^r(n, t_i; x_i; \alpha_i)\} &= \prod_{\kappa=2}^n \zeta(\alpha_{-t_n}(g_n) + \dots + \alpha_{-t_\kappa}(g_\kappa), \alpha_{-t_{\kappa-1}}(g_{\kappa-1})) \\ &\quad \zeta(g, \alpha_{-t_n}(g_n) + \dots + \alpha_{-t_1}(g_1)), \end{aligned} \tag{3.28}$$

where ζ is the multiplier defined in (3.2), and $g_i = (0, \alpha_i\chi_\kappa^{x_0})$. Moreover

$$G_0^r(n, t_i, x_i, \alpha_i) = \alpha_{-t_n}(g_n) + \dots + \alpha_{-t_1}(g_1).$$

Correspondingly as in Sect. 2, we have a similar expression for Ω_t^b .

Finally we recall the definition of the automorphism group α_t^H of Sect. 2, where here $H = H^{A,\kappa} = H_0 + V$:

$$\alpha_t^H = U_t^0 \Omega_t^\# \Omega_t^b, \tag{3.29}$$

with the right hand side defined correspondingly to Theorem 2.5. We have the following theorem:

Theorem 3.1. *The automorphism group α_t^H associated with the Hamiltonian $H^{A,\kappa}$ with trigonometric interaction V given by (3.17) maps linearly and boundedly $B_{am}(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s))$ into itself and gives a solution of the Schrödinger equation on the Banach space $B_{am}(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s))$:*

$$i\frac{\partial}{\partial t}(\alpha_t^H F)(g) = H_0 \alpha_t^H F(g) + \lambda \int_G d\varrho(g) \{\zeta(g, h) - \zeta(h, g)\} \alpha_t^H F(g+h),$$

with initial conditions F in

$$B_{am}(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s)) \cap C^1(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s)).$$

We have:

$$\alpha_t^H F(h_1, h_2) = \int P_{\lambda|v|}^{A,t}(d\omega_1) P_{\lambda|v|}^{A,t}(d\omega_2) \exp\{i\Phi(\omega_1, \omega_2)\} F_{\kappa,t}(h_1, h_2)(\omega_1, \omega_2),$$

where writing (m, s_i, x_i, α_i) for ω_1 and (n, t_j, y_j, β_j) for ω_2 :

$$\begin{aligned}
 & F_{\kappa, t}(h_1, h_2)((m, s_i, x_i, \alpha_i), (n, t_j, y_j, \beta_j)) \\
 &= \exp \left\{ \frac{i}{2} \sum_{i, i'} \alpha_i \alpha_{i'} (\chi_{\kappa} * \Delta_a^m * \chi_{\kappa})(s_i - s_{i'}, x_i - x_{i'}) \right\} \\
 & \cdot \exp \left\{ \frac{i}{2} \sum_{j, j'} \beta_j \beta_{j'} (\chi_{\kappa} * \Delta_r^m * \chi_{\kappa})(t_j - t_{j'}, y_j - y_{j'}) \right\} \\
 & \cdot \exp \left\{ \frac{i}{2} \sum_{i, j} \alpha_i \beta_j (\chi_{\kappa} * \Delta^m * \chi_{\kappa})(s_i - t_j; x_i - y_j) \right\} \\
 & \cdot \exp \left\{ \frac{i}{2} \left(\sum_{i=1}^m \alpha_i ((\partial_0 \chi_{\kappa} * \Delta^m * h_1)(s_i - t, x_i) - (\chi_{\kappa} * \Delta^m * h_2)(s_i - t, x_i)) \right) \right\} \\
 & \cdot \exp \left\{ - \frac{i}{2} \left(\sum_{j=1}^n \beta_j ((\partial_0 \chi_{\kappa} * \Delta^m * h_1)(t_j - t, y_j) - (\chi_{\kappa} * \Delta^m * h_2)(t_j - t, y_j)) \right) \right\} \\
 & \cdot F \left(h_{1, -t} + \sum_{i=1}^m \alpha_i (\partial_0 \chi_{\kappa} * \Delta^m)(s_i, x_i) + \sum_{j=1}^n \beta_j (\partial_0 \chi_{\kappa} * \Delta^m)(t_j, y_j), \right. \\
 & \quad \left. h_{2, -t} + \sum_{i=1}^m \alpha_i (\chi_{\kappa} * \Delta^m)(s_i, x_i) + \sum_{j=1}^n \beta_j (\chi_{\kappa} * \Delta^m)(t_j, y_j) \right),
 \end{aligned}$$

and the phase $\Phi(\omega_1, \omega_2)$ is equal to :

$$\Phi((m, s_i, x_i, \alpha_i), (n, t_j, y_j, \beta_j)) = \frac{\pi}{2} (m - n) + \sum_{i=1}^m \phi(\alpha_i) + \sum_{j=1}^n \phi(\beta_j),$$

$h_{i, -t}$ are defined as in (3.9). Moreover Δ_r^m , Δ_a^m , and Δ^m are the invariant distributions given by (3.13) and respectively by

$$\begin{aligned}
 \Delta_a^m(t, x) &\equiv \frac{1 - \theta(t)}{(2\pi)^s} \int_{\mathbb{R}^s} dp e^{i p x} \sin(\omega(p)t) / \omega(p), \\
 \Delta^m(t, x) &\equiv \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} dp e^{i p x} \sin(\omega(p)t) / \omega(p).
 \end{aligned}$$

Remark. In the case of nonrelativistic one dimensional quantum mechanics for a system with Hamiltonian H given by

$$H = - \frac{d^2}{dq^2} + q^2 + V(q)$$

with $V(q) = \int_{\mathbb{R}} \exp(-i\alpha q) dv(\alpha)$, the group G is \mathbb{R}^2 and the corresponding Schrödinger equation can be written from Theorem 2.1 as

$$i \frac{\partial}{\partial t} \alpha_i F(q, p) = H_0 \alpha_i F(q, p) + 2i \int_{\mathbb{R}} dv(\alpha) \sin(-\alpha q / 2) \alpha_i F(q, p - \alpha)$$

(cf. [33]).

4. Removal of the Cut-Offs

As we shall see the removal of the cut-off κ in the formulas of Theorem 3.1 is only possible on some but not all functionals F on $\mathcal{S}'_R(\mathbb{R}^s) \times \mathcal{S}'_R(\mathbb{R}^s)$. For instance if F is the well known vacuum functional associated with the Fock representation, then the argument of F in the formulas of Theorem 3.1 become singular as $\kappa \rightarrow \infty$ for all $s \geq 1$. For $s=1, m=0$, one can come over this difficulty by the usual renormalization procedure [16].

In this section, we shall prove that both limits $\kappa \rightarrow \infty$ and $A \uparrow \mathbb{R}^s$ exist for a large subset of functionals F . We consider first the functionals E_{k_1, k_2} defined as follows, for any pairs $k_1, k_2 \in \mathcal{S}'_R(\mathbb{R}^s)$ and $h_1, h_2 \in \mathcal{S}'_R(\mathbb{R}^s)$:

$$E_{k_1, k_2}(h_1, h_2) \equiv \exp \left\{ \frac{i}{2} \int_{\mathbb{R}^s} dx \{ h_1(x)k_2(x) - h_2(x)k_1(x) \} \right\}, \tag{4.1}$$

where the integral on the right hand side is interpreted in the sense of distributions. Obviously one has:

$$E_{k_1, k_2}(0, 0) = E_{0, 0}(h_1, h_2) = 1, \text{ for all } k_1, k_2 \in \mathcal{S}'_R(\mathbb{R}^s), \text{ and all } h_1, h_2 \in \mathcal{S}'_R(\mathbb{R}^s).$$

Remark. These functionals are not states; however in some cases they can be used to express states. Let us first look at the case of nonrelativistic quantum mechanics with one degree of freedom. Here we have for instance the formula:

$$(\Omega, \exp \{ i(pQ - qP) \} \Omega) = \frac{1}{4\pi} \int_{\mathbb{R}^2} dx \exp \left\{ -\frac{1}{2} \|x\|^2 + \frac{i}{2} (px_1 - qx_2) \right\},$$

where Ω is the ground state of the harmonic oscillator. Since $\exp \left(\frac{i}{2} (px_1 - qx_2) \right)$ plays the role in this case of the functionals E_{k_1, k_2} , this formula shows that we can describe the fundamental state of quantum mechanics by a superposition of functionals of the form E_{k_1, k_2} .

This integral representation extends also to the Fock vacuum state of relativistic free fields (see e.g. [41]), hence in the free field case one recovers the vacuum from the functionals E_{k_1, k_2} .

We shall call \mathcal{L} the linear subspace of $B(\mathcal{S}'_R(\mathbb{R}^s) \times \mathcal{S}'_R(\mathbb{R}^s))$ consisting of all finite linear combinations of E_{k_1, k_2} , i.e.

$$\mathcal{L} \equiv \left\{ \sum_{i=1}^N \alpha_i E_{k_1^i, k_2^i}, \alpha_i \in \mathbb{C}, k_1^i, k_2^i \in \mathcal{S}'_R(\mathbb{R}^s), i = 1, 2, \dots, N \right\}. \tag{4.2}$$

We remark that \mathcal{L} is invariant under the action of the Poincaré group associated with the free fields. Below we shall consider a subspace \mathcal{L}_0 of \mathcal{L} defined as \mathcal{L} with k_1, k_2 taken to be such that, in the sense of generalized functions, $\langle k_j^i, T \rangle$ are well defined, where T is either A^m or $\partial_0 A^m$.

It is also interesting to introduce another subspace of bounded functionals on $B(G)$, namely the linear subspace \mathfrak{M} consisting of finitely based functionals which are linear combinations of functionals $F_{H_{2n}}^f$ described as follows: Let $\{ \phi_n \}_{n \in \mathbb{N}}$ be a family of functions in $\mathcal{S}'_R(\mathbb{R}^s)$ which is separating in $L^2(\mathbb{R}^s, dx)$. Let H_{2n} be the subspace of $L^2_R(\mathbb{R}^s, dx)$ generated by $\{ \phi_i \}_{i=1, 2, \dots, 2n}$. Let f be an arbitrary function

on \mathbb{R}^{2n} , which is the Fourier transform of a bounded complex measure on \mathbb{R}^{2n} . The functional $F_{H_{2n}}^f$ evaluated on $(h_1, h_2) \in \mathcal{L}_R \times \mathcal{L}_R$ is by definition:

$$F_{H_{2n}}^f(h_1, h_2) \equiv \begin{cases} f((h_1, \phi_1), (h_2, \phi_2), \dots, (h_1, \phi_{2n-1}), (h_2, \phi_{2n})), & \text{if } h_1, h_2 \in H_{2n}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

By definition \mathfrak{M} is the complex linear space of the $F_{H_{2n}}^f$ for all n and all possible f . We remark that $F_{H_{2n}}^f$ can be written in the form:

$$F_{H_{2n}}^f(h_1, h_2) = \int d\mu(k_1, k_2) \exp \left\{ \frac{i}{2} [(h_1, k_2) - (h_2, k_1)] \right\}, \tag{4.4}$$

with

$$d\mu(k_1, k_2) = \int \delta_{-2 \sum_{j=1}^n p_{2j}\phi_{2j}, 2 \sum_{j=0}^{n-1} p_{2j+1}\phi_{2j+1}}(k_1, k_2) d\varrho_f(p), \tag{4.5}$$

where ϱ_f is the measure on \mathbb{R}^{2n} , whose Fourier transform is f and $p = (p_1, p_2, \dots, p_{2n})$. Note that $d\mu(k_1, k_2)$ is the measure on $\mathcal{L}_R(\mathbb{R}^s) \times \mathcal{L}_R(\mathbb{R}^s)$ defined by:

$$\int_{\mathcal{L}_R \times \mathcal{L}_R} g(k_1, k_2) d\mu(k_1, k_2) = \int_{\mathbb{R}^{2n}} g \left(2 \sum_{j=1}^n p_{2j}\phi_{2j}, 2 \sum_{j=0}^{n-1} p_{2j+1}\phi_{2j+1} \right) d\varrho_f(p), \tag{4.6}$$

for any bounded continuous g and one has

$$\|\mu\| \leq \varrho_f, \text{ with } \|\mu\| \text{ the total variation of } \mu. \tag{4.7}$$

We remark that the subspace \mathfrak{M} contains a subset of states (cf. [11]).

We shall now show that the ultraviolet cut-off κ in $\alpha_t^{A, \kappa}$ can be removed on $\mathcal{L}_0 \cup \mathfrak{M}$. We have:

Theorem 4.1. *The limit:*

$$\lim_{\kappa \rightarrow \infty} \alpha_t^{A, \kappa} = \alpha_t^A \tag{4.8}$$

exists on $\mathcal{L}_0 \cup \mathfrak{M}$ and defines an application of $\mathcal{L}_0 \cup \mathfrak{M}$ into $B_{sm}(\mathcal{L}_R(\mathbb{R}^s) \times \mathcal{L}_R(\mathbb{R}^s))$. One has for all $h_1, h_2 \in \mathcal{L}_R(\mathbb{R}^s)$:

$$\begin{aligned} \{\alpha_t^A E_{k_1, k_2}\}(h_1, h_2) &= \int_{\Omega \times \Omega} P_{|\lambda|v}^{A, t}(d\omega_1) P_{|\lambda|v}^{A, t}(d\omega_2) \\ &\cdot \exp \{i\Phi(\omega_1, \omega_2)\} \varepsilon_{k_1, k_2, t}^A(\omega_1, \omega_2), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} &\varepsilon_{k_1, k_2}^A((m, s_i, x_i, \alpha_i), (n, t_j, y_j, \beta_j)) \\ &= \exp \left\{ \frac{i}{2} \sum_{i \neq i'} \alpha_i \alpha_{i'} A_a^m(s_i - s_{i'}, x_i - x_{i'}) + \frac{i}{2} \sum_{j \neq j'} \beta_j \beta_{j'} A_r^m(t_j - t_{j'}, y_j - y_{j'}) \right\} \\ &\cdot \exp \left\{ \frac{i}{2} \sum_{i, j} \alpha_i \beta_j A^m(s_i - t_j, x_i - y_j) \right\} \end{aligned}$$

$$\begin{aligned}
& \cdot \exp \left\{ \frac{i}{2} \sum_{i=1}^m \alpha_i ((\partial_0 \Delta^m * h_1)(s_i - t, x_i) - (\Delta^m * h_2)(s_i - t, x_i)) \right\} \\
& \cdot \exp \left\{ -\frac{i}{2} \sum_{j=1}^n \beta_j ((\partial_0 \Delta^m * h_1)(t_j - t, y_j) - (\Delta^m * h_2)(t_j - t, y_j)) \right\} \\
& \cdot E_{k_1, k_2} \left(h_{1, -t} + \sum_{i=1}^m \alpha_i \partial_0 \Delta^m(s_i, x_i) + \sum_{j=1}^n \beta_j \partial_0 \Delta^m(t_j, y_j), \right. \\
& \left. h_{2, -t} + \sum_{i=1}^m \alpha_i \Delta^m(s_i, x_i) + \sum_{j=1}^n \beta_j \Delta^m(t_j, y_j) \right), \tag{4.10}
\end{aligned}$$

where Δ_a^m , Δ_r^m , and Δ^m are the invariant distributions previously defined and ∂_0 means the time derivative. α_t^A is a group of transformations on $\mathcal{L}_0 \cup \mathfrak{M}$, i.e.

$$\alpha_s^A \alpha_t^A = \alpha_{s+t}^A. \tag{4.11}$$

Moreover for any $F \in \mathcal{L}_0 \cup \mathfrak{M}$ and $h_i \in \mathcal{D}_{\mathbb{R}}(\mathbb{R}^s)$, α_t^A is a solution of the Schrödinger equation:

$$\begin{aligned}
i \frac{\partial}{\partial t} \{ \alpha_t^A F \} (h_1, h_2) &= i \int_{\mathbb{R}^s} dx \left\{ h_2(x) \frac{\delta}{\delta h_1(x)} - h_1(x) (-\Delta + m^2) \frac{\delta}{\delta h_2(x)} \right\} \alpha_t^A F(h_1, h_2) \\
&\quad - 2i\lambda \int_A dx \int_{\mathbb{R}} dv(\alpha) \sin \left(\frac{\alpha}{2} h_1(x) \right) \\
&\quad \cdot (\alpha_t^A F)(h_1, h_2 + \alpha \delta_x), \tag{4.12}
\end{aligned}$$

with initial condition:

$$\lim_{t \rightarrow 0} (\alpha_t^A F)(h_1, h_2) = F(h_1, h_2). \tag{4.13}$$

Remark. On right hand side of (4.12) the notation $(\alpha_t^A F)(h_1, h_2 + \alpha \delta_x)$ is of course only symbolic, since $\delta_x \notin \mathcal{S}'(\mathbb{R}^s)$. The interpretation of the notation follows from the fact that $E_{k_1, k_2} \in \mathcal{L}_0$.

Proof. The properties on elements in $\mathcal{L}_0 \cup \mathfrak{M}$ follow easily from those on elements of the form E_{k_1, k_2} . We shall now discuss the proof for these elements. From the support properties of Δ_a^m , Δ_r^m , and Δ^m , we have $(\chi_\kappa * \Delta' * \chi_\kappa)(0, x) = 0$, where Δ' stands for Δ_a^m , Δ_r^m , and Δ^m .

Hence:

$$\sum_{i,j} \alpha_i \alpha_j (\chi_\kappa * \Delta_a^m * \chi_\kappa)(s_i - s_j, x_i - x_j) = \sum_{i \neq j} \alpha_i \alpha_j (\chi_\kappa * \Delta_a^m * \chi_\kappa)(s_i - s_j, x_i - x_j),$$

and the same for α_i replaced by β_j and Δ_a^m replaced by Δ_r^m . Moreover $\chi_\kappa * \Delta' * \chi_\kappa(s, x)$ converges pointwise to $\Delta'(s, x)$ outside of the cone $|x|=s$. Hence by the Lebesgue dominated convergence theorem we have that the expression for $\{ \alpha_i^{A\kappa} E_{k_1, k_2} \} (h_1, h_2)$ of Theorem 3.1 converges as $\kappa \rightarrow \infty$ to the expression given by the present theorem, which we can express as a Poisson integral with respect to the measure $P_{\lambda|v|}^{A, t} \otimes P_{\lambda|v|}^{A, t}$, using Theorem 2.1. The group property is proven using the group property of $\alpha_t^{A\kappa}$ and the fact that $\alpha_t^A F$ is defined as the $\lim_{\kappa \rightarrow \infty} \alpha_t^{A, \kappa} F$ and $\alpha_{s+t}^A F$

by $\lim_{\kappa \rightarrow \infty} \alpha_s^{A,\kappa} \alpha_t^{A,\kappa} F$, which exist as seen from the fact that $\alpha_t^{A,\kappa}$ is defined on all $B_{am}(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s))$ and

$$\alpha_s^A F \in B_{am}(\mathcal{S}_R(\mathbb{R}^s) \times \mathcal{S}_R(\mathbb{R}^s)),$$

and using the explicit expression for $\alpha_t^{A,\kappa} G$, with $G = \alpha_s^A F$ (and once more the support properties of Δ' and the Lebesgue dominated convergence theorem).

The Schrödinger equation is obtained using the fact that $(\alpha_t^{A,\kappa} F)(h_1, h_2)$ as well as $\frac{\partial}{\partial t}(\alpha_t^{A,\kappa} F)(h_1, h_2)$ are uniformly bounded in κ , together with the convergence of both $(\alpha_t^{A,\kappa})(h_1, h_2)$ and $\frac{\partial}{\partial t} \alpha_t^{A,\kappa} F(h_1, h_2)$ as $\kappa \rightarrow \infty$, the compactness of the support of h_i 's and their smoothness which allows us to interchange the time derivative and the limit $\kappa \rightarrow \infty$. The explicit form of the right hand side of (4.12) is obtained using Theorem 3.1, (3.23) and the fact that $E_{k_1, k_2} \in \mathcal{L}_0$. \square

The next step is to remove the space cut-off Λ . First we consider the case where the initial conditions are smooth and of compact support. In this case we can prove the following theorem.

Theorem 4.2. *Let \mathcal{E} be the restriction of the function in $\mathcal{L}_0 \cup \mathfrak{M}$ to $\mathcal{D}_R(\mathbb{R}^s) \times \mathcal{D}_R(\mathbb{R}^s)$. Then $\lim_{\Lambda \uparrow \mathbb{R}^s} \alpha_t^A F = \alpha_t F$ exists for all $F \in \mathcal{E}$ and defines a relativistic flow on \mathcal{E} in the sense that*

$$\alpha_s \alpha_t = \alpha_{s+t}, \quad \text{for all } s, t > 0.$$

Proof. The theorem follows immediately from the fact that the Schrödinger equation (Theorem 4.1) implies $\alpha_t^A F(h_1, h_2) = \alpha_t^{A'} F(h_1, h_2)$ for all Λ and Λ' containing the support of h_1 . \square

Remark. It follows from the proof of the theorem that α_t satisfies the Schrödinger equation:

$$\begin{aligned} i \frac{\partial}{\partial t} \alpha_t F_{k_1, k_2}(h_1, h_2) &= i \int_{\mathbb{R}^s} dx \left\{ h_2(x) \frac{\delta}{\delta h_1(x)} - h_1(x) (-\Delta + m^2) \frac{\delta}{\delta h_2(x)} \right\} \\ &\cdot \alpha_t F_{k_1, k_2}(h_1, h_2) - 2i\lambda \int_{\mathbb{R}^s} dx \int_{\mathbb{R}} dv(\alpha) \\ &\cdot \sin\left(\frac{\alpha}{2} h_1(x)\right) \alpha_t F_{k_1, k_2}(h_1, h_2 + \alpha \delta_x), \end{aligned} \tag{4.14}$$

with initial condition $F_{k_1, k_2}(h_1, h_2)$, for any $h_1, h_2 \in \mathcal{D}_R(\mathbb{R}^s)$.

The right hand side of (4.14) is equal to $P_0^* \alpha_t F_{k_1, k_2}(h_1, h_2)$, where P_0^* is the infinitesimal generator of time translations on \mathcal{E} , given by the trigonometric interaction. The first term in (4.14) corresponds to the free evolution. The other generators P_i^* and M^{*ij} ($(i, j) = 1, 2, 3$) of the Poincaré group associated with the same interaction are the same as in the free case [7] whereas M^{*0i} is given by:

$$\begin{aligned} (M^{*0i} F)(h_1, h_2) &= -i \int_{\mathbb{R}^s} dx \left\{ h_2(x) x^i \frac{\delta}{\delta h_1(x)} - h_1(x) \left((-\Delta + m^2) x^i - \frac{\partial}{\partial x^i} \right) \frac{\delta}{\delta h_2(x)} \right\} F(h_1, h_2) \\ &- 2i\lambda \int_{\mathbb{R}^s} dx \int_{\mathbb{R}} dv(\alpha) x^i \sin\left(\frac{\alpha}{2} h_1(x)\right) F(h_1, h_2 + \alpha \delta_x). \end{aligned} \tag{4.15}$$

We shall now extend the existence of the infinite volume limit $A \uparrow \mathbb{R}^s$ to initial conditions in the larger class $\mathcal{L}_0 \cup \mathfrak{M}$, under the assumption of small coupling constant.

Theorem 4.3. *The limit*

$$\lim_{A \uparrow \mathbb{R}^s} \alpha_t^A = \alpha_t^\infty$$

exists on $\mathcal{L}_0 \cup \mathfrak{M}$ for $|\lambda| < e^{-2} c(t)^{-2}$, where $c(t)$ is a positive function of t (which tends to infinity for $t \rightarrow \infty$).

Proof. The proof is based on the following observations

i) Formula (4.9) can be looked upon as the grand partition function of the grand canonical ensemble of a gas of two types of particles α and β with the same charge distribution $dv(\alpha)$ and interacting in $s + 1$ dimensions through a two-body potential $i\Delta_r^m$ (interaction of two α -particles), $i\Delta_a^m$ (interaction of two β -particles) and $i\Delta^m$ (interaction of an α -particle and a β -particle) respectively, and with a purely imaginary magnetic field.

ii) The group property for all finite A

$$\alpha_s^A \alpha_t^A = \alpha_{s+t}^A, \quad \text{for all } s, t > 0$$

on the E_{k_1, k_2} holds.

iii) For all $k_1, k_2 \in \mathcal{L}_R(\mathbb{R}^s)$

$$\{\alpha_t^A E_{k_1, k_2}\}(0, 0) = 1, \quad \text{for all } t > 0. \tag{4.16}$$

This follows from the Schrödinger equation.

In addition the proof uses an application of the method of Kirkwood-Salzburg equations of statistical mechanics, see e.g. [36], along the lines used for ultraviolet cut-off interactions in [1]. Here we shall indicate how to define the relevant objects for the proof.

Again we remark that the convergence on arbitrary elements in $\mathcal{L}_0 \cup \mathfrak{M}$ follows easily from the one on elements of the form E_{k_1, k_2} and we discuss the proof for these elements.

Let R_ξ^v be the Banach vector space of double sequences of functions:

$$\Psi^{m, n}(s_1, x_1, \alpha_1, \dots, s_m, x_m, \alpha_m, t_1, y_1, \beta_1 \dots t_n, y_n, \beta_n), \tag{4.17}$$

$m, n \geq 1, s_i, t_i \geq 0, x_i, y_i \in \mathbb{R}^s, \alpha_i, \beta_i \in \mathbb{R}$, which are bounded measurable with respect to the measure:

$$d\bar{X} d\bar{Y} dv(A) dv(B) = \prod_{i=1}^m ds_i dx_i dv(\alpha_i) \prod_{j=1}^n dt_j dy_j dv(\beta_j), \tag{4.18}$$

and such that there exists a $\xi > 0$ such that:

$$\|\bar{\Psi}\|_\xi = \text{Sup}_{m, n} \xi^{-(m+n)} \text{ess. Sup}_{\substack{\bar{x}_i, \bar{y}_j \\ \alpha_i, \beta_j}} |\Psi^{m, n}(\bar{x}_i, \alpha_i, \bar{y}_j, \beta_j)|, \tag{4.19}$$

is finite, where $\bar{x} = (s, x)$ and $\bar{y} = (t, y)$.

We define two projections in R_{ξ}^v , viz. Π_A and Π_t on $\Psi \in R_{\xi}^v$ by:

$$\begin{aligned}
 (\Pi_A \Psi)^{m,n}(\bar{X}, A, \bar{Y}, B) &= 0, && \text{if at least one } x_i \notin A, \\
 & && \text{or at least one } y_j \notin A \\
 &= \Psi^{m,n}(\bar{X}, A, \bar{Y}, B), && \text{otherwise.}
 \end{aligned} \tag{4.20}$$

$$\begin{aligned}
 (\Pi_t \Psi)^{m,n}(\bar{X}, A, \bar{Y}, B) &= 0, && \text{if at least one } s_i \geq t, \\
 & && \text{or at least one } t_i \geq t \\
 &= \Psi^{m,n}(\bar{X}, A, \bar{Y}, B), && \text{otherwise,}
 \end{aligned} \tag{4.21}$$

where for shortness \bar{X} (respectively, \bar{Y}) stands for $\{\bar{x}_i\}_{i=1, \dots, m}$ (respectively, $\{\bar{y}_i\}_{i=1, \dots, n}$), A (respectively, B) for $\{\alpha_i\}_{i=1, \dots, m}$ (respectively, $\{\beta_j\}_{j=1, \dots, n}$).

Using these definitions we can rewrite the expression for $(\alpha_i^A E_{k_1, k_2})(h_1, h_2)$ given by formulas (4.9) and (4.10) as:

$$\begin{aligned}
 \{\alpha_i^A E_{k_1 k_2}\}(h_1, h_2) &= E_{k_1, k_2}(\alpha_{-t}(h_1, h_2)) \\
 &\cdot \left(1 + \sum_{\substack{m, n \in \mathbb{N}^2 \\ (m, n) \neq (0, 0)}} \frac{1}{m!n!} \int d\bar{X} \int d\bar{Y} \int dv(A) \int dv(B) R_{A,t}^{m,n}(\bar{X}, A, \bar{Y}, B) \right. \\
 &\cdot \left. \prod_{i \in \{1, \dots, m\}} (D_1(\bar{x}_i, \alpha_i) - 1) \prod_{j \in \{1, \dots, n\}} (D_2(\bar{y}_j, \beta_j) - 1) \right),
 \end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
 R_{A,t}^{m,n}(\bar{X}, A, \bar{Y}, B) &= \sum_{r,s=0}^{\infty} \frac{\lambda^{m+n+r+s}}{r!s!} i^{m+r-n-s} \\
 &\cdot \int d\bar{X}' \int dY' \int dv(A') \int dv(B') G_{A,t}(\bar{X} \cup \bar{X}', A \cup A', \bar{Y} \cup \bar{Y}', B \cup B'),
 \end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
 G_{A,t}(\bar{X}, A, \bar{Y}, B) &= \exp\left(\frac{i}{2} \sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j \Delta^m(\bar{x}_i - \bar{y}_j)\right) \\
 &\cdot \exp\left\{\frac{i}{2} \sum_{i,j} \alpha_i \alpha_j \Delta_a^m(\bar{x}_i - \bar{x}_j) + \frac{i}{2} \sum_{k,\ell} \beta_k \beta_\ell \Delta_r^m(\bar{y}_k - \bar{y}_\ell)\right\}
 \end{aligned} \tag{4.24}$$

if all $x_i, y_j \in A$ and all $s_i, t_j \in [0, t]$ and zero otherwise. Furthermore

$$\begin{aligned}
 D_1(\bar{x}, \alpha) &= \exp\left\{\frac{i}{2} \alpha((\partial_0 \Delta^m * h_1)(s-t, x) - (\Delta^m * h_2)(s-t, x))\right\} \\
 &\cdot \exp\left\{\frac{i}{2} \alpha((\partial_0 \Delta^m * k_2)(s, x) - (\Delta^m * k_1)(s_1, x))\right\},
 \end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
 D_2(\bar{x}, \alpha) &= \exp\left\{-\frac{i}{2} \alpha(\partial_0 \Delta^m * h_1)(s-t, x) - (\Delta^m * h_2)(s-t, x)\right\} \\
 &\cdot \exp\left\{\frac{i}{2} \alpha((\partial_0 \Delta^m * k_2)(s, x) - (\Delta^m * k_1)(s, x))\right\}.
 \end{aligned} \tag{4.26}$$

As an statistical mechanics we can define an operator K on R_ξ^v by:

$$\begin{aligned} \{K\Psi\}^{n',n}(\bar{X}, A, \bar{Y}, B) = & \exp\left\{\frac{i}{2} \sum_{j=2}^{n'} \alpha_1 \alpha_j \Delta_a^m(\bar{x}_1 - \bar{x}_j) + \frac{i}{2} \sum_{k=2}^n \beta_1 \beta_k \Delta_r^m(\bar{y}_1 - \bar{y}_j)\right\} \\ & \cdot \exp\left\{\frac{i}{4} \sum_{k=1}^n \alpha_1 \beta_k \Delta^m(\bar{x}_1 - \bar{y}_k) + \frac{i}{4} \sum_{j=1}^{n'} \alpha_j \beta_1 \Delta^m(\bar{x}_j - \bar{y}_1)\right\} \\ & \cdot \Psi^{n'-1, n-1}(\bar{X}, A, \bar{Y}, B \setminus \{\bar{x}_1, \alpha_1, \bar{y}_1, \beta_1\}) (1 - \delta_{n',1}) (1 - \delta_{n,1}) \\ & + \sum_{r,s=1}^{\infty} \frac{1}{r!s!} \int d\bar{X}' \int d\bar{Y}' \int dv(A') \int dv(B') \\ & \cdot \prod_{i=1}^r \left(\exp\left\{\frac{i}{2} \alpha_1' \alpha_i' \Delta_a^m(\bar{x}_1 - \bar{x}_i') - \frac{i}{4} \beta_1 \alpha_i' \Delta^m(\bar{y}_1 - \bar{x}_i')\right\} - 1 \right) \\ & \cdot \prod_{j=1}^s \left(\exp\left\{\frac{i}{2} \beta_1' \beta_j' \Delta_r^m(\bar{y}_1 - \bar{y}_j') + \frac{i}{4} \alpha_1 \beta_j' \Delta^m(\bar{x}_1 - \bar{y}_j')\right\} - 1 \right) \\ & \cdot \Psi^{m+r-1, n+s-1}(\bar{X} \cup \bar{X}', A \cup A', \bar{Y} \cup \bar{Y}', B \cup B' \setminus \{\bar{x}_1, \alpha_1, \bar{y}_1, \beta_1\}), \end{aligned} \tag{4.27}$$

where as previously \bar{X}' (respectively, \bar{Y}') stands for $\{\bar{x}_i'\}_{i=1, \dots, r}$ (respectively, $\{\bar{y}_i'\}_{i=1, \dots, s}$) and A' (respectively, B') for $\{\alpha_i'\}_{i=1, \dots, r}$ (respectively, $\{\beta_i'\}_{i=1, \dots, s}$), and $\delta_{n,k} = 1$ if $n=k$, $\delta_{n,k} = 0$ if $n \neq k$.

The norm of the operator $\Pi_t K \Pi_t$ can be readily estimated:

$$\|\Pi_t K \Pi_t\|_\xi \leq \xi^{-2} \exp\{2\xi C(t)\}, \tag{4.28}$$

where

$$C(t) = \text{Sup}_{\alpha, \beta} \text{Sup}_{\bar{y} \in [0, 2t] \times_{\mathbb{R}^s}} \int_0^t \int_{\mathbb{R}} ds \int d|v|(\gamma) \left| \exp\left(\frac{i\alpha\gamma}{2} \Delta_a^m(\bar{x}) + \frac{i\beta\gamma}{4} \Delta^m(\bar{x} - \bar{y})\right) - 1 \right|. \tag{4.29}$$

Obviously one has

$$C(t) \leq 4|v|(\mathbb{R})t|V_s(t)|. \tag{4.29}$$

This expression is finite since for a given \bar{y} it is the integral of a function bounded by $2|v|(\mathbb{R})$ over a domain which is contained in $V_s(t)$

$$V_s(t) = \{x; |x| < t\} \cup \{x; |x - y| < t\}. \tag{4.30}$$

This follows from the support properties of Δ^m , Δ_a^m , and Δ_r^m .

The best choice of ξ in view of Eq. (4.28) is

$$\xi = C(t)^{-1}, \tag{4.31}$$

corresponding to the estimate for the norm of $\Pi_t K \Pi_t$:

$$\|\Pi_t K \Pi_t\| \leq C(t)^2 e^2. \tag{4.32}$$

Furthermore the operator K allows us to write an integral equation for $R_{A,t}$:

$$R_{A,t} = \lambda^2 \Pi_A \Pi_t S + \lambda^2 \Pi_A \Pi_t K R_{A,t}, \tag{4.33}$$

where

$$S_{ij} = 0, \quad (i, j) \neq (1, 1), \quad S_{11} = 1. \tag{4.34}$$

Equation (4.33) can be solved by the Neumann series provided that:

$$\lambda^2 < C(t)^{-2} e^{-2}. \tag{4.35}$$

With these elements the rest of the proof follows the line of [1]. \square

We shall now give a result which shows that, at least for Λ bounded, an asymptotics in time to a free evolution holds.

Theorem 4.4. *If $m > 0$, for any bounded $\Lambda \subset \mathbb{R}^s$, for $\lambda < \lambda_0$, which depends on Λ , the limit*

$$\lim_{t \rightarrow \pm\infty} \alpha_t^\Lambda = \alpha_\pm^\Lambda$$

exists on $\mathcal{L}_0 \cup \mathfrak{M}$. Moreover with $F^{+, \Lambda}(h_1, h_2) \equiv \lim_{t \rightarrow +\infty} (\Omega_t^\# \Omega_t^b)(h_1, h_2)$ and $F \in \mathcal{L}_0 \cup \mathfrak{M}$, $h_1, h_2 \in \mathcal{S}_R(\mathbb{R}^s)$, we have: $\alpha_\pm^\Lambda F^{+, \Lambda} = U_t^0 F^{+, \Lambda}$, where U_t^0 is the free evolution defined in (3.15). The limit $\alpha_\pm^\Lambda F^{+, \Lambda}$ is invariant under α_t^Λ .

Proof. In a quite similar way as in the proof of Theorem 4.3 one can estimate the norm of $\Pi_\Lambda K \Pi_\Lambda$:

$$\|\Pi_\Lambda K \Pi_\Lambda\| \leq c'(\Lambda)^2 e^2, \tag{4.36}$$

where

$$c'(\Lambda) \equiv \sup_{\alpha, \beta} \sup_{y \in \Lambda \times \mathbb{R}_+} \int_{\Lambda \times \mathbb{R}_+} d\bar{x}' \int_{\mathbb{R}} d|v|(\alpha') \left| \exp\left(\frac{i}{2} \alpha \alpha' \Delta_a^m(\bar{x}') - \frac{i}{4} \alpha' \beta \Delta^m(\bar{x}' - \bar{y})\right) - 1 \right| \tag{4.37}$$

is finite, due to the exponential decrease of Δ_a^m and Δ^m for large time. Consequently for $\lambda < \lambda_0 = c'(\Lambda)^{-1} e^{-1}$ the Kirkwood-Salzburg equation has a solution. The rest follows immediately. \square

Acknowledgements. We are very grateful to Prof. H. Araki for the numerous detailed valid suggestions he made to improve the presentation of this paper. We also thank the Referees who went carefully through the manuscript and pointed out a number of misprints and suggested improvements.

One of us (M.S.) wants to express his deep thanks to the members of the Physics Department of the University of Bielefeld for their warm hospitality during the Wintersemester 1980–1981 when most of this work was done. We also gratefully acknowledge support received from the Universities of Aix-Marseille II, Bielefeld, Bochum, Oslo and the Centre de Physique Théorique, CNRS, Marseille, during visits which made possible the completion of this work.

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Communicated by H. Araki

Received February 15, 1982; in revised form March 9, 1983

